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Escaping Subsets of Cosine Functions with the Given Hausdorff Dimension

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Abstract. The escaping set is the important object studied in dynamics of transcendental entire functions. As exponential function is the most typical transcendental entire function, its escaping set has been deeply studied. It is well known that if the function is slightly disturbed, the properties of its dynamical system may vary greatly. We can't easily study different functions in the same way. Contrasting exponential function, we pay our main attention to the cosine function in this paper. We construct some escaping subsets of cosine function by Devaney-Krych codes so that the Hausdorff dimension of the subsets is equal to the given number in the interval $(1, 2)$.

1. Introduction

Julia set (see [9]) is one of the main objects studied in complex dynamics. The escaping set (see [2]), which is closely related to the Julia set, also draws people's increasing interest. These two kinds of points sets usually have very rich structures even for simple transcendental entire functions.

For example, about the exponential functions dynamics, Misiurewicz proved that the Julia set of e^z is the whole complex plane $\mathbb{C}$ (see [10]). Changing the coefficient slightly, Devaney and Krych proved the Julia set of λe^z, which $0 < \lambda < \frac{1}{e}$, is a Cantor set of curves in $\mathbb{C}$ (see [1]). Later, Karpińska proved the Cantor set of curves for λe^z with $0 < \lambda < \frac{1}{e}$ has a peculiar phenomenon of "dimension paradox". That is the Hausdorff dimension (see [3]) of the hairs without endpoints is 1, however, the Hausdorff dimension of the set of endpoints is 2 (see [5, 6]). Schleicher and Zimmer turned to study the escaping set of λe^z with $\lambda \neq 0$ and proved that the phenomenon of "dimension paradox" also exists for exponential function escaping set (see [16]). Further more, Schleiher, Forster, Rempe, Bailesteanu and Balan proved that the escaping parameters set of exponential functions family also has the properties of Cantor bundle structure and "dimension paradox" (see [11, 14, 15, 17]). Such strange fractal structure has aroused the interest of many people. For the escaping points set of an arbitrary exponential function, Karpińska and Urbański investigated the finer fractal structure of the set of escaping points and even provided an exact formulas

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showing how sensitively the Hausdorff dimension depends on the rate of growth of Devaney-Krych codes (see [1, 7]).

This paper mainly discusses the Hausdorff dimension of escaping subsets of cosine function $ae^z + be^{-z}$, where $ab \neq 0$. Cosine function is transcendental entire functions closely related to exponential function. McMullen made an in-depth comparative study of them as early as 1986 (see [8]), which has triggered ongoing research about cosine function. The Cantor bundle structure and "dimension paradox" of escaping set of cosine function and escaping parameters set of cosine functions family with single parameter are proved respectively by papers [12, 13] and papers [4, 11, 18]. From these studies, we can see that the different term between cosine function and exponential function often brings us some new phenomena and difficulties in the iterative process. We often need some skills to apply the exponential function research method to the study of cosine function dynamics. In this paper, the escaping subsets of cosine function will be constructed with Devaney-Krych codes by using the method of Karpińska and Urbański for exponential function so that the Hausdorff dimension of them is equal to the given number which is in (1,2) interval.

In the following Section 2, we show some notations and the main conclusion of this paper. In Section 3, we make some preparations for the proof of the main conclusion. In Section 4, we prove the main conclusion of this paper.

2. Symbols and main result

The entire function $S(z) := ae^z + be^{-z}$, where $a, b \in \mathbb{C}$ and $ab \neq 0$, is called cosine function. Denote $S^n(z)$ as its n-fold iterate, where n is positive integer. The escaping set of $S(z)$ is denoted as $I(S)$ or I, which is defined below.

$$I(S) = I := \{z : |S^n(z)| \to \infty \text{ as } n \to \infty\}.$$

For $S(z) = ae^z + be^{-z}$, we have

$$|S^n(z)| \leq |a| \exp(ReS^{n-1}(z)) + |b| \exp(-ReS^{n-1}(z)),$$

so the escaping set of $S(z)$ has the following equivalent definition

$$I = \{z : |Re(S^n(z))| \to \infty \text{ as } n \to \infty\}.$$

In addition, we denote

$$I^q := \{z \in I : |Re(z)| \geq q \textit{ and } |Re(S^n(z))| \geq q \text{ for all } n \geq 1\}.$$

From the above equivalent definition of escaping set, we infer that the set I^q will not be empty.

If we divide the plane $\mathbb{C}$ into infinitely many strips:

$$P_j = \{z \in \mathbb{C} : (2j-1)\pi \leq Imz < (2j+1)\pi\} \quad \text{where } j \in \mathbb{Z}.$$

then every point z, under iteration of cosine function $S(z)$ (Let $S^0(z) = z$), has uniquely defined sequence of integers $s(z) = (s_0, s_1, \cdots)$ such that

$$s_k = j \text{ if and only if } S^k(z) \in P_j.$$

The sequence $s(z)$ is called the Devaney-Krych codes of the point z (see [1]).

We can choose some escaping subsets according to the growth characteristics of Devaney-krych codes and to study the Hausdorff dimension of this sets. Therefore, we first give the function $h_\epsilon(x)$ as below.

For any given $\epsilon > 0$, let $h_\epsilon(x) := \frac{x}{(\log x)^\epsilon}$ (see [7]). Evidently, if $x > q$ and $q > 0$ is large enough, then

(a) $h_\epsilon(x)$ be an increasing function such that $\lim_{x\to\infty} h_\epsilon(x) = +\infty$;

(b) $3\pi \leq h_\epsilon(x)$, $3h_\epsilon(x) + 2\pi \leq \frac{x}{8e^\pi}$;

(c) $h_\epsilon(2e^\pi x) \leq 2e^\pi h_\epsilon(x)$.

Let

$$D^q(h_\epsilon) := \{z \in I^q : 2\pi|s_n(z)| \le h_\epsilon(|S^n(z)|) \text{ for all } n \ge 0\}.$$

It is obvious that

$$D^q(h_\epsilon) \subseteq D^q(2h_\epsilon) \subseteq D^q(3h_\epsilon).$$

Since the Hausdorff dimension is a good measure to describe complicated or irregular set and the paper intends to discuss the Hausdorff dimension of the above set $D^q(h_\epsilon)$, we turn to briefly introduce the concept of Hausdorff dimension (see [3]). For any set U, denote the diameter of U by

$$|U| := \sup\{|z - w| : z, w \in U\}.$$

Let X be a set and s a positive real number. Define s-dimensional measure $H^s(X)$ of X by

$$H^s(X) := \lim_{\delta\to 0} \inf\Big\{\sum_{i=1}^{\infty} |U_i|^s : |U_i| < \delta, X \subseteq \bigcup_i U_i\Big\},$$

and define the Hausdorff dimension $HD(X)$ of X by

$$HD(X) := \inf\Big\{s \ge 0 : H^s(X) = 0\Big\} = \sup\Big\{s \ge 0 : H^s(X) = \infty\Big\}.$$

We claim that the Hausdorff dimension of the set $D^q(h_\epsilon)$ is sensitively depending on ϵ of function $h_\epsilon(x)$. The following result holds.

Theorem 2.1. *Let $S(z) = ae^z + be^{-z}$, where $a, b \in \mathbb{C}$ and $ab \neq 0$. For any $\epsilon > 0$, if $q > 0$ is a large enough number, then the Hausdorff dimension*

$$HD(D^q(h_\epsilon)) = 1 + \frac{1}{1+\epsilon},$$

where $D^q(h_\epsilon)$ is defined as above.

3. Preliminaries

3.1. *Simple properties of cosine function*

Lemma 3.1. *Let $S(z) = ae^z + be^{-z}$, where $a, b \in \mathbb{C}$ and $ab \neq 0$, m, n are nonnegative integers, if $q > 0$ is sufficiently large and $|Rez| \ge q$, then*

(a) $S^{(m)}(z) \neq 0$, where $S^{(m)}(z)$ is the m-order derivative, $S^{(0)}(z) = S(z)$;

(b) the horizontal strip domain with width smaller than 2π and the real part no less than q (or no more than $-q$) is the univalent domain of $S^{(m)}(z)$;

(c) $e < \frac{2}{3}\min\{|a|, |b|\}e^{|Rez|} < |S^{(m)}(z)| < \frac{3}{2}\max\{|a|, |b|\}e^{|Rez|}$;

(d) $\frac{1}{2e^\pi} < \frac{|S^{(m)}(z_1)|}{|S^{(n)}(z_2)|} < 2e^\pi$, where $|Rez_1 - Rez_2| < \pi$ and $|Rez_i| \ge q, i = 1, 2$.

Proof. (a) Obviously, $S^{(m)}(z) = ae^z \pm be^{-z}$, If $S^{(m)}(z) = 0$, then $z = \frac{1}{2}\log|\frac{b}{a}| + \frac{i}{2}Arg(\pm\frac{b}{a})$. Because $|Rez| \ge q > |\frac{1}{2}\log|\frac{b}{a}||$, then $S^{(m)}(z) \neq 0$.

