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Riemann and Ricci Bourguignon Solitons on Three-Dimensional Quasi-Sasakian Manifolds

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Abstract. The aim of the present article is to analyze three-dimensional quasi-Sasakian manifolds admitting Riemann solitons and Ricci Bourguignon solitons.

1. Introduction

A non-linear pseudo parabolic evolution equation given by

$$\frac{\partial}{\partial t}g(x,t) = -2S(g(x,t)), \quad t \in [0,T), \quad g(x,0) = g_0 \tag{1}$$

is called Ricci flow [11] satisfied by the metric $g(x,t)$. In harmonic local coordinates around a point p, the Ricci tensor takes the form $S_{ij} = -\frac{1}{2}\Delta(g_{ij})(p)$. g_{ij} is local expression of the metric tensor g. Thus Ricci flow is analogous to heat flow.

It is well known that a fixed solution of a Ricci flow, upto diffeomorphisms and scaling, is known as a Ricci soliton given by the following formulation

$$S(g) + \frac{1}{2}\pounds_X g + \lambda g = 0, \tag{2}$$

where λ is a real number. The initial metric $g(x,0) = g_0$ is called the profile of the solution. The solution is called shrinking, steady or expanding according as $\lambda < 0, \lambda = 0, \lambda > 0$. If λ is a C^∞ function on the manifold, the Ricci soliton is called Ricci almost soliton.

The theory of Ricci soliton have become a topic of growing interest due to the fundamental work of Perelman [16] to solve Poincare conjecture. The geometric aspects of Ricci solitons and other properties have been critically analyzed by a large number of authors in the context of several types of geometric structures. For instance, we refer [17] to [23] and [28–31]. Some remarks on Kinematical aspects of Ricci flow and Ricci solitons have been added in the literature by Hiraca and Udriste [12].

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In [26] and [27] Udriste analyzed the Kinematical aspects of Riemann flow after its successful introduction. It was further studied in [24]. In [10], the authors studied Riemann solitons on K-contact and Sasakian manifolds. On a Riemannian manifold M with a Riemann metric g and a smooth vector field V, the Riemann soliton is given by

$$2R + \lambda g \owedge g + g \owedge \pounds_V g = 0, \tag{3}$$

where R is the Riemann curvature tensor field of type $(0, 4)$, g is Riemann metric, λ is a real number, $\pounds$ is Lie derivative operator and $\owedge$ is Kulkarni-Nomizu product [1] defined by

$$\begin{aligned}(p \owedge q)(X, Y, U, V) &= p(X, W)q(Y, W) + p(Y, U)q(X, W) \\ &- p(X, U)q(Y, W) - p(Y, W)q(X, U).\end{aligned}$$

It is evident that a Riemann soliton is a kind of generalization of manifolds of constant curvatures. Likewise Ricci solitons, a Riemann soliton is a fixed solution, upto diffeomorphisms and scaling, of Riemann flow [26, 27] given by

$$\frac{\partial}{\partial t} G(t)_{ijkl} = -2R_{ijkl}(t), \quad t \in [0, \epsilon), \tag{4}$$

with the initial condition $g(0) = g_0$. Here $G = g \owedge g$ and R_{ijkl} denote components of Riemann curvature tensor of type $(0, 4)$. A Riemann soliton expressed by (3) is called shrinking, steady or expanding according as $\lambda < 0, \lambda = 0$, or $\lambda > 0$. If the vector field V is gradient of a C^∞ function on M, then the Riemann soliton is called gradient Riemann soliton given by

$$2R + \lambda g \owedge g + 2g \owedge \mathrm{Hess} f = 0. \tag{5}$$

$\mathrm{Hess} f$ denotes Hessain of f. If in the above formulation λ is taken as a C^∞ function on M, instead of a real number, then a Riemann soliton is called an almost Riemann soliton and a gradient Riemann soliton is called a gradient almost Riemann soliton.

Another important generalization of Ricci flow is Ricci Bourguignon flow and a soliton associated with Ricci Bourguignon flow is known as a Ricci Bourguignon soliton [7, 8].

