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Positive Solutions for a Class of Nonlinear p-Laplacian Hadamard Fractional Differential Systems with Coupled Nonlocal Riemann-Stieltjes Integral Boundary Conditions

Wengui Yang[a,b]

[a] *Ministry of Public Education, Sanmenxia Polytechnic, Sanmenxia 472000, China*
[b] *College of Applied Engineering, Henan University of Science and Technology, Sanmenxia 472000, China*

Abstract. This paper investigates a class of nonlinear p-Laplacian Hadamard fractional differential systems with coupled nonlocal Riemann-Stieltjes integral boundary conditions. First, we obtain the corresponding Green's function for the considered boundary value problems and some of its properties. Then, by using the Guo-Krasnosel'skii fixed point theorem, some sufficient conditions for existence and nonexistence of positive solutions for the addressed systems are obtained under the different intervals of the parameters μ and ν. As applications, some examples are presented to show the effectiveness of the main results.

1. Introduction

In this paper, we consider the system of nonlinear Hadamard fractional differential equations with p_1-Laplacian and p_2-Laplacian operators

$$\mathscr{D}^{\alpha_1}\varphi_{p_1}(\mathscr{D}^{\beta_1}x(t)) = \mu f_1(t, x(t), y(t)), \quad \mathscr{D}^{\alpha_2}\varphi_{p_2}(\mathscr{D}^{\beta_2}y(t)) = \nu f_2(t, x(t), y(t)), \quad t \in (1, e) \tag{1.1}$$

subject to the following coupled nonlocal Riemann-Stieltjes integral boundary conditions

$$\begin{aligned}
&\delta x(1) = \delta^2 x(1) = \cdots = \delta^{n_1-2}x(1) = 0, \ \ \mathscr{D}^{\gamma_0}x(e) = \sum_{i=1}^{q_2}\int_1^{\theta_i} h_i(t)\mathscr{D}^{\eta_i}y(t)\frac{dH_i(t)}{t},\\
&\delta y(1) = \delta^2 y(1) = \cdots = \delta^{n_2-2}y(1) = 0, \ \ \mathscr{D}^{\eta_0}y(e) = \sum_{j=1}^{q_1}\int_1^{\vartheta_j} k_j(t)\mathscr{D}^{\gamma_j}x(t)\frac{dK_j(t)}{t},\\
&\mathscr{D}^{\beta_1}x(1) = \mathscr{D}^{\beta_1}x(e) = \delta(\varphi_{p_1}(\mathscr{D}^{\beta_1}x(1))) = 0, \ \ \mathscr{D}^{\beta_2}y(1) = \mathscr{D}^{\beta_2}y(e) = \delta(\varphi_{p_2}(\mathscr{D}^{\beta_2}y(1))) = 0,
\end{aligned} \tag{1.2}$$

where $\delta^l = (td/dt)^l$ for $l \in \mathbb{N}^+$, $\mathscr{D}^k$ denotes the Hadamard fractional derivative of the order k for $k = \alpha_m, \beta_m, \gamma_0, \gamma_1, \ldots, \gamma_{q_1}, \eta_0, \eta_1, \ldots, \eta_{q_2}$, $\alpha_m \in (2, 3]$, $\beta_m \in (n_m - 1, n_m]$, $n_m, q_m \in \mathbb{N}$, $n_m \geqslant 3$, $0 \leqslant \gamma_1 < \gamma_2 < \cdots < \gamma_{q_1} \leqslant \gamma_0 <$

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Email address: wgyang0617@yahoo.com (Wengui Yang)

β_1-1, $\delta_0 \geqslant 1$, $0 \leqslant \eta_1 < \eta_2 < \cdots < \eta_{q_2} \leqslant \eta_0 < \beta_2-1$, $\gamma_0 \geqslant 1$, $p_m > 1$, $\varphi_{p_m}(s) = |s|^{p_m-2}s$, $\varphi_{p_m}^{-1} = \varphi_{r_m}$, $1/p_m + 1/r_m = 1$, $m = 1, 2$, $\mu, \nu > 0$, the nonlinear functions $f_m \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ ($\mathbb{R}^+ = [0, +\infty)$, $m = 1, 2$). $h_i, k_j \in C(1, e)$ are nonnegative and $\theta_i, \vartheta_j \in (1, e)$, the integrals $\int_1^{\theta_i} h_i(t)\mathscr{D}^{\eta_i} y(t)dH_i(t)/t$ and $\int_1^{\vartheta_j} k_j(t)\mathscr{D}^{\gamma_j} x(t)dK_j(t)/t$ from (1.2) are Riemann-Stieltjes integrals with H_i and K_j functions of bounded variation, where $H_i, K_j : [1, e] \to \mathbb{R}$ are nondecreasing functions for $i = 1, \ldots, q_2$ and $j = 1, \ldots, q_1$, there exist $i_0 \in \{1, 2, \ldots, q_2\}$ and $j_0 \in \{1, 2, \ldots, q_1\}$ such that $H_{i_0}(e) \geqslant H_{i_0}(1)$ and $K_{j_0}(e) \geqslant K_{j_0}(1)$.

Over the last couple of decades, more and more researchers show solicitude for the development of fractional calculus since it has shown its importance in many fields, not only pure and applied mathematics but also physics, control theory, chemistry, economics, etc, we can refer the reader to see [1, 5, 13, 17, 21, 29, 32, 33, 38, 41, 45, 64]. There exist many papers dealing with the fractional differential equation boundary value problems for different kinds of multi-point/integral boundary conditions. Based on fixed point theorem of the mixed-monotone operator, the upper-lower solution methods, the priori estimate method with a maximal principle, the Banach contraction mapping principle and the Guo-Krasnose'skii fixed point theorem, the authors ([15, 16, 60, 62, 63], [2, 3, 11, 58], [10, 14], [4, 6, 50, 65]) investigated extensively the existence and uniqueness theorems of solutions/positive solutions for boundary value problems of nonlinear fractional differential equations with kinds of boundary conditions, respectively. A great deal of systems of fractional differential equations involving various multi-point or Riemann-Stieltjes integral boundary conditions have been also studied in [8, 9, 20, 22, 23, 25–27, 35, 37, 46, 49, 56, 57]. For example, By using some fixed point theorems, Henderson and Luca [24], Henderson *et al.* [28] and Luca [36] obtained the existence, nonexistence and multiplicity of positive solutions for Riemann-Liouville fractional differential equations with uncoupled and coupled multipoint boundary conditions, respectively. By using the Guo-Krasnosel'skii fixed point theorem, Hao and Wang [18] and Hao *et al.* [19] gave the existence of positive solutions for the system of semipositone singular fractional differential equations with parameters and nonlocal integral/multi-point boundary conditions, respectively. Tudorache and Luca [42–44] considered the existence and nonexistence of positive solutions for a system of Riemann-Liouville fractional differential equations involving uncoupled and coupled nonlocal Riemann-Stieltjes integral boundary conditions without and with p-Laplacian operators, respectively.

In particular, Hadamard type fractional differential equations have attracted more and more people's attentions. A large number of boundary value problems for Hadamard type fractional differential equations with various boundary value conditions have been deliberate widely, the reader can see [12, 30, 34, 47, 52–54, 59, 61]. By means of Leray-Schauder alternative and Banach's contraction principle, Ahmad and Ntouyas [7] presented the existence and uniqueness of solutions for a coupled system of Hadamard type fractional differential equations involving integral boundary conditions. Yang [51] and Jiang *et al.* [31] considered the system of nonlinear Hadamard fractional differential equations with coupled integral boundary conditions, respectively. By combining the monotone iterative technique with Avery-Henderson fixed point theorem, Xu *et al.* [48] studied the existence of positive solutions for a class of Hadamard fractional-order three-point boundary value problems with p-Laplacian operator. By applying the Guo-Krasnosel'skii fixed point theorem, the author [55] investigated eigenvalue problems for a class of nonlinear Hadamard fractional differential equations with p-laplacian operator and three-point boundary conditions. Based on the standard fixed point theorems, Rao *et al.* [39, 40] researched the existence of multiple positive solutions for a system of Hadamard fractional differential equations with p-Laplacian operators, respectively.

From the previous literature review, there exist some references on Hadamard fractional differential equations with coupled nonlocal boundary conditions, however, no work has been done to study the eigenvalue problems for positive solutions for nonlinear p-Laplacian Hadamard fractional differential systems with coupled nonlocal Riemann-Stieltjes integral boundary conditions. Motivated by the mentioned papers above, the main purpose of this paper is to investigate the existence and nonexistence of positive solutions for the addressed system (1.1)-(1.2). The main results of this paper can be seen as a supplement to the existing literature. First, we present the Green's functions of the considered systems and their properties in Section 2. Second, the different intervals of the parameters μ and ν for existence and nonexistence of positive solutions for the addressed systems are obtained based on Guo-Krasnosel'skii fixed point theorem in Section 3. Then, some examples are give to show the availability of the main results in Section 4. At last, some conclusions are drawn in Section 5.

2. Preliminaries

For the convenience of the reader, we firstly present some basic concepts of Hadamard type fractional calculus to facilitate analysis of problem (1.1).

Definition 2.1. *[33]* The Hadamard fractional integral of order q for a function $g:[1,\infty)\to\mathbb{R}$ is defined as

$$I^q g(t)=\frac{1}{\Gamma(q)}\int_1^t\left(\log\frac{t}{s}\right)^{q-1}g(s)\frac{ds}{s},\quad q>0, \tag{2.1}$$

provided the integral exists, $\Gamma(q)$ denotes the Gamma function $\Gamma(q)=\int_0^{+\infty}t^{q-1}e^{-t}dt$, and $\log(\cdot)=\log_e(\cdot)$.

Definition 2.2. *[33]* The Hadamard derivative of fractional order q for a function $g:[1,\infty)\to\mathbb{R}$ is defined as

$$D^q g(t)=\frac{1}{\Gamma(n-q)}\left(t\frac{d}{dt}\right)^n\int_1^t\left(\log\frac{t}{s}\right)^{n-q-1}g(s)\frac{ds}{s},\quad n-1<q<n, \tag{2.2}$$

where $n=[q]+1$, $[q]$ denotes the integer part of the real number q.

Let $\varphi_{p_1}(\mathscr{D}^{\beta_1}x(t))=u(t)$ and $\varphi_{p_2}(\mathscr{D}^{\beta_2}y(t))=v(t)$. Then system (1.1) can be transformed into the following three problems:

$$\mathscr{D}^{\alpha_1}u(t)=\mu f_1(t,x(t),y(t)),\quad t\in(1,e),\quad u(1)=u(e)=\delta u(1)=0, \tag{2.3}$$
$$\mathscr{D}^{\alpha_2}v(t)=\nu f_2(t,x(t),y(t)),\quad t\in(1,e),\quad v(1)=v(e)=\delta v(1)=0, \tag{2.4}$$

and

$$\begin{aligned}
&\mathscr{D}^{\beta_1}x(t)=\varphi_{r_1}(u(t)),\quad \mathscr{D}^{\beta_2}y(t)=\varphi_{r_2}(v(t)),\quad t\in(1,e),\\
&\delta x(1)=\delta^2x(1)=\cdots=\delta^{n_1-2}x(1)=0,\ \ \mathscr{D}^{\gamma_0}x(e)=\sum_{i=1}^{q_2}\int_1^{\theta_i}h_i(t)\mathscr{D}^{\eta_i}y(t)\frac{dH_i(t)}{t},\\
&\delta y(1)=\delta^2y(1)=\cdots=\delta^{n_2-2}y(1)=0,\ \ \mathscr{D}^{\eta_0}y(e)=\sum_{j=1}^{q_1}\int_1^{\vartheta_j}k_j(t)\mathscr{D}^{\gamma_j}x(t)\frac{dK_j(t)}{t}.
\end{aligned} \tag{2.5}$$

It follows from the reference [30, Lemma 1] that problems (2.3) and (2.4) have the unique solutions $u\in C[1,e]$ and $v\in C[1,e]$, respectively

$$u(t)=-\mu\int_1^e G_{\alpha_1,\alpha_1}(t,s)f_1(s,x(s),y(s))\frac{ds}{s}, \tag{2.6}$$
$$v(t)=-\nu\int_1^e G_{\alpha_2,\alpha_2}(t,s)f_2(s,x(s),y(s))\frac{ds}{s}, \tag{2.7}$$

where the Green's functions of problems (2.3) and (2.4) have the following forms

$$G_{\iota,\kappa}(t,s)=\frac{1}{\Gamma(\iota)}\begin{cases}(\log t)^{\iota-1}(1-\log s)^{\kappa-1}-(\log(t/s))^{\iota-1}, & 1\leqslant s\leqslant t\leqslant e,\\ (\log t)^{\iota-1}(1-\log s)^{\kappa-1}, & 1\leqslant t\leqslant s\leqslant e,\end{cases}\quad \iota=\kappa=\alpha_1,\alpha_2. \tag{2.8}$$

Let $\Delta=\Gamma(\beta_1)\Gamma(\beta_2)/(\Gamma(\beta_1-\gamma_0)\Gamma(\beta_2-\eta_0))-\Delta_1\Delta_2$, where

$$\Delta_1=\sum_{j=1}^{q_1}\frac{\Gamma(\beta_1)}{\Gamma(\beta_1-\gamma_j)}\int_1^{\vartheta_j}(\log t)^{\beta_1-\gamma_j-1}k_j(t)\frac{dK_j(t)}{t},\quad \Delta_2=\sum_{i=1}^{q_2}\frac{\Gamma(\beta_2)}{\Gamma(\beta_2-\eta_i)}\int_1^{\theta_i}(\log t)^{\beta_2-\eta_i-1}h_i(t)\frac{dH_i(t)}{t}. \tag{2.9}$$

Lemma 2.3. *Let $\Delta \neq 0$. The system (2.5) has an integral representation*

$$\begin{aligned} x(t) &= -\int_1^e M_1(t,s)\varphi_{r_1}(u(s))\frac{ds}{s} - \int_1^e N_2(t,s)\varphi_{r_2}(v(s))\frac{ds}{s}, \\ y(t) &= -\int_1^e M_2(t,s)\varphi_{r_2}(v(s))\frac{ds}{s} - \int_1^e N_1(t,s)\varphi_{r_1}(u(s))\frac{ds}{s}, \end{aligned} \tag{2.10}$$

where

$$M_1(t,s) = G_{\beta_1,\beta_1-\gamma_0}(t,s) + \frac{\Delta_2}{\Delta}(\log t)^{\beta_1-1}\left(\sum_{j=1}^{q_1}\int_1^{\vartheta_j} k_j(t)G_{\beta_1-\gamma_j,\beta_1-\gamma_0}(t,s)\frac{dK_j(t)}{t}\right), \tag{2.11a}$$

$$M_2(t,s) = G_{\beta_2,\beta_2-\eta_0}(t,s) + \frac{\Delta_1}{\Delta}(\log t)^{\beta_2-1}\left(\sum_{i=1}^{q_2}\int_1^{\theta_i} h_i(t)G_{\beta_2-\eta_i,\beta_2-\eta_0}(t,s)\frac{dH_i(t)}{t}\right), \tag{2.11b}$$

$$N_1(t,s) = \frac{\Gamma(\beta_1)}{\Delta\Gamma(\beta_1-\gamma_0)}(\log t)^{\beta_2-1}\left(\sum_{j=1}^{q_1}\int_1^{\vartheta_j} k_j(t)G_{\beta_1-\gamma_j,\beta_1-\gamma_0}(t,s)\frac{dK_j(t)}{t}\right), \tag{2.11c}$$

$$N_2(t,s) = \frac{\Gamma(\beta_2)}{\Delta\Gamma(\beta_2-\eta_0)}(\log t)^{\beta_1-1}\left(\sum_{i=1}^{q_2}\int_1^{\theta_i} h_i(t)G_{\beta_2-\eta_i,\beta_2-\eta_0}(t,s)\frac{dH_i(t)}{t}\right). \tag{2.11d}$$

Proof. Based on the reference [64], the solution of problem (2.5) can be written the following form

$$\begin{aligned} x(t) &= c_{11}(\log t)^{\beta_1-1} + c_{12}(\log t)^{\beta_1-2} + \cdots + c_{1n_1}(\log t)^{\beta_1-n_1} + \frac{1}{\Gamma(\beta_1)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta_1-1}\varphi_{r_1}(u(s))\frac{ds}{s}, \\ y(t) &= c_{21}(\log t)^{\beta_2-1} + c_{22}(\log t)^{\beta_2-2} + \cdots + c_{2n_2}(\log t)^{\beta_2-n_2} + \frac{1}{\Gamma(\beta_2)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta_2-1}\varphi_{r_2}(v(s))\frac{ds}{s}, \end{aligned} \tag{2.12}$$

where $c_{11}, c_{12}, \ldots, c_{1n_1}, c_{21}, c_{22}, \ldots, c_{2n_2} \in \mathbb{R}$. From $\delta x(1) = \delta^2 x(1) = \cdots = \delta^{n_1-2}x(1) = 0$ and $\delta y(1) = \delta^2 y(1) = \cdots = \delta^{n_1-2}y(1) = 0$, we have $c_{in_i} = c_{i(n_i-1)} = \cdots = c_{i2} = 0$ $(i = 1, 2)$. Thus, (2.12) can be represented as

$$\begin{aligned} x(t) &= c_{11}(\log t)^{\beta_1-1} + \frac{1}{\Gamma(\beta_1)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta_1-1}\varphi_{r_1}(u(s))\frac{ds}{s}, \\ y(t) &= c_{21}(\log t)^{\beta_2-1} + \frac{1}{\Gamma(\beta_2)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta_2-1}\varphi_{r_2}(v(s))\frac{ds}{s}. \end{aligned} \tag{2.13}$$

From the above obtained functions x and y (2.13), we can obtain

$$\begin{aligned} \mathscr{D}^{\gamma}x(t) &= c_{11}\frac{\Gamma(\beta_1)}{\Gamma(\beta_1-\gamma)}(\log t)^{\beta_1-\gamma-1} + \frac{1}{\Gamma(\beta_1-\gamma)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta_1-\gamma-1}\varphi_{r_1}(u(s))\frac{ds}{s}, \quad \gamma = \gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_{q_1}, \\ \mathscr{D}^{\eta}y(t) &= c_{21}\frac{\Gamma(\beta_2)}{\Gamma(\beta_2-\eta)}(\log t)^{\alpha_2-\eta-1} + \frac{1}{\Gamma(\beta_2-\eta)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta_2-\eta-1}\varphi_{r_2}(v(s))\frac{ds}{s}, \quad \eta = \eta_0, \eta_1, \eta_2, \ldots, \eta_{q_2}. \end{aligned} \tag{2.14}$$

By applying equation (2.14) to the boundary conditions $\mathscr{D}^{\gamma_0}x(e) = \sum_{i=1}^{q_2}\int_1^{\theta_i} h_i(t)\mathscr{D}^{\eta_i}y(t)dH_i(t)/t$ and $\mathscr{D}^{\eta_0}y(e) = \sum_{j=1}^{q_1}\int_1^{\vartheta_j} k_j(t)\mathscr{D}^{\gamma_j}x(t)dK_j(t)/t$ from (2.5), we get the equations with respect to c_{11} and c_{21} as follows:

$$\begin{aligned} c_{11} &= \frac{\Gamma(\beta_1-\gamma_0)}{\Gamma(\beta_1)}\sum_{i=1}^{q_2}\int_1^{\theta_i} h_i(t)\mathscr{D}^{\eta_i}y(t)\frac{dH_i(t)}{t} - \frac{1}{\Gamma(\beta_1)}\int_1^e (1-\log s)^{\beta_1-\gamma_0-1}\varphi_{r_1}(u(s))\frac{ds}{s}, \\ c_{21} &= \frac{\Gamma(\beta_2-\eta_0)}{\Gamma(\beta_2)}\sum_{j=1}^{q_1}\int_1^{\vartheta_j} k_j(t)\mathscr{D}^{\gamma_j}x(t)\frac{dK_j(t)}{t} - \frac{1}{\Gamma(\beta_2)}\int_1^e (1-\log s)^{\beta_2-\eta_0-1}\varphi_{r_2}(v(s))\frac{ds}{s}. \end{aligned} \tag{2.15}$$