(b) Note that $S^{(m)}(z) = ae^z \pm be^{-z} = \sqrt{ab}(\sqrt{\frac{a}{b}}e^z \pm \sqrt{\frac{b}{a}}e^{-z})$. If $S^{(m)}(z_1) = S^{(m)}(z_2)$, then

$$\sqrt{\frac{a}{b}}e^{z_1} = \sqrt{\frac{a}{b}}e^{z_2} \text{ or } |\sqrt{\frac{a}{b}}e^{z_1} \cdot \sqrt{\frac{a}{b}}e^{z_2}| = 1.$$

Since q is large enough such that $|\sqrt{\frac{a}{b}}e^{z_1} \cdot \sqrt{\frac{a}{b}}e^{z_2}| \neq 1$, we have $\sqrt{\frac{a}{b}}e^{z_1} = \sqrt{\frac{a}{b}}e^{z_2}$ and then $z_2 = z_1$ (width of strip $< 2\pi$).

(c) Suppose $Rez > q > 0$, as q is large enough, then

$$\begin{aligned}|S^{(m)}(z)| &\geq ||a|e^{Rez} - |b|e^{-Rez}| > |a|e^{Rez} - \frac{1}{3}|a|e^{Rez} \\ &= \frac{2}{3}|a|e^{Rez} \geq \frac{2}{3}\min\{|a|,|b|\}e^{|Rez|} > e, \\ |S^{(m)}(z)| &\leq |a|e^{Rez} + |b|e^{-Rez} < |a|e^{Rez} + \frac{1}{2}|a|e^{Rez} \\ &= \frac{3}{2}|a|e^{|Rez|} \leq \frac{3}{2}\max\{|a|,|b|\}e^{|Rez|}.\end{aligned}$$

The prove is completely similar when $Rez < -q < 0$.

(d)Without losing generality, suppose $Rez > q > 0$. It can be proved similarly when $Rez < -q < 0$.

$$\frac{||a|e^{Rez_1} - |b|e^{-Rez_1}|}{|a|e^{Rez_2} + |b|e^{-Rez_2}} \leq \frac{|S^{(m)}(z_1)|}{|S^{(n)}(z_2)|} \leq \frac{|a|e^{Rez_1} + |b|e^{-Rez_1}}{||a|e^{Rez_2} - |b|e^{-Rez_2}|}.$$

As $q > 0$ is large enough and $|Rez_1 - Rez_2| < \pi$, then Rez_1 and Rez_2 are positive large enough. So

$$\frac{1}{2}e^{-\pi} < \frac{|S^{(m)}(z_1)|}{|S^{(n)}(z_2)|} \approx e^{Rez_1 - Rez_2} < 2e^{\pi}.$$

□

Lemma 3.2. *If q is large enough, then $|Re(S^n(z))|$ tends to infinity uniformly on $D^q(3h_\epsilon)$.*

Proof. As $q > 0$ is large enough, we have that

$$\frac{2}{3}\min\{|a|,|b|\}\exp(\sqrt{3}x/2) \geq 2x \text{ for every } x \geq q.$$

For any given $z \in D^q(3h_\epsilon)$ and $n \geq 0$, by the definition of Devaney-Krych codes and the properties of function $h_\epsilon(x)$, we have

$$\begin{aligned}|Re(S^n(z))| &= \sqrt{|S^n(z)|^2 - (|ImS^n(z)|)^2} \\ &\geq \sqrt{|S^n(z)|^2 - (3h_\epsilon(|S^n(z)|) + 2\pi)^2} \\ &\geq \sqrt{|S^n(z)|^2 - (\frac{|S^n(z)|}{2})^2} \\ &= \frac{\sqrt{3}}{2}|S^n(z)|.\end{aligned} \tag{1}$$

According to lemma 3.1 (c), we get

$$\begin{aligned}|S^{n+1}(z)| &= |ae^{S^n(z)} + be^{-S^n(z)}| \\ &\geq \frac{2}{3}\min\{|a|,|b|\}e^{|ReS^n(z)|} \\ &\geq \frac{2}{3}\min\{|a|,|b|\}\exp(\sqrt{3}|S^n(z)|/2) \\ &\geq 2|S^n(z)|.\end{aligned}$$

Hence,

$$\begin{aligned}|Re(S^{n+1}(z))| &\geq \frac{\sqrt{3}}{2}|S^{n+1}(z)| \geq \frac{\sqrt{3}}{2}2|S^n(z)| \\ &\geq \sqrt{3}|Re(S^n(z))| \geq \cdots \geq (\sqrt{3})^{n+1}q.\end{aligned}$$

□

Lemma 3.3. *For any given $\alpha > 0$ and $T > 0$, there exist $K_0 > 0$ and $n_0 \geq 0$ such that for every $n \geq n_0$,*

$$|(S^{n+1})'(z)| \geq K_0 |(S^n)'(z)|^\alpha$$

for all $z \in D^q(3h_\epsilon) \cap B(0,T)$.

Proof. By lemma 3.2, for any given $\alpha > 0$, there is $n_0 \geq 0$ such that

$$\frac{1}{2e^\pi}\frac{2}{3}\min\{|a|,|b|\}e^{\sqrt{3}|S^{n+1}(z)|/2} \geq (2e^\pi)^\alpha |S^{n+1}(z)|^\alpha$$

for all $z \in D^q(3h_\epsilon)$ when $n \geq n_0$.

We claim that

$$\inf_{z\in \overline{D^q(3h_\epsilon)\bigcap B(0,T)}} \frac{|(S^{n_0+1})'(z)|}{|(S^{n_0})'(z)|^\alpha} \neq 0.$$

If there no exist $j \in \{0, 1, \cdots n_0\}$ and $z_0 \in \overline{D^q(3h_\epsilon) \cap B(0,T)}$ such that $S'(S^j(z_0)) = 0$, $\frac{|(S^{n_0+1})'(z)|}{|(S^{n_0})'(z)|^\alpha}$ is a positive continuous function on bounded closed sets, the claim holds. Suppose there exist $j \in \{0, 1, \cdots n_0\}$ and $z_0 \in \overline{D^q(3h_\epsilon) \cap B(0,T)}$ such that $S'(S^j(z_0)) = 0$, then exist $\{z_n\} \subseteq D^q(3h_\epsilon) \cap B(0,T)$ such that $z_n \to z_0$ or $z_n \equiv z_0$. By lemma 3.1 (d)

$$|S'(S^j(z_0))| \leftarrow |S'(S^j(z_n))| \geq \frac{1}{2e^\pi}|S^{j+1}(z_n)| \geq \frac{1}{2e^\pi}q,$$

which contradicts to $S'(S^j(z_0)) = 0$.

Let K_0 be the infimum of the function $z \mapsto |(S^{n_0+1})'(z)||(S^{n_0})'(z)|^{-\alpha}$ in $D^q(3h_\epsilon) \cap B(0,T)$, then K_0 is a positive number. Proof by induction. According to the definition of K_0, the lemma holds when $n = n_0$. Suppose it is true for $n \geq n_0$, so

$$\begin{aligned}|(S^{n+2})'(z)| &= |(S'(S^{n+1}(z))| \cdot |(S^{n+1})'(z)| \\ &\geq K_0 |(S'(S^{n+1}(z))| \cdot |(S^n)'(z)|^\alpha.\end{aligned}$$

By lemma 3.1 (d) (c) and (1)

$$\begin{aligned}|(S'(S^{n+1}(z))| &\geq \frac{1}{2e^\pi}|S^{n+2}(z)| \geq \frac{1}{2e^\pi}\frac{2}{3}\min\{|a|,|b|\}e^{|ReS^{n+1}(z)|} \\ &\geq \frac{1}{2e^\pi}\frac{2}{3}\min\{|a|,|b|\}e^{\sqrt{3}|S^{n+1}(z)|/2} \geq (2e^\pi)^\alpha |S^{n+1}(z)|^\alpha \\ &\geq (2e^\pi)^\alpha \cdot (\frac{1}{2e^\pi})^\alpha |S'(S^n(z))|^\alpha = |S'(S^n(z))|^\alpha.\end{aligned}$$

Therefore

$$|(S^{n+2})'(z)| \geq K_0 |S'(S^n(z))|^\alpha \cdot |(S^n)'(z)|^\alpha = K_0 |(S^{n+1})'(z)|^\alpha.$$

□

3.2. Construction of mass distribution and some supplementary symbols

Let $H_q^R := \{z \in \mathbb{C} : Rez \geq q\}$ and $H_q^L := \{z \in \mathbb{C} : Rez \leq -q\}$. The following discussion is restricted on $H_q := H_q^R \cup H_q^L$.