The theory of quasi-Sasakian structures bears its own importance due to its association with string theory [2–4]. In 1967, D. E. Blair [5] introduced the theory of quasi-Sasakian structures in order to generalize Sasakian and co-symplectic structures. The theory was further rectified and developed by Tanno [25]. He gave example of a proper quasi-Sasakian structure which is neither Sasakian nor cosymplectic. In [15], Olszak characterized three-dimensional quasi-Sasakian structures. Three-dimensional quasi-Sasakian manifolds, i.e., three-dimensional Riemannian manifolds admitting quasi-Sasakian structures have also been studied in [9, 15]. In [10], Riemann solitons on K-contact and Sasakian manifolds have been studied. Since a quasi-Sasakian manifold is not necessarily Sasakian or K-contact, we naturally motivate to analyze some aspects of quasi-Sasakian manifolds admitting Riemann solitons. We also go through Ricci Bourguignon solitons on such manifolds. We consider three-dimensional manifolds due to some strikingly interesting properties possessed by three-dimensional manifolds which are not found in higher dimensions, in general.

The present paper is organized as follows: In Section 2, we recall some known results that will be required in subsequent sections. In Section 3, we study Riemann solitons on three-dimensional quasi-Sasakian manifolds by considering some specific vector fields and provide relevant examples. The last section is devoted to study Ricci Bourguignon solitons.

2. Preliminaries

A C^∞ manifold M of dimension $(2n + 1)$ is called an almost contact manifold [6] if there exist a $(1, 1)$ tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \tag{6}$$

where $X \in \chi(M)$, $\chi(M)$ being the set of all vector fields on M. The manifold is called almost contact metric manifold if there exists a Riemannian metric g on M such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{7}$$

where $X, Y \in \chi(M)$. For such a manifold we also have

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad g(X, \xi) = \eta(X), \tag{8}$$

where $X, Y \in \chi(M)$. The fundamental 2-form of an almost contact metric manifold is given by

$$\Phi(X, Y) = g(X, \phi Y), \quad X, Y \in \chi(M).$$

If $d\eta(X, Y) = \Phi(X, Y)$, the almost contact metric manifold is called contact metric manifold. An almost contact metric structure is called normal if

$$[\phi, \phi](X, Y) + d\eta(X, Y)\xi = 0.$$

A normal almost contact metric structure is called quasi-Sasakian if the fundamental 2-form Φ is closed. The rank of a quasi-Sasakian structure is always odd. It is 1 if the structure is cosymplectic and $2n + 1$ when the structure is Sasakian. The Reeb vector field ξ of a quasi-Sasakian structure is always Killing.

For a three-dimensional quasi-Sasakian manifold, we always have[15]

$$\nabla_X \xi = -\beta\phi X, \quad X \in \chi(M), \tag{9}$$

β being a C^∞ function on M and ∇ is Levi-Civita connection. As a consequence of (9) one obtains

$$\xi\beta = 0. \tag{10}$$

Again on a three-dimensional quasi-Sasakian manifold

$$(\nabla_X \phi)Y = \beta(g(X, Y)\xi - \eta(Y)X), \quad X, Y \in \chi(M), \tag{11}$$

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y) = -\beta g(\phi X, Y), \tag{12}$$

$$(\nabla_X \eta)\xi = -\beta\eta(\phi X) = 0. \tag{13}$$

The Ricci tensor S of a three-dimensional quasi-Sasakian manifold is given by

$$S(Y, Z) = (\frac{r}{2} - \beta^2)g(Y, Z) + (3\beta^2 - \frac{r}{2})\eta(Y)\eta(Z) - \eta(Y)d\beta(\phi Z) - \eta(Z)d\beta(\phi Y) \tag{14}$$

for $X, Y, Z \in \chi(M)$ and r is the scalar curvature of the manifold. As a consequence of (14) we have the Ricci operator Q as follows:

$$QY = (\frac{r}{2} - \beta^2)Y + (3\beta^2 - \frac{r}{2})\eta(Y)\xi + \eta(Y)\phi\,\mathrm{grad}\beta - g(\mathrm{grad}\beta, \phi Y)\xi. \tag{15}$$

By a straightforward consequence of (9) one gets the $(0, 3)$ type Riemann curvature as

$$R(X, Y)\xi = \beta^2(\eta(Y)X - \eta(X)Y) - (X\beta)\phi Y + (Y\beta)\phi X. \tag{16}$$

Now we conclude the preliminary section by citing the following example of a three-dimensional quasi-Sasakian manifold which is not Sasakian.