Substituting (2.15) into (2.14), we obtain

$$\mathscr{D}^{\gamma_j}x(t) =(\log t)^{\beta_1-\gamma_j-1}\left(\frac{\Gamma(\beta_1-\gamma_0)}{\Gamma(\beta_1-\gamma_j)}\sum_{i=1}^{q_2}\int_1^{\theta_i} h_i(t)\mathscr{D}^{\eta_i}y(t)\frac{dH_i(t)}{t} - \int_1^e \frac{(1-\log s)^{\beta_1-\gamma_0-1}}{\Gamma(\beta_1-\gamma_j)}\varphi_{r_1}(u(s))\frac{ds}{s}\right)$$
$$+\frac{1}{\Gamma(\beta_1-\gamma_j)}\int_1^t\left(\log\frac{t}{s}\right)^{\beta_1-\gamma_j-1}\varphi_{r_1}(u(s))\frac{ds}{s}$$
$$=-\int_1^e G_{\beta_1-\gamma_j,\beta_1-\gamma_0}(t,s)\varphi_{r_1}(u(s))\frac{ds}{s}+\frac{\Gamma(\beta_1-\gamma_0)}{\Gamma(\beta_1-\gamma_j)}(\log t)^{\beta_1-\gamma_j-1}\sum_{i=1}^{q_2}\int_1^{\theta_i} h_i(t)\mathscr{D}^{\eta_i}y(t)\frac{dH_i(t)}{t}, \tag{2.16a}$$
$$\mathscr{D}^{\eta_i}y(t) =(\log t)^{\beta_2-\eta_i-1}\left(\frac{\Gamma(\beta_2-\eta_0)}{\Gamma(\beta_2-\eta_i)}\sum_{j=1}^{q_1}\int_1^{\vartheta_j} k_j(t)\mathscr{D}^{\gamma_j}x(t)\frac{dK_j(t)}{t} - \int_1^e \frac{(1-\log s)^{\beta_2-\eta_0-1}}{\Gamma(\beta_2-\eta_i)}\varphi_{r_2}(v(s))\frac{ds}{s}\right)$$
$$+\frac{1}{\Gamma(\beta_2-\eta_i)}\int_1^t\left(\log\frac{t}{s}\right)^{\beta_2-\eta_i-1}\varphi_{r_2}(v(s))\frac{ds}{s}$$
$$=-\int_1^e G_{\beta_2-\eta_i,\beta_2-\eta_0}(t,s)\varphi_{r_2}(v(s))\frac{ds}{s}+\frac{\Gamma(\beta_2-\eta_0)}{\Gamma(\beta_2-\eta_i)}(\log t)^{\beta_2-\eta_i-1}\sum_{j=1}^{q_1}\int_1^{\vartheta_j} k_j(t)\mathscr{D}^{\gamma_j}x(t)\frac{dK_j(t)}{t}. \tag{2.16b}$$

Multiplying by $k_j(t)$ and $h_i(t)$ on both sides of (2.16), integrating with respect to t from 1 to ϑ_j and θ_i and summing the obtained results with respect to j and i from 1 to q_2 and q_1, respectively, we write

$$\sum_{j=1}^{q_1}\int_1^{\vartheta_j} k_j(t)\mathscr{D}^{\gamma_j}x(t)\frac{dK_j(t)}{t}=-\sum_{j=1}^{q_1}\int_1^{\vartheta_j}k_j(t)\int_1^e G_{\beta_1-\gamma_j,\beta_1-\gamma_0}(t,s)\varphi_{r_1}(u(s))\frac{ds}{s}\frac{dK_j(t)}{t}$$
$$+\left(\sum_{j=1}^{q_1}\frac{\Gamma(\beta_1-\gamma_0)}{\Gamma(\beta_1-\gamma_j)}\int_1^{\vartheta_j}(\log t)^{\beta_1-\gamma_j-1}g_2(t)\frac{dK_i(t)}{t}\right)\left(\sum_{i=1}^{q_2}\int_1^{\theta_i}h_i(t)\mathscr{D}^{\eta_i}y(t)\frac{dH_i(t)}{t}\right), \tag{2.17a}$$
$$\sum_{i=1}^{q_2}\int_1^{\theta_i} h_i(t)\mathscr{D}^{\eta_i}y(t)\frac{dH_i(t)}{t}=-\sum_{i=1}^{q_2}\int_1^{\theta_i}h_i(t)\int_1^e G_{\beta_2-\eta_i,\beta_2-\eta_0}(t,s)\varphi_{r_2}(v(s))\frac{ds}{s}\frac{dH_i(t)}{t}$$
$$+\left(\sum_{i=1}^{q_2}\frac{\Gamma(\beta_2-\eta_0)}{\Gamma(\beta_2-\eta_i)}\int_1^{\theta_i}(\log t)^{\beta_2-\eta_i-1}h_i(t)\frac{dH_i(t)}{t}\right)\left(\sum_{j=1}^{q_1}\int_1^{\vartheta_j}k_j(t)\mathscr{D}^{\gamma_j}x(t)\frac{dK_j(t)}{t}\right). \tag{2.17b}$$

Solving the equations (2.17), we can have the following results

$$\frac{\Gamma(\beta_2-\eta_0)}{\Gamma(\beta_2)}\sum_{j=1}^{q_1}\int_1^{\vartheta_j} k_j(t)\mathscr{D}^{\gamma_j}x(t)\frac{dK_j(t)}{t}=-\frac{\Gamma(\beta_1)}{\Delta\Gamma(\beta_1-\gamma_0)}\sum_{j=1}^{q_1}\int_1^{\vartheta_j}k_j(t)\int_1^e G_{\beta_1-\gamma_j,\beta_1-\gamma_0}(t,s)\varphi_{r_1}(u(s))\frac{ds}{s}\frac{dK_j(t)}{t}$$
$$-\Delta_1\left(\sum_{i=1}^{q_2}\int_1^{\theta_i}h_i(t)\int_1^e G_{\beta_2-\eta_i,\beta_2-\eta_0}(t,s)\varphi_{r_2}(v(s))\frac{ds}{s}\frac{dH_i(t)}{t}\right), \tag{2.18a}$$
$$\frac{\Gamma(\beta_1-\gamma_0)}{\Gamma(\beta_1)}\sum_{i=1}^{q_2}\int_1^{\theta_i} h_i(t)\mathscr{D}^{\gamma_i}y(t)\frac{dH_i(t)}{t}=-\frac{\Gamma(\beta_2)}{\Delta\Gamma(\beta_2-\eta_0)}\sum_{i=1}^{q_2}\int_1^{\theta_i}h_i(t)\int_1^e G_{\beta_2-\eta_i,\beta_2-\eta_0}(t,s)\varphi_{r_2}(v(s))\frac{ds}{s}\frac{dH_i(t)}{t}$$
$$-\Delta_2\left(\sum_{j=1}^{q_1}\int_1^{\vartheta_j}k_j(t)\int_1^e G_{\beta_1-\gamma_j,\beta_1-\gamma_0}(t,s)\varphi_{r_1}(u(s))\frac{ds}{s}\frac{dK_j(t)}{t}\right). \tag{2.18b}$$

Substituting (2.18) and (2.15) into (2.13), we get

$$x(t)=-(\log t)^{\beta_1-1}\frac{1}{\Gamma(\beta_1)}\int_1^e(1-\log s)^{\beta_1-\gamma_0-1}\varphi_{r_1}(u(s))\frac{ds}{s}+\frac{1}{\Gamma(\beta_1)}\int_1^t\left(\log\frac{t}{s}\right)^{\beta_1-1}\varphi_{r_1}(u(s))\frac{ds}{s}$$

$$
\begin{aligned}
&-\Bigg(\Delta_2\Bigg(\sum_{j=1}^{q_1}\int_1^{\vartheta_j} k_j(t)\int_1^e G_{\beta_1-\gamma_j,\beta_1-\gamma_0}(t,s)\varphi_{r_1}(u(s))\frac{ds}{s}\frac{dK_j(t)}{t}\Bigg)\\
&-\frac{\Gamma(\beta_2)}{\Delta\Gamma(\beta_2-\eta_0)}\sum_{i=1}^{q_2}\int_1^{\theta_i} h_i(t)\int_1^e G_{\beta_2-\eta_i,\beta_2-\eta_0}(t,s)\varphi_{r_2}(v(s))\frac{ds}{s}\frac{dH_i(t)}{t}\Bigg)(\log t)^{\beta_1-1}\\
&=-\int_1^e M_1(t,s)\varphi_{r_1}(u(s))\frac{ds}{s}-\int_1^e N_2(t,s)\varphi_{r_2}(v(s))\frac{ds}{s},
\end{aligned}
\tag{2.19a}
$$

$$
\begin{aligned}
y(t)=&-(\log t)^{\beta_2-1}\frac{1}{\Gamma(\beta_2)}\int_1^e(1-\log s)^{\beta_2-\eta_0-1}\varphi_{r_2}(v(s))\frac{ds}{s}+\frac{1}{\Gamma(\beta_2)}\int_1^t\left(\log\frac{t}{s}\right)^{\beta_2-1}\varphi_{r_2}(v(s))\frac{ds}{s}\\
&-\Bigg(\Delta_1\Bigg(\sum_{i=1}^{q_2}\int_1^{\theta_i} h_i(t)\int_1^e G_{\beta_2-\eta_i,\beta_2-\eta_0}(t,s)\varphi_{r_2}(v(s))\frac{ds}{s}\frac{dH_i(t)}{t}\Bigg)\\
&-\frac{\Gamma(\beta_1)}{\Delta\Gamma(\beta_1-\gamma_0)}\sum_{j=1}^{q_1}\int_1^{\vartheta_j} k_j(t)\int_1^e G_{\beta_1-\gamma_j,\beta_1-\gamma_0}(t,s)\varphi_{r_1}(u(s))\frac{ds}{s}\frac{dK_j(t)}{t}\Bigg)(\log t)^{\beta_2-1}\\
&=-\int_1^e M_2(t,s)\varphi_{r_2}(v(s))\frac{ds}{s}-\int_1^e N_1(t,s)\varphi_{r_1}(u(s))\frac{ds}{s},
\end{aligned}
\tag{2.19b}
$$

This completes the proof of the lemma. □

It follows from (2.6), (2.7) and (2.10) that we observe the following lemma.

Lemma 2.4. *Let $\Delta\neq 0$. The the unique solution $(x,y)\in C[1,e]^2$ of system (1.1)-(1.2) is given by*

$$
\begin{aligned}
x(t)=&\mu^{r_1-1}\int_1^e M_1(t,s)\varphi_{r_1}\left(\int_1^e G_{\alpha_1,\alpha_1}(s,\tau)f_1(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&+\nu^{r_2-1}\int_1^e N_2(t,s)\varphi_{r_2}\left(\int_1^e G_{\alpha_2,\alpha_2}(s,\tau)f_2(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s},\quad t\in[1,e],
\end{aligned}
\tag{2.20a}
$$

$$
\begin{aligned}
y(t)=&\nu^{r_2-1}\int_1^e M_2(t,s)\varphi_{r_2}\left(\int_1^e G_{\alpha_2,\alpha_2}(s,\tau)f_2(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&+\mu^{r_1-1}\int_1^e N_1(t,s)\varphi_{r_1}\left(\int_1^e G_{\alpha_1,\alpha_1}(s,\tau)f_1(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s},\quad t\in[1,e].
\end{aligned}
\tag{2.20b}
$$

Lemma 2.5. *[52] The function $G_{\kappa,\kappa}(t,s)$ defined by (2.8) has the following properties:* (I) *$G_{\kappa,\kappa}(t,s)$ is continuous function on $(t,s)\in[1,e]^2$ and $G_{\kappa,\kappa}(t,s)>0$ for $t,s\in(1,e)$;* (II) *$\Gamma(\kappa)\varrho_\kappa(t)\rho_\kappa(s)\le G_{\kappa,\kappa}(t,s)\le(\kappa-1)\rho_\kappa(s)$ for $t,s\in[1,e]$;* (III) *$\Gamma(\kappa)\varrho_\kappa(t)\rho_\kappa(s)\le G_{\kappa,\kappa}(t,s)\le(\kappa-1)\varrho_\kappa(t)$ for $t,s\in[1,e]$, where $\varrho_\kappa(t)=(\log t)^{\kappa-1}(1-\log t)/\Gamma(\kappa)$ and $\rho_\kappa(t)=(1-\log t)^{\kappa-1}\log t/\Gamma(\kappa)$ for $\kappa\in(2,3]$ and $t\in[1,e]$.*

Lemma 2.6. *Let $\kappa>1$. Then $x^\kappa\leqslant 1-(1-x)^\kappa$ for $x\in[0,1]$.*

Proof. For $\kappa>1$ and $x\in[0,1]$, then

$$
x^\kappa\leqslant 1-(1-x)^\kappa\Leftrightarrow x^\kappa+(1-x)^\kappa\leqslant 1.
\tag{2.21}
$$

Let $g(x)=x^\kappa+(1-x)^\kappa$. Then $g(x)$ is a continuous function on $[0,1]$ for $\kappa>1$. Along the absolute maximum and minimum values of a continuous function on a closed interval, then the absolute maximum of $g(x)$ is 1 at $x=0,1$, that is $g(x)\leqslant 1$. This completes the proof of the lemma. □

Lemma 2.7. *For $i=1,\dots,q_2$ and $j=1,\dots,q_1$, the functions $G_{\beta_1,\beta_1-\gamma_0}(t,s)$, $G_{\beta_2,\beta_2-\eta_0}(t,s)$, $G_{\beta_1-\gamma_j,\beta_1-\gamma_0}(t,s)$ and $G_{\beta_2-\eta_i,\beta_2-\eta_0}(t,s)$ given by (2.11) have the properties:*

(A1) *The function* $G_{\beta_1,\beta_1-\gamma_0}(t,s)$ *is continuous on* $[1,e]^2$; $G_{\beta_1,\beta_1-\gamma_0}(t,s) \geqslant 0$ *for all* $(t,s) \in [1,e]^2$; $G_{\beta_1,\beta_1-\gamma_0}(t,s) > 0$ *for all* $(t,s) \in (1,e)^2$; $(\log t)^{\beta_1-1}\mathscr{G}_{\gamma_0}(s) \leqslant G_{\beta_1,\beta_1-\gamma_0}(t,s) \leqslant \mathscr{G}_{\gamma_0}(s)$ *for all* $(t,s) \in [1,e]^2$, *where* $\mathscr{G}_{\gamma_0}(s) = (1-\log s)^{\beta_1-\gamma_0-1}(1-(1-\log s)^{\gamma_0})/\Gamma(\beta_1)$.

(A2) *The function* $G_{\beta_2,\beta_2-\eta_0}(t,s)$ *is continuous on* $[1,e]^2$; $G_{\beta_2,\beta_2-\eta_0}(t,s) \geqslant 0$ *for all* $(t,s) \in [1,e]^2$; $G_{\beta_2,\beta_2-\eta_0}(t,s) > 0$ *for all* $(t,s) \in (1,e)^2$; $(\log t)^{\beta_2-1}\mathscr{G}_{\eta_0}(s) \leqslant G_{\beta_2,\beta_2-\eta_0}(t,s) \leqslant \mathscr{G}_{\eta_0}(s)$ *for all* $(t,s) \in [1,e]^2$, *where* $\mathscr{G}_{\eta_0}(s) = (1-\log s)^{\beta_2-\eta_0-1}(1-(1-\log s)^{\eta_0})/\Gamma(\beta_2)$.

(A3) *The function* $G_{\beta_1-\gamma_j,\beta_1-\gamma_0}(t,s)$ *is continuous on* $[1,e]^2$; $G_{\beta_1-\gamma_j,\beta_1-\gamma_0}(t,s) \geqslant 0$ *for all* $(t,s) \in [1,e]^2$; $G_{\beta_1-\gamma_j,\beta_1-\gamma_0}(t,s) > 0$ *for all* $(t,s) \in (1,e)^2$; $(\log t)^{\beta_1-\gamma_j-1}\mathscr{G}_{\gamma_j}(s) \leqslant G_{\beta_1-\gamma_j,\beta_1-\gamma_0}(t,s) \leqslant (\log t)^{\beta_1-\gamma_j-1}(1-\log s)^{\beta_1-\gamma_0-1}$ *for all* $(t,s) \in [1,e]^2$, *where* $\mathscr{G}_{\gamma_j}(s) = (1-\log s)^{\beta_1-\gamma_0-1}(1-(1-\log s)^{\gamma_0-\gamma_j})/\Gamma(\beta_1-\gamma_j)$.

(A4) *The function* $G_{\beta_2-\eta_i,\beta_2-\eta_0}(t,s)$ *is continuous on* $[1,e]^2$; $G_{\beta_2-\eta_i,\beta_2-\eta_0}(t,s) \geqslant 0$ *for all* $(t,s) \in [1,e]^2$; $G_{\beta_2-\eta_i,\beta_2-\eta_0}(t,s) > 0$ *for all* $(t,s) \in (1,e)^2$; $(\log t)^{\beta_2-\eta_i-1}\mathscr{G}_{\eta_i}(s) \leqslant G_{\beta_2-\eta_i,\beta_2-\eta_0}(t,s) \leqslant (\log t)^{\beta_2-\eta_i-1}(1-\log s)^{\beta_2-\eta_0-1}$ *for all* $(t,s) \in [1,e]^2$, *where* $\mathscr{G}_{\eta_i}(s) = (1-\log s)^{\beta_2-\eta_0-1}(1-(1-\log s)^{\eta_0-\eta_i})/\Gamma(\beta_2-\eta_i)$.

Proof. (A1) It's quite obvious that the function $G_{\beta_1,\beta_1-\gamma_0}(t,s)$ is continuous on $[1,e]^2$. For $1 \leqslant s \leqslant t \leqslant e$, then

$$\begin{aligned}
G_{\beta_1,\beta_1-\gamma_0}(t,s) =& \frac{1}{\Gamma(\beta_1)}[(\log t)^{\beta_1-1}(1-\log s)^{\beta_1-\gamma_0-1} - (\log t-\log s)^{\beta_1-1}] \\
=& \frac{1}{\Gamma(\beta_1)}(\log t)^{\beta_1-1}[(1-\log s)^{\beta_1-\gamma_0-1} - (1-\log s/\log t)^{\beta_1-1}] \\
\geqslant& \frac{1}{\Gamma(\beta_1)}(\log t)^{\beta_1-1}[(1-\log s)^{\beta_1-\gamma_0-1} - (1-\log s)^{\beta_1-1}] = (\log t)^{\beta_1-1}\mathscr{G}_{\gamma_0}(s), \qquad (2.22)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial t}G_{\beta_1,\beta_1-\gamma_0}(t,s) =& \frac{1}{\Gamma(\beta_1)}\frac{\partial}{\partial t}[(\log t)^{\beta_1-1}(1-\log s)^{\beta_1-\gamma_0-1} - (\log t-\log s)^{\beta_1-1}] \\
=& \frac{1}{t\Gamma(\beta_1-1)}[(\log t)^{\beta_1-2}(1-\log s)^{\beta_1-\gamma_0-1} - (\log t-\log s)^{\beta_1-2}] \\
\geqslant& \frac{1}{t\Gamma(\beta_1-1)}[(\log t)^{\beta_1-2}(1-\log s)^{\beta_1-2} - (\log t-\log s)^{\beta_1-2}] \\
=& \frac{1}{t\Gamma(\beta_1-1)}[(\log t-\log t\log s)^{\beta_1-2} - (\log t-\log s)^{\beta_1-2}] \geqslant 0, \qquad (2.23)
\end{aligned}$$

which implies that $G_{\beta_1,\beta_1-\gamma_0}(t,s)$ is increasing with respect to $t \in [1,e]$ for $1 \leqslant s \leqslant t \leqslant e$. Hence, we can obtain $(\log t)^{\beta_1-1}\mathscr{G}_{\gamma_0}(s) \leqslant G_{\beta_1,\beta_1-\gamma_0}(t,s) \leqslant G_{\beta_1,\beta_1-\gamma_0}(e,s) = \mathscr{G}_{\gamma_0}(s)$ for all $(t,s) \in [1,e]^2$ and $1 \leqslant s \leqslant t \leqslant e$.

For $1 \leqslant t \leqslant s \leqslant e$, then we can get

$$G_{\beta_1,\beta_1-\gamma_0}(t,s) \geqslant \frac{1}{\Gamma(\beta_1)}(\log t)^{\beta_1-1}[(1-\log s)^{\beta_1-\gamma_0-1} - (1-\log s)^{\beta_1-1}] = (\log t)^{\beta_1-1}\mathscr{G}_{\gamma_0}(s), \qquad (2.24)$$

and

$$\frac{\partial}{\partial t}G_{\beta_1,\beta_1-\gamma_0}(t,s) = \frac{1}{\Gamma(\beta_1)}\frac{\partial}{\partial t}[(\log t)^{\beta_1-1}(1-\log s)^{\beta_1-\gamma_0-1}] = \frac{1}{t\Gamma(\beta_1-1)}(\log t)^{\beta_1-2}(1-\log s)^{\beta_1-\gamma_0-1} \geqslant 0, \quad (2.25)$$

which implies that $G_{\beta_1,\beta_1-\gamma_0}(t,s)$ is increasing with respect to $t \in [1,e]$ for $1 \leqslant t \leqslant s \leqslant e$. Combining with Lemma 2.6, then we have

$$G_{\beta_1,\beta_1-\gamma_0}(t,s) \leqslant G_{\beta_1,\beta_1-\gamma_0}(s,s) = \frac{1}{\Gamma(\beta_1)}(\log s)^{\beta_1-1}(1-\log s)^{\beta_1-\gamma_0-1}$$

$$\leqslant \frac{1}{\Gamma(\beta_1)}(\log s)^{\gamma_0}(1-\log s)^{\beta_1-\gamma_0-1} \leqslant \frac{1}{\Gamma(\beta_1)}(1-\log s)^{\beta_1-\gamma_0-1}(1-(1-\log s)^{\gamma_0}) = \mathscr{G}_{\gamma_0}(s) \tag{2.26}$$

for all $(t,s) \in [1,e]^2$ and $1 \leqslant t \leqslant s \leqslant e$. Hence, we can obtain $(\log t)^{\beta_1-1}\mathscr{G}_{\gamma_0}(s) \leqslant G_{\beta_1,\beta_1-\gamma_0}(t,s) \leqslant G_{\beta_1,\beta_1-\gamma_0}(s,s) \leqslant G_{\beta_1,\beta_1-\gamma_0}(e,s) = \mathscr{G}_{\gamma_0}(s)$ for all $(t,s) \in [1,e]^2$ and $1 \leqslant t \leqslant s \leqslant e$. It follows from (2.22)-(2.26) that all conclusions of (A1) hold. Along the proof of (A1), we can also obtain (A2)-(A4). This completes the proof of the lemma. □

Based on Lemmas 2.3 and 2.7, the properties for the functions M_m and N_m $(m = 1,2)$ are established.