Divide H_q as follows

$$H_q = \cup_{k=-\infty}^{k=\infty}(S_k^R \cup S_k^L \cup S_k^{*R} \cup S_k^{*L}),$$

where for every $k \in \mathbb{Z}$

$$S_k^R := \{z \in H_q^R : -\frac{\pi}{2} + 2k\pi - \arg a \le Imz < \frac{\pi}{2} + 2k\pi - \arg a\},$$
$$S_k^L := \{z \in H_q^L : -\frac{\pi}{2} + 2k\pi + \arg b \le Imz < \frac{\pi}{2} + 2k\pi + \arg b\},$$
$$S_k^{*R} := \{z \in H_q^R : \frac{\pi}{2} + 2k\pi - \arg a \le Imz < \frac{3\pi}{2} + 2k\pi - \arg a\},$$
$$S_k^{*L} := \{z \in H_q^L : \frac{\pi}{2} + 2k\pi + \arg b \le Imz < \frac{3\pi}{2} + 2k\pi + \arg b\}.$$

Then divide S_k^R, S_k^L, S_k^{*R} and S_k^{*L} into squares,

$$S_k^R := \cup_{j=0}^{j=\infty} B_{j,Rk}$$
$$:= \cup_{j=0}^{j=\infty} \{z \in S_k^R : q + j\pi \le Rez < q + (j+1)\pi\},$$
$$S_k^L := \cup_{j=0}^{j=\infty} B_{j,Lk}$$
$$:= \cup_{j=0}^{j=\infty} \{z \in S_k^L : -q - (j+1)\pi < Rez \le -q - j\pi\},$$
$$S_k^{*R} := \cup_{j=0}^{j=\infty} B_{j,*Rk}$$
$$:= \cup_{j=0}^{j=\infty} \{z \in S_k^{*R} : q + j\pi \le Rez < q + (j+1)\pi\},$$
$$S_k^{*L} := \cup_{j=0}^{j=\infty} B_{j,*Lk}$$
$$:= \cup_{j=0}^{j=\infty} \{z \in S_k^{*L} : -q - (j+1)\pi < Rez \le -q - j\pi\}.$$

The family of all squares $B_{j,Rk}, B_{j,Lk}, B_{j,*Rk}B_{j,*Lk}$, $j = 0, 1, \cdots, \infty$ are denoted by $\mathfrak{B}$ and the squares above are sometimes denoted as B for convenience. See the following figure 1.

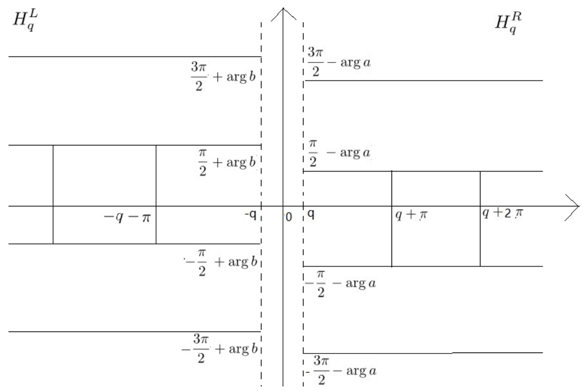

Figure 1. division of the complex plane

For any given small positive number θ, As long as $q > 0$ is large enough, we have that

$$\max\{|a|, |b|\}e^{-|Rez|} < \theta. \tag{2}$$

So we can observe that $S(z) \approx ae^z$ in H_q^R and $S(z) \approx be^{-z}$ in H_q^L.

Take $B = B_{j,Rk}$ for example, $S(B)$ contains a half-annulus with inner radius of $|a|e^{q+j\pi} + \theta$ and outer radius of $|a|e^{q+(j+1)\pi} - \theta$. At the same time, $S(B)$ included in a half-annulus with inner radius of $|a|e^{q+j\pi} - \theta$ and outer radius of $|a|e^{q+(j+1)\pi} + \theta$. As the positive number θ is very small, $S(B)$ can be viewed as 'approximate-half-annulus'. Furthermore, the 'approximate-half-annulus' $S(B)$ lies in the half plane $\{z \in \mathbb{C} : Rez > -\theta\}$.

Similarly, for any given B above, $S(B)$ is a 'approximate-half-annulus' centered at the origin. And the 'approximate-half-annulus' $S(B)$ lies either in $\{z \in \mathbb{C} : Rez > -\theta\}$ when $B \subseteq \cup_{k=-\infty}^{+\infty}(S_k^R \cup S_k^L)$ or in $\{z \in \mathbb{C} : Rez < \theta\}$ when $B \subseteq \cup_{k=-\infty}^{+\infty}(S_k^{*R} \cup S_k^{*L})$.

Let

$$R(S(B)) := \sup |S(B) \cap H_q|, \; r(S(B)) := \inf |S(B) \cap H_q|.$$

and $\widetilde{A}(r(S(B)), R(S(B))) := S(B) \cap H_q$.

Denote $\widetilde{A}(a_0 r(S(B)) + a_1, b_0 R(S(B)) + b_1)$ as the 'approximate-half-annulus' in H_q, which is enclosed by the image of inner and outer boundary of $S(B)$ under linear transformation $a_0 z + a_1$ and $b_0 z + b_1$ respectively along radial direction, where a_0, a_1, b_0, b_1 are real number.

When $z \in I^q$, then the whole orbit $\{z, S^1(z), S^2(z), \cdots\}$ of z stays in H_q, so for every $n \geq 0$ there exists a unique square $B_n(z) \in \mathfrak{B}$ such that $S^n(z) \in B_n(z)$. If necessary, we can ask q to be sufficiently large that the above conclusions hold when $|Rez| > \frac{q}{2}$. It follows immediately from lemma 3.1 (c) that there exists a unique holomorphic inverse branch $S_z^{-n} : B_n(z) \to H_{q-\pi}$ of S^n sending $S^n(z)$ to z. Denote $K_n(z) = S_z^{-n}(B_n(z))$.

In order to estimate the Hausdorff dimension of $D^q(h_\epsilon)$ by mass distribution principles, We need to construct measure μ and its support set X_∞ in two different ways, which comes from the ideas of [7, 8]. For a given square $B_0 \in \mathfrak{B}$, we shall construct inductively a sequence $\{\mathfrak{B}^n\}_{n=0}^{\infty}$ of subfamilies of $\mathfrak{B}$. Put $\mathfrak{B}^0 = \{B_0\}$ and suppose that the family $\mathfrak{B}^n$ has been constructed. We choose $\mathfrak{B}^{n+1}$ in two different ways. For any $B \in \mathfrak{B}^n$,

the first construction:

select from $\mathfrak{B}$ all of the squares Q that are contained in the approximate-half-annulus

$$\widetilde{A}(2r(S(B)), \frac{2}{3}R(S(B))) \tag{3}$$

with the property that

$$2\pi + \max_{z \in Q}\{|Imz|\} \leq \inf_{z \in Q}\{h_\epsilon(|z|)\}. \tag{4}$$

the second construction:

select from $\mathfrak{B}$ all of the squares Q that are contained in the approximate-half-annulus

$$\widetilde{A}(r(S(B)), R(S(B))) \tag{5}$$

with the property that

$$\min_{z \in Q}\{|Imz|\} \leq \sup_{z \in Q}\{2h_\epsilon(|z|)\} + 2\pi. \tag{6}$$

These squares will be called the successors of B. The family $\mathfrak{B}^{n+1}$ consists of all successors of all squares from $\mathfrak{B}^n$. See the following figure 2.

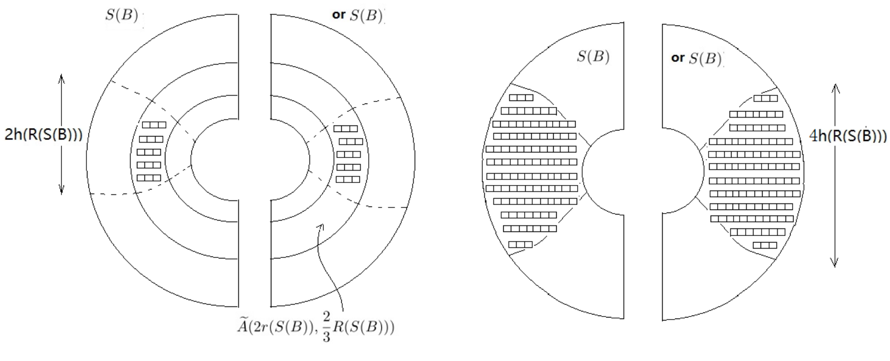

Figure 2. construction of the sequence $\{\mathfrak{B}^n\}_{n=0}^{\infty}$

By lemma 3.1 (c), if $B_i \in \mathfrak{B}^i$, $0 \le i \le n$, and B_{i+1} is a successor of B_i, then there exists a unique holomorphic inverse branch $S_*^{-n} : B_n \to B_0$ of S^n such that $S^i(S_*^{-n}(B_n)) \subseteq B_i$ for all $i = 0, 1, \cdots, n$. Denote $S^{-n}(\mathfrak{B}^n)$ as the family of all sets $S_*^{-n}(B_n)$, where $B_n \in \mathfrak{B}^n$. If $K_{n+1} \in S^{-(n+1)}(\mathfrak{B}^{n+g})$, then there exists a unique $K_n \in S^{-n}(\mathfrak{B}^n)$ such that $K_{n+1} \subseteq K_n$. We call the set K_{n+1} be a child of K_n and denote $ch(K_n)$ as the family of all children of K_n.