Example 2.1.[25] Consider the three-dimensional Euclidean space E^3 with (x, y, z) as coordinates, and define the structure tensors (ϕ, ξ, η, g) by

$$\phi = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & y & 0 \end{pmatrix}$$

$$\xi = (0, 0, 2),$$

$$2\eta = (-y, 0, 1)$$

and

$$4g = \begin{pmatrix} 1+y^2 & 0 & -y \\ 0 & 1 & 0 \\ -y & 0 & 1 \end{pmatrix}.$$

Then it is well known that (ϕ, ξ, η, g) is a three-dimensional Sasakian structure. Suppose β is a non-constant positive function of x and y. Define the metric g^* by

$$g^* = \beta g + (1-\beta)\eta \otimes \eta.$$

Then (ϕ, ξ, η, g^*) is a normal almost contact metric structure and

$$\Phi^* = \beta\Phi = \frac{1}{2}\beta d\eta = \frac{1}{4}\beta dx \wedge dy.$$

Since $d\beta$ is a function of x and y, from above it follows that $d\Phi^* = 0$, and $E^3(\phi, \xi, \eta, g^*)$ is a quasi-Sasakian manifold of rank 3, which is not Sasakian.

3. Riemann solitons on three-dimensional quasi-Sasakian manifolds

In this section we intend to study Riemann solitons on three-dimensional quasi-Sasakian manifolds.

Lemma 3.1. In a three-dimensional quasi-Sasakian manifold admitting a Riemann soliton the relation $(\pounds_V \phi)Y = 2\eta(Y)\phi \text{grad}\beta$ holds.

Proof. Suppose a three-dimensional quasi-Sasakian manifold admits a Riemann soliton. Then from (3) one obtains

$$\begin{aligned} 2R(X,Y,U,W) &+ 2\lambda\Big(g(X,W)g(Y,U) - g(X,U)g(Y,W)\Big) \\ &+ \Big(g(X,W)(\pounds_V g)(Y,U) + g(Y,U)(\pounds_V g)(X,W) \\ &- g(X,U)(\pounds_V g)(Y,W) - g(Y,W)(\pounds_V g)(X,U)\Big). \end{aligned} \tag{17}$$

Contracting X and W we infer that

$$(\pounds_V g)(Y,U) + 2S(Y,U) + 2(2\lambda + \text{div}V)g(Y,U) = 0. \tag{18}$$

In (18), putting $U = \phi Y$ and using (14) one obtains

$$(\pounds_V g)(Y, \phi Y) - 2\eta(Y)d\beta(\phi Y) = 0.$$

The above equation yields

$$g(Y, (\pounds_V \phi)Y) + 2\eta(Y)d\beta(\phi Y) = 0.$$

Consequently, we have

$$(\pounds_V \phi)Y = 2\eta(Y)\phi \text{grad}\beta.$$

This completes the proof. □

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Lemma 3.2. A Riemann soliton on a three-dimensional quasi-Sasakian manifold reduces to a Ricci almost soliton.

Proof. In the equation (18), set $(2\lambda + \text{div}V) = \mu$. Obviously μ is a C^∞ function on the manifold. Consequently (18) reduces to

$$(\pounds_V g)(Y,U) + 2S(Y,U) + 2\mu g(Y,U) = 0.$$

Clearly, the above equation represents Ricci almost soliton. □

In [18], Sarkar studied Ricci almost solitons on three-dimensional quasi-Sasakian manifolds. In view of Lemma 4.1 and Lemma 4.2 of [18] we state the following:

Lemma 3.3. If a three-dimensional quasi-Sasakian manifold admits a Riemann soliton, then its structure function β is constant.

Lemma 3.4. The scalar curvature r of a three-dimensional quasi-Sasakian manifold admitting a Riemann soliton is given by $r = 6\beta^2$.