Lemma 2.8. *Let $\Delta > 0$. the functions $M_m(t,s)$ and $N_m(t,s)$ $(m = 1,2)$ given by (2.11) have the properties:*

(B1) *The function $M_1(t,s)$ is continuous nonnegative on $[1,e]^2$; $(\log t)^{\beta_1-1}\mathscr{M}_1(s) \leqslant M_1(t,s) \leqslant \mathscr{M}_1(s)$ for all $(t,s) \in [1,e]^2$, where $\mathscr{M}_1(s) = \mathscr{G}_{\gamma_0}(s) + (\Delta_2/\Delta)\sum_{j=1}^{q_1}\int_1^{\vartheta_j} k_j(t)G_{\beta_1-\gamma_j,\beta_1-\gamma_0}(t,s)dK_j(t)/t$.*

(B2) *The function $M_2(t,s)$ is continuous nonnegative on $[1,e]^2$; $(\log t)^{\beta_2-1}\mathscr{M}_2(s) \leqslant M_2(t,s) \leqslant \mathscr{M}_2(s)$ for all $(t,s) \in [1,e]^2$, where $\mathscr{M}_2(s) = \mathscr{G}_{\eta_0}(s) + (\Delta_1/\Delta)\sum_{i=1}^{q_2}\int_1^{\theta_i} h_i(t)G_{\beta_2-\eta_i,\beta_2-\eta_0}(t,s)dH_i(t)/t$.*

(B3) *The function $N_1(t,s)$ is continuous nonnegative on $[1,e]^2$; $(\log t)^{\beta_2-1}\mathscr{N}_1(s) = N_1(t,s) \leqslant \mathscr{N}_1(s)$ for all $(t,s) \in [1,e]^2$, where $\mathscr{N}_1(s) = (\Gamma(\beta_1)/(\Delta\Gamma(\beta_1-\gamma_0)))\sum_{j=1}^{q_1}\int_1^{\vartheta_j} k_j(t)G_{\beta_1-\gamma_j,\beta_1-\gamma_0}(t,s)dK_j(t)/t$.*

(B4) *The function $N_2(t,s)$ is continuous nonnegative on $[1,e]^2$; $(\log t)^{\beta_1-1}\mathscr{N}_2(s) = N_2(t,s) \leqslant \mathscr{N}_2(s)$ for all $(t,s) \in [1,e]^2$, where $\mathscr{N}_2(s) = (\Gamma(\beta_2)/(\Delta\Gamma(\beta_2-\eta_0)))\sum_{i=1}^{q_2}\int_1^{\theta_i} h_i(t)G_{\beta_2-\eta_i,\beta_2-\eta_0}(t,s)dH_i(t)/t$.*

Here let $\mathcal{X} = C[1,e]$. Then $\mathcal{X}$ and $\mathcal{Z} = \mathcal{X}\times\mathcal{X}$ are two Banach spaces with the supremum norm $\|x\| = \sup_{t\in[1,e]}|x(t)|$ and the norm $\|(x,y)\|_{\mathcal{Z}} = \|x\| + \|x\|$, respectively. Define the cones $\mathcal{Y}_1 = \{x \in \mathcal{X} | x(t) \geqslant (\log t)^{\beta_1-1}\|x\|, \forall t \in [1,e]\} \subset \mathcal{X}$ and $\mathcal{Y}_2 = \{y \in \mathcal{X} | y(t) \geqslant (\log t)^{\beta_2-1}\|y\|, \forall t \in [1,e]\} \subset \mathcal{X}$, then $\mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2 \subset \mathcal{Z}$.

We define the operator $\mathscr{A}(x,y)(t) = (\mathscr{A}_1(x,y)(t), \mathscr{A}_2(x,y)(t))$ from $\mathcal{Z}$ to $\mathcal{Z}$, where $\mathscr{A}_1(x,y)(t), \mathscr{A}_2(x,y)(t) : \mathcal{Z} \to \mathcal{X}$ are given by

$$\begin{aligned}\mathscr{A}_1(x,y)(t) =& \mu^{r_1-1}\int_1^e M_1(t,s)\varphi_{r_1}\left(\int_1^e G_{\alpha_1,\alpha_1}(s,\tau)f_1(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\ &+ \nu^{r_2-1}\int_1^e N_2(t,s)\varphi_{r_2}\left(\int_1^e G_{\alpha_2,\alpha_2}(s,\tau)f_2(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}, \quad t \in [1,e], \end{aligned} \tag{2.27a}$$

$$\begin{aligned}\mathscr{A}_2(x,y)(t) =& \nu^{r_2-1}\int_1^e M_2(t,s)\varphi_{r_2}\left(\int_1^e G_{\alpha_2,\alpha_2}(s,\tau)f_2(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\ &+ \mu^{r_1-1}\int_1^e N_1(t,s)\varphi_{r_1}\left(\int_1^e G_{\alpha_1,\alpha_1}(s,\tau)f_1(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}, \quad t \in [1,e]. \end{aligned} \tag{2.27b}$$

Then the pair (x,y) is a fixed point of the operator $\mathscr{A}$ if and only if (x,y) is a solution of system (1.1)-(1.2).

Lemma 2.9. *Under the assumptions of system (1.1)-(1.2), then $\mathscr{A} : \mathcal{Y} \to \mathcal{Y}$ is a completely continuous operator.*

Proof. For $(x,y) \in \mathcal{Y}$, by using Lemma 2.8, we can get

$$\begin{aligned}\mathscr{A}_1(x,y)(t) \leqslant& \mu^{r_1-1}\int_1^e \mathscr{M}_1(s)\varphi_{r_1}\left(\int_1^e G_{\alpha_1,\alpha_1}(s,\tau)f_1(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\ &+ \nu^{r_2-1}\int_1^e \mathscr{N}_2(s)\varphi_{r_2}\left(\int_1^e G_{\alpha_2,\alpha_2}(s,\tau)f_2(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}, \quad t \in [1,e], \end{aligned} \tag{2.28a}$$

$$\begin{aligned}\mathscr{A}_2(x,y)(t) \leqslant& \nu^{r_2-1}\int_1^e \mathscr{M}_2(s)\varphi_{r_2}\left(\int_1^e G_{\alpha_2,\alpha_2}(s,\tau)f_2(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\ &+ \mu^{r_1-1}\int_1^e \mathscr{N}_1(s)\varphi_{r_1}\left(\int_1^e G_{\alpha_1,\alpha_1}(s,\tau)f_1(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}, \quad t \in [1,e]. \end{aligned} \tag{2.28b}$$

that is,

$$\|\mathscr{A}_1(x,y)\| \leqslant \mu^{r_1-1}\int_1^e \mathscr{M}_1(s)\varphi_{r_1}\left(\int_1^e G_{\alpha_1,\alpha_1}(s,\tau)f_1(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} + \nu^{r_2-1}\int_1^e \mathscr{N}_2(s)\varphi_{r_2}\left(\int_1^e G_{\alpha_2,\alpha_2}(s,\tau)f_2(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}, \tag{2.29a}$$

$$\|\mathscr{A}_2(x,y)\| \leqslant \nu^{r_2-1}\int_1^e \mathscr{M}_2(s)\varphi_{r_2}\left(\int_1^e G_{\alpha_2,\alpha_2}(s,\tau)f_2(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} + \mu^{r_1-1}\int_1^e \mathscr{N}_1(s)\varphi_{r_1}\left(\int_1^e G_{\alpha_1,\alpha_1}(s,\tau)f_1(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}. \tag{2.29b}$$

On the other hand, For $(x,y)\in\mathscr{Y}$, by using Lemma 2.8, we can obtain

$$\mathscr{A}_1(x,y)(t) \geqslant \mu^{r_1-1}(\log t)^{\beta_1-1}\int_1^e \mathscr{M}_1(s)\varphi_{r_1}\left(\int_1^e G_{\alpha_1,\alpha_1}(s,\tau)f_1(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} + \nu^{r_2-1}(\log t)^{\beta_1-1} \times\int_1^e \mathscr{N}_2(s)\varphi_{r_2}\left(\int_1^e G_{\alpha_2,\alpha_2}(s,\tau)f_2(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \geqslant (\log t)^{\beta_1-1}\|\mathscr{A}_1(x,y)\|,\ t\in[1,e], \tag{2.30a}$$

$$\mathscr{A}_2(x,y)(t) \geqslant \nu^{r_2-1}(\log t)^{\beta_2-1}\int_1^e \mathscr{M}_2(s)\varphi_{r_2}\left(\int_1^e G_{\alpha_2,\alpha_2}(s,\tau)f_2(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} + \mu^{r_1-1}(\log t)^{\beta_2-1} \times\int_1^e \mathscr{N}_1(s)\varphi_{r_1}\left(\int_1^e G_{\alpha_1,\alpha_1}(s,\tau)f_1(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \geqslant (\log t)^{\beta_2-1}\|\mathscr{A}_2(x,y)\|,\ t\in[1,e]. \tag{2.30b}$$

Hence, $\mathscr{A}(x,y) = (\mathscr{A}_1(x,y),\mathscr{A}_2(x,y)) \in \mathscr{Y}$ for $(x,y)\in\mathscr{Y}$, that is, $\mathscr{A}(\mathscr{Y})\subset\mathscr{Y}$. Based on the continuity of the functions f_m, G_{α_m,α_m}, $M_m(t,s)$ and $N_m(t,s)$ $(m=1,2)$ and the Ascoli-Arzela theorem, it is easy to see that $\mathscr{A}_1$ and $\mathscr{A}_2$ are completely continuous operators. Furthermore, we can prove that $\mathscr{A}$ is a completely continuous operator. This completes the proof of the lemma. □

3. Main results

For $[\sigma_1,\sigma_2]\subset[1,e]$ with $1<\sigma_1<\sigma_2\leqslant e$, we define the extreme limits as follows:

$$\mathfrak{F}^s_{m0} = \limsup_{x,y\geqslant 0, x+y\to 0^+}\max_{t\in[1,e]}\frac{f_m(t,x,y)}{\varphi_{p_m}(x+y)},\qquad \mathfrak{F}^i_{m0} = \liminf_{x,y\geqslant 0, x+y\to 0^+}\min_{t\in[\sigma_1,\sigma_2]}\frac{f_m(t,x,y)}{\varphi_{p_m}(x+y)}, \tag{3.1a}$$

$$\mathfrak{F}^s_{m\infty} = \limsup_{x,y\geqslant 0, x+y\to \infty}\max_{t\in[1,e]}\frac{f_m(t,x,y)}{\varphi_{p_m}(x+y)},\qquad \mathfrak{F}^i_{m\infty} = \liminf_{x,y\geqslant 0, x+y\to \infty}\min_{t\in[\sigma_1,\sigma_2]}\frac{f_m(t,x,y)}{\varphi_{p_m}(x+y)},\ m=1,2, \tag{3.1b}$$

and we give the following denotations

$$\mathfrak{M}_m = \int_1^e \mathscr{M}_m(s)\varphi_{r_m}\left(\int_1^e(\alpha_m-1)\rho_{\alpha_m}(\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s},\quad \mathfrak{M}'_m = \int_{\sigma_1}^{\sigma_2} \mathscr{M}_m(s)\varphi_{r_m}(\varrho_{\alpha_m}(s))\varphi_{r_m}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_m)\rho_{\alpha_m}(\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s},$$

$$\mathfrak{N}_m = \int_1^e \mathscr{N}_m(s)\varphi_{r_m}\left(\int_1^e(\alpha_m-1)\rho_{\alpha_m}(\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s},\quad \mathfrak{N}'_m = \int_{\sigma_1}^{\sigma_2} \mathscr{N}_m(s)\varphi_{r_m}(\varrho_{\alpha_m}(s))\varphi_{r_m}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_m)\rho_{\alpha_m}(\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s},$$

where $\varrho_\kappa(s)$, $\rho_\kappa(s)$ and $\mathscr{M}_m(s)$, $\mathscr{N}_m(s)$ are defined in Lemmas 2.5 and 2.8, respectively, $\kappa=\alpha_m$, $m=1,2$.

First, for $\mathfrak{F}^s_{m0}, \mathfrak{F}^i_{m\infty}\in(0,\infty)$ $(m=1,2)$, there exist $a_{mn},\omega_m\in[0,1]$ and $b_{mn},\pi_m\in(0,1)$ $(m,n=1,2)$ such that $a_{1n}+a_{2n}=1$, $b_{1n}+b_{2n}=1$ $(n=1,2)$, $\omega_1+\omega_2=1$ and $\pi_1+\pi_2=1$. We introduce the following constants

$$\mathfrak{U}_1 = \frac{1}{\mathfrak{F}^i_{1\infty}}\max\left\{\varphi_{p_1}\left(\frac{\omega_1 a_{11}}{\varsigma\varsigma_1\mathfrak{M}'_1}\right),\varphi_{p_1}\left(\frac{\omega_2 a_{12}}{\varsigma\varsigma_2\mathfrak{N}'_1}\right)\right\},\qquad \mathfrak{U}_2 = \frac{1}{\mathfrak{F}^i_{2\infty}}\max\left\{\varphi_{p_2}\left(\frac{\omega_1 a_{21}}{\varsigma\varsigma_1\mathfrak{N}'_2}\right),\varphi_{p_2}\left(\frac{\omega_2 a_{22}}{\varsigma\varsigma_2\mathfrak{M}'_1}\right)\right\}, \tag{3.2a}$$

$$\mathfrak{L}_1 = \frac{1}{\mathfrak{F}^s_{10}} \min\left\{\varphi_{p_1}\left(\frac{\pi_1 b_{11}}{\mathfrak{M}_1}\right), \varphi_{p_1}\left(\frac{\pi_2 b_{12}}{\mathfrak{N}_1}\right)\right\}, \qquad \mathfrak{L}_2 = \frac{1}{\mathfrak{F}^s_{20}} \min\left\{\varphi_{p_2}\left(\frac{\pi_1 b_{21}}{\mathfrak{N}_2}\right), \varphi_{p_2}\left(\frac{\pi_2 b_{22}}{\mathfrak{M}_2}\right)\right\}, \tag{3.2b}$$

$$\mathfrak{L}'_1 = \frac{1}{\mathfrak{F}^s_{10}} \min\left\{\varphi_{p_1}\left(\frac{\pi_1}{\mathfrak{M}_1}\right), \varphi_{p_1}\left(\frac{\pi_2}{\mathfrak{N}_1}\right)\right\}, \qquad \mathfrak{L}'_2 = \frac{1}{\mathfrak{F}^s_{20}} \min\left\{\varphi_{p_2}\left(\frac{\pi_1}{\mathfrak{N}_2}\right), \varphi_{p_2}\left(\frac{\pi_2}{\mathfrak{M}_2}\right)\right\}, \tag{3.2c}$$

where $\varsigma_1 = (\log \sigma_1)^{\beta_1 - 1}$, $\varsigma_2 = (\log \sigma_1)^{\beta_2 - 1}$ and $\varsigma = \min\{\varsigma_1, \varsigma_2\}$.

Theorem 3.1. *Let $\Delta > 0$, $[\sigma_1, \sigma_2] \subset [1, e]$ with $1 < \sigma_1 < \sigma_2 \leqslant e$. there exist $a_{mn}, \omega_m \in [0,1]$ and $b_{mn}, \pi_m \in (0,1)$ $(m, n = 1, 2)$ such that $a_{1n} + a_{2n} = 1$, $b_{1n} + b_{2n} = 1$ $(n = 1, 2)$, $\omega_1 + \omega_2 = 1$ and $\pi_1 + \pi_2 = 1$.*

(C1) *If $\mathfrak{F}^s_{m0}, \mathfrak{F}^i_{m\infty} \in (0,\infty)$ and $\mathfrak{U}_m < \mathfrak{L}_m$ $(m = 1,2)$, then system (1.1)-(1.2) has at least one positive solution $(x(t), y(t))$, $t \in [1, e]$, for each $\mu \in (\mathfrak{U}_1, \mathfrak{L}_1)$ and $\nu \in (\mathfrak{U}_2, \mathfrak{L}_2)$.*

(C2) *If $\mathfrak{F}^s_{10} = 0$, $\mathfrak{F}^s_{20}, \mathfrak{F}^i_{1\infty}, \mathfrak{F}^i_{2\infty} \in (0,\infty)$ and $\mathfrak{U}_2 < \mathfrak{L}'_2$, then system (1.1)-(1.2) has at least one positive solution $(x(t), y(t))$, $t \in [1, e]$, for each $\mu \in (\mathfrak{U}_1, \infty)$ and $\nu \in (\mathfrak{U}_2, \mathfrak{L}'_2)$.*

(C3) *If $\mathfrak{F}^s_{20} = 0$, $\mathfrak{F}^s_{10}, \mathfrak{F}^i_{1\infty}, \mathfrak{F}^i_{2\infty} \in (0,\infty)$ and $\mathfrak{U}_1 < \mathfrak{L}'_1$, then system (1.1)-(1.2) has at least one positive solution $(x(t), y(t))$, $t \in [1, e]$, for each $\mu \in (\mathfrak{U}_1, \mathfrak{L}'_1)$ and $\nu \in (\mathfrak{U}_2, \infty)$.*

(C4) *If $\mathfrak{F}^s_{10} = \mathfrak{F}^s_{20} = 0$ and $\mathfrak{F}^i_{1\infty}, \mathfrak{F}^i_{2\infty} \in (0,\infty)$, then system (1.1)-(1.2) has at least one positive solution $(x(t), y(t))$, $t \in [1, e]$, for each $\mu \in (\mathfrak{U}_1, \infty)$ and $\nu \in (\mathfrak{U}_2, \infty)$.*

(C5) *If $\mathfrak{F}^s_{10}, \mathfrak{F}^s_{20} \in (0,\infty)$ and at least one of $\mathfrak{F}^i_{1\infty}, \mathfrak{F}^i_{2\infty}$ is ∞, then system (1.1)-(1.2) has at least one positive solution $(x(t), y(t))$, $t \in [1, e]$, for each $\mu \in (0, \mathfrak{U}_1)$ and $\nu \in (0, \mathfrak{U}_2)$.*

(C6) *If $\mathfrak{F}^s_{10} = 0$, $\mathfrak{F}^s_{20} \in (0,\infty)$ and at least one of $\mathfrak{F}^i_{1\infty}, \mathfrak{F}^i_{2\infty}$ is ∞, then system (1.1)-(1.2) has at least one positive solution $(x(t), y(t))$, $t \in [1, e]$, for each $\mu \in (0, \infty)$ and $\nu \in (0, \mathfrak{L}'_2)$.*

(C7) *If $\mathfrak{F}^s_{20} = 0$, $\mathfrak{F}^s_{10} \in (0,\infty)$ and at least one of $\mathfrak{F}^i_{1\infty}, \mathfrak{F}^i_{2\infty}$ is ∞, then system (1.1)-(1.2) has at least one positive solution $(x(t), y(t))$, $t \in [1, e]$, for each $\mu \in (0, \mathfrak{L}'_1)$ and $\nu \in (0, \infty)$.*

(C8) *If $\mathfrak{F}^s_{10} = \mathfrak{F}^s_{20} = 0$ and at least one of $\mathfrak{F}^i_{1\infty}, \mathfrak{F}^i_{2\infty}$ is ∞, then system (1.1)-(1.2) has at least one positive solution $(x(t), y(t))$, $t \in [1, e]$, for each $\mu \in (0, \infty)$ and $\nu \in (0, \infty)$.*

Proof. Here we will prove only the cases (C1) and (C6) since the proofs of other cases are similar.