Define X_n to be union of closures of all elements of $S^{-n}(\mathfrak{B}^n)$ for every $n \ge 0$. It is clearly that $X_{n+1} \subseteq X_n$. We construct the sequence $\{\mu_n\}_{n=0}^{\infty}$ of Borel measures on the sets X_n as follows. Let μ_0 be the normalized Lebesgue measure on $X_0 = \overline{B_0}$. Suppose now that the measure μ_n on X_n has been defined. The measure μ_{n+1} on X_{n+1} is defined on each $K_{n+1} \in S^{-(n+1)}(\mathfrak{B}^{n+g})$ as follows:

$$\mu_{n+1}|_{K_{n+1}} = \frac{area(K_{n+1})}{\sum_{K \in ch(K_n)} area(K)} \cdot \mu_n|_{K_n}, \tag{7}$$

where K_n is the unique element of $S^{-n}(\mathfrak{B}^n)$ containing K_{n+1}. So $\mu_{n+1}(K_n \cap X_{n+1}) = \mu_n(K_n)$ and therefore,

$$\mu_k(K_n) = \mu_n(K_n)$$

for all $k \ge n$ and every $K_n \in S^{-n}(\mathfrak{B}^n)$. Then, it derives a unique measure μ on the set $X_\infty = \cap_{n\ge 0} X_n = \cap_{n \ge k} X_n$. Denote them by X_∞^1 and X_∞^2, respectively, corresponding to first and second construction, which all satisfy

$$\mu(K_n) = \mu_n(K_n) \quad \text{for every} \quad K_n \in S^{-n}(\mathfrak{B}^n). \tag{8}$$

In addition, we need to explain some symbols. Denote $R(S(B_{n-1}(z)))$ and $r(S(B_{n-1}(z)))$, i.e. $R(S^n(K_{n-1}(z)))$ and $r(S^n(K_{n-1}(z)))$, respectively, by $R_n(z)$ and $r_n(z)$. Denote the ball centered at z of radius r by $B(z, r)$. For real valued functions $f(z)$, $g(z)$, the symbol $f(z) \asymp g(z)$ means that there exists $C \ge 1$ such that $C^{-1}g(z) \le f(z) \le Cg(z)$ for all z. We go on to give the concept of "edge points set" and call the below set,

$$\partial_\infty := \left\{ z \in I^q : B_{j+1}(z) \cap \partial S(B_j(z)) \ne \emptyset \text{ for infinitely many j} \right\},$$

"edge points set". Correspondingly, let

$$\partial_\infty^n := \left\{ z \in I^q : B_{j+1}(z) \cap \partial S(B_j(z)) = \emptyset, \ \text{ for all } j \ge n \right\},$$

where j, n are nonnegative integers.

Lemma 3.4. *If q is large enough, then $D^q(h_\epsilon) \supseteq X_\infty^1$, $D^q(2h_\epsilon) = [D^q(2h_\epsilon) \cap \partial_\infty] \cup [\cup_{n\ge 0}(D^q(2h_\epsilon) \cap \partial_\infty^n)]$ and $B_0 \cap D^q(2h_\epsilon) \cap \partial_\infty^0 \subseteq X_\infty^2$.*

Proof. Assume $z \in X_\infty^1$, by (4),

$$\begin{aligned} 2\pi|s_n(z)| &= 2\pi(|s_n(z)| - 1) + 2\pi \le \max_{w \in B_n(z)} \{|Imw|\} + 2\pi \\ &\le \inf_{w \in B_n(z)} h_\epsilon(|w|) \le h_\epsilon(|S^n(z)|). \end{aligned}$$

So, $z \in D^q(h_\epsilon)$.

It is obvious that $\partial_\infty = \cap_{n \ge 0}(I^q \setminus \partial_\infty^n)$ and $I^q = \partial_\infty \cup (\cup_{n\ge 0} \partial_\infty^n)$, so

$$D^q(2h_\epsilon) = D^q(2h_\epsilon) \cap I^q = [D^q(2h_\epsilon) \cap \partial_\infty] \cup [\cup_{n \ge 0}(D^q(2h_\epsilon) \cap \partial_\infty^n)].$$

If $z \in B_0 \cap D^q(2h_\epsilon) \cap \partial_\infty^0$, then $z \in \partial_\infty^0$ and $B_n(z)$ does not intersect the boundary of $S(B_{n-1}(z))$. Since $z \in D^q(2h_\epsilon)$, by the properties of function $h_\epsilon(x)$, which is bounded by a region with a small opening angle, we get that $B_n(z) \subseteq S(B_{n-1}(z)) \cap H_q$ and

$$\begin{aligned} \min_{w \in B_n(z)} \{|Imw|\} &\le |ImS^n(z)| \le 2\pi(|s_n(z)| + 1) \\ &\le 2h_\epsilon(|S^n(z)|) + 2\pi \le \sup_{w \in B_n(z)} 2h_\epsilon(|w|) + 2\pi. \end{aligned}$$

By (6), and $B_n(z) \in \mathfrak{B}^n$ for all n, consequently, $z \in X_\infty^2$.

□

3.3. Distortion theorem and some estimates

Lemma 3.5. *If q is large enough and $z \in I^q$, then*

(*a*) $S^k(K_n(z)) \subseteq \{z \in \mathbb{C} : |Rez| > q - \pi\}$ *for every* $0 \le k \le n$;

(*b*) $diam(K_n(z)) \le \sqrt{2}\pi e^{-n}$;

(*c*) *there exists constants K_1 and K_2 independent on n and z such that*

$$\frac{|(S_z^{-n})'(x)|}{|(S_z^{-n})'(y)|} \le K_1$$

for all $x, y \in B_n(z)$, and

$$\frac{|(S^n)'(x)|}{|(S^n)'(y)|} \le K_2$$

for all $x, y \in K_{n-1}(z)$ i.e. $S^{n-1}(x), S^{n-1}(y) \in B_{n-1}(z)$.

Proof. (a)(b) are obvious.

(c) Denote $\widetilde{B_i(z)} \supset B_i(z)$ as the open square of side length 2π with sides parallel to $B_i(z)$ and center coincident with $B_i(z)$. By lemma 3.1 (b), we know that $S(z)$ is univalent on $\widetilde{B_i(z)}$ and the $S(\widetilde{B_i(z)})$ contains $\widetilde{B_{i+1}(z)}$ for $i = 0, 1, 2, \cdots$. See the following figure 3.

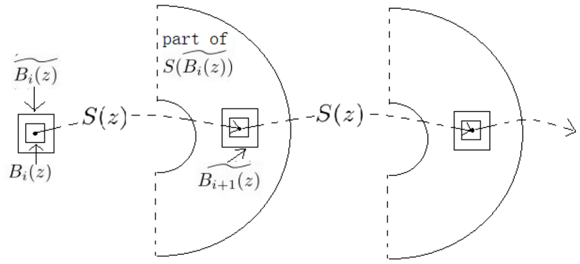

Figure 3. expanding property of $S(z)$

The module of $\widetilde{B_i(z)} \setminus B_i(z)$ is constant, by distortion theorem, for all $x, y \in B_n(z)$

$$\frac{|(S_z^{-n})'(x)|}{|(S_z^{-n})'(y)|} \le K_1.$$

By lemma 3.1 (d)

$$\frac{|(S^n)'(x)|}{|(S^n)'(y)|} = \frac{|S'(S^{n-1}(x))|}{|S'(S^{n-1}(y))|} \cdot \frac{|(S^{n-1})'(x)|}{|(S^{n-1})'(y)|} \le 2e^{\pi} K_1 = K_2.$$

□

Lemma 3.6. *If q is large enough and $z \in I^q$, then for every $n \in \mathbb{N}$ there exist K_3, K_4, K_5 such that*

(*a*) $K_3^{-1}|S'(S^{n-1}(z))| \le r_n(z) \le R_n(z) \le K_3|S'(S^{n-1}(z))|$;

(*b*) $K_3^{-n}\prod_{j=1}^n r_j(z) \le |(S^n)'(z)| \le K_3^n \prod_{j=1}^n r_j(z)$;

(*c*) $diam K_n(z) \le K_4^n(\prod_{j=1}^n r_j(z))^{-1}$;

(*d*) *exist $B(z_0, r)$ such that $B(z_0, r) \subseteq K_n(z)$ and*

$$diam K_n(z) > r \ge K_5^{-n}(\prod_{j=1}^n r_j(z))^{-1},$$

where $z_0 = S^{-n}(w_0)$, w_0 is the center of $B_n(z)$.

Proof. (a) By lemma 3.1 (d), and $S^{n-1}(z) \in B_{n-1}(z)$, we have

$$1 \leq \frac{R_n(z)}{r_n(z)} \leq 2e^{\pi}, \frac{1}{2e^{\pi}} \leq \frac{r_n(z)}{S'(S^{n-1}(z))} \leq 2e^{\pi}, \frac{1}{2e^{\pi}} \leq \frac{R_n(z)}{S'(S^{n-1}(z))} \leq 2e^{\pi},$$

then

$$(2e^{\pi})^{-1}|S'(S^{n-1}(z))| \leq r_n(z) \leq R_n(z) \leq 2e^{\pi}|S'(S^{n-1}(z))|.$$

(b) Note that $|(S^n)'(z)| = |(S)'(S^{n-1}(z))||(S)'(S^{n-2}(z))| \cdots |S'(z)|$, so

$$(2e^{\pi})^{-n}\prod_{j=1}^{n} r_j(z) \leq |(S^n)'(z)| \leq (2e^{\pi})^{n}\prod_{j=1}^{n} r_j(z).$$

(c) For any $z_1, z_1 \in K_n(z)$, corresponding $w_1, w_2 \in B_n(z)$, by lemma 3.5 (c), we have

$$|z_2 - z_1| = |\int_{w_1}^{w_2} (S^{-n})' dw| \leq K_1|(S^{-n})'||w_2 - w_1|,$$

then

$$diam K_n(z) \leq \sqrt{2}\pi K_1 (2e^{\pi})^n (\prod_{j=1}^{n} r_j(z))^{-1}.$$

(d) Let $S^n(z_0)$ is the center of $B_n(z)$, $z_0 \in K_n(z)$, there exist $z_* \in \partial K_n(z)$ such that

$$\min_{t\in\partial K_n(z)} |t - z_0| = |z_* - z_0|.$$

Let $r = |z_* - z_0|$, then $B(z_0, r) \subseteq K_n(z)$.
Denote by l the straight line segment $\overrightarrow{z_* z_0}$, L is $S^n(l)$, then

$$\begin{aligned} r = |z_* - z_0| &= \int_l |dz| = \int_L |dS^{-n}(w)| \\ &= \int_L |(S^{-n}(w))'||dw| \\ &\geq \frac{1}{K_1}(2e^{\pi})^{-n}(\prod_{j=1}^{n} r_j(z))^{-1}\frac{\pi}{2}. \end{aligned}$$