Theorem 3.1. A three-dimensional quasi-Sasakian manifold admitting a Riemann soliton is a manifold of constant curvature β^2.

Proof. In view of equation (14) and Lemma 3.4, we have $S(X,Y) = 2\beta^2 g(X,Y)$. Hence the manifold is Einstein. Since every three dimensional Einstein manifold is manifold of constant curvature, we easily conclude that the manifold is of constant curvature β^2. □

Theorem 3.2. If a three-dimensional quasi-Sasakian manifold admits Riemann soliton, then the soliton is shrinking and the soliton vector field is Killing.

Proof. Since β is a constant, from (16), we have

$$R(X,\xi)\xi = \beta^2(X - \eta(X)\xi). \tag{19}$$

Now, contracting (18), we have

$$\text{div}V = -\frac{r + 6\lambda}{4}.$$

Combining (18) and (14), one obtains

$$(\pounds_V g)(Y,U) - (\frac{4\beta^2 - r - 2\lambda}{2})g(Y,U) - (6\beta^2 - r)\eta(Y)\eta(U) = 0. \tag{20}$$

Using Lemma 3.3 in the above equation, we see that

$$(\pounds_V g)(Y,U) = -(\lambda + \beta^2)g(Y,U). \tag{21}$$

Differentiating the above equation with respect to X, we have

$$(\nabla_X \pounds_V g)(Y,U) = 0. \tag{22}$$

From Yano [32], it is well known that

$$2g((\pounds_V \nabla)(X,Y),U) = (\nabla_X \pounds_V g)(Y,U) + (\nabla_Y \pounds_V g)(U,X) - (\nabla_U \pounds_V g)(X,Y). \tag{23}$$

By virtue of (22) and (23)

$$g((\pounds_V \nabla)(X,Y),U) = 0.$$

The above equation gives

$$(\pounds_V \nabla)(X,Y) = 0. \tag{24}$$

Differentiating (24), we have

$$(\nabla_Z \pounds_V \nabla)(X,Y) = 0. \tag{25}$$

Again from Yano [32], it is well known that

$$(\pounds_V R)(X, Y)Z = (\nabla_X \pounds_V \nabla)(Y, Z) - (\nabla_Y \pounds_V \nabla)(X, Z). \tag{26}$$

By virtue of (25) and (26)

$$(\pounds_V R)(X, \xi)\xi = 0. \tag{27}$$

In view of (19)

$$\begin{aligned}(\pounds_V R)(X, \xi)\xi &= -\beta^2(\eta(X)\pounds_V \xi + (\pounds_V \eta)(X)\xi) \\ &\quad -R(X, \pounds_V \xi)\xi - R(X, \xi)\pounds_V \xi.\end{aligned} \tag{28}$$

By virtue of (27) and (28) we have

$$g(R(X, \xi)\pounds_V \xi, \xi) = -\beta^2(\eta(X)g(\pounds_V \xi, \xi) - (\pounds_V \eta)X).$$

Applying (19) in the above equation we have

$$g(X, \pounds_V \xi) - 2\eta(X)g(\pounds_V \xi, \xi) = -(\pounds_V \eta)X.$$

For $X = \xi$, the above equation gives

$$\eta(\pounds_V \xi) = -\eta(\pounds_V \xi).$$

Consequently,

$$\eta(\pounds_V \xi) = 0. \tag{29}$$

But for $Y = U = \xi$, (21) gives

$$\eta(\pounds_V \xi) = -\frac{\lambda + \beta^2}{2}. \tag{30}$$

On the basis of (29) and (30), we conclude $\lambda = -\beta^2$. Hence, the soliton is shrinking. Consequently, by the Lemma 3.4 and the equation (20), we infer $(\pounds_V g)(Y, U) = 0$. This completes the proof. □

By Corollary 4.6 of the paper [5], we know that a quasi-Sasakian manifold of strictly positive constant curvature is Sasakian. Hence, by Theorem 3.1, we obtain the following:

Corollary 3.1. A three-dimensional quasi-Sasakian manifold admitting Riemann soliton is a Sasakian manifold.