Case (C1). We have $\mathfrak{F}^s_{m0}, \mathfrak{F}^i_{m\infty} \in (0,\infty)$ and $\mathfrak{U}_m < \mathfrak{L}_m$ $(m = 1,2)$. For $\mu \in (\mathfrak{U}_1, \mathfrak{L}_1)$ and $\nu \in (\mathfrak{U}_2, \mathfrak{L}_2)$, there exist a small number $\varepsilon > 0$ such that $\mathfrak{F}^i_{m\infty} > \varepsilon$ $(m = 1, 2)$ and

$$\frac{1}{\mathfrak{F}^i_{1\infty} - \varepsilon} \max\left\{\varphi_{p_1}\left(\frac{\omega_1 a_{11}}{\varsigma\varsigma_1 \mathfrak{M}'_1}\right), \varphi_{p_1}\left(\frac{\omega_2 a_{12}}{\varsigma\varsigma_2 \mathfrak{N}'_1}\right)\right\} \leqslant \mu \leqslant \frac{1}{\mathfrak{F}^s_{10} + \varepsilon} \min\left\{\varphi_{p_1}\left(\frac{\pi_1 b_{11}}{\mathfrak{M}_1}\right), \varphi_{p_1}\left(\frac{\pi_2 b_{12}}{\mathfrak{N}_1}\right)\right\}, \tag{3.3a}$$

$$\frac{1}{\mathfrak{F}^i_{2\infty} - \varepsilon} \max\left\{\varphi_{p_2}\left(\frac{\omega_1 a_{21}}{\varsigma\varsigma_1 \mathfrak{N}'_2}\right), \varphi_{p_2}\left(\frac{\omega_2 a_{22}}{\varsigma\varsigma_2 \mathfrak{M}'_2}\right)\right\} \leqslant \nu \leqslant \frac{1}{\mathfrak{F}^s_{20} + \varepsilon} \min\left\{\varphi_{p_2}\left(\frac{\pi_1 b_{21}}{\mathfrak{N}_2}\right), \varphi_{p_2}\left(\frac{\pi_2 b_{22}}{\mathfrak{M}_2}\right)\right\}. \tag{3.3b}$$

According to the definitions of $\mathfrak{F}^s_{m0}$ $(m = 1, 2)$, we observe that there exists $\mathcal{R}_1 > 0$ such that

$$f_m(t, x, y) \leqslant (\mathfrak{F}^s_{m0} + \varepsilon)\varphi_{p_m}(x + y) \ \text{ for all } t \in [1, e] \text{ and } x, y \geqslant 0,\ x + y \leqslant \mathcal{R}_1,\ m = 1, 2. \tag{3.4}$$

Now we define the set $\Omega_1 = \{(x, y) \in \mathcal{Z}, \|(x, y)\|_{\mathcal{Z}} < \mathcal{R}_1\}$. Let $(x, y) \in \mathcal{Y} \cap \partial\Omega_1$, namely $(x, y) \in \mathcal{Y}$ with $\|(x, y)\|_{\mathcal{Z}} = \mathcal{R}_1$ or $\|x\| + \|y\| = \mathcal{R}_1$. Then $x(t) + y(t) \leqslant \mathcal{R}_1$ for $t \in [1, e]$. From Lemmas 2.5 and 2.8, we obtain

$$\begin{aligned}\mathscr{A}_1(x, y)(t) \leqslant & \mu^{r_1 - 1} \int_1^e \mathscr{M}_1(s)\varphi_{r_1}\left(\int_1^e (\alpha_1 - 1)\rho_{\alpha_1}(\tau) f_1(\tau, x(\tau), y(\tau)) \frac{d\tau}{\tau}\right) \frac{ds}{s} \\ & + \nu^{r_2 - 1} \int_1^e \mathscr{N}_2(s)\varphi_{r_2}\left(\int_1^e (\alpha_2 - 1)\rho_{\alpha_2}(\tau) f_2(\tau, x(\tau), y(\tau)) \frac{d\tau}{\tau}\right) \frac{ds}{s}\end{aligned}$$

$$\begin{aligned}
&\leqslant\mu^{r_1-1}\int_1^e \mathscr{M}_1(s)\varphi_{r_1}\left(\int_1^e(\alpha_1-1)\rho_{\alpha_1}(\tau)(\mathfrak{F}_{10}^s+\varepsilon)\varphi_{p_1}(x(\tau)+y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&\quad+\nu^{r_2-1}\int_1^e \mathscr{N}_2(s)\varphi_{r_2}\left(\int_1^e(\alpha_2-1)\rho_{\alpha_2}(\tau)(\mathfrak{F}_{20}^s+\varepsilon)\varphi_{p_2}(x(\tau)+y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&\leqslant\mu^{r_1-1}\varphi_{r_1}(\mathfrak{F}_{10}^s+\varepsilon)\int_1^e \mathscr{M}_1(s)\varphi_{r_1}\left(\int_1^e(\alpha_1-1)\rho_{\alpha_1}(\tau)\varphi_{p_1}(\|x\|+\|y\|)\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&\quad+\nu^{r_2-1}\varphi_{r_2}(\mathfrak{F}_{20}^s+\varepsilon)\int_1^e \mathscr{N}_2(s)\varphi_{r_2}\left(\int_1^e(\alpha_2-1)\rho_{\alpha_2}(\tau)\varphi_{p_2}(\|x\|+\|y\|)\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&=[\varphi_{r_1}(\mu(\mathfrak{F}_{10}^s+\varepsilon))\mathfrak{M}_1+\varphi_{r_2}(\nu(\mathfrak{F}_{20}^s+\varepsilon))\mathfrak{N}_2]\|(x,y)\|_{\mathcal{Z}}\leqslant\pi_1\|(x,y)\|_{\mathcal{Z}},\quad\forall t\in[1,e],
\end{aligned}\tag{3.5a}$$

$$\begin{aligned}
\mathscr{A}_2(x,y)(t)&\leqslant\nu^{r_2-1}\int_1^e \mathscr{M}_2(s)\varphi_{r_2}\left(\int_1^e(\alpha_2-1)\rho_{\alpha_2}(\tau)f_2(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&\quad+\mu^{r_1-1}\int_1^e \mathscr{N}_1(s)\varphi_{r_1}\left(\int_1^e(\alpha_1-1)\rho_{\alpha_1}(\tau)f_1(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&\leqslant\nu^{r_2-1}\int_1^e \mathscr{M}_2(s)\varphi_{r_2}\left(\int_1^e(\alpha_2-1)\rho_{\alpha_2}(\tau)(\mathfrak{F}_{20}^s+\varepsilon)\varphi_{p_2}(x(\tau)+y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&\quad+\mu^{r_1-1}\int_1^e \mathscr{N}_1(s)\varphi_{r_1}\left(\int_1^e(\alpha_1-1)\rho_{\alpha_1}(\tau)(\mathfrak{F}_{10}^s+\varepsilon)\varphi_{p_1}(x(\tau)+y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&\leqslant\nu^{r_2-1}\varphi_{r_2}(\mathfrak{F}_{20}^s+\varepsilon)\int_1^e \mathscr{M}_2(s)\varphi_{r_2}\left(\int_1^e(\alpha_2-1)\rho_{\alpha_2}(\tau)\varphi_{p_2}(\|x\|+\|y\|)\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&\quad+\mu^{r_1-1}\varphi_{r_1}(\mathfrak{F}_{10}^s+\varepsilon)\int_1^e \mathscr{N}_1(s)\varphi_{r_1}\left(\int_1^e(\alpha_1-1)\rho_{\alpha_1}(\tau)\varphi_{p_1}(\|x\|+\|y\|)\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&=[\varphi_{r_2}(\nu(\mathfrak{F}_{20}^s+\varepsilon))\mathfrak{M}_2+\varphi_{r_1}(\mu(\mathfrak{F}_{10}^s+\varepsilon))\mathfrak{N}_1]\|(x,y)\|_{\mathcal{Z}}\leqslant\pi_2\|(x,y)\|_{\mathcal{Z}},\quad\forall t\in[1,e].
\end{aligned}\tag{3.5b}$$

Hence, it follows from (3.5) that we deduce

$$\begin{aligned}
\|\mathscr{A}(x,y)\|_{\mathcal{Z}}&=\|\mathscr{A}_1(x,y)\|+\|\mathscr{A}_2(x,y)\|=\sup_{t\in[1,e]}|\mathscr{A}_1(x,y)(t)|+\sup_{t\in[1,e]}|\mathscr{A}_2(x,y)(t)|\\
&\leqslant\pi_1\|(x,y)\|_{\mathcal{Z}}+\pi_2\|(x,y)\|_{\mathcal{Z}}=\|(x,y)\|_{\mathcal{Z}}\ \text{ for all }(x,y)\in\mathcal{Y}\cap\partial\Omega_1.
\end{aligned}\tag{3.6}$$

According to the definitions of $\mathfrak{F}_{m\infty}^i$ $(m=1,2)$, we observe that there exists $\mathcal{R}_2'>0$ such that

$$f_m(t,x,y)\geqslant(\mathfrak{F}_{m\infty}^i-\varepsilon)\varphi_{p_m}(x+y)\ \text{ for all } t\in[\sigma_1,\sigma_2] \text{ and } x,y\geqslant0,\ x+y\geqslant\mathcal{R}_2',\ m=1,2.\tag{3.7}$$

We introduce the set $\Omega_2=\{(x,y)\in\mathcal{Z},\|(x,y)\|_{\mathcal{Z}}<\mathcal{R}_2\}$ with $\mathcal{R}_2=\max\{2\mathcal{R}_1,\mathcal{R}_2'/\varsigma\}$. Then for $(x,y)\in\mathcal{Y}\cap\partial\Omega_2$, we have by using Lemma 2.8

$$\begin{aligned}
x(t)+y(t)&\geqslant\min_{t\in[\sigma_1,\sigma_2]}\{(\log t)^{\beta_1-1}\}\|x\|+\min_{t\in[\sigma_1,\sigma_2]}\{(\log t)^{\beta_2-1}\}\|y\|=(\log\sigma_1)^{\beta_1-1}\|x\|+(\log\sigma_1)^{\beta_2-1}\|y\|\\
&=\varsigma_1\|x\|+\varsigma_2\|y\|\geqslant\varsigma\|(x,y)\|_{\mathcal{Z}}=\varsigma\mathcal{R}_2\geqslant\mathcal{R}_2'\ \text{ for }\forall t\in[\sigma_1,\sigma_2].
\end{aligned}\tag{3.8}$$

Therefore, it follows from (3.3),(3.7),(3.8) and Lemmas 2.5 and 2.8 that we obtain

$$\begin{aligned}
\mathscr{A}_1(x,y)(\sigma_1)&\geqslant\mu^{r_1-1}\int_1^e(\log\sigma_1)^{\beta_1-1}\mathscr{M}_1(s)\varphi_{r_1}\left(\int_1^e\Gamma(\alpha_1)\varrho_{\alpha_1}(s)\rho_{\alpha_1}(\tau)f_1(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&\quad+\nu^{r_2-1}\int_1^e(\log\sigma_1)^{\beta_1-1}\mathscr{N}_2(s)\varphi_{r_2}\left(\int_1^e\Gamma(\alpha_2)\varrho_{\alpha_2}(s)\rho_{\alpha_2}(\tau)f_2(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&\geqslant\mu^{r_1-1}\int_{\sigma_1}^{\sigma_2}(\log\sigma_1)^{\beta_1-1}\mathscr{M}_1(s)\varphi_{r_1}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_1)\varrho_{\alpha_1}(s)\rho_{\alpha_1}(\tau)(\mathfrak{F}_{1\infty}^i-\varepsilon)\varphi_{p_1}(x(\tau)+y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}
\end{aligned}$$

$$
\begin{aligned}
&+\nu^{r_2-1}\int_{\sigma_1}^{\sigma_2}(\log\sigma_1)^{\beta_1-1}\mathscr{N}_2(s)\varphi_{r_2}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_2)\varrho_{\alpha_2}(s)\rho_{\alpha_2}(\tau)(\mathfrak{F}^i_{2\infty}-\varepsilon)\varphi_{p_2}(x(\tau)+y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
\geqslant&\varsigma_1\mu^{r_1-1}\varphi_{r_1}(\mathfrak{F}^i_{1\infty}-\varepsilon)\int_{\sigma_1}^{\sigma_2}\mathscr{M}_1(s)\varphi_{r_1}(\varrho_{\alpha_1}(s))\varphi_{r_1}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_1)\rho_{\alpha_1}(\tau)\varphi_{p_1}(\varsigma\|(x,y)\|_{\mathcal{Z}})\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&+\varsigma_1\nu^{r_2-1}\varphi_{r_2}(\mathfrak{F}^i_{2\infty}-\varepsilon)\int_{\sigma_1}^{\sigma_2}\mathscr{N}_2(s)\varphi_{r_2}(\varrho_{\alpha_2}(s))\varphi_{r_2}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_2)\rho_{\alpha_2}(\tau)\varphi_{p_2}(\varsigma\|(x,y)\|_{\mathcal{Z}})\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
=&[\varsigma\varsigma_1\varphi_{r_1}(\mu(\mathfrak{F}^i_{1\infty}-\varepsilon))\mathfrak{M}'_1+\varsigma\varsigma_1\varphi_{r_2}(\nu(\mathfrak{F}^i_{2\infty}-\varepsilon))\mathfrak{N}'_2]\|(x,y)\|_{\mathcal{Z}}\geqslant\omega_1\|(x,y)\|_{\mathcal{Z}}, \qquad (3.9a)\\
\mathscr{A}_2(x,y)(\sigma_1)\geqslant&\nu^{r_2-1}\int_1^e(\log\sigma_1)^{\beta_2-1}\mathscr{M}_2(s)\varphi_{r_2}\left(\int_1^e\Gamma(\alpha_2)\varrho_{\alpha_2}(s)\rho_{\alpha_2}(\tau)f_2(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&+\mu^{r_1-1}\int_1^e(\log\sigma_1)^{\beta_2-1}\mathscr{N}_1(s)\varphi_{r_1}\left(\int_1^e\Gamma(\alpha_1)\varrho_{\alpha_1}(s)\rho_{\alpha_1}(\tau)f_1(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
\geqslant&\nu^{r_2-1}\int_{\sigma_1}^{\sigma_2}(\log\sigma_1)^{\beta_2-1}\mathscr{M}_2(s)\varphi_{r_2}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_2)\varrho_{\alpha_2}(s)\rho_{\alpha_2}(\tau)(\mathfrak{F}^i_{2\infty}-\varepsilon)\varphi_{p_2}(x(\tau)+y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&+\mu^{r_1-1}\int_{\sigma_1}^{\sigma_2}(\log\sigma_1)^{\beta_2-1}\mathscr{N}_1(s)\varphi_{r_1}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_1)\varrho_{\alpha_1}(s)\rho_{\alpha_1}(\tau)(\mathfrak{F}^i_{1\infty}-\varepsilon)\varphi_{p_1}(x(\tau)+y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
\geqslant&\varsigma_2\nu^{r_2-1}\varphi_{r_2}(\mathfrak{F}^i_{2\infty}-\varepsilon)\int_{\sigma_1}^{\sigma_2}\mathscr{M}_2(s)\varphi_{r_2}(\varrho_{\alpha_2}(s))\varphi_{r_2}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_2)\rho_{\alpha_2}(\tau)\varphi_{p_2}(\varsigma\|(x,y)\|_{\mathcal{Z}})\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&+\varsigma_2\mu^{r_1-1}\varphi_{r_1}(\mathfrak{F}^i_{1\infty}-\varepsilon)\int_{\sigma_1}^{\sigma_2}\mathscr{N}_1(s)\varphi_{r_1}(\varrho_{\alpha_1}(s))\varphi_{r_1}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_1)\rho_{\alpha_1}(\tau)\varphi_{p_1}(\varsigma\|(x,y)\|_{\mathcal{Z}})\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
=&[\varsigma\varsigma_1\varphi_{r_2}(\nu(\mathfrak{F}^i_{2\infty}-\varepsilon))\mathfrak{M}'_2+\varsigma\varsigma_1\varphi_{r_1}(\mu(\mathfrak{F}^i_{1\infty}-\varepsilon))\mathfrak{N}'_1]\|(x,y)\|_{\mathcal{Z}}\geqslant\omega_2\|(x,y)\|_{\mathcal{Z}}. \qquad (3.9b)
\end{aligned}
$$

Hence, it follows from (3.9) that we deduce

$$
\begin{aligned}
\|\mathscr{A}(x,y)\|_{\mathcal{Z}}=&\|\mathscr{A}_1(x,y)\|+\|\mathscr{A}_2(x,y)\|\geqslant\mathscr{A}_1(x,y)(\sigma_1)+\mathscr{A}_2(x,y)(\sigma_1)\\
\geqslant&\omega_1\|(x,y)\|_{\mathcal{Z}}+\omega_2\|(x,y)\|_{\mathcal{Z}}=\|(x,y)\|_{\mathcal{Z}}\ \text{ for all }(x,y)\in\mathcal{Y}\cap\partial\Omega_2. \qquad (3.10)
\end{aligned}
$$

It follows from (3.6), (3.10), Lemma 2.9 and Guo-Krasnosel'skii fixed point theorem that $\mathscr{A}$ has a fixed point $(x,y)\in\mathcal{Y}\cap(\overline{\Omega}_2\setminus\Omega_1)$ with $\mathcal{R}_1\leqslant\|(x,y)\|_{\mathcal{Z}}\leqslant\mathcal{R}_2$, $x(t)\geqslant(\log t)^{\beta_1-1}\|x\|$ and $y(t)\geqslant(\log t)^{\beta_2-1}\|y\|$ for all $t\in[1,e]$. If $\|x\|>0$ and $\|y\|>0$, then $x(t)>0$ and $y(t)>0$ for all $t\in(1,e]$, respectively. Therefore, system (1.1)-(1.2) has at least one positive solution $(x(t),y(t))$, $t\in[1,e]$.

Case (C6). We consider the conditions $\mathfrak{F}^s_{10}=0$, $\mathfrak{F}^s_{20}\in(0,\infty)$ and $\mathfrak{F}^i_{1\infty}=\infty$. For $\mu\in(0,\infty)$ and $\nu\in(0,\mathfrak{L}'_2)$, we choose $b'_{21}\in(\varphi_{r_2}(\nu\mathfrak{F}^s_{20})\mathfrak{N}_2/\pi_1,1)$ and $b'_{22}\in(\varphi_{r_2}(\nu\mathfrak{F}^s_{20})\mathfrak{M}_2/\pi_2,1)$ such that $b'_{1n}+b'_{2n}=1$ $(n=1,2)$ with $b'_{mn}\in(0,1)$ $(m,n=1,2)$. Let $\varepsilon>0$ be such that

$$
\begin{aligned}
\varepsilon\varphi_{p_1}\left(\frac{1}{\varsigma\varsigma_1\mathfrak{M}'_1}\right)\leqslant\mu\leqslant&\frac{1}{\varepsilon}\min\left\{\varphi_{p_1}\left(\frac{\pi_1b'_{11}}{\mathfrak{M}_1}\right),\varphi_{p_1}\left(\frac{\pi_2b'_{12}}{\mathfrak{N}_1}\right)\right\}, \qquad (3.11a)\\
\nu\leqslant&\frac{1}{\mathfrak{F}^s_{20}+\varepsilon}\min\left\{\varphi_{p_2}\left(\frac{\pi_1b'_{21}}{\mathfrak{N}_2}\right),\varphi_{p_2}\left(\frac{\pi_2b'_{22}}{\mathfrak{M}_2}\right)\right\}. \qquad (3.11b)
\end{aligned}
$$

Based on the definitions of $\mathfrak{F}^s_{m0}$ $(m=1,2)$, we observe that there exists $\mathcal{R}_1>0$ such that

$$
f_1(t,x,y)\leqslant\varepsilon\varphi_{p_1}(x+y)\text{ and }f_2(t,x,y)\leqslant(\mathfrak{F}^s_{20}+\varepsilon)\varphi_{p_2}(x+y)\ \text{ for all }t\in[1,e],\ x,y\geqslant0,\ x+y\leqslant\mathcal{R}_1. \qquad (3.12)
$$

We define the set $\Omega_1=\{(x,y)\in\mathcal{Z},\|(x,y)\|_{\mathcal{Z}}<\mathcal{R}_1\}$. Along the proof of case (C1), for any $(x,y)\in\mathcal{Y}\cap\partial\Omega_1$, then we conclude by applying (3.11) and Lemma 2.8

$$
\begin{aligned}
\mathscr{A}_1(x,y)(t)\leqslant&[\varphi_{r_1}(\mu\varepsilon)\mathfrak{M}_1+\varphi_{r_2}(\nu(\mathfrak{F}^s_{20}+\varepsilon))\mathfrak{N}_2]\|(x,y)\|_{\mathcal{Z}}\leqslant\pi_1\|(x,y)\|_{\mathcal{Z}},\quad\forall t\in[1,e], \qquad (3.13a)\\
\mathscr{A}_2(x,y)(t)\leqslant&[\varphi_{r_2}(\nu(\mathfrak{F}^s_{20}+\varepsilon))\mathfrak{M}_2+\varphi_{r_1}(\mu\varepsilon)\mathfrak{N}_1]\|(x,y)\|_{\mathcal{Z}}\leqslant\pi_2\|(x,y)\|_{\mathcal{Z}},\quad\forall t\in[1,e]. \qquad (3.13b)
\end{aligned}
$$

