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On Some Generalized Contractive Maps in Partial Metric Space and Related Fixed Point Theorems

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Abstract. In this paper we generalize some results of fixed point theory to partial metric spaces by using metric methods in the context of a new extension of Ekeland's variational principle. We provide some corollaries to unify our results with other existing results in the literature along the same vein. Then, we show that our results require existence assumptions weaker than those for some well-known contractive maps, including the ones in the sense of Banach, Ćirić, Song, Kannan and Hardy Rogers. Also, we provide some estimates for the distance to the fixed point set in partial metric space. In order to illustrate the strength of our fixed point theorem, we use it in order to derive a new result on coupled fixed points.

In memory of Prof. Fernando Lobo Pereira

1. Introduction

Fixed point theory is a vast body of results that have been witnessing a huge number of developments over the years, [6, 9, 10, 29], along which, a number of powerful results have emerged to address a wide variety of problems from integral and differential equations, and inclusions to optimization, game theory, optimal control, dynamic programming, reinforcement learning and fractal geometry, among others.

The existence of fixed points for various single valued contractive, and set-valued contractive mappings had been studied by many authors, under different conditions. In [24] the famous Banach contraction principle was extended from single valued to set valued mappings in the metric spaces setting. Since then, this result was extended further in different directions by many authors, notably [16, 23, 35], whose results entailed new variational methods derived directly from Ekeland's variational principle. In [10] a general result on the existence of fixed points for a very general class of multifunctions, and consequently, for single

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valued mappings, generalizing some recent results, and sharp estimates for the distance to the set fixed points were obtained in standard metric spaces.

Several generalizations of standard metric spaces have been provided. In [21] Mattews introduced the concept of partial metric space to formulate problems of computer science, especially, in domain theory and semantics, by considering a metric-like structure in which the self distance may not to be zero. Additionally, partial metric spaces have been used to model metric spaces via domain theory in different areas such as integral equations, fixed-point theorems for single-valued maps, fixed-point theorems for set-valued maps and Ekeland's variational principle, [14]. Among the most important past results on partial metric spaces, the concept of a partial Hausdorff metric was introduced in [6], and Nadler's fixed point theorem to partial metric spaces using the partial Hausdorff metric was extended. In [13] a fixed point theorem was established on partial metric spaces by using a version of Ekeland's variational principle, thus enabling the generalization of results in [6, 10].

In this paper we establish a new version of Ekeland's variational principle and exploit it in the proof of very general fixed point theorems in partial metric spaces. In fact, we show that our results require existence assumptions weaker than those for some well-known contractive maps, including the ones in the sense of Banach, Ćirić, Song, Kannan, Hadry-Rogers, [2, 20, 21]. We also provide some estimates for the distance to the fixed point set in partial metric space.

This article is organized as follows. In the first section, 2, a number of general definitions and results on which the developments of this article are based, are given and discussed. Then, in section 3, a new extension of Ekeland's variational principle for partial metric spaces and some related results are presented, discussed and proved. In section 4, six generalized contractive mappings are presented and shown to be sufficient for a certain inequality condition related to Ekeland's variational principle shown in the previous section. Using these results, two theorems and a corollary on fixed points in partial metric spaces are obtained in section 5. These results are illustrated with an example in section 6 and a simple application. Finally, some conclusions are drawn in the section 7.

2. Preliminaries

In this section we state some results that will provide the basis on which the contributions of this article are built. First, let us start with a simple form of Ekeland's variational principle.

Theorem 2.1. *[10] Let (X, d) be a metric space. Then, the following proprieties are equivalent:*

(i) (X, d) *is complete;*

(ii) every proper, lower semi-continuous, and bounded from below function $f : X \to \mathbb{R}$ admits a d-point, that is, a point $x \in X$ satisfying:

$$f(x) < f(y) + d(x, y) \quad \forall\ y \in X,\ y \neq x. \tag{1}$$

Now, let us recall some definitions and properties of the partial metric space (see [21, 22]).

Definition 2.2. *[21] Let X be a nonempty set. The function $p : X \times X \to X$ is called a partial metric if, for all $x, y, z \in X$, it satisfies the following conditions:*

p_1. $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;
p_2. $p(x, x) \leq p(y, x)$;
p_3. $p(x, y) = p(y, x)$;
p_4. $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

The pair (X, p) is called a partial metric space.

Clearly, from p_1, and p_2, it follows that if $p(x, y) = 0$, then $x = y$. However, $p(x, x)$ is not a necessarily zero. Thus p is not a metric. Now let us give some examples of partial metric spaces.