A consequence of the above result is:

Corollary 3.2. A non-Sasakian quasi-Sasakian manifold of dimension three does not admit Riemann soliton.

The above corollary is an important tool to verify whether a three-dimensional quasi-Sasakian manifold admits a Riemann soliton or not. Let us now mention some examples of three-dimensional quasi-Sasakian manifolds which does not admit Riemann soliton

In Example 2.1, we cited a non-Sasakian three-dimensional quasi-Sasakian manifold. Such a manifold does not admit Riemann soliton by Corollary 3.2.

Example 3.1. Consider the three-dimensional manifold $M = \{(x, y, z) \in R^3, z \neq 0\}$, where (x, y, z) are the standard coordinates in R^3 . The vector fields

$$e_1 = \frac{\partial}{\partial z} - y\frac{\partial}{\partial x}, \qquad e_2 = \frac{\partial}{\partial y}, \qquad e_3 = 2\frac{\partial}{\partial x}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0, \qquad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$. Let ϕ be the (1,1) tensor field defined by $\phi(e_1) = -e_2$, $\phi(e_2) = e_1$, $\phi(e_3) = 0$. Then following [13] it is easy to show that the manifold is quasi-Sasakian for $\beta = \frac{1}{4}$. Obviously, the manifold is not Sasakian and so by Corollary 3.2, it does not admit Riemann soliton.

Now we shall present an example of a manifold from [10] which will admit a Riemann soliton.

Example 3.2. It is known that [6] the unit sphere $S^{2n+1} \subset R^{2n+1}$ admits a standard Sasakian structure. Keeping in mind the well known Obatta's theorem [14], let us take a non-trivial smooth function ψ such that $\nabla\nabla\psi = -\psi g$. Take

$$V = -D\psi + cg, \tag{31}$$

where c is a constant. Since for a Sasakian manifold $\nabla_X \xi = -\phi X$, we have from (31)

$$\nabla_X V = \psi g - c\phi X,$$

which yields $\pounds_V g = 2\psi g$. This reveals that $(S^{2n+1}, g, V, \lambda)$ is an almost Riemann soliton for $\lambda = 2(1 - \psi)$. For $\psi = \frac{1}{2}$, it gives an example of a Riemann soliton. For $c = 0$ and $\psi = \frac{1}{2}$, it gives an example of gradient Riemann soliton.

Let us consider the situation when the soliton vector field is a gradient vector field. Let us go to prove the following:

Theorem 3.3. A non-cosymplectic three-dimensional quasi-Sasakian manifold does not admit proper gradient Riemann soliton.

Proof. For a gradient Riemann soliton (18) yields

$$(\nabla_Y Df) = -((2\lambda + \mathrm{div}Df)Y + QY).$$

As a consequence of the above equation

$$R(X, Y)Df = (\nabla_Y Q)X - (\nabla_X Q)Y + \nabla_Y(\mathrm{div}Df)X - \nabla_X(\mathrm{div}Df)Y. \tag{32}$$

The above expression leads us to

$$\begin{aligned} g(R(X, Df)\xi, Y) &= g((\nabla_\xi Q)Y, X) - g((\nabla_V Q)\xi, X) \\ &+ g(\nabla_\xi(\mathrm{div}Df), X) - g(\nabla_Y(\mathrm{div}Df), X). \end{aligned} \tag{33}$$

Since, by Lemma 3.3, β is constant, by virtue of (15) and (16), it follows that

$$\beta^2(\eta(Df)g(X, Y) - \eta(X)g(Df, Y)) = 0. \tag{34}$$

Contracting X and Y, we get

$$\beta^2\eta(Df) = 0.$$

Considering $\beta \neq 0$, we find $\eta(Df) = 0$. So, in (34) putting $X = \xi$, one obtains

$$g(Df, Y) = 0.$$

Since Y is arbitrary, $Df = 0$. For $\beta = 0$, the manifold is cosymplectic. Thus, the theorem follows. □

Stepanov [24] studied Riemann soliton considering the soliton vector field as an infinitesimal contact transformation or simply as a contact transformation on a contact manifold and obtained interesting geometric consequences. Since the contact form η is called almost contact form in almost contact manifolds, we shall call the analogue of contact transformations in almost contact manifolds as almost contact transformations.