□

Lemma 3.7. *If q is large enough and $z \in I^q$, then there exist K_6, K_7 such that*

$$\begin{aligned} \mu(K_n(z)) &= c_1^n(z)(\prod_{j=1}^{n} r_j(z))^{-2}\prod_{i=1}^{n}\frac{r_i^2(z)}{h_\epsilon(r_i(z))r_i(z)} \\ &= c_1^n(z)(\prod_{j=1}^{n} r_j(z))^{-(1+\delta)}\prod_{i=1}^{n}\frac{r_i^\delta(z)}{h_\epsilon(r_i(z))} \\ &= c_2^n(z) diam(K_n(z))^{1+\delta}\prod_{i=1}^{n}\frac{r_i^\delta(z)}{h_\epsilon(r_i(z))}, \end{aligned}$$

where $c_1(z) \in [K_6^{-1}, K_6]$, $c_2(z) \in [K_7^{-1}, K_7]$.

Proof. It follows from (7), (8) that

$$\begin{aligned}\mu(K_n(z)) &= \mu_n(K_n) = \frac{area(K_n)}{\sum_{K\in ch(K_{n-1})} area(K)} \cdot \mu_{n-1}(K_{n-1}) \\ &= area(K_n(z)) \cdot \prod_{i=1}^{n} \frac{area(K_{i-1}(z))}{\sum_{K\in ch(K_{i-1}(z))} area(K)}.\end{aligned}$$

and from lemma 3.6 (c) (d) that

$$\pi(K_5^{-n}(\prod_{j=1}^{n} r_j(z))^{-1})^2 \le area(K_n(z)) \le \pi\left(\frac{K_4^n(\prod_{j=1}^{n} r_j(z))^{-1}}{2}\right)^2,$$

then

$$\mu(K_n(z)) = c(z)^n(\prod_{j=1}^{n} r_j(z))^{-2} \prod_{i=1}^{n} \frac{area(K_{i-1}(z))}{\sum_{K\in ch(K_{i-1}(z))} area(K)},$$

with $c(z) \in [K_6^{-1}, K_6]$.

By $(areaS^i(K_{i-1}(z))) = \int\int_{K_{i-1}(z)} |(S^i)'|^2 dxdy$ and lemma 3.5 (c), we have

$$\frac{area(K_{i-1}(z))}{\sum_{K\in ch(K_{i-1}(z))} area(K)} \asymp \frac{areaS^i(K_{i-1}(z))}{\sum_{K\in ch(K_{i-1}(z))} area(S^i(K))}.$$

Note that $S^i(K_{i-1}(z)) \cap H_q$ is an approximate-half-annulus and $S^i(K)$ is square in $S^i(K_{i-1}(z)) \cap H_q$ satisfies (4) or (6), then

$$\frac{areaS^i(K_{i-1}(z))}{\sum_{K\in ch(K_{i-1}(z))} area(S^i(K))} \asymp \frac{r_i^2(z)}{h_\epsilon(r_i(z))r_i(z)}.$$

Hence, according to lemma 3.6 (c) (d), there exists a constant K_7 with $c_1(z)$, $c_2(z) \in [K_7^{-1}, K_7]$ such that

$$\begin{aligned}\mu(K_n(z)) &= c_1^n(z)(\prod_{j=1}^{n} r_j(z))^{-2} \prod_{i=1}^{n} \frac{r_i^2(z)}{h_\epsilon(r_i(z))r_i(z)} \\ &= c_1^n(z)(\prod_{j=1}^{n} r_j(z))^{-(1+\delta)} \prod_{i=1}^{n} \frac{r_i^\delta(z)}{h_\epsilon(r_i(z))} \\ &= c_2^n(z) diam(K_n(z))^{1+\delta} \prod_{i=1}^{n} \frac{r_i^\delta(z)}{h_\epsilon(r_i(z))}.\end{aligned}$$

□

For convenient, let $K > K_j$ $j = 1, 2, 3, 4, 5, 6, 7$ and we can replace K_j by the same K in all the above inequalities.

Lemma 3.8. *If q is large enough and $z \in D^q(3h_\epsilon)$, $n \in \mathbb{N}$, then $r_n(z) \asymp \log r_{n+1}(z)$*

Proof. By lemma 3.1 (d) (c) and (1)

$$\begin{aligned}r_{n+1}(z) &\ge \frac{1}{2e^\pi}|S^{n+1}(z)| \ge \frac{1}{2e^\pi}\frac{2}{3}\min\{|a|, |b|\}\exp(|ReS^n(z)|) \\ &\ge C\exp(\sqrt{3}|S^n(z)|/2) \ge C\exp(\sqrt{3}r_n(z)/2),\end{aligned}$$

where $C = \frac{1}{2e^{\pi}}\frac{2}{3}\min\{|a|, |b|\}$. Then we have

$$\begin{aligned}\log r_{n+1}(z) &\geq \log C\exp(\sqrt{3}r_n(z)/2)\\ &\geq (\log C + \frac{\sqrt{3}-1}{2}r_n(z)) + \frac{1}{2}r_n(z)\\ &\geq \frac{1}{3e^{\pi}}r_n(z)\end{aligned}$$

and

$$\begin{aligned}\log r_{n+1}(z) &\leq \log(|a|e^{|S^n(z)|} + |b|e^{|-S^n(z)|})\\ &\leq \log(2\max\{|a|, |b|\}e^{2e^{\pi}r_n(z)})\\ &\leq 3e^{\pi}r_n(z).\end{aligned}$$

□

Lemma 3.9. *Let q is large enough and $\delta \in (0,1)$, if $z \in D^q(h_\epsilon)$ and $\delta < 1/(1+\epsilon)$, then the sequence $\{r_n(z)\}_{n=1}^{\infty}$ satisfies that for every $c > 0$ there exists n_0 such that for every $n \geq n_0$ the following inequality holds*

$$c^n\frac{r_1^{\delta}(z)\cdots r_{n-1}^{\delta}(z)}{h_\epsilon(r_1(z))\cdots h_\epsilon(r_{n-1}(z))}\cdot(\frac{r_n(z)}{h_\epsilon(r_n(z))})^{\delta} \leq 1. \tag{9}$$

If $z \in D^q(3h_\epsilon)$ and $\delta > 1/(1+\epsilon)$, then the sequence $\{r_n(z)\}_{n=1}^{\infty}$ satisfies that for every $c > 0$ there exists n_0 such that for every $n \geq n_0$ the following inequality holds

$$c^n\frac{r_1^{\delta}(z)\cdots r_{n-1}^{\delta}(z)}{3h_\epsilon(r_1(z))\cdots 3h_\epsilon(r_{n-1}(z))}\cdot(\frac{r_n(z)}{3h_\epsilon(r_n(z))})^{\delta} > 1. \tag{10}$$

Proof. The inequality (9) is equivalent to the following

$$\frac{r_1^{1-\delta}}{c(\log r_1)^{\epsilon}}\cdots\frac{r_{n-2}^{1-\delta}}{c(\log r_{n-2})^{\epsilon}}\cdot\{\frac{r_{n-1}^{1-\delta}}{c(\log r_{n-1})^{\epsilon}}\cdot\frac{1}{c(\log r_n)^{\epsilon\delta}}\} \geq 1.$$

Since $1-\delta-\epsilon\delta > 0$, it follows from lemma 3.2 and lemma 3.8 that

$$\{\frac{r_{n-1}^{1-\delta}}{c(\log r_{n-1})^{\epsilon}}\cdot\frac{1}{c(\log r_n)^{\epsilon\delta}}\} \geq \frac{r_{n-1}^{1-\delta-\epsilon\delta}}{c^2(3e^{\pi})^{\epsilon\delta}(\log r_{n-1})^{\epsilon}} \to \infty$$

as $n \to \infty$, and $\frac{r_{n-2}^{1-\delta}}{c(\log r_{n-2})^{\epsilon}} \to \infty$ as $n \to \infty$.