Hence, it follows from (3.13) that we deduce

$$\|\mathscr{A}(x,y)\|_{\mathcal{Z}} = \|\mathscr{A}_1(x,y)\| + \|\mathscr{A}_2(x,y)\| \leqslant (\pi_1+\pi_2)\|(x,y)\|_{\mathcal{Z}} = \|(x,y)\|_{\mathcal{Z}} \text{ for all } (x,y) \in \mathcal{Y} \cap \partial\Omega_1. \tag{3.14}$$

Based on the definitions of $\mathfrak{F}^i_{1\infty}$, we observe that there exists $\mathcal{R}_2' > 0$ such that

$$f_1(t,x,y) \geqslant \varphi_{p_1}(x+y)/\varepsilon \text{ for all } t \in [\sigma_1,\sigma_2] \text{ and } x,y \geqslant 0,\ x+y \geqslant \mathcal{R}_2'. \tag{3.15}$$

We introduce the set $\Omega_2 = \{(x,y) \in \mathcal{Z}, \|(x,y)\|_{\mathcal{Z}} < \mathcal{R}_2\}$ with $\mathcal{R}_2 = \max\{2\mathcal{R}_1, \mathcal{R}_2'/\varsigma\}$. Along the proof of case (C1), then for $(x,y) \in \mathcal{Y} \cap \partial\Omega_2$, we have $x(t)+y(t) \geqslant \varsigma_1\|x\| + \varsigma_2\|y\| \geqslant \varsigma\|(x,y)\|_{\mathcal{Z}} = \varsigma\mathcal{R}_2 \geqslant \mathcal{R}_2'$ for $\forall t \in [\sigma_1,\sigma_2]$. Furthermore, by using the inequality (3.15) and Lemmas 2.5 and 2.8, we obtain

$$\begin{aligned}
\mathscr{A}_1(x,y)(\sigma_1) \geqslant & \mu^{r_1-1}\int_1^e (\log\sigma_1)^{\beta_1-1}\mathscr{M}_1(s)\varphi_{r_1}\left(\int_1^e \Gamma(\alpha_1)\varrho_{\alpha_1}(s)\rho_{\alpha_1}(\tau)f_1(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\
& + \nu^{r_2-1}\int_1^e (\log\sigma_1)^{\beta_1-1}\mathscr{N}_2(s)\varphi_{r_2}\left(\int_1^e \Gamma(\alpha_2)\varrho_{\alpha_2}(s)\rho_{\alpha_2}(\tau)f_2(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\
\geqslant & \mu^{r_1-1}\int_{\sigma_1}^{\sigma_2} (\log\sigma_1)^{\beta_1-1}\mathscr{M}_1(s)\varphi_{r_1}\left(\int_{\sigma_1}^{\sigma_2} \Gamma(\alpha_1)\varrho_{\alpha_1}(s)\rho_{\alpha_1}(\tau)\frac{1}{\varepsilon}\varphi_{p_1}(x(\tau)+y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\
\geqslant & \varsigma_1\mu^{r_1-1}\varphi_{r_1}(1/\varepsilon)\int_{\sigma_1}^{\sigma_2} \mathscr{M}_1(s)\varphi_{r_1}(\varrho_{\alpha_1}(s))\varphi_{r_1}\left(\int_{\sigma_1}^{\sigma_2} \Gamma(\alpha_1)\rho_{\alpha_1}(\tau)\varphi_{p_1}(\varsigma\|(x,y)\|_{\mathcal{Z}})\frac{d\tau}{\tau}\right)\frac{ds}{s} \\
= & \varsigma\varsigma_1\varphi_{r_1}(\mu/\varepsilon)\mathfrak{M}_1'\|(x,y)\|_{\mathcal{Z}} \geqslant \|(x,y)\|_{\mathcal{Z}},
\end{aligned} \tag{3.16}$$

which implies

$$\|\mathscr{A}(x,y)\|_{\mathcal{Z}} = \|\mathscr{A}_1(x,y)\| + \|\mathscr{A}_2(x,y)\| \geqslant \mathscr{A}_1(x,y)(\sigma_1) \geqslant \|(x,y)\|_{\mathcal{Z}} \text{ for all } (x,y) \in \mathcal{Y} \cap \partial\Omega_2. \tag{3.17}$$

It follows from (3.14), (3.17) and Case (C6) that system (1.1)-(1.2) has at least one positive solution $(x(t),y(t))$, $t \in [1,e]$. Hence, all conclusions of the theorem are obtained. □

Second, for $\mathfrak{F}^i_{m0}, \mathfrak{F}^s_{m\infty} \in (0,\infty)$ $(m=1,2)$, there exist $a_{mn}, \omega_m \in [0,1]$ and $b_{mn}, \pi_m \in (0,1)$ $(m,n=1,2)$ such that $a_{1n}+a_{2n}=1$, $b_{1n}+b_{2n}=1$ $(n=1,2)$, $\omega_1+\omega_2=1$ and $\pi_1+\pi_2=1$. We introduce the following constants

$$\widetilde{\mathfrak{U}}_1 = \frac{1}{\mathfrak{F}^i_{10}}\max\left\{\varphi_{p_1}\left(\frac{\omega_1 a_{11}}{\varsigma\varsigma_1\mathfrak{M}_1'}\right), \varphi_{p_1}\left(\frac{\omega_2 a_{12}}{\varsigma\varsigma_2\mathfrak{N}_1'}\right)\right\}, \quad \widetilde{\mathfrak{U}}_2 = \frac{1}{\mathfrak{F}^i_{20}}\max\left\{\varphi_{p_2}\left(\frac{\omega_1 a_{21}}{\varsigma\varsigma_1\mathfrak{N}_2'}\right), \varphi_{p_2}\left(\frac{\omega_2 a_{22}}{\varsigma\varsigma_2\mathfrak{M}_1'}\right)\right\}, \tag{3.18a}$$

$$\widetilde{\mathfrak{L}}_1 = \frac{1}{\mathfrak{F}^s_{1\infty}}\min\left\{\varphi_{p_1}\left(\frac{\pi_1 b_{11}}{\mathfrak{M}_1}\right), \varphi_{p_1}\left(\frac{\pi_2 b_{12}}{\mathfrak{N}_1}\right)\right\}, \quad \widetilde{\mathfrak{L}}_2 = \frac{1}{\mathfrak{F}^s_{2\infty}}\min\left\{\varphi_{p_2}\left(\frac{\pi_1 b_{21}}{\mathfrak{N}_2}\right), \varphi_{p_2}\left(\frac{\pi_2 b_{22}}{\mathfrak{M}_2}\right)\right\}, \tag{3.18b}$$

$$\widetilde{\mathfrak{L}}_1' = \frac{1}{\mathfrak{F}^s_{1\infty}}\min\left\{\varphi_{p_1}\left(\frac{\pi_1}{\mathfrak{M}_1}\right), \varphi_{p_1}\left(\frac{\pi_2}{\mathfrak{N}_1}\right)\right\}, \quad \widetilde{\mathfrak{L}}_2' = \frac{1}{\mathfrak{F}^s_{2\infty}}\min\left\{\varphi_{p_2}\left(\frac{\pi_1}{\mathfrak{N}_2}\right), \varphi_{p_2}\left(\frac{\pi_2}{\mathfrak{M}_2}\right)\right\}. \tag{3.18c}$$

Theorem 3.2. *Let $\Delta > 0$, $[\sigma_1,\sigma_2] \subset [1,e]$ with $1 < \sigma_1 < \sigma_2 \leqslant e$. there exist $a_{mn}, \omega_m \in [0,1]$ and $b_{mn}, \pi_m \in (0,1)$ $(m,n=1,2)$ such that $a_{1n}+a_{2n}=1$, $b_{1n}+b_{2n}=1$ $(n=1,2)$, $\omega_1+\omega_2=1$ and $\pi_1+\pi_2=1$.*

(D1) *If $\mathfrak{F}^i_{m0}, \mathfrak{F}^s_{m\infty} \in (0,\infty)$ and $\widetilde{\mathfrak{U}}_m < \widetilde{\mathfrak{L}}_m$ $(m=1,2)$, then system (1.1)-(1.2) has at least one positive solution $(x(t),y(t))$, $t \in [1,e]$, for each $\mu \in \left(\widetilde{\mathfrak{U}}_1, \widetilde{\mathfrak{L}}_1\right)$ and $\nu \in \left(\widetilde{\mathfrak{U}}_2, \widetilde{\mathfrak{L}}_2\right)$.*

(D2) *If $\mathfrak{F}^s_{1\infty} = 0$, $\mathfrak{F}^s_{2\infty}, \mathfrak{F}^i_{10}, \mathfrak{F}^i_{20} \in (0,\infty)$ and $\widetilde{\mathfrak{U}}_2 < \widetilde{\mathfrak{L}}_2'$, then system (1.1)-(1.2) has at least one positive solution $(x(t),y(t))$, $t \in [1,e]$, for each $\mu \in \left(\widetilde{\mathfrak{U}}_1, \infty\right)$ and $\nu \in \left(\widetilde{\mathfrak{U}}_2, \widetilde{\mathfrak{L}}_2'\right)$.*

(D3) *If $\mathfrak{F}^s_{2\infty} = 0$, $\mathfrak{F}^s_{1\infty}, \mathfrak{F}^i_{10}, \mathfrak{F}^i_{20} \in (0,\infty)$ and $\widetilde{\mathfrak{U}}_1 < \widetilde{\mathfrak{L}}_1'$, then system (1.1)-(1.2) has at least one positive solution $(x(t),y(t))$, $t \in [1,e]$, for each $\mu \in \left(\widetilde{\mathfrak{U}}_1, \widetilde{\mathfrak{L}}_1'\right)$ and $\nu \in \left(\widetilde{\mathfrak{U}}_2, \infty\right)$.*

(D4) *If* $\mathfrak{F}^s_{1\infty} = \mathfrak{F}^s_{2\infty} = 0$ *and* $\mathfrak{F}^i_{10}, \mathfrak{F}^i_{20} \in (0,\infty)$, *then system* (1.1)-(1.2) *has at least one positive solution* $(x(t), y(t))$, $t \in [1,e]$, *for each* $\mu \in \left(\widetilde{\mathfrak{U}}_1, \infty\right)$ *and* $\nu \in \left(\widetilde{\mathfrak{U}}_2, \infty\right)$.

(D5) *If* $\mathfrak{F}^s_{1\infty}, \mathfrak{F}^s_{2\infty} \in (0,\infty)$ *and at least one of* $\mathfrak{F}^i_{10}, \mathfrak{F}^i_{20}$ *is* ∞, *then system* (1.1)-(1.2) *has at least one positive solution* $(x(t), y(t))$, $t \in [1,e]$, *for each* $\mu \in (0, \widetilde{\mathfrak{U}}_1)$ *and* $\nu \in (0, \widetilde{\mathfrak{U}}_2)$.

(D6) *If* $\mathfrak{F}^s_{1\infty} = 0$, $\mathfrak{F}^s_{2\infty} \in (0,\infty)$ *and at least one of* $\mathfrak{F}^i_{10}, \mathfrak{F}^i_{20}$ *is* ∞, *then system* (1.1)-(1.2) *has at least one positive solution* $(x(t), y(t))$, $t \in [1,e]$, *for each* $\mu \in (0,\infty)$ *and* $\nu \in \left(0, \widetilde{\mathfrak{L}}'_2\right)$.

(D7) *If* $\mathfrak{F}^s_{2\infty} = 0$, $\mathfrak{F}^s_{1\infty} \in (0,\infty)$ *and at least one of* $\mathfrak{F}^i_{10}, \mathfrak{F}^i_{20}$ *is* ∞, *then system* (1.1)-(1.2) *has at least one positive solution* $(x(t), y(t))$, $t \in [1,e]$, *for each* $\mu \in \left(0, \widetilde{\mathfrak{L}}'_1\right)$ *and* $\nu \in (0,\infty)$.

(D8) *If* $\mathfrak{F}^s_{1\infty} = \mathfrak{F}^s_{2\infty} = 0$ *and at least one of* $\mathfrak{F}^i_{10}, \mathfrak{F}^i_{20}$ *is* ∞, *then system* (1.1)-(1.2) *has at least one positive solution* $(x(t), y(t))$, $t \in [1,e]$, *for each* $\mu \in (0,\infty)$ *and* $\nu \in (0,\infty)$.

Proof. Here we will prove only the cases (D1) and (D6) since the proofs of other cases are similar.

Case (D1). We have $\mathfrak{F}^i_{m0}, \mathfrak{F}^s_{m\infty} \in (0,\infty)$ and $\widetilde{\mathfrak{U}}_m < \widetilde{\mathfrak{L}}_m$ $(m = 1,2)$. For $\mu \in \left(\widetilde{\mathfrak{U}}_1, \widetilde{\mathfrak{L}}_1\right)$ and $\nu \in \left(\widetilde{\mathfrak{U}}_2, \widetilde{\mathfrak{L}}_2\right)$, there exist a small number $\varepsilon > 0$ such that $\mathfrak{F}^i_{m0} > \varepsilon$ $(m = 1,2)$ and

$$\frac{1}{\mathfrak{F}^i_{10} - \varepsilon} \max\left\{\varphi_{p_1}\left(\frac{\omega_1 a_{11}}{\varsigma\varsigma_1 \mathfrak{M}'_1}\right), \varphi_{p_1}\left(\frac{\omega_2 a_{12}}{\varsigma\varsigma_2 \mathfrak{N}'_1}\right)\right\} \leqslant \mu \leqslant \frac{1}{\mathfrak{F}^s_{1\infty} + \varepsilon} \min\left\{\varphi_{p_1}\left(\frac{\pi_1 b_{11}}{\mathfrak{M}_1}\right), \varphi_{p_1}\left(\frac{\pi_2 b_{12}}{\mathfrak{N}_1}\right)\right\}, \tag{3.19a}$$

$$\frac{1}{\mathfrak{F}^i_{20} - \varepsilon} \max\left\{\varphi_{p_2}\left(\frac{\omega_1 a_{21}}{\varsigma\varsigma_1 \mathfrak{N}'_2}\right), \varphi_{p_2}\left(\frac{\omega_2 a_{22}}{\varsigma\varsigma_2 \mathfrak{M}'_2}\right)\right\} \leqslant \nu \leqslant \frac{1}{\mathfrak{F}^s_{2\infty} + \varepsilon} \min\left\{\varphi_{p_2}\left(\frac{\pi_1 b_{21}}{\mathfrak{N}_2}\right), \varphi_{p_2}\left(\frac{\pi_2 b_{22}}{\mathfrak{M}_2}\right)\right\}. \tag{3.19b}$$

According to the definitions of $\mathfrak{F}^i_{m0}$ $(m = 1,2)$, we observe that there exists $\mathcal{R}_3 > 0$ such that

$$f_m(t,x,y) \geqslant (\mathfrak{F}^i_{m0} - \varepsilon)\varphi_{p_m}(x+y) \ \text{ for all } t \in [\sigma_1,\sigma_2] \text{ and } x,y \geqslant 0,\ x+y \leqslant \mathcal{R}_3,\ m = 1,2. \tag{3.20}$$

Now we define the set $\Omega_3 = \{(x,y) \in \mathcal{Z}, \|(x,y)\|_{\mathcal{Z}} < \mathcal{R}_3\}$. Let $(x,y) \in \mathcal{Y} \cap \partial\Omega_3$, namely $(x,y) \in \mathcal{Y}$ with $\|(x,y)\|_{\mathcal{Z}} = \mathcal{R}_3$ or $\|x\| + \|y\| = \mathcal{R}_3$. Similar to the proof of case (C1), then $x(t) + y(t) \leqslant \mathcal{R}_1$ for $\forall t \in [1,e]$ and $x(t) + y(t) \geqslant \varsigma\|(x,y)\|_{\mathcal{Z}}$ for $\forall t \in [\sigma_1,\sigma_2]$. By using (3.19), (3.20) and Lemmas 2.5 and 2.8, we can deduce

$$\begin{aligned}
\mathscr{A}_1(x,y)(\sigma_1) \geqslant&\mu^{r_1-1}\int_1^e (\log\sigma_1)^{\beta_1-1}\mathscr{M}_1(s)\varphi_{r_1}\left(\int_1^e \Gamma(\alpha_1)\varrho_{\alpha_1}(s)\rho_{\alpha_1}(\tau)f_1(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&+\nu^{r_2-1}\int_1^e (\log\sigma_1)^{\beta_1-1}\mathscr{N}_2(s)\varphi_{r_2}\left(\int_1^e \Gamma(\alpha_2)\varrho_{\alpha_2}(s)\rho_{\alpha_2}(\tau)f_2(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
\geqslant&\mu^{r_1-1}\int_{\sigma_1}^{\sigma_2} (\log\sigma_1)^{\beta_1-1}\mathscr{M}_1(s)\varphi_{r_1}\left(\int_{\sigma_1}^{\sigma_2} \Gamma(\alpha_1)\varrho_{\alpha_1}(s)\rho_{\alpha_1}(\tau)(\mathfrak{F}^i_{10}-\varepsilon)\varphi_{p_1}(x(\tau)+y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&+\nu^{r_2-1}\int_{\sigma_1}^{\sigma_2} (\log\sigma_1)^{\beta_1-1}\mathscr{N}_2(s)\varphi_{r_2}\left(\int_{\sigma_1}^{\sigma_2} \Gamma(\alpha_2)\varrho_{\alpha_2}(s)\rho_{\alpha_2}(\tau)(\mathfrak{F}^i_{20}-\varepsilon)\varphi_{p_2}(x(\tau)+y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
\geqslant&\varsigma_1\mu^{r_1-1}\varphi_{r_1}(\mathfrak{F}^i_{10}-\varepsilon)\int_{\sigma_1}^{\sigma_2}\mathscr{M}_1(s)\varphi_{r_1}(\varrho_{\alpha_1}(s))\varphi_{r_1}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_1)\rho_{\alpha_1}(\tau)\varphi_{p_1}(\varsigma\|(x,y)\|_{\mathcal{Z}})\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&+\varsigma_1\nu^{r_2-1}\varphi_{r_2}(\mathfrak{F}^i_{20}-\varepsilon)\int_{\sigma_1}^{\sigma_2}\mathscr{N}_2(s)\varphi_{r_2}(\varrho_{\alpha_2}(s))\varphi_{r_2}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_2)\rho_{\alpha_2}(\tau)\varphi_{p_2}(\varsigma\|(x,y)\|_{\mathcal{Z}})\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
=&[\varsigma\varsigma_1\varphi_{r_1}(\mu(\mathfrak{F}^i_{10}-\varepsilon))\mathfrak{M}'_1 + \varsigma\varsigma_1\varphi_{r_2}(\nu(\mathfrak{F}^i_{20}-\varepsilon))\mathfrak{N}'_2]\|(x,y)\|_{\mathcal{Z}} \geqslant \omega_1\|(x,y)\|_{\mathcal{Z}},
\end{aligned}\tag{3.21a}$$

$$\mathscr{A}_2(x,y)(\sigma_1) \geqslant\nu^{r_2-1}\int_1^e (\log\sigma_1)^{\beta_2-1}\mathscr{M}_2(s)\varphi_{r_2}\left(\int_1^e \Gamma(\alpha_2)\varrho_{\alpha_2}(s)\rho_{\alpha_2}(\tau)f_2(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}$$

$$+\mu^{r_1-1}\int_1^e(\log\sigma_1)^{\beta_2-1}\mathscr{N}_1(s)\varphi_{r_1}\left(\int_1^e\Gamma(\alpha_1)\varrho_{\alpha_1}(s)\rho_{\alpha_1}(\tau)f_1(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}$$

$$\geqslant\nu^{r_2-1}\int_{\sigma_1}^{\sigma_2}(\log\sigma_1)^{\beta_2-1}\mathscr{M}_2(s)\varphi_{r_2}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_2)\varrho_{\alpha_2}(s)\rho_{\alpha_2}(\tau)(\mathfrak{F}_{20}^i-\varepsilon)\varphi_{p_2}(x(\tau)+y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}$$

$$+\mu^{r_1-1}\int_{\sigma_1}^{\sigma_2}(\log\sigma_1)^{\beta_2-1}\mathscr{N}_1(s)\varphi_{r_1}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_1)\varrho_{\alpha_1}(s)\rho_{\alpha_1}(\tau)(\mathfrak{F}_{10}^i-\varepsilon)\varphi_{p_1}(x(\tau)+y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}$$

$$\geqslant\varsigma_2\nu^{r_2-1}\varphi_{r_2}(\mathfrak{F}_{20}^i-\varepsilon)\int_{\sigma_1}^{\sigma_2}\mathscr{M}_2(s)\varphi_{r_2}(\varrho_{\alpha_2}(s))\varphi_{r_2}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_2)\rho_{\alpha_2}(\tau)\varphi_{p_2}(\varsigma\|(x,y)\|_Z)\frac{d\tau}{\tau}\right)\frac{ds}{s}$$

$$+\varsigma_2\mu^{r_1-1}\varphi_{r_1}(\mathfrak{F}_{10}^i-\varepsilon)\int_{\sigma_1}^{\sigma_2}\mathscr{N}_1(s)\varphi_{r_1}(\varrho_{\alpha_1}(s))\varphi_{r_1}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_1\rho_{\alpha_1}(\tau)\varphi_{p_1}(\varsigma\|(x,y)\|_Z)\frac{d\tau}{\tau}\right)\frac{ds}{s}$$

$$=[\varsigma\varsigma_1\varphi_{r_2}(\nu(\mathfrak{F}_{20}^i-\varepsilon))\mathfrak{M}_2'+\varsigma\varsigma_1\varphi_{r_1}(\mu(\mathfrak{F}_{10}^i-\varepsilon))\mathfrak{N}_1']\|(x,y)\|_Z\geqslant\omega_2\|(x,y)\|_Z.\tag{3.21b}$$