Example 2.3. *The pair* $(\mathbb{R}^+, p)$ *with* $p(x, y) = \max\{x, y\}$*, where* $\mathbb{R}^+ = [0, +\infty)$ *is clearly a simple example of a partial metric space.*

Example 2.4. *Let* (X, d) *and* (X, p) *bea metric space and, respectively, a partial metric space. Let* $a \geq 0$ *and* $\omega : X \to \mathbb{R}^+$ *be an arbitrary function and let the functions* $\rho_i : X \times X \to \mathbb{R}^+$*,* $i \in \{1, 2, 3\}$ *be given by*

$$\begin{aligned} \rho_1(x, y) &= d(x, y) + p(x, y); \\ \rho_2(x, y) &= d(x, y) + \max\{\omega(x), \omega(y)\}; \\ \rho_3(x, y) &= d(x, y) + a. \end{aligned}$$

Then (X, ρ_i)*,* $i \in \{1, 2, 3\}$ *is a partial metric spaces.*

Example 2.5. *Let* $X = \{x \in [a, b] : a, b \in \mathbb{R}, a \leq b\}$ *and define* $p : X \times X \to \mathbb{R}$ *by:*

$$p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\},$$

where $\mathbb{R} = (-\infty, +\infty)$*. Then* (X, p) *is a partial metric space.*

Each partial metric on X generates a T_0 topology τ_p on X which has, as a basis, the family of open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) - p(x, x) < \varepsilon\}$ for all $x \in X$, and $\varepsilon > 0$. Similarly, the closed p-balls are defined by $B_p[x, \varepsilon] = \{y \in X : p(x, y) - p(x, x) \leq \varepsilon\}$.

Remark that, if p is a partial metric on X the function $p^s : X \times X \to [0, +\infty)$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y),$$

is a metric on X.

Definition 2.6. *Given a partial metric space* (X, p) *and* $\{x_n\}$ *a sequence in* X*. We say that*

(*a*) $\{x_n\}$ *converges to* $x \in X$ *if* $\lim_{n\to+\infty} p(x_n, x) = p(x, x)$*;*

(*b*) $\{x_n\}$ *is a Cauchy sequence if* $\lim_{n,m\to+\infty} p(x_n, x_m)$ *exists, and is finite;*

(*c*) *The partial metric space* (X, p) *is complete if every Cauchy sequence* $\{x_n\} \subset X$ *converges to a point* $x \in X$ *with respect to* τ_p *if*

$$p(x, x) = \lim_{n,m\to+\infty} p(x_n, x_m).$$

Notice that, the limit of sequence is not necessarily unique. For example consider the sequence $\{\frac{1}{1+n}\}_{n\in\mathbb{N}}$ in the partial metric space defined in example 2.3. Clearly, we have that

$$p(1, 1) = \lim_{n\to\infty} p(1, \frac{1}{1+n}), \text{ and } p(2, 2) = \lim_{n\to\infty} p(2, \frac{1}{1+n}).$$

Lemma 2.7. *Let* (X, p) *be a partial metric space. The following equivalences hold:*

(*i*) $\{x_n\}$ *is a Cauchy sequence in* (X, p) *if and only if it is a Cauchy sequence in the metric space* (X, p^s)*;*

(*ii*) (X, p) *is complete partial metric space if and only if* (X, p^s) *is complete metric space.*

Example 2.8. *Let* $X = [0, 1] \cup [2, 3]$ *and* $p : X \times X \to \mathbb{R}$ *defined by:*

$$p(x, y) = \begin{cases} \max\{x, y\}, & \text{if } \{x, y\} \cap [2, 3] \neq \emptyset; \\ \|x - y\|, & \text{if } \{x, y\} \subset [0, 1]. \end{cases}$$

Then, (X, p) *is a complete partial metric space.*

3. Ekeland's variational principle on partial metric space

In this section, we present an extension of Ekeland's variational principle in partial metric space. In [7], H. Aydi, E. Karapinar, and C. Vetro extended Ekeland's variational principle to the class of partial metric spaces by requiring only the lower semi-continuity of the considered functions. They derive both, the strong and the weak, statements of Ekeland's variational principle. In [13], M. Djedidi, A. Mansour and K. Nachi established a new version of Ekeland's variational principle on partial metric space by using a metric method and the concept of p-point.

We introduce a new statement of Ekeland's variational principle in partial metric space, and also apply it to prove the existence of a fixed point. In order to extend the variational principle of Ekeland (see [14]) to the partial metric space, we recall the notion of the p-point introduced in [13] and we introduce the notion of partial sub-level sets of function defined in such type of space.