If $\delta > 1/(1+\epsilon)$, then $1-\delta-\epsilon < 1-\delta-\epsilon\delta < 0$, therefore for n large enough

$$\begin{aligned}&\frac{3^{\delta}}{c(\log r_1)^{\epsilon}}\frac{3r_1^{1-\delta-\epsilon}}{c(\frac{1}{3e^{\pi}})^{\epsilon}}\cdots\frac{3r_{n-2}^{1-\delta-\epsilon}}{c(\frac{1}{3e^{\pi}})^{\epsilon}}\cdot\frac{3r_{n-1}^{1-\delta-\epsilon\delta}}{c(\frac{1}{3e^{\pi}})^{\epsilon\delta}} < 1 &&\Rightarrow\\ &\frac{3^{\delta}}{c(\log r_1)^{\epsilon}}\frac{3r_1^{1-\delta}}{c(\frac{1}{3e^{\pi}}r_1)^{\epsilon}}\cdots\frac{3r_{n-2}^{1-\delta}}{c(\frac{1}{3e^{\pi}}r_{n-2})^{\epsilon}}\cdot\frac{3r_{n-1}^{1-\delta}}{c(\frac{1}{3e^{\pi}}r_{n-1})^{\epsilon\delta}} < 1 &&\Rightarrow\\ &\frac{3^{\delta}}{c(\log r_1)^{\epsilon}}\frac{3r_1^{1-\delta}}{c(\log r_2)^{\epsilon}}\cdots\frac{3r_{n-2}^{1-\delta}}{c(\log r_{n-1})^{\epsilon}}\cdot\frac{3r_{n-1}^{1-\delta}}{c(\log r_n)^{\epsilon\delta}} < 1 &&\Rightarrow (10).\end{aligned}$$

□

4. The proof of theorem

Based on the above preliminaries, we can begin to prove the main result of this paper.

4.1. The lower bound of Hausdorff dimension

Claim 1: If $z \in D^q(h_\epsilon)$ and $0 < \delta < 1/(1+\epsilon)$, then $HD(D^q(h_\epsilon)) \geq 1+\delta$.

Proof. By lemma 3.4, we only need to prove

$$HD(X^1_\infty) \geq 1+\delta.$$

Take an arbitrary point $z \in X^1_\infty$. We should show that if r is small then $\mu(B(z,r)) \leq constant \cdot r^{1+\delta}$.

Take the least $n \geq 1$ such that

$$diamK_n(z) \leq r. \tag{11}$$

If $r > 0$ is sufficiently small, then $n \geq 1$ is large enough, and $r_n(z)$ is large enough. By (3), (4), we know that the distance (less than $\sqrt{2}\pi$) between the boundary of square $S^n(K_n(z))$ and $S^n(z) \ll$ the distance($\asymp r_n(z)$) between the boundary of approximate-half-annulus $S^n(K_{n-1}(z)) \cap H_q$ and $S^n(z)$. Using lemma 3.5 (c), we have $r \asymp \frac{\sqrt{2}\pi}{|(S^n)'(z)|}$ and $\text{dist}(z, \partial K_{n-1}(z)) \asymp \frac{r_n(z)}{|(S^n)'(z)|}$, then $B(z,r) \subseteq K_{n-1}(z)$. Note the construction of μ, computing $\mu(B(z,r))$, we only need to consider the $Q \in S^{-n}(\mathfrak{B}^n)$ such that $Q \cap B(z,r) \neq \emptyset$, denote by $F(z,r)$ the family of all sets $Q \in S^{-n}(\mathfrak{B}^n)$ intersecting $B(z,r)$, consequently, $Q \subseteq K_{n-1}(z)$.

By the construction of μ, lemma 3.5 (c) and $areaQ = \int\int |(S^{-n}(z))'|^2 dxdy$, we have

$$\frac{areaQ}{areaK_n(z)} \leq K^4, \text{ then } \mu(Q) \leq K^4\mu(K_n(z)).$$

It follows from lemma 3.7 and (9) that

$$\begin{aligned}
\mu(Q) &\leq K^4 c_2^n(z) \prod_{i=1}^{n} \frac{r_i^\delta(z)}{h_\epsilon(r_i(z))} (diamK_n(z))^{1+\delta} \\
&\leq K^{6+2\delta} c_2^n(z) \prod_{i=1}^{n-1} \frac{r_i^\delta(z)}{h_\epsilon(r_i(z))} (\frac{r_n(z)}{h_\epsilon(r_n(z))})^\delta (diamQ)^{1+\delta} \\
&\leq (diamQ)^{1+\delta}.
\end{aligned} \tag{12}$$

In addition, by lemma 3.5 (c) and (11), we have

$$K^{-2}diamK_n(z) \leq diamQ \leq K^2 diamK_n(z) \leq K^2 r. \tag{13}$$

Applying (12) and (13), we get that

$$\mu(Q) \leq (diamQ)^{1+\delta} \leq K^{2(1+\delta)}(diamK_n(z))^{1+\delta} \leq K^{2(1+\delta)} r^{1+\delta}. \tag{14}$$

Then the estimation of $F(z,r)$ is critical for computing $\mu(B(z,r))$. We shall consider several cases. Fix a constant $D \geq 2$.

Case 1: $r \leq DdiamK_n(z)$.

Since $z \in X^1_\infty$ and $diamK_n(z) \leq K|(S^{-n})'(z)|^{-1}\sqrt{2}\pi$ (lemma 3.5 (c) and $|z_1 - z_2| = |\int_{w_1}^{w_2} (S^{-n})'(w)dw|$), we get

$$\begin{aligned}
K_{n-1}(z) &\supseteq S_z^{-n}(\widetilde{A}(r_n(z), R_n(z))) \supseteq S_z^{-n}(B(S^n(z), r_n(z))) \\
&\supseteq B(z, \frac{1}{4}K^{-1}|(S^n)'(z)|^{-1} r_n(z)) \quad Koebe - \frac{1}{4} theorem \\
&\supseteq B(z, \frac{1}{4}(\sqrt{2}\pi K^2)^{-1} r_n(z) diamK_n(z)) \\
&\supseteq B(z, D(K^2+1)diamK_n(z)) \supseteq B(z, (K^2+1)r).
\end{aligned} \tag{15}$$

The $r_n(z)$ could be larger than $4\sqrt{2}\pi K^2(K^2+1)D$ as n is sufficiently large. By (13) and (15), if $Q \in F(z,r)$, then $Q \subset B(z,(K^2+1)r) \subset K_{n-1}(z)$. Since $diam(S^n(B(z,(K^2+1)r))$ is less than $2K(K^2+1)r|(S^n)'(z)|$, the number of squares $S^n(Q)$, $Q \in F(z,r)$ is smaller than $K^2(K^2+1)^2\pi^{-1}r^2|(S^n)'(z)|^2$. Since

$$r \leq D diam K_n(z) \leq DK\sqrt{2}\pi|(S^n)'(z)|^{-1},$$

we get

$$\sharp\{S^n(Q) : Q \in F(z,r)\} \leq 2K^4D^2\pi(K^2+1)^2.$$

However, $\sharp\{S^n(Q) : Q \in F(z,r)\} = \sharp F(z,r)$. Then by (14) we get that

$$\begin{aligned}\mu(B(z,r)) &\leq \sum_{Q\in F(z,r)} \mu(Q) \leq \sum_{Q\in F(z,r)} K^{2(1+\delta)}r^{1+\delta} \\ &\leq 2\pi K^{6+2\delta}(K^2+1)^2D^2r^{1+\delta}.\end{aligned}$$

Case 2: $D diam K_n(z) \leq r \leq D^{-1}diam(K_{n-1}(z))$.

Take $D \geq 4\sqrt{2}\pi K^3(K^2+1)$ large enough, since $z \in X^1_\infty$, using lemma 3.6 (a), we get that

$$\begin{aligned}K_{n-1}(z) &\supseteq B(z, \frac{1}{4}K^{-1}|(S^n)'(z)|^{-1}r_n(z)) \\ &\supseteq B(z, \frac{1}{4}K^{-2}|(S^{n-1})'(z)|^{-1}) \\ &\supseteq B(z, \frac{1}{4}(\sqrt{2}\pi K^3)^{-1} diam K_{n-1}(z)) \\ &\supseteq B(z,(K^2+1)r). \qquad (16)\end{aligned}$$

By (13) and (16), if $Q \in F(z,r)$, then $Q \subseteq B(z,(K^2+1)r) \subseteq K_{n-1}(z)$. Hence, using lemma 3.7, we get that

$$\begin{aligned}\mu(B(z,r)) &\leq \sum_{Q\in F(z,r)} \mu(Q) \\ &\leq \sum_{Q\in F(z,r)} K^4c_1^n(z)(\prod_{i=1}^n r_i(z))^{-(1+\delta)} \prod_{i=1}^n \frac{r_i^\delta(z)}{h_\epsilon(r_i(z))} \\ &= \sharp F(z,r)K^4 \cdot c_1^n(z) \prod_{i=1}^n \frac{r_i^{-1}(z)}{h_\epsilon(r_i(z))}. \qquad (17)\end{aligned}$$

It will be discussed in two subcases.