Hence, it follows from (3.21) that we deduce

$$\|\mathscr{A}(x,y)\|_Z\geqslant\mathscr{A}_1(x,y)(\sigma_1)+\mathscr{A}_2(x,y)(\sigma_1)\geqslant(\omega_1+\omega_2)\|(x,y)\|_Z=\|(x,y)\|_Z\ \text{ for all }(x,y)\in\mathcal{Y}\cap\partial\Omega_3.\tag{3.22}$$

Let $\mathcal{F}_m:[1,e]\times\mathbb{R}^+\to\mathbb{R}^+$ be defined by $\mathcal{F}_m(t,u)=\max_{0\leqslant x+y\leqslant u}f_m(t,x,y)$ for all $t\in[1,e]$ and $u\in\mathbb{R}^+$, $m=1,2$. Then $f_m(t,x,y)\leqslant\mathcal{F}_m(t,u)$ for all $t\in[1,e]$ and $x,y\geqslant 0$, $x+y\leqslant u$, $m=1,2$. Then, the function $\mathcal{F}_m(t,*)$ are nondecreasing with respect to $t\in[1,e]$ satisfying the condition $\limsup_{u\to\infty}\max_{t\in[1,e]}(\mathcal{F}_m(t,u)/\varphi_{p_m}(u))=\mathfrak{F}_{m\infty}^s$, $m=1,2$. Furthermore, for $\varepsilon>0$, there exists $\mathcal{R}_4'>0$ such that for all $t\in[1,e]$ and $u\geqslant\mathcal{R}_4'$, we have

$$\frac{\mathcal{F}_m(t,u)}{\varphi_{p_m}(u)}\leqslant\limsup_{u\to\infty}\max_{t\in[1,e]}\frac{\mathcal{F}_m(t,u)}{\varphi_{p_m}(u)}+\varepsilon=\mathfrak{F}_{m\infty}^s+\varepsilon\Rightarrow\mathcal{F}_m(t,u)\leqslant(\mathfrak{F}_{m\infty}^s+\varepsilon)\varphi_{p_m}(u),\quad m=1,2.\tag{3.23}$$

We consider the set $\Omega_4=\{(x,y)\in\mathcal{Y},\|(x,y)\|_Z<\mathcal{R}_4\}$ with $\mathcal{R}_4=\max\{2\mathcal{R}_3,\mathcal{R}_4'\}$. According to the definition of $\mathcal{F}_m$ ($m=1,2$), for $(x,y)\in\mathcal{Y}\cap\partial\Omega_4$, we can observe

$$f_m(t,x(t),y(t))\leqslant\mathcal{F}_m(t,\|(x,y)\|_Z)\leqslant(\mathfrak{F}_{m\infty}^s+\varepsilon)\varphi_{p_m}(\|(x,y)\|_Z)\ \text{ for all }t\in[1,e],\quad m=1,2.\tag{3.24}$$

By applying the inequality (3.24) and Lemmas 2.5 and 2.8, then we deduce the following results

$$\mathscr{A}_1(x,y)(t)\leqslant\mu^{r_1-1}\int_1^e\mathscr{M}_1(s)\varphi_{r_1}\left(\int_1^e(\alpha_1-1)\rho_{\alpha_1}(\tau)f_1(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}$$

$$+\nu^{r_2-1}\int_1^e\mathscr{N}_2(s)\varphi_{r_2}\left(\int_1^e(\alpha_2-1)\rho_{\alpha_2}(\tau)f_2(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}$$

$$\leqslant\mu^{r_1-1}\int_1^e\mathscr{M}_1(s)\varphi_{r_1}\left(\int_1^e(\alpha_1-1)\rho_{\alpha_1}(\tau)(\mathfrak{F}_{1\infty}^s+\varepsilon)\varphi_{p_1}(\|(x,y)\|_Z)\frac{d\tau}{\tau}\right)\frac{ds}{s}$$

$$+\nu^{r_2-1}\int_1^e\mathscr{N}_2(s)\varphi_{r_2}\left(\int_1^e(\alpha_2-1)\rho_{\alpha_2}(\tau)(\mathfrak{F}_{2\infty}^s+\varepsilon)\varphi_{p_2}(\|(x,y)\|_Z)\frac{d\tau}{\tau}\right)\frac{ds}{s}$$

$$=[\varphi_{r_1}(\mu(\mathfrak{F}_{1\infty}^s+\varepsilon))\mathfrak{M}_1+\varphi_{r_2}(\nu(\mathfrak{F}_{2\infty}^s+\varepsilon))\mathfrak{N}_2]\|(x,y)\|_Z\leqslant\pi_1\|(x,y)\|_Z,\quad\forall t\in[1,e],\tag{3.25a}$$

$$\mathscr{A}_2(x,y)(t)\leqslant\nu^{r_2-1}\int_1^e\mathscr{M}_2(s)\varphi_{r_2}\left(\int_1^e(\alpha_2-1)\rho_{\alpha_2}(\tau)f_2(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}$$

$$+\mu^{r_1-1}\int_1^e\mathscr{N}_1(s)\varphi_{r_1}\left(\int_1^e(\alpha_1-1)\rho_{\alpha_1}(\tau)f_1(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}$$

$$\leqslant\nu^{r_2-1}\int_1^e\mathscr{M}_2(s)\varphi_{r_2}\left(\int_1^e(\alpha_2-1)\rho_{\alpha_2}(\tau)(\mathfrak{F}_{2\infty}^s+\varepsilon)\varphi_{p_2}(\|(x,y)\|_Z)\frac{d\tau}{\tau}\right)\frac{ds}{s}$$

$$+\mu^{r_1-1}\int_1^e\mathscr{N}_1(s)\varphi_{r_1}\left(\int_1^e(\alpha_1-1)\rho_{\alpha_1}(\tau)(\mathfrak{F}_{1\infty}^s+\varepsilon)\varphi_{p_1}(\|(x,y)\|_Z)\frac{d\tau}{\tau}\right)\frac{ds}{s}$$

$$=[\varphi_{r_2}(\nu(\mathfrak{F}^s_{2\infty}+\varepsilon))\mathfrak{M}_2+\varphi_{r_1}(\mu(\mathfrak{F}^s_{1\infty}+\varepsilon))\mathfrak{N}_1]\|(x,y)\|_{\mathcal{Z}}\leqslant\pi_2\|(x,y)\|_{\mathcal{Z}},\quad\forall t\in[1,e].\tag{3.25b}$$

Hence, it follows from (3.25) that we deduce

$$\|\mathscr{A}(x,y)\|_{\mathcal{Z}}=\|\mathscr{A}_1(x,y)\|+\|\mathscr{A}_2(x,y)\|\leqslant(\pi_1+\pi_2)\|(x,y)\|_{\mathcal{Z}}=\|(x,y)\|_{\mathcal{Z}}\ \text{ for all }(x,y)\in\mathcal{Y}\cap\partial\Omega_4.\tag{3.26}$$

It follows from (3.22), (3.26), Lemma 2.9 and Guo-Krasnosel'skii fixed point theorem that $\mathscr{A}$ has a fixed point $(x,y)\in\mathcal{Y}\cap(\overline{\Omega}_4\setminus\Omega_3)$ with $\mathcal{R}_3\leqslant\|(x,y)\|_{\mathcal{Z}}\leqslant\mathcal{R}_4$, $x(t)\geqslant(\log t)^{\beta_1-1}\|x\|$ and $y(t)\geqslant(\log t)^{\beta_2-1}\|y\|$ for all $t\in[1,e]$. If $\|x\|>0$ and $\|y\|>0$, then $x(t)>0$ and $y(t)>0$ for all $t\in(1,e]$, respectively. Therefore, system (1.1)-(1.2) has at least one positive solution $(x(t),y(t))$, $t\in[1,e]$.

Case (D6). We consider the conditions $\mathfrak{F}^s_{1\infty}=0$, $\mathfrak{F}^s_{2\infty}\in(0,\infty)$ and $\mathfrak{F}^i_{10}=\infty$. For $\mu\in(0,\infty)$ and $\nu\in\left(0,\widetilde{\mathfrak{L}}'_2\right)$, we choose $b'_{21}\in(\varphi_{r_2}(\nu\mathfrak{F}^s_{2\infty})\mathfrak{N}_2/\pi_1,1)$ and $b'_{22}\in(\varphi_{r_2}(\nu\mathfrak{F}^s_{2\infty})\mathfrak{M}_2/\pi_2,1)$ such that $b'_{1n}+b'_{2n}=1$ $(n=1,2)$ with $b'_{mn}\in(0,1)$ $(m,n=1,2)$. Let $\varepsilon>0$ be such that

$$\varepsilon\varphi_{p_1}\left(\frac{1}{\varsigma\varsigma_1\mathfrak{M}'_1}\right)\leqslant\mu\leqslant\frac{1}{\varepsilon}\min\left\{\varphi_{p_1}\left(\frac{\pi_1 b'_{11}}{\mathfrak{M}_1}\right),\varphi_{p_1}\left(\frac{\pi_2 b'_{12}}{\mathfrak{N}_1}\right)\right\},\tag{3.27a}$$

$$\nu\leqslant\frac{1}{\mathfrak{F}^s_{2\infty}+\varepsilon}\min\left\{\varphi_{p_2}\left(\frac{\pi_1 b'_{21}}{\mathfrak{N}_2}\right),\varphi_{p_2}\left(\frac{\pi_2 b'_{22}}{\mathfrak{M}_2}\right)\right\}.\tag{3.27b}$$

Based on the definitions of $\mathfrak{F}^i_{10}$, we observe that there exists $\mathcal{R}_3>0$ such that

$$f_1(t,x,y)\geqslant\varphi_{p_1}(x+y)/\varepsilon\ \text{ for all }t\in[\sigma_1,\sigma_2]\text{ and }x,y\geqslant0,\ x+y\leqslant\mathcal{R}_3.\tag{3.28}$$

We introduce the set $\Omega_3=\{(x,y)\in\mathcal{Z},\|(x,y)\|_{\mathcal{Z}}<\mathcal{R}_3\}$. Along the proof of case (C1), then for $(x,y)\in\mathcal{Y}\cap\partial\Omega_3$, we have $x(t)+y(t)\geqslant\varsigma_1\|x\|+\varsigma_2\|y\|\geqslant\varsigma\|(x,y)\|_{\mathcal{Z}}$ for $\forall t\in[\sigma_1,\sigma_2]$. Furthermore, we give

$$\begin{aligned}\mathscr{A}_1(x,y)(\sigma_1)\geqslant&\mu^{r_1-1}\int_1^e(\log\sigma_1)^{\beta_1-1}\mathscr{M}_1(s)\varphi_{r_1}\left(\int_1^e\Gamma(\alpha_1)\varrho_{\alpha_1}(s)\rho_{\alpha_1}(\tau)f_1(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\&+\nu^{r_2-1}\int_1^e(\log\sigma_1)^{\beta_1-1}\mathscr{N}_2(s)\varphi_{r_2}\left(\int_1^e\Gamma(\alpha_1)\varrho_{\alpha_2}(s)\rho_{\alpha_2}(\tau)f_2(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\\geqslant&\mu^{r_1-1}\int_{\sigma_1}^{\sigma_2}(\log\sigma_1)^{\beta_1-1}\mathscr{M}_1(s)\varphi_{r_1}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_1)\varrho_{\alpha_1}(s)\rho_{\alpha_1}(\tau)\frac{1}{\varepsilon}\varphi_{p_1}(x(\tau)+y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\\geqslant&\varsigma_1\mu^{r_1-1}\varphi_{r_1}(1/\varepsilon)\int_{\sigma_1}^{\sigma_2}\mathscr{M}_1(s)\varphi_{r_1}(\varrho_{\alpha_1}(s))\varphi_{r_1}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_1)\rho_{\alpha_1}(\tau)\varphi_{p_1}(\varsigma\|(x,y)\|_{\mathcal{Z}})\frac{d\tau}{\tau}\right)\frac{ds}{s}\\=&\varsigma\varsigma_1\varphi_{r_1}(\mu/\varepsilon)\mathfrak{M}'_1\|(x,y)\|_{\mathcal{Z}}\geqslant\|(x,y)\|_{\mathcal{Z}},\end{aligned}\tag{3.29}$$

which implies

$$\|\mathscr{A}(x,y)\|_{\mathcal{Z}}=\|\mathscr{A}_1(x,y)\|+\|\mathscr{A}_2(x,y)\|\geqslant\mathscr{A}_1(x,y)(\sigma_1)\geqslant\|(x,y)\|_{\mathcal{Z}}\ \text{ for all }(x,y)\in\mathcal{Y}\cap\partial\Omega_3.\tag{3.30}$$

Along the proof of case (D1), the function $\mathcal{F}_m$ $(m=1,2)$ defined in case (D1) satisfies the conditions

$$\limsup_{u\to\infty}\max_{t\in[1,e]}\frac{\mathcal{F}_1(t,u)}{\varphi_{p_1}(u)}=0\ \text{ and }\ \limsup_{u\to\infty}\max_{t\in[1,e]}\frac{\mathcal{F}_2(t,u)}{\varphi_{p_2}(u)}=\mathfrak{F}^s_{2\infty}.\tag{3.31}$$

From (3.31), for $\varepsilon>0$, there exists $\mathcal{R}'_4>0$ such that for all $t\in[1,e]$ and $u\geqslant\mathcal{R}'_4$, we have

$$\frac{\mathcal{F}_1(t,u)}{\varphi_{p_1}(u)}\leqslant\limsup_{u\to\infty}\max_{t\in[1,e]}\frac{\mathcal{F}_1(t,u)}{\varphi_{p_1}(u)}+\varepsilon=\varepsilon\Rightarrow\mathcal{F}_1(t,u)\leqslant\varepsilon\varphi_{p_m}(u),\tag{3.32a}$$

$$\frac{\mathcal{F}_2(t,u)}{\varphi_{p_2}(u)}\leqslant\limsup_{u\to\infty}\max_{t\in[1,e]}\frac{\mathcal{F}_2(t,u)}{\varphi_{p_2}(u)}+\varepsilon=\mathfrak{F}^s_{2\infty}+\varepsilon\Rightarrow\mathcal{F}_2(t,u)\leqslant(\mathfrak{F}^s_{2\infty}+\varepsilon)\varphi_{p_2}(u).\tag{3.32b}$$

We consider the set $\Omega_4 = \{(x,y) \in \mathcal{Y}, \|(x,y)\|_{\mathcal{Z}} < \mathcal{R}_4\}$ with $\mathcal{R}_4 = \max\{2\mathcal{R}_3, \mathcal{R}'_4\}$. According to the definition of $\mathcal{F}_m$ $(m = 1,2)$, for $(x,y) \in \mathcal{Y} \cap \partial\Omega_4$, we can observe

$$f_1(t,x(t),y(t)) \leqslant \mathcal{F}_1(t,\|(x,y)\|_{\mathcal{Z}}) \leqslant \varepsilon\varphi_{p_1}(\|(x,y)\|_{\mathcal{Z}}) \text{ for all } t \in [1,e], \tag{3.33a}$$
$$f_2(t,x(t),y(t)) \leqslant \mathcal{F}_2(t,\|(x,y)\|_{\mathcal{Z}}) \leqslant (\mathfrak{F}^s_{2\infty} + \varepsilon)\varphi_{p_2}(\|(x,y)\|_{\mathcal{Z}}) \text{ for all } t \in [1,e], \tag{3.33b}$$

By applying the inequality (3.33) and Lemmas 2.5 and 2.8, then we deduce the following results

$$\mathscr{A}_1(x,y)(t) \leqslant [\varphi_{r_1}(\mu\varepsilon)\mathfrak{M}_1 + \varphi_{r_2}(\nu(\mathfrak{F}^s_{2\infty} + \varepsilon))\mathfrak{N}_2]\|(x,y)\|_{\mathcal{Z}} \leqslant \pi_1\|(x,y)\|_{\mathcal{Z}}, \quad \forall t \in [1,e], \tag{3.34a}$$
$$\mathscr{A}_2(x,y)(t) \leqslant [\varphi_{r_2}(\nu(\mathfrak{F}^s_{2\infty} + \varepsilon))\mathfrak{M}_2 + \varphi_{r_1}(\mu\varepsilon)\mathfrak{N}_1]\|(x,y)\|_{\mathcal{Z}} \leqslant \pi_2\|(x,y)\|_{\mathcal{Z}}, \quad \forall t \in [1,e]. \tag{3.34b}$$

Hence, it follows from (3.34) that we deduce

$$\|\mathscr{A}(x,y)\|_{\mathcal{Z}} = \|\mathscr{A}_1(x,y)\| + \|\mathscr{A}_2(x,y)\| \leqslant (\pi_1 + \pi_2)\|(x,y)\|_{\mathcal{Z}} = \|(x,y)\|_{\mathcal{Z}} \text{ for all } (x,y) \in \mathcal{Y} \cap \partial\Omega_4. \tag{3.35}$$

It follows from (3.30), (3.35) and case (D6) that system (1.1)-(1.2) has at least one positive solution $(x(t), y(t))$, $t \in [1,e]$. Hence, all conclusions of the theorem are obtained. □

Next, by using the proof by contradiction, we present some sufficient conditions for nonexistence of positive solution of addressed system (1.1)-(1.2) under the different intervals of μ and v.

Theorem 3.3. *Let $\Delta > 0$, $[\sigma_1, \sigma_2] \subset [1,e]$ with $1 < \sigma_1 < \sigma_2 \leqslant e$.*

(E1) *If there exist two positive constants Λ_m such that $f_m(t,x,y) \leqslant \Lambda_m\varphi_{p_m}(x+y)$ for $\forall t \in [1,e]$ and $x,y \geqslant 0$, $m = 1,2$, then there exist two positive constants μ_0 and ν_0 such that system (1.1)-(1.2) has no positive solution for every $\mu \in (0,\mu_0)$ and $\nu \in (0,\nu_0)$.*

(E2) *If there exist a positive constant Θ_1 such that $f_1(t,x,y) \geqslant \Theta_1\varphi_{p_1}(x+y)$ for $\forall t \in [\sigma_1,\sigma_2]$ and $x,y \geqslant 0$, then there exist a positive constant μ'_0 such that system (1.1)-(1.2) has no positive solution for every $\mu \in (\mu'_0, \infty)$ and $\nu \in (0,\infty)$.*

(E3) *If there exist a positive constant Θ_2 such that $f_2(t,x,y) \geqslant \Theta_2\varphi_{p_2}(x+y)$ for $\forall t \in [\sigma_1,\sigma_2]$ and $x,y \geqslant 0$, then there exist a positive constant ν'_0 such that system (1.1)-(1.2) has no positive solution for every $\mu \in (0,\infty)$ and $\nu \in (\nu'_0, \infty)$.*

(E4) *If there exist two positive constants Θ_m such that $f_m(t,x,y) \geqslant \Theta_m\varphi_{p_m}(x+y)$ for $\forall t \in [\sigma_1,\sigma_2]$ and $x,y \geqslant 0$, $m = 1,2$, then there exist two positive constants μ^*_0 and ν^*_0 such that system (1.1)-(1.2) has no positive solution for every $\mu \in (\mu^*_0, \infty)$ and $\nu \in (\nu^*_0, \infty)$.*

(E5) *If $\mathfrak{F}^s_{m0}, \mathfrak{F}^s_{m\infty} \in [0,\infty)$ $(m = 1,2)$, then there exist two positive constants μ_0 and ν_0 such that system (1.1)-(1.2) has no positive solution for every $\mu \in (0,\mu_0)$ and $\nu \in (0,\nu_0)$.*

(E6) *If $\mathfrak{F}^i_{10}, \mathfrak{F}^i_{1\infty} \in (0,\infty]$, $f_1(t,x,y) > 0$ for $\forall t \in [\sigma_1,\sigma_2]$ and $x,y \geqslant 0$ with $x+y > 0$, then there exist a positive constant μ'_0 such that system (1.1)-(1.2) has no positive solution for every $\mu \in (\mu'_0,\infty)$ and $\nu \in (0,\infty)$.*

(E7) *If $\mathfrak{F}^i_{20}, \mathfrak{F}^i_{2\infty} \in (0,\infty]$, $f_2(t,x,y) > 0$ for $\forall t \in [\sigma_1,\sigma_2]$ and $x,y \geqslant 0$ with $x+y > 0$, then there exist a positive constant μ'_0 such that system (1.1)-(1.2) has no positive solution for every $\mu \in (0,\infty)$ and $\nu \in (\nu'_0,\infty)$.*

(E8) *If $\mathfrak{F}^i_{m0}, \mathfrak{F}^i_{m\infty} \in (0,\infty]$, $f_m(t,x,y) > 0$ for $\forall t \in [\sigma_1,\sigma_2]$ and $x,y \geqslant 0$ with $x+y > 0$, $m = 1,2$, then there exist two positive constants μ^*_0 and ν^*_0 such that system (1.1)-(1.2) has no positive solution for every $\mu \in (\mu^*_0,\infty)$ and $\nu \in (\nu^*_0,\infty)$.*

Proof. Case (E1) Let μ_0 and ν_0 be defined by

$$\mu_0 = \frac{1}{\Lambda_1}\min\left\{\varphi_{p_1}\left(\frac{\pi_1 b_{11}}{\mathfrak{M}_1}\right), \varphi_{p_1}\left(\frac{\pi_2 b_{12}}{\mathfrak{N}_1}\right)\right\} \text{ and } \nu_0 = \frac{1}{\Lambda_2}\min\left\{\varphi_{p_2}\left(\frac{\pi_1 b_{21}}{\mathfrak{N}_2}\right), \varphi_{p_2}\left(\frac{\pi_2 b_{22}}{\mathfrak{M}_2}\right)\right\}, \tag{3.36}$$

where $b_{mn}, \pi_m \in (0,1)$ $(m,n = 1,2)$ such that $b_{1n} + b_{2n} = 1$ $(n = 1,2)$ and $\pi_1 + \pi_2 = 1$. In other word, we will prove that system (1.1)-(1.2) has no positive solution for every $\mu \in (0,\mu_0)$ and $\nu \in (0,\nu_0)$.