For $\lambda \in \mathbb{R}$, and $\mu \in \mathbb{R} \cup \{+\infty\}$ let us define

$$\begin{aligned} [f \leq \lambda]_p &:= \{x \in X : f(x) - p(x,x) \leq \lambda\} \text{ and} \\ [f < \mu]_p &:= \{x \in X : f(x) - p(x,x) < \mu\}, \end{aligned}$$

respectively, the closed, and the partial open sub-level sets of f in (X, p). If $\lambda < \mu$, we further, denote the "slice" between λ, and μ by

$$[\lambda < f < \mu]_p := \{x \in X : \lambda < f(x) - p(x,x) < \mu\} := [f < \mu]_p \setminus [f \leq \lambda]_p.$$

If $\mu = +\infty$, we shall rather write:

$$[f > \lambda]_p := [\lambda < f < +\infty]_p \text{ and } dom f := [f < +\infty]_p$$

and, as usual, we say that f is proper if $dom f \neq \emptyset$. Note that, if P is a metric, i.e., $p := d$, then these sets coincide with the usual sub-level sets of f.

Definition 3.1. *Let (X, p) be a partial metric space and a function $f : X \to \mathbb{R} \cup \{+\infty\}$. A point $x \in dom f = f^{-1}(\mathbb{R})$ is said to be a p-point of f if*

$$f(x) + p(x,x) < f(y) + p(x,y) \quad \forall y \in X,\ y \neq x \ \text{ s.t. } \ p(y,y) = p(x,x). \tag{2}$$

Observe that p-points are in $dom f$, and that global minima are p-points.

Definition 3.2 (Lower-semicontinuity). *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function defined on (X, p). We say that f is lower semi-continuous on X if, for all $x \in X$ and any sequence $\{x_n\}$ in X, one has*

$$p(x_n, x) \to p(x,x) \Rightarrow f(x) \leq \liminf f(x_n).$$

Moreover, a function is said to be proper if the pre-image of a compact set is a compact set.

Further, let us give a variational principle of Ekeland's type in partial metric space.

Theorem 3.3 (Theorem 8, [13]). *Let (X, p) be a complete partial metric space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semi-continuous, and bounded from below function, with $x \in dom f$ and $\alpha > 0$. Then, there exists $u \in X$ such that $p(u,u) = p(x,x)$ and*

(1) $f(u) + \alpha p(x,u) \leq f(x) + \alpha p(x,x)$,

(2) $f(u) + \alpha p(u,u) < f(y) + \alpha p(u,y)$, for all $y \in X$, $y \neq x$, such that $p(y,y) = p(u,u)$.

This result asserts the existence of a p-point of f. It is proved in [13] through a simple argument based on the construction of an iterative procedure involving a sequence of closed sets of the type:

$$\begin{aligned} S(x) &:= \{y \in X : p(y,y) = p(x,x), \text{ and } f(y) + p(x,y) \leq f(x) + p(x,x)\} \\ &\subset [f < f(x)]_p \end{aligned}$$

Next we prove the following proposition.

Proposition 3.4. *If $\bar{x}$ is a p-point of the restriction f to $S(x)$, designated by $f_{|S(x)}$, then $\bar{x}$ is a p-point of f on X.*

Proof. Given $\bar{x}$ which is a p-point of $f_{|S(x)}$, and $y \in X$ with $y \neq \bar{x}$ such that $p(y,y) = p(\bar{x},\bar{x})$. Hence, $p(y,y) = p(x,x)$. If $y \in S(x)$ then (2) is satisfied for $\bar{x}$. Suppose now that $y \notin S(x)$ and, as $\bar{x} \in S(x)$, the conclusion is obtained since

$$f(\bar{x}) + p(x,\bar{x}) \leq f(x) + p(x,x) < f(y) + p(y,x).$$

□

Thus, we have the following result:

Corollary 3.5. *Let (X,p) be a complete partial metric space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semi-continuous, and bounded from below. Then f admits a p-point in $S(x)$.*

Proof. Since $S(x)$ is a closed set, thus complete, it follows that the restriction of f to $S(x)$ admits a p-point which, in turn, due to Proposition 3.4, is also a p-point on the whole space. □

The following two lemmas are the main results of this article.