Definition 3.1. A vector field V on an almost contact metric manifold is called almost contact transformation if it satisfies

$$\pounds_V \eta = \rho\eta \tag{35}$$

for a smooth function ρ on the manifold. In the following, we shall prove

Theorem 3.4. If the soliton vector field V of a Riemann soliton on a three-dimensional quasi-Sasakian manifold is an almost contact transformation, then it leaves the almost contact form η invariant, upto scaling.

Proof. Suppose the soliton vector field of a Riemann soliton on a three-dimensional quasi-Sasakian manifold is an almost contact transformation.

Now, from (13), we have

$$d\eta(X,Y) = 2\beta g(X,\phi Y).$$

Taking Lie derivative in both sides of the above equation

$$(£_V d\eta)(X,Y) = 2\beta\Big((£_V g)(X,Y) + g(X,(£_V\phi)Y)\Big) + 2d\beta(Y)g(X,\phi Y).$$

Using equation (18) and Lemma 3.1 in the above equation one can establish

$$\begin{aligned}(£_V d\eta)(X,Y) = &- 2\beta\Big(S(X,\phi Y) + 2(2\lambda + \mathrm{div}V)g(X,\phi Y)\\ &- 2\eta(Y)g(X,\phi\mathrm{grad}\beta)\Big) + d\beta(Y)g(X,\phi Y).\end{aligned} \tag{36}$$

By virtue of (35) and (13) one gets

$$(£_V d\eta)(X,Y) = 2\rho\beta g(X,\phi Y) + \frac{1}{2}\Big(d\rho(X)\eta(Y) - d\rho(Y)\eta(X)\Big). \tag{37}$$

For $Y = \xi$, (36) and (37) jointly yields

$$4\beta g(X,\phi\mathrm{grad}\beta) = \frac{1}{2}\Big(d\rho(X) - d\rho(\xi)\eta(X)\Big). \tag{38}$$

Since β is a constant

$$g(\mathrm{grad}\rho, X) = g(X,(\xi\rho)\xi).$$

Hence

$$D\rho = (\xi\rho)\xi. \tag{39}$$

As a consequence of the above equation, one obtains

$$\nabla_X D\rho = X(\xi\rho)\xi - \beta(\xi\rho)\phi X.$$

Taking inner product in the above equation, we have

$$g(\nabla_X D\rho, Y) = X(\xi\rho)\eta(Y) - \beta(\xi\rho)g(\phi X, Y).$$

Antisymmetrizing the above equation and using $g(\nabla_X D\rho, Y) = g(\nabla_Y D\rho, X)$ one can deduce

$$(X(\xi\rho) - Y(\xi\rho))(\eta(Y) - \eta(X)) - 2\beta(\xi\rho)g(\phi X, Y) = 0.$$

Replacing X by ϕX and Y by ϕY in the above equation we obtain

$$(\xi\rho)g(X,\phi Y) = 0.$$

Let $\{e_1, e_2, \xi\}$ be a ϕ basis. Then putting $X = e_1$ and $Y = e_2$ in the above, we infer

$$\xi\rho = 0. \tag{40}$$

By virtue of (39) and (40) we conclude that ρ is constant. Hence η is invariant, upto scaling, under Lie derivative with respect to V. This completes the proof. □

4. Ricci Bourguignon solitons on three-dimensional quasi-Sasakian manifolds

The Ricci Bourguignon flow [7]

$$\frac{\partial}{\partial t}g_{ij} = -2S_{ij} + 2lrg_{ij} \tag{41}$$

was introduced by Jean-Pierre Bourguignon in 1981 taking l as a real number. Here r being the scalar curvature of the manifold. Equation (41) represents a family of geometric flows of which one is Ricci flow for $l = 0$. Again, by a suitable rescaling in time, when l is non-positive, the flows can be interpreted as an interpolation between the Ricci flow and the Yamabe flow. It is to be observed that for special values of the constant l, the tensor $S_{ij} - lrg_{ij}$ in the right hand side of (41) is of special interest. It is noted that on a manifold of dimension d the tensor $S_{ij} - lrg_{ij}$ is

• Einstein for $l = \frac{1}{2}$.
• Trace less Ricci tensor for $l = \frac{1}{d}$.
• The Schouten tensor when $l = \frac{1}{d-1}$.