For $\mu \in (0,\mu_0)$ and $\nu \in (0,\nu_0)$, we assume that system (1.1)-(1.2) has a positive solution $(x(t), y(t))$, $t \in [1,e]$. From Lemmas 2.5 and 2.8, we deduce

$$\begin{aligned}
x(t) = \mathscr{A}_1(x,y)(t) \leqslant & \mu^{r_1-1}\int_1^e \mathscr{M}_1(s)\varphi_{r_1}\left(\int_1^e (\alpha_1-1)\rho_{\alpha_1}(\tau)\Lambda_1\varphi_{p_1}(x(\tau)+y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\
& + \nu^{r_2-1}\int_1^e \mathscr{N}_2(s)\varphi_{r_2}\left(\int_1^e (\alpha_2-1)\rho_{\alpha_2}(\tau)\Lambda_2\varphi_{p_2}(x(\tau)+y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\
< & \mu_0^{r_1-1}\varphi_{r_1}(\Lambda_1)\int_1^e \mathscr{M}_1(s)\varphi_{r_1}\left(\int_1^e (\alpha_1-1)\rho_{\alpha_1}(\tau)\varphi_{p_1}(\|x\|+\|y\|)\frac{d\tau}{\tau}\right)\frac{ds}{s} \\
& + \nu_0^{r_2-1}\varphi_{r_2}(\Lambda_2)\int_1^e \mathscr{N}_2(s)\varphi_{r_2}\left(\int_1^e (\alpha_2-1)\rho_{\alpha_2}(\tau)\varphi_{p_2}(\|x\|+\|y\|)\frac{d\tau}{\tau}\right)\frac{ds}{s} \\
= & [\varphi_{r_1}(\mu_0\Lambda_1)\mathfrak{M}_1 + \varphi_{r_2}(\nu_0\Lambda_2)\mathfrak{N}_2]\|(x,y)\|_Z \leqslant \pi_1\|(x,y)\|_Z, \quad \forall t\in[1,e],
\end{aligned} \tag{3.37a}$$

$$\begin{aligned}
y(t) = \mathscr{A}_2(x,y)(t) \leqslant & \nu^{r_2-1}\int_1^e \mathscr{M}_2(s)\varphi_{r_2}\left(\int_1^e (\alpha_2-1)\rho_{\alpha_2}(\tau)\Lambda_2\varphi_{p_2}(x(\tau)+y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\
& + \mu^{r_1-1}\int_1^e \mathscr{N}_1(s)\varphi_{r_1}\left(\int_1^e (\alpha_1-1)\rho_{\alpha_1}(\tau)\Lambda_1\varphi_{p_1}(x(\tau)+y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\
< & \nu_0^{r_2-1}\varphi_{r_2}(\Lambda_2)\int_1^e \mathscr{M}_2(s)\varphi_{r_2}\left(\int_1^e (\alpha_2-1)\rho_{\alpha_2}(\tau)\varphi_{p_2}(\|x\|+\|y\|)\frac{d\tau}{\tau}\right)\frac{ds}{s} \\
& + \mu_0^{r_1-1}\varphi_{r_1}(\Lambda_1)\int_1^e \mathscr{N}_1(s)\varphi_{r_1}\left(\int_1^e (\alpha_1-1)\rho_{\alpha_1}(\tau)\varphi_{p_1}(\|x\|+\|y\|)\frac{d\tau}{\tau}\right)\frac{ds}{s} \\
= & [\varphi_{r_2}(\nu_0\Lambda_2)\mathfrak{M}_2 + \varphi_{r_1}(\mu_0\Lambda_1)\mathfrak{N}_1]\|(x,y)\|_Z \leqslant \pi_2\|(x,y)\|_Z, \quad \forall t\in[1,e].
\end{aligned} \tag{3.37b}$$

Hence, it follows from (3.37) that we obtain

$$\|(x,y)\|_Z = \|\mathscr{A}_1(x,y)\| + \|\mathscr{A}_2(x,y)\| < \pi_1\|(x,y)\|_Z + \pi_2\|(x,y)\|_Z = \|(x,y)\|_Z \ \text{ for } \forall t\in[1,e], \tag{3.38}$$

which is a contradiction. Hence, system (1.1)-(1.2) has no positive solution for every $\mu \in (0,\mu_0)$ and $\nu \in (0,\nu_0)$.

Case (E2) Let μ_0' be defined by

$$\mu_0' = \min\left\{\varphi_{p_1}\left(1/(\varsigma\varsigma_1\mathfrak{M}_1')\right), \varphi_{p_1}\left(1/(\varsigma\varsigma_2\mathfrak{N}_1')\right)\right\}/\Theta_1. \tag{3.39}$$

In other word, we will prove that system (1.1)-(1.2) has no positive solution for every $\mu \in (\mu_0',\infty)$ and $\nu \in (0,\infty)$. For $\mu \in (\mu_0',\infty)$ and $\nu \in (0,\infty)$, we assume that system (1.1)-(1.2) has a positive solution $(x(t), y(t))$, $t \in [1,e]$. If $\varsigma_1\mathfrak{M}_1' > \varsigma_2\mathfrak{N}_1'$, then $\mu_0' = (1/\Theta_1)\varphi_{p_1}(1/(\varsigma\varsigma_1\mathfrak{M}_1'))$. From Lemmas 2.5 and 2.8, we observe

$$\begin{aligned}
x(\sigma_1) = \mathscr{A}_1(x,y)(\sigma_1) \geqslant & \mu^{r_1-1}\int_1^e (\log\sigma_1)^{\beta_1-1}\mathscr{M}_1(s)\varphi_{r_1}\left(\int_1^e \Gamma(\alpha_1)\varrho_{\alpha_1}(s)\rho_{\alpha_1}(\tau)f_1(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\
> & \varsigma_1(\mu_0')^{r_1-1}\varphi_{r_1}(\Theta_1)\int_{\sigma_1}^{\sigma_2}\mathscr{M}_1(s)\varphi_{r_1}(\varrho_{\alpha_1}(s))\varphi_{r_1}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_1)\rho_{\alpha_1}(\tau)\varphi_{p_1}(\varsigma\|(x,y)\|_Z)\frac{d\tau}{\tau}\right)\frac{ds}{s} \\
= & \varsigma\varsigma_1\varphi_{r_1}(\mu_0'\Theta_1)\mathfrak{M}_1'\|(x,y)\|_Z \geqslant \|(x,y)\|_Z,
\end{aligned} \tag{3.40}$$

which implies $\|(x,y)\|_Z = \|\mathscr{A}_1(x,y)\| + \|\mathscr{A}_2(x,y)\| \geqslant \|\mathscr{A}_1(x,y)\| > \|(x,y)\|_Z$, which is a contradiction. If $\varsigma_1\mathfrak{M}_1' < \varsigma_2\mathfrak{N}_1'$, then $\mu_0' = (1/\Theta_1)\varphi_{p_1}(1/(\varsigma\varsigma_2\mathfrak{N}_1'))$. From Lemmas 2.5 and 2.8, we observe

$$y(\sigma_1) = \mathscr{A}_2(x,y)(\sigma_1) \geqslant \mu^{r_1-1}\int_1^e (\log\sigma_1)^{\beta_2-1}\mathscr{N}_1(s)\varphi_{r_1}\left(\int_1^e \Gamma(\alpha_1)\varrho_{\alpha_1}(s)\rho_{\alpha_1}(\tau)f_1(\tau,x(\tau),y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}$$

$$\geqslant \mu^{r_1-1}\int_{\sigma_1}^{\sigma_2}(\log\sigma_1)^{\beta_2-1}\mathscr{N}_1(s)\varphi_{r_1}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_1)\varrho_{\alpha_1}(s)\rho_{\alpha_1}(\tau)\Theta_1\varphi_{p_1}(x(\tau)+y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}$$
$$>\varsigma_2(\mu_0')^{r_1-1}\varphi_{r_1}(\Theta_1)\int_{\sigma_1}^{\sigma_2}\mathscr{N}_1(s)\varphi_{r_1}(\varrho_{\alpha_1}(s))\varphi_{r_1}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_1)\rho_{\alpha_1}(\tau)\varphi_{p_1}(\varsigma\|(x,y)\|_Z)\frac{d\tau}{\tau}\right)\frac{ds}{s}$$
$$=\varsigma\varsigma_2\varphi_{r_1}(\mu_0'\Theta_1)\mathfrak{N}_1'\|(x,y)\|_Z \geqslant \|(x,y)\|_Z, \tag{3.41}$$

which implies $\|(x,y)\|_Z = \|\mathscr{A}_1(x,y)\| + \|\mathscr{A}_2(x,y)\| \geqslant \|\mathscr{A}_2(x,y)\| > \|(x,y)\|_Z$, which is a contradiction. Hence, system (1.1)-(1.2) has no positive solution for every $\mu \in (\mu_0',\infty)$ and $\nu \in (0,\infty)$.

Case (E3) Let ν_0' be defined by

$$\nu_0' = \min\left\{\varphi_{p_2}\left(1/(\varsigma\varsigma_2\mathfrak{M}_2')\right), \varphi_{p_2}\left(1/(\varsigma\varsigma_1\mathfrak{N}_2')\right)\right\}/\Theta_2. \tag{3.42}$$

In other word, we will prove that system (1.1)-(1.2) has no positive solution for every $\mu \in (0,\infty)$ and $\nu \in (\nu_0',\infty)$. Similar to the proof of case (E2), we can easily obtain that the conclusion of case (E3) holds.

Case (E4) Let μ_0^* and ν_0^* be defined by

$$\mu_0^* = \frac{1}{\Theta_1}\min\left\{\varphi_{p_1}\left(\frac{\omega_1 a_{11}}{\varsigma\varsigma_1\mathfrak{M}_1'}\right), \varphi_{p_1}\left(\frac{\omega_2 a_{12}}{\varsigma\varsigma_2\mathfrak{N}_1'}\right)\right\} \text{ and } \nu_0^* = \frac{1}{\Theta_2}\min\left\{\varphi_{p_2}\left(\frac{\omega_1 a_{21}}{\varsigma\varsigma_1\mathfrak{N}_2'}\right), \varphi_{p_2}\left(\frac{\omega_2 a_{22}}{\varsigma\varsigma_2\mathfrak{M}_2'}\right)\right\}, \tag{3.43}$$

where $a_{mn}, \omega_m \in [0,1]$ $(m,n=1,2)$ such that $a_{1n}+a_{2n}=1$ $(n=1,2)$ and $\omega_1+\omega_2=1$. In other word, we will prove that system (1.1)-(1.2) has no positive solution for every $\mu \in (0,\mu_0^*)$ and $\nu \in (0,\nu_0^*)$. For $\mu \in (0,\mu_0^*)$ and $\nu \in (0,\nu_0^*)$, we assume that system (1.1)-(1.2) has a positive solution $(x(t),y(t))$, $t\in[1,e]$. From Lemmas 2.5 and 2.8, we have

$$x(\sigma_1) = \mathscr{A}_1(x,y)(\sigma_1) \geqslant \mu^{r_1-1}\int_{\sigma_1}^{\sigma_2}(\log\sigma_1)^{\beta_1-1}\mathscr{M}_1(s)\varphi_{r_1}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_1)\varrho_{\alpha_1}(s)\rho_{\alpha_1}(\tau)\Theta_1\varphi_{p_1}(x(\tau)+y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}$$
$$+\nu^{r_2-1}\int_{\sigma_1}^{\sigma_2}(\log\sigma_1)^{\beta_1-1}\mathscr{N}_2(s)\varphi_{r_2}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_2)\varrho_{\alpha_2}(s)\rho_{\alpha_2}(\tau)\Theta_2\varphi_{p_2}(x(\tau)+y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}$$
$$>\varsigma_1(\mu_0^*)^{r_1-1}\varphi_{r_1}(\Theta_1)\int_{\sigma_1}^{\sigma_2}\mathscr{M}_1(s)\varphi_{r_1}(\varrho_{\alpha_1}(s))\varphi_{r_1}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_1)\rho_{\alpha_1}(\tau)\varphi_{p_1}(\varsigma\|(x,y)\|_Z)\frac{d\tau}{\tau}\right)\frac{ds}{s}$$
$$+\varsigma_1(\nu_0^*)^{r_2-1}\varphi_{r_2}(\Theta_2)\int_{\sigma_1}^{\sigma_2}\mathscr{N}_2(s)\varphi_{r_2}(\varrho_{\alpha_2}(s))\varphi_{r_2}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_2)\rho_{\alpha_2}(\tau)\varphi_{p_2}(\varsigma\|(x,y)\|_Z)\frac{d\tau}{\tau}\right)\frac{ds}{s}$$
$$=[\varsigma\varsigma_1\varphi_{r_1}(\mu_0^*\Theta_1)\mathfrak{M}_1' + \varsigma\varsigma_1\varphi_{r_2}(\nu_0^*\Theta_2)\mathfrak{N}_2']\|(x,y)\|_Z \geqslant \omega_1\|(x,y)\|_Z, \tag{3.44a}$$

$$y(\sigma_1) = \mathscr{A}_2(x,y)(\sigma_1) \geqslant \nu^{r_2-1}\int_{\sigma_1}^{\sigma_2}(\log\sigma_1)^{\beta_2-1}\mathscr{M}_2(s)\varphi_{r_2}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_2)\varrho_{\alpha_2}(s)\rho_{\alpha_2}(\tau)\Theta_2\varphi_{p_2}(x(\tau)+y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}$$
$$+\mu^{r_1-1}\int_{\sigma_1}^{\sigma_2}(\log\sigma_1)^{\beta_2-1}\mathscr{N}_1(s)\varphi_{r_1}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_1)\varrho_{\alpha_1}(s)\rho_{\alpha_1}(\tau)\Theta_1\varphi_{p_1}(x(\tau)+y(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}$$
$$>\varsigma_2(\nu_0^*)^{r_2-1}\varphi_{r_2}(\Theta_2)\int_{\sigma_1}^{\sigma_2}\mathscr{M}_2(s)\varphi_{r_2}(\varrho_{\alpha_2}(s))\varphi_{r_2}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_2)\rho_{\alpha_2}(\tau)\varphi_{p_2}(\varsigma\|(x,y)\|_Z)\frac{d\tau}{\tau}\right)\frac{ds}{s}$$
$$+\varsigma_2(\mu_0^*)^{r_1-1}\varphi_{r_1}(\Theta_1)\int_{\sigma_1}^{\sigma_2}\mathscr{N}_1(s)\varphi_{r_1}(\varrho_{\alpha_1}(s))\varphi_{r_1}\left(\int_{\sigma_1}^{\sigma_2}\Gamma(\alpha_1\rho_{\alpha_1}(\tau)\varphi_{p_1}(\varsigma\|(x,y)\|_Z)\frac{d\tau}{\tau}\right)\frac{ds}{s}$$
$$=[\varsigma\varsigma_2\varphi_{r_2}(\nu_0^*\Theta_2)\mathfrak{M}_2' + \varsigma\varsigma_2\varphi_{r_1}(\mu_0^*\Theta_1)\mathfrak{N}_1']\|(x,y)\|_Z \geqslant \omega_2\|(x,y)\|_Z. \tag{3.44b}$$

Hence, it follows from (3.44) that we obtain

$$\|(x,y)\|_Z = \|\mathscr{A}_1(x,y)\| + \|\mathscr{A}_2(x,y)\| > \omega_1\|(x,y)\|_Z + \omega_2\|(x,y)\|_Z = \|(x,y)\|_Z, \tag{3.45}$$

which is a contradiction. Hence, system (1.1)-(1.2) has no positive solution for every $\mu \in (\mu_0^*,\infty)$ and $\nu \in (\nu_0^*,\infty)$.

Case (E5) Based on the proofs of Theorems 3.1 and 3.2, from the conditions $\mathfrak{F}^s_{m0}, \mathfrak{F}^s_{m\infty} \in [0,\infty)$ $(m = 1,2)$, we can obtain easily that there exist two positive constants Λ_m such that $f_m(t,x,y) \leqslant \Lambda_m\varphi_{p_m}(x+y)$ for $\forall t \in [1,e]$ and $x,y \geqslant 0$, $m = 1,2$, then the assumptions of case (E1) hold. Furthermore, we have the conclusion of case (E1).

Case (E6) Based on the proofs of Theorems 3.1 and 3.2, from the conditions $\mathfrak{F}^i_{10}, \mathfrak{F}^i_{1\infty} \in (0,\infty]$, $f_1(t,x,y) > 0$ for $\forall t \in [\sigma_1,\sigma_2]$ and $x,y \geqslant 0$ with $x+y>0$, we can obtain easily that there exist a positive constant Θ_1 such that $f_1(t,x,y) \geqslant \Theta_1\varphi_{p_1}(x+y)$ for $\forall t \in [\sigma_1,\sigma_2]$ and $x,y \geqslant 0$, then the assumptions of case (E2) hold. Furthermore, we have the conclusion of case (E2).

Case (E7) Based on the proofs of Theorems 3.1 and 3.2, from the conditions $\mathfrak{F}^i_{20}, \mathfrak{F}^i_{2\infty} \in (0,\infty]$, $f_2(t,x,y) > 0$ for $\forall t \in [\sigma_1,\sigma_2]$ and $x,y \geqslant 0$ with $x+y>0$, we can obtain easily that there exist a positive constant Θ_2 such that $f_2(t,x,y) \geqslant \Theta_2\varphi_{p_2}(x+y)$ for $\forall t \in [\sigma_1,\sigma_2]$ and $x,y \geqslant 0$, then the assumptions of case (E3) hold. Furthermore, we have the conclusion of case (E3).