Case 2a: $diam(S^n(B(z,(K^2+1)r))) \leq 2h_\epsilon(r_n(z))$. Since

$$\sharp F(z,r) \leq \frac{area(S^n(B(z,(K^2+1)r)))}{\pi^2},$$

by lemma 3.6 (b), the (17) could be as follows:

$$\begin{aligned}\mu(B(z,r)) &\leq \frac{K^4}{\pi^2}c_1^n(z)area(S^n(B(z,(K^2+1)r)) \prod_{i=1}^n \frac{r_i^{-1}(z)}{h_\epsilon(r_i(z))} \\ &\leq K^4\pi^{-1}K^{2n+2}(K^2+1)^2c_1^n(z)r^2 \prod_{i=1}^n r_i^2(z) \prod_{i=1}^n \frac{r_i^{-1}(z)}{h_\epsilon(r_i(z))} \\ &\leq K^4\pi^{-1}K^{2n+2}(K^2+1)^2K^nr^2 \prod_{i=1}^n \frac{r_i(z)}{h_\epsilon(r_i(z))} \\ &= K^4\pi^{-1}K^{2n+2}(K^2+1)^2K^nr^{1+\delta}r^{1-\delta} \prod_{i=1}^n \frac{r_i(z)}{h_\epsilon(r_i(z))}.\end{aligned}$$

However, $diam(S^n(B(z,(K^2+1)r))) \leq 2h_\epsilon(r_n(z))$, so using lemma 3.6 (b) again, we get that

$$r \cdot \prod_{i=1}^{n} r_i(z) \leq \frac{K^{n+1}}{K^2+1} h_\epsilon(r_n(z)).$$

Therefore, by (9), we get

$$\begin{aligned}
&\mu(B(z,r)) \\
&\leq K^4\pi^{-1}K^{2n+2}(K^2+1)^2K^n r^{1+\delta}\left(\frac{K^{n+1}}{K^2+1}\frac{h_\epsilon(r_n(z))}{r_1(z)\cdots r_n(z)}\right)^{1-\delta} \cdot \prod_{i=1}^{n}\frac{r_i(z)}{h_\epsilon(r_i(z))} \\
&= r^{1+\delta}\pi^{-1}K^{7-\delta}(K^2+1)^{1+\delta}(K^{4-\delta})^n \prod_{i=1}^{n-1}\frac{r_i(z)^\delta}{h_\epsilon(r_i(z))} \cdot \left(\frac{r_n(z)}{h_\epsilon(r_n(z))}\right)^\delta \\
&\leq r^{1+\delta}.
\end{aligned}$$

and the claim 1 holds in this case.

Case 2b: $diam(S^n(B(z,(K^2+1)r))) > 2h_\epsilon(r_n(z))$. It need to estimate the cardinality of $F(z,r)$ in a different way in this case.

By (4), lemma 3.6 (b) and the properties of function $h_\epsilon(x)$, we get

$$\begin{aligned}
\sharp F(z,r) &\leq \frac{area(S^n(B(z,(K^2+1)r)) \cap \{w : |Imw| \leq h_\epsilon(R_n(z))\})}{\pi^2} \\
&\leq \frac{2diam(S^n(B(z,(K^2+1)r))h_\epsilon(R_n(z))}{\pi^2} \\
&\leq 4\pi^{-2}e^\pi h_\epsilon(r_n(z)) \cdot 2rK^{n+1}(K^2+1) \cdot \prod_{i=1}^{n} r_i(z).
\end{aligned}$$

Hence, the (17) could be as follows:

$$\begin{aligned}
\mu(B(z,r)) &\leq K^4 8\pi^{-2}e^\pi K^{n+1}(K^2+1)(c_1(z))^n r\frac{h_\epsilon(r_n(z))}{h_\epsilon(r_1(z))\cdots h_\epsilon(r_n(z))} \\
&\leq 8\pi^{-2}e^\pi(K^2+1)K^5(K^2)^n r^{1+\delta}r^{-\delta}\prod_{i=1}^{n-1}(h_\epsilon(r_i(z)))^{-1}.
\end{aligned}$$

Since $diam(S^n(B(z,(K^2+1)r))) > 2h_\epsilon(r_n(z))$, by lemma 3.6 (b), we get that

$$r \cdot r_1(z)\cdots r_n(z) \geq \frac{1}{(K^2+1)K^{n+1}} h_\epsilon(r_n(z)).$$

Thus,

$$\mu(B(z,r)) \leq 8\pi^{-2}e^\pi K^{5+\delta}(K^2+1)^{1+\delta}(K^{2+\delta})^n r^{1+\delta} \cdot \prod_{i=1}^{n-1}\frac{r_i(z)^\delta}{h_\epsilon(r_i(z))} \cdot \left(\frac{r_n(z)}{h_\epsilon(r_n(z))}\right)^\delta.$$

By (9), we get

$$\mu(B(z,r)) \leq r^{1+\delta}.$$

Case 3: $r > D^{-1}diam(K_{n-1}(z))$. That is $diam(K_{n-1}(z)) < Dr$. However, by our choice of n, $diam(K_{n-1}(z)) > r =$

$D^{-1}(Dr)$. As discussed above

$$\begin{aligned} K_{n-2}(z) &\supseteq S_z^{-(n-1)}(\widetilde{A}(r_{n-1}(z), R_{n-1}(z))) \supseteq S_z^{-(n-1)}(B(S^{n-1}(z), r_{n-1}(z))) \\ &\supseteq B(z, \frac{1}{4}K^{-1}|(S^{n-1})'(z)|^{-1}r_{n-1}(z)) \\ &\supseteq B(z, \frac{1}{4}(\sqrt{2}\pi K^2)^{-1}r_{n-1}(z) diam K_{n-1}(z)) \\ &\supseteq B(z, D diam K_{n-1}(z)) \\ &\supseteq B(z, Dr). \end{aligned}$$

This means that $diam(K_{n-2}(z)) > Dr$. So, $n-1$ is the number ascribed to the radius Dr as in the beginning of the proof and Case 1 holds. Therefore,

$$\mu(B(z,r)) \leq \mu(B(z,Dr)) \leq 2\pi K^{6+2\delta}(K^2+1)^2 D^2 (Dr)^{1+\delta}.$$

□

4.2. The Hausdorff dimension of edge points set

Claim 2: Let ∂_∞ be the edge points set, then $HD(\partial_\infty \cap D^q(3h_\epsilon)) \leq 1$.

Proof. Let

$$\partial_n := \cup_{z\in I^q} S_z^{-n}(\widetilde{A}(r_n(z), r_n(z)+2\pi) \cup \widetilde{A}(R_n(z)-2\pi, R_n(z))).$$

Then ∂_∞ can be covered by the set $\cup_{n\geq k}\partial_n$ for every $k \geq 0$ and the approximate-half-annuli $\widetilde{A}(r_n(z), r_n(z)+2\pi) \cup \widetilde{A}(R_n(z)-2\pi, R_n(z))$ can be covered by $M_1 r_n(z)$ squares with diameters less than 1, where M_1 is a constant. Therefore, according to lemma 3.5 (c), $K_{n-1}(z) \cap \partial_n$ can be covered with no more than $M_1 r_n(z)$ sets $J_{i,n}(z)$ of diameters less than $K|(S^n)'(z)|^{-1}$. Let $T \geq 2q$. Note that any two sets $K_{n-1}(z)$ and $K_{n-1}(z')$ are either disjoint or equal, so we can find a set $Z_n \subset I^q$ such that $K_{n-1}(z)$ and $K_{n-1}(z')$ are disjoint for $z, z' \in Z_n, z \neq z'$ and

$$\partial_n \cap B(0,T) \subset \cup_{z\in Z_n} K_{n-1}(z) \subset B(0,2T).$$

For the given $\epsilon > 0$, let n be large enough such that lemma 3.3 is satisfied for $\alpha = 2/\epsilon$ and $2T$. Using lemma 3.1 (d), lemma 3.3 and (2), we get

$$\begin{aligned} \sum_{z\in Z_n}\sum_{J_{i,n}}(diam J_{i,n}(z))^{1+\epsilon} &\leq \sum_{z\in Z_n} M_1 K^{1+\epsilon} r_n(z)|(S^n)'(z)|^{-(1+\epsilon)} \\ &\leq 2e^\pi M_1 K^{1+\epsilon}\sum_{z\in Z_n}|S'(S^{n-1}(z))||(S^n)'(z)|^{-(1+\epsilon)} \\ &\leq 2e^\pi M_1 K^{1+\epsilon}\sum_{z\in Z_n}|S'(S^{n-1}(z))||S'(S^{n-1}(z))|^{-(1+\epsilon)}|(S^{n-1})'(z)|^{-(1+\epsilon)} \\ &\leq 2e^\pi M_1 K^{1+\epsilon}\sum_{z\in Z_n}|S'(S^{n-1}(z))|^{-\epsilon}|(S^{n-1})'(z)|^{-\epsilon}|(S^{n-1})'(z)|^{-1} \\ &\leq 2e^\pi M_1 K^{1+\epsilon}\sum_{z\in Z_n}|(S^n)'(z)|^{-\epsilon}|(S^{n-1})'(z))|^{-1} \\ &\leq 2e^\pi M_1 K^{1+\epsilon}\sum_{z\in Z_n}L^{-\epsilon}|(S^{n-1})'(z)|^{-2}|(S^{n-1})'(z))|^{-1} \\ &\leq 2e^\pi L^{-\epsilon} M_1 K^{1+\epsilon} e^{-(n-1)}\sum_{z\in Z_n}|(S^{n-1})'(z)|^{-2}. \end{aligned}$$

Because $K_{n-1}(z)$ and $K_{n-1}(z')$ are disjoint and the Lebesgue measure of each set of the form $K_{n-1}(z)$ is proportional to $|(S^{n-1})'(z)|^{-2}$, we get that there exists a constant $M_2 > 0$ such that the last term in the above inequality is no more than $M_2 e^{-(n-1)} \cdot area(B(0,2T))$.