For $d = 2$, the tensor $S_{ij} - lrg_{ij}$ is zero. Hence, the flow is static.

In 2017, Catino et al [8] proved the short time existence and uniqueness for solution of the flow in the time interval $[0, T)$. A constant solution of Ricci Bourguignon flow, upto diffeomorphisms and scaling, is known as Ricci Bourguignon soliton. In the following, we study three-dimensional quasi-Sasakian manifold admitting a Ricci Bourguignon soliton.

In view of (41), we obtain Ricci Bourguignon soliton as a metric satisfying the following equation:

$$(\pounds_V g)(X, Y) + 2S(X, Y) + 2(\lambda - lr)g(X, Y) = 0, \tag{42}$$

where λ and l are constants. A Ricci Bourguignon soliton expressed by (42) is called shrinking, steady or expanding according as $\lambda < 0, \lambda = 0$, or $\lambda > 0$. If V is a gradient of a smooth function f, then

$$\nabla^2 f + S = (\lambda - lr)g, \tag{43}$$

where $\nabla^2 f$ is the Hessain of f. Suppose a three dimensional quasi-Sasakian manifold admits a Ricci Bourguignon soliton. Since we take λ and l as constants, (42) yields that the Ricci Bourguignon soliton becomes an almost Ricci soliton. Hence, we have

Lemma 4.1. A Ricci Bourguignon soliton on a three-dimensional quasi-Sasakian manifold reduces to a Ricci almost soliton.

In view of the above lemma, the Lemma 4.1 and Lemma 4.2 of the paper [18], as the previous section, we obtain the following:

Lemma 4.2. The structure function of a three-dimensional quasi-Sasakian manifold admitting a Ricci Bourguignon soliton is constant.

Lemma 4.3. The scalar curvature r and the structure function β of a three-dimensional quasi-Sasakian manifold admitting a Ricci Bourguignon soliton are related by $r = 6\beta^2$.

Theorem 4.1. A three-dimensional quasi-Sasakian manifold admitting Ricci Bourguignon soliton is a manifold of constant curvature β^2.

Proof. In view of equation (14) and Lemma 4.3, we have $S(X, Y) = 2\beta^2 g(X, Y)$. Hence the manifold is Einstein. Since every three dimensional Einstein manifold is manifold of constant curvature, we infer that the manifold is of constant curvature β^2. This completes the proof. □

By Corollary 4.6 of the paper [5], we know that a quasi-Sasakian manifold of strictly positive constant curvature is Sasakian. Hence, by Theorem 4.1, we obtain the following:

Corollary 4.1. A three-dimensional quasi-Sasakian manifold admitting a Ricci Bourguignon soliton is a Sasakian manifold.

A consequence of the above result is:

Corollary 4.2. A non-Sasakian quasi-Sasakian manifold of dimension three does not admit a Ricci Bourguignon soliton.

Let us now prove the following:

Theorem 4.2. The soliton vector field of a Ricci Bourguignon soliton in a three-dimensional quasi-Sasakian manifold is Killing.

Proof. From (42) and (14), one obtains

$$(\pounds_V g)(Y,U) + 2(2\beta^2 + \lambda - lr)g(Y,U) = 0. \tag{44}$$

By Lemma 4.3, r is constant. So, by covariant differentiation of the above equation, we infer

$$(\nabla_X \pounds_V g)(Y,U) = 0. \tag{45}$$

From Yano [32], it is well known that

$$2g((\pounds_V\nabla)(X,Y),U) = (\nabla_X \pounds_V g)(Y,U) + (\nabla_Y \pounds_V g)(U,X) - (\nabla_U \pounds_V g)(X,Y). \tag{46}$$