Case (E8) Based on the proofs of Theorems 3.1 and 3.2, from the conditions $\mathfrak{F}^i_{m0}, \mathfrak{F}^i_{m\infty} \in (0,\infty]$, $f_m(t,x,y) > 0$ for $\forall t \in [\sigma_1,\sigma_2]$ and $x,y \geqslant 0$ with $x+y>0$, $m=1,2$, we can obtain easily that there exist two positive constants Λ_m such that $f_m(t,x,y) \leqslant \Lambda_m\varphi_{p_m}(x+y)$ for $\forall t \in [1,e]$ and $x,y \geqslant 0$, $m=1,2$, then the assumptions of case (E4) hold. Furthermore, we have the conclusion of case (E4). This completes the proofs of theorem. □

4. Some examples

Consider the nonlinear Hadamard fractional differential systems with p-Laplacian operators

$$\mathscr{D}^{\frac{5}{2}}\varphi_{p_1}(\mathscr{D}^{\frac{7}{2}}x(t)) = \mu f_1(t,x(t),y(t)), \quad \mathscr{D}^{\frac{8}{3}}\varphi_{p_2}(\mathscr{D}^{\frac{14}{3}}y(t)) = \nu f_2(t,x(t),y(t)), \quad t \in (1,e) \tag{4.1}$$

with the following coupled nonlocal Riemann-Stieltjes integral boundary conditions

$$\begin{aligned} &\delta x(1) = \delta^2 x(1) = 0, \ \mathscr{D}^{\frac{5}{3}}x(e) = \int_1^{\sqrt{e}} \log t\,\mathscr{D}^{\frac{3}{2}}y(t)\frac{dH_1(t)}{t} + \int_1^{e^{2/3}} t(\log t)^{\frac{1}{6}}\mathscr{D}^{\frac{5}{2}}y(t)\frac{dH_2(t)}{t},\\ &\delta y(1) = \delta^2 y(1) = \delta^3 y(1) = 0, \ \mathscr{D}^{\frac{7}{2}}y(e) = \int_1^{e^{3/4}} 3t(\log t)^{\frac{1}{6}}\mathscr{D}^{\frac{2}{3}}x(t)\frac{dK_1(t)}{t},\\ &\mathscr{D}^{\frac{7}{2}}x(1) = \mathscr{D}^{\frac{7}{2}}x(e) = \delta(\varphi_{p_1}(\mathscr{D}^{\frac{7}{2}}x(1))) = 0, \ \mathscr{D}^{\beta_2}y(1) = \mathscr{D}^{\frac{14}{3}}y(e) = \delta(\varphi_{p_2}(\mathscr{D}^{\frac{14}{3}}y(1))) = 0, \end{aligned} \tag{4.2}$$

where $p_1 = 4$, $r_1 = 4/3$, $p_2 = 3$, $r_2 = 3/2$, $H_1(t) = 7t$ for all $t \in [1,e]$, $H_2(t) = \{0, t \in [1,\sqrt{e}); 2, t \in [\sqrt{e},e]\}$, $K_1(t) = \{1, t \in [1,\sqrt{e}); 4, t \in [\sqrt{e},e]\}$, $\Delta_1 = (\Gamma(\beta_1)/\Gamma(\beta_1-\gamma_1))\int_1^{\vartheta_1}(\log t)^{\beta_1-\gamma_1-1}k_1(t)dK_1(t)/t = 9\Gamma(\frac{7}{2})/(4\Gamma(\frac{17}{6})) \approx 4.3360$, $\Delta_2 = (\Gamma(\beta_2)/\Gamma(\beta_2-\eta_1))\int_1^{\theta_1}(\log t)^{\beta_2-\eta_1-1}h_1(t)(dH_1(t)/t) + (\Gamma(\beta_2)/\Gamma(\beta_2-\eta_2))\int_1^{\theta_2}(\log t)^{\beta_2-\eta_2-1}h_2(t)(dH_2(t)/t) = 7\Gamma(\frac{14}{3})/(2^{\frac{19}{6}}\Gamma(\frac{19}{6})) + \frac{1}{2}\Gamma(\frac{14}{3})/\Gamma(\frac{17}{6}) \approx 9.1556$, $\Delta = \Gamma(\beta_1)\Gamma(\beta_2)/(\Gamma(\beta_1-\gamma_0)\Gamma(\beta_2-\eta_0)) - \Delta_1\Delta_2 = \Gamma(\frac{7}{2})\Gamma(\frac{14}{3})/(\Gamma(\frac{11}{6})\Gamma(\frac{7}{6})) - \Delta_1\Delta_2 \approx 16.3266 > 0$, and

$$G_{\alpha_1,\alpha_1}(t,s) = \frac{1}{\Gamma(\frac{5}{2})}\begin{cases} (\log t)^{\frac{3}{2}}(1-\log s)^{\frac{3}{2}} - (\log(t/s))^{\frac{3}{2}}, & 1 \leqslant s \leqslant t \leqslant e,\\ (\log t)^{\frac{3}{2}}(1-\log s)^{\frac{3}{2}}, & 1 \leqslant t \leqslant s \leqslant e, \end{cases} \quad \alpha_1 = \frac{5}{2}, \tag{4.3a}$$

$$G_{\alpha_2,\alpha_2}(t,s) = \frac{1}{\Gamma(\frac{8}{3})}\begin{cases} (\log t)^{\frac{5}{3}}(1-\log s)^{\frac{5}{3}} - (\log(t/s))^{\frac{5}{3}}, & 1 \leqslant s \leqslant t \leqslant e,\\ (\log t)^{\frac{5}{3}}(1-\log s)^{\frac{5}{3}}, & 1 \leqslant t \leqslant s \leqslant e, \end{cases} \quad \alpha_2 = \frac{8}{3}, \tag{4.3b}$$

$$G_{\beta_1,\beta_1-\gamma_0}(t,s) = \frac{1}{\Gamma(\frac{7}{2})}\begin{cases} (\log t)^{\frac{5}{2}}(1-\log s)^{\frac{5}{6}} - (\log(t/s))^{\frac{5}{2}}, & 1 \leqslant s \leqslant t \leqslant e,\\ (\log t)^{\frac{5}{2}}(1-\log s)^{\frac{5}{6}}, & 1 \leqslant t \leqslant s \leqslant e, \end{cases} \quad \beta_1 = \frac{7}{2}, \quad \gamma_0 = \frac{5}{3}, \tag{4.3c}$$

$$G_{\beta_2,\beta_2-\eta_0}(t,s) = \frac{1}{\Gamma(\frac{14}{3})}\begin{cases} (\log t)^{\frac{11}{3}}(1-\log s)^{\frac{1}{6}} - (\log(t/s))^{\frac{11}{3}}, & 1 \leqslant s \leqslant t \leqslant e,\\ (\log t)^{\frac{11}{3}}(1-\log s)^{\frac{1}{6}}, & 1 \leqslant t \leqslant s \leqslant e, \end{cases} \quad \beta_2 = \frac{14}{3}, \quad \eta_0 = \frac{7}{2}, \tag{4.3d}$$

$$G_{\beta_1-\gamma_1,\beta_1-\gamma_0}(t,s) = \frac{1}{\Gamma(\frac{17}{6})}\begin{cases} (\log t)^{\frac{11}{6}}(1-\log s)^{\frac{5}{6}} - (\log(t/s))^{\frac{11}{6}}, & 1 \leqslant s \leqslant t \leqslant e,\\ (\log t)^{\frac{11}{6}}(1-\log s)^{\frac{5}{6}}, & 1 \leqslant t \leqslant s \leqslant e, \end{cases} \quad \gamma_0 = \frac{5}{3}, \quad \gamma_1 = \frac{2}{3}, \tag{4.3e}$$

$$G_{\beta_2-\eta_1,\beta_2-\eta_0}(t,s)=\frac{1}{\Gamma(\frac{19}{6})}\begin{cases}(\log t)^{\frac{13}{6}}(1-\log s)^{\frac{1}{6}}-(\log(t/s))^{\frac{13}{6}}, & 1\leqslant s\leqslant t\leqslant e,\\ (\log t)^{\frac{13}{6}}(1-\log s)^{\frac{1}{6}}, & 1\leqslant t\leqslant s\leqslant e,\end{cases}\quad \eta_0=\frac{7}{2},\quad \eta_1=\frac{3}{2},\tag{4.3f}$$

$$G_{\beta_2-\eta_2,\beta_2-\eta_0}(t,s)=\frac{1}{\Gamma(\frac{17}{6})}\begin{cases}(\log t)^{\frac{11}{6}}(1-\log s)^{\frac{1}{6}}-(\log(t/s))^{\frac{11}{6}}, & 1\leqslant s\leqslant t\leqslant e,\\ (\log t)^{\frac{11}{6}}(1-\log s)^{\frac{1}{6}}, & 1\leqslant t\leqslant s\leqslant e,\end{cases}\quad \eta_0=\frac{7}{2},\quad \eta_2=\frac{11}{6}.\tag{4.3g}$$

From (4.3), then we have $\varrho_{\alpha_1}(t)=(\log t)^{\frac{3}{2}}(1-\log t)/\Gamma(\frac{3}{2})$, $\rho_{\alpha_1}(t)=(1-\log t)^{\frac{3}{2}}\log t/\Gamma(\frac{5}{2})$, $\varrho_{\alpha_2}(t)=(\log t)^{\frac{5}{3}}(1-\log t)/\Gamma(\frac{8}{3})$, $\rho_{\alpha_2}(t)=(1-\log t)^{\frac{5}{3}}\log t/\Gamma(\frac{8}{3})$, $\mathscr{G}_{\gamma_0}(s)=(1-\log s)^{\frac{5}{6}}(1-(1-\log s)^{\frac{5}{3}})/\Gamma(\frac{7}{2})$, $\mathscr{G}_{\eta_0}(s)=(1-\log s)^{\frac{1}{6}}(1-(1-\log s)^{\frac{7}{2}})/\Gamma(\frac{14}{3})$. Furthermore, from (4.3) and Lemma 2.8, we can deduce

$$\mathscr{M}_1(s)=\mathscr{G}_{\gamma_0}(s)+\frac{3\Delta_2\left(\frac{1}{2}\right)^{\frac{1}{6}}}{\Delta\Gamma(\frac{17}{6})}\begin{cases}\left(\frac{1}{2}\right)^{\frac{11}{6}}(1-\log s)^{\frac{5}{6}}-\left(\frac{1}{2}-\log s\right)^{\frac{11}{6}}, & 1\leqslant s<\sqrt{e},\\ \left(\frac{1}{2}\right)^{\frac{11}{6}}(1-\log s)^{\frac{5}{6}}, & \sqrt{e}\leqslant s\leqslant e,\end{cases}\tag{4.4a}$$

$$\begin{aligned}\mathscr{M}_2(s)=\mathscr{G}_{\eta_0}(s)&+\frac{21\Delta_1}{25\Delta\Gamma(\frac{19}{6})}\left(\left(\frac{1}{2}\right)^{\frac{19}{6}}(1-\log s)^{\frac{1}{6}}-\frac{1}{19}(12\log s+19)\left(\frac{1}{2}-\log s\right)^{\frac{19}{6}}\right)\\&+\frac{2\Delta_1\left(\frac{1}{2}\right)^{\frac{1}{6}}}{\Delta\Gamma(\frac{17}{6})}\begin{cases}\left(\frac{1}{2}\right)^{\frac{11}{6}}(1-\log s)^{\frac{1}{6}}-\left(\frac{1}{2}-\log s\right)^{\frac{11}{6}}, & 1\leqslant s<\sqrt{e},\\ \left(\frac{1}{2}\right)^{\frac{11}{6}}(1-\log s)^{\frac{1}{6}}, & \sqrt{e}\leqslant s\leqslant e,\end{cases}\end{aligned}\tag{4.4b}$$

$$\mathscr{N}_1(s)=\frac{3\Gamma(\frac{7}{2})\left(\frac{1}{2}\right)^{\frac{1}{6}}}{\Delta\Gamma(\frac{11}{6})\Gamma(\frac{17}{6})}\begin{cases}\left(\frac{1}{2}\right)^{\frac{11}{6}}(1-\log s)^{\frac{5}{6}}-\left(\frac{1}{2}-\log s\right)^{\frac{11}{6}}, & 1\leqslant s<\sqrt{e},\\ \left(\frac{1}{2}\right)^{\frac{11}{6}}(1-\log s)^{\frac{5}{6}}, & \sqrt{e}\leqslant s\leqslant e,\end{cases}\tag{4.4c}$$

$$\begin{aligned}\mathscr{N}_2(s)=&\frac{21\Gamma(\frac{14}{3})}{25\Delta\Gamma(\frac{7}{6})\Gamma(\frac{19}{6})}\left(\left(\frac{1}{2}\right)^{\frac{19}{6}}(1-\log s)^{\frac{1}{6}}-\frac{1}{19}(12\log s+19)\left(\frac{1}{2}-\log s\right)^{\frac{19}{6}}\right)\\&+\frac{2\Gamma(\frac{14}{3})\left(\frac{1}{2}\right)^{\frac{1}{6}}}{\Delta\Gamma(\frac{7}{6})\Gamma(\frac{17}{6})}\begin{cases}\left(\frac{1}{2}\right)^{\frac{11}{6}}(1-\log s)^{\frac{1}{6}}-\left(\frac{1}{2}-\log s\right)^{\frac{11}{6}}, & 1\leqslant s<\sqrt{e},\\ \left(\frac{1}{2}\right)^{\frac{11}{6}}(1-\log s)^{\frac{1}{6}}, & \sqrt{e}\leqslant s\leqslant e.\end{cases}\end{aligned}\tag{4.4d}$$

Here let $\sigma_1=\sqrt[4]{e}$ and $\sigma_2=\sqrt[4]{e^3}$, then $\left[\sqrt[4]{e},\sqrt[4]{e^3}\right]\subset[1,e]$, $\varsigma_1=(1/4)^{5/2}$, $\varsigma_2=(1/4)^{11/3}$ and $\varsigma=(1/4)^{11/3}$. It follows from (4.4) that $\mathfrak{M}_1\approx 0.084950$, $\mathfrak{M}_2\approx 0.033247$, $\mathfrak{N}_1\approx 0.017544$, $\mathfrak{N}_2\approx 0.059091$, $\mathfrak{M}_1'\approx 0.027963$, $\mathfrak{M}_2'\approx 0.009042$, $\mathfrak{N}_1'\approx 0.005792$ and $\mathfrak{N}_2'\approx 0.011025$. Let $a_{mn}=b_{mn}=\omega_m=\pi_m=1/2$ $(m,n=1,2)$. Then $\max\{\varphi_{p_1}(\omega_1 a_{11}/(\varsigma\varsigma_1\mathfrak{M}_1')),\varphi_{p_1}(\omega_2 a_{12}/(\varsigma\varsigma_2\mathfrak{N}_1'))\}\approx\max\{9.8215\times 10^{13},1.4147\times 10^{18}\}=1.4147\times 10^{18}$, $\min\{\varphi_{p_1}(\pi_1 b_{11}/\mathfrak{M}_1),\varphi_{p_1}(\pi_2 b_{12}/\mathfrak{N}_1)\}\approx\min\{25.4877,2.8936\times 10^{3}\}=25.4877$, $\max\{\varphi_{p_2}(\omega_1 a_{21}/(\varsigma\varsigma_1\mathfrak{N}_2')),\varphi_{p_2}(\omega_2 a_{22}/(\varsigma\varsigma_2\mathfrak{M}_1'))\}\approx\max\{1.3694\times 10^{10},5.1709\times 10^{11}\}=5.1709\times 10^{11}$, $\min\{\varphi_{p_2}(\pi_1 b_{21}/\mathfrak{N}_2),\varphi_{p_2}(\pi_2 b_{22}/\mathfrak{M}_2)\approx\min\{17.8994,56.5425\}=17.8994$.

Example 4.1. Consider the Hadamard fractional differential systems (4.1) with

$$f_1(t,x,y)=(t+1)^{\hbar}(x^4+y^4),\quad f_2(t,x,y)=(2-\log t)^{\ell}(e^{(x+y)^2}-1)\ \text{ for all } t\in[1,e] \text{ and } x,y\geqslant 0,\tag{4.5}$$

where $\hbar,\ell>0$. From (3.1) and (4.5), we can know that $\mathfrak{F}^s_{10}=0$, $\mathfrak{F}^i_{1\infty}=\infty$, $\mathfrak{F}^s_{20}=2^{\ell}$, $\mathfrak{F}^i_{2\infty}=\infty$.

(F1) Let $\pi_1=\pi_2=1/2$. From case (C6) of Theorem 3.1, system (4.1)-(4.2) with (4.5) has at least one positive solution $(x(t),y(t))$, $t\in[1,e]$, for each $\mu\in(0,\infty)$ and $\nu\in(0,\mathfrak{L}_2')$, where $\mathfrak{L}_2'=1/(\mathfrak{F}^s_{20}(2\mathfrak{M}_2)^3)$.

(F2) From (4.5), we deduce that $f_2(t,x,y)\geqslant\Theta_2(x+y)^2$ for all $t\in\left[\sqrt[4]{e},\sqrt[4]{e^3}\right]$ and $x,y\geqslant 0$ with $\Theta_2=(5/4)^{\ell}$. From case (E3) of Theorem 3.3, system (4.1)-(4.2) with (4.5) has no positive solution for every $\mu\in(0,\infty)$ and $\nu\in(\nu_0',\infty)$, where $\nu_0'=1/(\Theta_2(\varsigma\varsigma_1\mathfrak{N}_2')^2)$ with $\varsigma_1=(1/4)^{5/2}$ and $\varsigma=(1/4)^{11/3}$.

Example 4.2. Consider the Hadamard fractional differential systems (4.1) with

$$f_1(t,x,y) = (\log t + 1)\frac{(2\times 10^{18}(x^3+y^3)^2 + (x+y)^3)(2+\sin(x^3+y^3))}{1+x^3+y^3}, \tag{4.6a}$$

$$f_2(t,x,y) = (2-\log t)\frac{(8\times 10^{11}(x^2+y^2)^2 + (x+y)^2)(3+\tan(x^2+y^2))}{1+x^2+y^2}, \tag{4.6b}$$

for all $t \in [1,e]$ and $x, y \geqslant 0$. From (3.1) and (4.6), we can know that $\mathfrak{F}_{10}^s = 4$, $\mathfrak{F}_{1\infty}^s = 1.2\times 10^{19}$, $\mathfrak{F}_{10}^i = 2$, $\mathfrak{F}_{1\infty}^i = 2\times 10^{18}$, $\mathfrak{F}_{20}^s = 6$, $\mathfrak{F}_{2\infty}^s = 6.4\times 10^{12}$, $\mathfrak{F}_{20}^i = 3$, $\mathfrak{F}_{2\infty}^i = 8\times 10^{11}$, $2\varphi_{p_1}(x+y) \leqslant f_1(t,x,y) \leqslant 1.2\times 10^{19}\varphi_{p_1}(x+y)$, $3\varphi_{p_2}(x+y) \leqslant f_2(t,x,y) \leqslant 6.4\times 10^{12}\varphi_{p_2}(x+y)$.

(G1) From case (C1) of Theorem 3.1, system (4.1)-(4.2) with (4.6) has at least one positive solution $(x(t), y(t))$, $t \in [1,e]$, for each $\mu \in (0.707350, 6.371925)$ and $\nu \in (0.646363, 2.983233)$.

(G2) From case (E1) of Theorem 3.3, system (4.1)-(4.2) with (4.6) has no positive solution for every $\mu \in (0, 2.123975\times 10^{-18})$ and $\nu \in (0, 2.796781\times 10^{-12})$.

(G3) From case (E4) of Theorem 3.3, system (4.1)-(4.2) with (4.6) has no positive solution for every $\mu \in (7.073500\times 10^{17}, \infty)$ and $\nu \in (1.723633\times 10^{11}, \infty)$.

Example 4.3. Consider the Hadamard fractional differential systems (4.1) with

$$f_1(t,x,y) = (\log t + 1)\frac{((x^3+y^3)^2 + (x+y)^3)(3\times 10^{13}+\tan(x^3+y^3))}{1+2\times 10^5(x^3+y^3)}, \tag{4.7a}$$

$$f_2(t,x,y) = (2-\log t)\frac{((x^2+y^2)^2 + (x+y)^2)(2\times 10^5+\sin(x^2+y^2))}{1+2\times 10^3(x^2+y^2)}, \tag{4.7b}$$

for all $t \in [1,e]$ and $x, y \geqslant 0$. From (3.1) and (4.7), we can know that $\mathfrak{F}_{10}^s = 6\times 10^{13}$, $\mathfrak{F}_{1\infty}^s = 3\times 10^5 + 1\times 10^{-5}$, $\mathfrak{F}_{10}^i = 3\times 10^{13}$, $\mathfrak{F}_{1\infty}^i = 5\times 10^{-6}$, $\mathfrak{F}_{20}^s = 4\times 10^5$, $\mathfrak{F}_{2\infty}^s = 3\times 10^2 + 1\times 10^{-3}$, $\mathfrak{F}_{20}^i = 2\times 10^5$, $\mathfrak{F}_{2\infty}^i = 5\times 10^{-4}$, $5\times 10^{-6}\varphi_{p_1}(x+y) \leqslant f_1(t,x,y) \leqslant 6\times 10^{13}\varphi_{p_1}(x+y)$, $5\times 10^{-4}\varphi_{p_2}(x+y) \leqslant f_2(t,x,y) \leqslant 4\times 10^5\varphi_{p_2}(x+y)$.

(H1) From case (D1) of Theorem 3.2, system (4.1)-(4.2) with (4.7) has at least one positive solution $(x(t), y(t))$, $t \in [1,e]$, for each $\mu \in (4.715667\times 10^4, 3.058524\times 10^6)$ and $\nu \in (2.585450\times 10^3, 3.579880\times 10^4)$.

(H2) From case (E1) of Theorem 3.3, system (4.1)-(4.2) with (4.7) has no positive solution for every $\mu \in (0, 4.247950\times 10^{13})$ and $\nu \in (0, 4.474850\times 10^5)$.

(H3) From case (E4) of Theorem 3.3, system (4.1)-(4.2) with (4.7) has no positive solution for every $\mu \in (2.829400\times 10^{23}, \infty)$ and $\nu \in (1.0341800\times 10^{15}, \infty)$.

5. Conclusions

In this paper, a class of nonlinear p-Laplacian Hadamard fractional differential systems with coupled nonlocal Riemann-Stieltjes integral boundary conditions have been investigated. First, we established the Green's functions of the considered systems and their properties. Then, by means of Guo-Krasnosel'skii fixed point theorem, some sufficient conditions for existence and nonexistence of positive solutions for the addressed systems have been obtained under the different intervals of the parameters μ and ν. Finally, some examples have been presented to show the effectiveness of the main results. Based on main results in this paper, nonlinear p-Laplacian/generalized p-Laplacian/variable Laplacian Hadamard/ψ-Hilfer fractional differential systems with other coupled nonlocal Riemann-Stieltjes integral/multi-points boundary conditions will be one of our future research topics.

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