Hence,

$$\sum_{n=k}^{\infty}\sum_{z\in Z_n}\sum_{J_{i,n}}(diamJ_{i,n}(z))^{1+\epsilon} \le M_2\cdot area(B(0,2T))\sum_{n=k}^{\infty}e^{-(n-1)} = 4\pi T^2M_2\frac{e^{-k+2}}{e-1}.$$

Let $k\to\infty$, then $4\pi T^2M_2\frac{e^{-k+2}}{e-1}\to 0$. That is, for any given $\epsilon>0$, the $(1+\epsilon)$-dimensional Hausdorff measure of $\partial_\infty\cap D^q(3h_\epsilon)\cap B(0,T)$ is equal to zero. Hence,

$$HD(\partial_\infty\cap D^q(3h_\epsilon))\le 1.$$

□

4.3. The upper bound of Hausdorff dimension

Claim 3: If $z\in D^q(2h_\epsilon)$ and $\delta>1/(1+\epsilon)$, then $HD(D^q(2h_\epsilon))\le 1+\delta$.

Proof. By lemma 3.4, $D^q(2h_\epsilon)=[D^q(2h_\epsilon)\cap\partial_\infty]\cup[\cup_{n\ge 0}(D^q(2h_\epsilon)\cap\partial_\infty^n)]$. Using Claim 2, we get $HD(D^q(2h_\epsilon)\cap\partial_\infty)\le 1$. We only need to show that $HD(D^q(2h_\epsilon)\cap\partial^n_\infty)\le 1+\delta$ for every $n\ge 1$ and as $S^n(D^q(2h_\epsilon)\cap\partial^n_\infty)\subseteq D^q(2h_\epsilon)\cap\partial^0_\infty$, it is sufficient to prove that $HD(B_0\cap D^q(2h_\epsilon)\cap\partial^0_\infty)\le 1+\delta$.

By lemma 3.4, in fact, it suffices to demonstrate that

$$HD(X^2_\infty)\le 1+\delta.$$

Let μ be the measure on X^2_∞ constructed above. Cover X^2_∞ by countably many mutually disjoint sets $K_{n-1}(z_j)$ such that $z_j\in X^2_\infty$ for all j. Fix $z_j=z$.

For an arbitrary set $F\subseteq S(B_{n-1}(z))=S^n(K_{n-1}(z))$. Then by lemma 3.6 (b) and lemma 3.5 (c), we have that

$$diam(S_z^{-n}(F))\le K^{n+1}(\prod_{k=1}^{n}r_k(z))^{-1}\cdot diamF \tag{18}$$

where $S_z^{-n}:S(B_{n-1}(z))\to K_{n-1}(z)$ is the unique holomorphic inverse branch of S^n defined on $S(B_{n-1}(z))$ and sending $S^n(z)$ to z. Let $\mathfrak{G}^n_z$ is the covering of $S(B_{n-1}(z))\cap\{w:|Imw|\le 3h_\epsilon(R_n(z))\}$, which consists of squares G with the following properties. See the following figure 4:

- the length of each edge of G is equal to $3h_\epsilon(R_n(z))$;
- one of the horizontal edges of G is contained in the real axis;
- at least two of the edges of G are contained in $S(B_{n-1}(z))\cap H_q$.

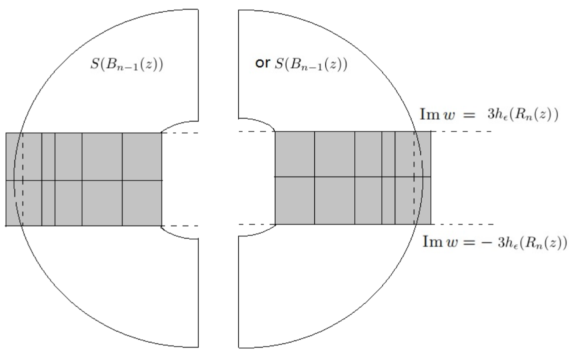

Figure 4. the covering of $S(B_{n-1}(z))\cap\{w:|Imw|\le 3h_\epsilon(R_n(z))\}$

Using lemma 3.7 we have

$$\mu(K_n(z)) = c_1^n(z)(\prod_{j=1}^{n} r_j(z))^{-(1+\delta)} \prod_{i=1}^{n} \frac{r_i^\delta(z)}{h_\epsilon(r_i(z))}. \tag{19}$$

Let $\widetilde{G} = G \cap S(B_{n-1}(z))$ and denote $\widetilde{\mathfrak{G}_z^n} := \{G \cap S(B_{n-1}(z)) : G \in \mathfrak{G}_z^n\}$. Let n is large enough such that $h(R_n(z))$ is as large as the necessary. Look at figure 3, it's obvious there exists a universal constant $\kappa \in (0, 1)$ such that $\widetilde{G}$ contains at least $\frac{9\kappa}{\pi^2} h_\epsilon^2(R_n(z)) \geq \frac{9\kappa}{\pi^2} h_\epsilon^2(r_n(z))$ squares from $\mathfrak{B}^n$. Since

$$diam\widetilde{G} \leq 3\sqrt{2}h_\epsilon(R_n(z)) \leq 6e^\pi \sqrt{2}h_\epsilon(r_n(z)),$$

using (19) and (18) we therefore get

$$\begin{aligned}
\mu(S_z^{-n}(\widetilde{G})) &\geq \frac{9\kappa}{\pi^2} h_\epsilon^2(r_n(z)) c_1^n(z)(\prod_{j=1}^{n} r_j(z))^{-(1+\delta)} \prod_{i=1}^{n} \frac{r_i^\delta(z)}{h_\epsilon(r_i(z))} \\
&\geq \frac{9\kappa}{\pi^2}(6e^\pi\sqrt{2})^{-(1+\delta)} c_1^n(z) 3^{n-1+\delta} \\
&\quad (\frac{6e^\pi\sqrt{2}h_\epsilon(r_n(z))}{\prod_{i=1}^{n} r_i(z)})^{1+\delta} (\frac{r_n(z)}{3h_\epsilon(r_n(z))})^\delta \cdot \prod_{i=1}^{n-1} \frac{r_i^\delta(z)}{3h_\epsilon(r_i(z))} \\
&\geq \frac{9\kappa 3^\delta}{3\pi^2(4e^\pi\sqrt{2}K)^{1+\delta}} (\frac{3}{K^{2+\delta}})^n \\
&\quad (diam(S_z^{-n}(\widetilde{G})))^{1+\delta} (\frac{r_n(z)}{3h_\epsilon(r_n(z))})^\delta \cdot \prod_{i=1}^{n-1} \frac{r_i^\delta(z)}{3h_\epsilon(r_i(z))}.
\end{aligned}$$

The last inequality is because

$$\begin{aligned}
(diam(S_z^{-n}(\widetilde{G})))^{1+\delta} &\leq K^{(n+1)(1+\delta)}(\prod_{j=1}^{n} r_j(z))^{-(1+\delta)} (diam(\widetilde{G}))^{1+\delta} \\
&\leq K^{(n+1)(1+\delta)}(\prod_{j=1}^{n} r_j(z))^{-(1+\delta)} (6e^\pi\sqrt{2}h_\epsilon(r_n))^{1+\delta}.
\end{aligned}$$

By (10), we thus get that

$$\mu(S_z^{-n}(\widetilde{G})) \geq (diam(S_z^{-n}(\widetilde{G})))^{1+\delta}. \tag{20}$$

The squares G may overlap, but we can choose the covering $\mathfrak{G}_z^n$ so that its multiplicity does not exceed 2. By (6), it is easy to know that the union of all squares $\widetilde{G} \in \widetilde{\mathfrak{G}_z^n}$ covers all the successors of $S^{n-1}(K_{n-1}(z))$, so the set $S_z^{-n}(\cup_{\widetilde{G}\in\widetilde{\mathfrak{G}_z^n}} \widetilde{G})$ covers all the children of $K_{n-1}(z)$, and then covers $K_{n-1}(z) \cap X_\infty^2$. Therefore

$$\cup_j \cup_{\widetilde{G}\in\widetilde{\mathfrak{G}_{z_j}^n}} S_{z_j}^{-n}(\widetilde{G}) \supseteq X_\infty^2.$$

By (20)

$$\begin{aligned}
\sum_j \sum_{\widetilde{G}\in\widetilde{\mathfrak{G}_{z_j}^n}} (diam(S_{z_j}^{-n}(\widetilde{G})))^{1+\delta} &\leq \sum_j \sum_{\widetilde{G}\in\widetilde{\mathfrak{G}_{z_j}^n}} \mu(S_{z_j}^{-n}(\widetilde{G})) \\
&\leq \sum_j 2\mu(K_{n-1}(z_j)) \leq 2\mu(B_0).
\end{aligned}$$

According to lemma 3.2, the diameters of the sets $S_{z_j}^{-n}(\widetilde{G})$, $j \geq 1$,$\widetilde{G} \in \widetilde{\mathfrak{G}_{z_j}^n}$ converge uniformly to zero as $n \to \infty$. so the $(1+\delta)$-dimensional Hausdorff measure of the set X_∞^2 is less than or equal to 2. Hence, $HD(X_\infty^2) \leq 1+\delta$. □

Therefore, we have

$$HD(D^q(h_\epsilon)) \leq HD(2D^q(h_\epsilon) \leq 1+\delta, \text{ when } \delta > \frac{1}{1+\epsilon}.$$

As well as, we have

$$HD(D^q(h_\epsilon)) \geq 1+\delta, \text{ when } 0 < \delta < \frac{1}{1+\epsilon}.$$

Let δ tends to $\frac{1}{1+\epsilon}$, we obtain

$$HD(D^q(h_\epsilon)) = 1 + \frac{1}{1+\epsilon}.$$

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