By virtue of (45) and (46)

$$g((\pounds_V\nabla)(X,Y),U) = 0.$$

The above equation gives

$$(\pounds_V\nabla)(X,Y) = 0. \tag{47}$$

Differentiating (47), we have

$$(\nabla_Z \pounds_V\nabla)(X,Y) = 0. \tag{48}$$

Again from Yano [32], it is well known that

$$(\pounds_V R)(X,Y)Z = (\nabla_X \pounds_V\nabla)(Y,Z) - (\nabla_Y \pounds_V\nabla)(X,Z). \tag{49}$$

By virtue of (48) and (49)

$$(\pounds_V R)(X,\xi)\xi = 0. \tag{50}$$

In view of (16)

$$\begin{aligned}(\pounds_V R)(X,\xi)\xi &= -\beta^2(\eta(X)\pounds_V\xi + (\pounds_V\eta)(X)\xi)\\ &\quad -R(X,\pounds_V\xi)\xi - R(X,\xi)\pounds_V\xi.\end{aligned} \tag{51}$$

By virtue of (50) and (51) we have

$$g(R(X,\xi)\pounds_V\xi,\xi) = -\beta^2(\eta(X)g(\pounds_V\xi,\xi) - (\pounds_V\eta)X).$$

Applying (16) in the above equation we have

$$g(X,\pounds_V\xi) - 2\eta(X)g(\pounds_V\xi,\xi) = -(\pounds_V\eta)X.$$

For $X = \xi$, the above equation gives

$$\eta(\pounds_V\xi) = -\eta(\pounds_V\xi).$$

Consequently,

$$\eta(\pounds_V\xi) = 0. \tag{52}$$

But for $Y = U = \xi$, (44) gives

$$\eta(\pounds_V \xi) = 2\beta^2 + \lambda - lr. \tag{53}$$

Hence,

$$2\beta^2 + \lambda - lr = 0. \tag{54}$$

Hence from (42) $(\pounds_V g)(Y, U) = 0$. Thus, V is Killing. This completes the proof. □

From (54)

$$\lambda = 2\beta^2(3l - 1). \tag{55}$$

So, $l = \frac{1}{3}$ if and only if $\lambda = 0$, provided β is non-zero. If the flow is steady, then $l = \frac{1}{3}$ and the right hand side of (41) is trace less Ricci tensor. Thus, A steady Ricci Bourguignon soliton reduces to traceless Ricci soliton.

In view of (55), we obtain the following:

Corollary 4.3. A Ricci Bourguignon soliton on a three-dimensional quasi-Sasakian manifold is shrinking, steady or expanding according as $l < \frac{1}{3}, l = \frac{1}{3}, l > \frac{1}{3}$, and if it is steady, it is a trace less Ricci soliton.

Remark 4.1. By the above corollary, we see that on a three-dimensional quasi-Sasakian manifold, the solitons corresponding to Einstein flow and Schouten flow are expanding since in these cases $l = \frac{1}{2}$, while the soliton for traceless Ricci flow is steady. For the Ricci flow $l = 0$. So the soliton for Ricci flow is shrinking.

Now, we prove the following:

Theorem 4.3. A non-cosymplectic three-dimensional quasi-Sasakian manifold does not admit proper gradient Ricci Bourguignon soliton.

Proof. If the soliton is gradient

$$\nabla_Y Df = (\lambda - lr)Y - QY.$$

Since λ, l and r are constants, as a consequence of the above equation

$$R(X, Y)Df = (\nabla_Y Q)X - (\nabla_X Q)Y.$$

Putting $X = \xi$, we have from above

$$R(\xi, Y)Df = (\nabla_Y Q)\xi - (\nabla_\xi Q)Y).$$

Using (15),

$$R(\xi, Y)Df = 0. \tag{56}$$

Contracting Y, we have

$$S(Df, \xi) = 0.$$

By virtue of (15), the above equation gives

$$\eta(Df) = 0. \tag{57}$$

Now, in view of (16)

$$g(R(X, Y)Df, \xi) = -\beta^2(\eta(Y)g(X, Df) - \eta(X)g(Y, Df)). \tag{58}$$

Putting $X = \xi$ in (58) and using (56) and (57) we have for $\beta \neq 0$

$$g(Y, Df) = 0.$$

Since Y is arbitrary, it follows that $Df = 0$. Hence the result follows. □

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