Lemma 3.6. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function defined on the complete partial metric space (X,p) and let $0 \leq \lambda \leq \mu \leq +\infty$ be such that $[f < \mu]_p \neq \emptyset$.*

We assume that, for each $x \in [\lambda < f < \mu]_p$, there exists $y \neq x$ such that

$$f(y) + p(x,y) \leq f(x) + p(x,x) \quad \textit{with} \quad p(x,x) = p(y,y). \tag{3}$$

Then $[f \leq \lambda]_p \neq \emptyset$ and for each $x \in [f < \mu]_p$, there exists $y \in [f \leq \lambda]_p$ such that $p(x,y) \leq f(x) + p(x,x)$.

Proof. First, we show that, for a given $x \in [f < \mu]_p$, we have $S(x) \subset [f < \mu]_p$. Since $y \in S(x)$ we have $f(y) + p(x,y) \leq f(x) + p(x,x)$ and $p(x,x) = p(y,y)$. This implies that

$$\begin{aligned} f(y) &\leq f(x) + p(x,x) - p(x,y) \\ &\leq \mu + p(x,x) + [p(x,x) - p(x,y)] \\ &\leq \mu + p(x,x). \end{aligned}$$

This means that $y \in [f < \mu]_p$ and, thus, $S(x) \subset [f < \mu]_p$.

Hypothesis (3) states that every $x \in [\lambda < f < \mu]_p$ is not a p-point. Then, a p-point y which belongs to $S(x)$, and whose existence is assured by Corollary 3.5 is necessarily in $[f \leq \lambda]_p$ and satisfies

$$p(x,y) \leq f(y) + p(x,y) \leq f(x) + p(x,x).$$

□

Lemma 3.7. *Let $T : X \to X$ be a proper and continuous function defined on a complete partial metric space (X,p). Suppose that:*

a) There exists $\alpha > 0$ such that, for all $x \in X$ with $x \neq T(x)$, there exists $u \in X \setminus \{x\}$ such that:

$$p(u,T(u)) + \alpha p(x,u) \leq p(x,T(x)) + \alpha p(x,x). \qquad [H_p]$$

b) The application $x \mapsto p(x,T(x))$ is lower semi-continuous.

Then,

(1) there exists $\alpha > 0$ such that $[H_p]$ is satisfied with u such that $p(u,u) = p(x,x)$; and
(2) there exists u such that $p(u,T(u)) = \alpha p(u,u)$.

Proof. By setting $f(x) = p(x, T(x))$, we have a function f which is proper, lower semi-continuous and bounded from below. By Theorem 3.3 we get, for all $x \in X$ and for a given $\alpha > 0$, there exists $u \in X$ such that $p(u, u) = p(x, x)$ and

$$p(u, T(u)) + \alpha p(x, u) \leq p(x, T(x)) + \alpha p(x, x).$$

Remark that, this is the assumption $[H_p]$ with an u such that $p(u, u) = p(x, x)$ and we get the same hypotheses of Lemma 3.6 with $\widehat{p} = \alpha p$.

Now, applying Lemma 3.6 with $f(x) = p(x, T(x))$, $\lambda = 0$ and $\mu = +\infty$, we obtain

$$f(u) + \alpha p(x, u) \leq f(x) + \alpha p(x, x), \quad \text{for all } x \in X.$$

Then, it follows the existence of a p-point $u \in [f \leq 0]_{\alpha p}$, i.e.,

$$f(u) = p(u, T(u)) = \alpha p(u, u).$$

□

4. Some generalized contractive mappings in partial metric space

Let $T : X \to X$ be a function defined in a complete partial metric space (X, p) and consider the following definitions of contractive mappings:

1. $[H_1]$(Banach,[21]). There exist a with $0 < a < 1$ such that, for all x and $u \in X$, we have
$$p(T(x), T(u)) \leq ap(x, u).$$
2. $[H_2]$(Chatterjee, [20]). There exists $a > 0$ with $a \in \left(0, \frac{1}{2}\right)$ such that, $\forall\, x, u \in X$, we have
$$p(T(x), T(u)) \leq a[p(x, T(u)) + p(u, T(x))].$$
3. $[H_3]$ (Kannan, [2]). There exists $a, b \geq 0$ not all zero, with $a + b < 1$, such that, for all $x, u \in X$, we have
$$p(T(x), T(u)) \leq ap(x, T(x)) + bp(u, T(u)).$$
4. $[H_4]$ (Reich, [2]). There exists $a, b, c \geq 0$ not all zero, with $a + b + c < 1$, such that, for all $x, u \in X$, we have
$$p(T(x), T(u)) \leq ap(x, T(x)) + bp(u, T(u)) + cp(x, u).$$
5. $[H_5]$ (Ćirić). There exists nonnegative functions q, r, s, t and λ and μ such that $0 \leq \mu < \lambda < 1$ satisfying
$$\sup_{x,u \in X} \{q(x, u) + r(x, u) + s(x, u) + 2t(x, u)\} \leq \lambda$$
and, for all $x, u \in X$, $s(x, u) + t(x, u) \leq \mu$, such that, for all $x, u \in X$, we have
$$\begin{aligned} p(T(x), T(u)) \quad &\leq \quad q(x, u)p(x, u) + r(x, u)p(x, T(x)) \\ &+ \quad s(x, u)p(u, T(u)) + t(x, u)[p(x, T(u)) + p(u, T(x))]. \end{aligned}$$
6. $[H_6]$ (Hadry Rogers, [2]). There exists $a,\ b,\ c,\ m,\ e \geq 0$ such that, if $m \geq e$ then $a + b + c + m + e < 1$ and if $m < e$, then $a + b + c + m + 2e < 1$ such that, for all $x, u \in X$, we have
$$\begin{aligned} p(T(x), T(u)) \quad &\leq \quad ap(x, u) + bp(x, T(x)) + cp(u, T(u)) \\ &\qquad +mp(x, T(u)) + ep(u, T(x)). \end{aligned}$$

In order to prove the existence of fixed points for applications, we establish a relationship between the hypotheses $[H_i]$ for $i \in \{1, \ldots, 6\}$ and the main hypothesis $[H_p]$.

These relationships are summarized in the following Lemma.

Lemma 4.1. *For all $i \in \{1, \dots, 6\}$, we have $[H_i] \Rightarrow [H_p]$.*

Proof. To obtain $[H_i] \Rightarrow [H_p]$, for each i, we judiciously put $u = T(x)$ and perform the manipulations required to obtain $[H_p]$ for some constant α depending on the data of each one of the $[H_i]$'s.

1. From $[H_1]$ we readily obtain $p(u, T(u)) \leq ap(x, u)$. Thus $p(u, T(u)) + p(x, u) \leq ap(x, u) + p(x, u)$. Then, we get
$$\begin{aligned} p(u, T(u)) + \alpha p(x, u) &\leq p(x, T(x)) \\ &\leq p(x, T(x)) + \alpha p(x, x). \end{aligned}$$
Hence, we have $[H_p]$ with $0 < \alpha = 1 - a < 1$.
2. From $[H_2]$ we get $p(u, T(u)) \leq a[p(x, T(u)) + p(u, u)]$ and, as a consequence,
$$\begin{aligned} &p(u, T(u)) + p(x, u) \\ &\qquad p(x, T(x)) + a[p(x, u) + p(u, T(u)) - p(u, u) + p(u, u)]. \end{aligned}$$
After some calculations, we conclude that $(1 - a)p(u, T(u)) + (1 - 2a)p(x, u) \leq (1 - a)p(x, T(x))$. Then,
$$p(u, T(u)) + \alpha p(x, u) \leq p(x, T(x)) \leq p(x, T(x)) + \alpha p(x, x),$$
and, thus, we have $[H_p]$ with $\alpha = \dfrac{1 - 2a}{1 - a} \in (0, 1)$.
3. From $[H_3]$ we obtain $(1 - b)p(u, T(u)) \leq ap(x, u)$, i.e.,
$$p(u, T(u)) \leq a(1 - b)^{-1} p(x, u).$$
Then, by adding, and subtracting the same quantities, we conclude that
$$\begin{aligned} p(u, T(u)) + \alpha p(x, u) &\leq p(x, T(x)) \\ &\leq p(x, T(x)) + \alpha p(x, x). \end{aligned}$$
Thus, $[H_p]$ holds with $\alpha = (1 - (a + b))(1 - b)^{-1} \in (0, 1)$.
4. Similarly to the previous item, from $[H_4]$, we get
$$p(u, T(u)) + \alpha p(x, u) \leq p(x, T(x)) + \alpha p(x, x).$$
Thus, $[H_p]$ holds with $\alpha = \dfrac{1 - (a + b + c)}{1 - b} \in (0, 1)$.
5. From $[H_5]$ we get (for the sake of simplicity, we omit the arguments of the functions q, r, s, t)
$$p(u, T(u)) \leq (q + r)p(x, u) + sp(u, T(u)) + t[p(x, T(u)) + p(u, u)],$$
and, thus,
$$\begin{aligned} p(u, T(u)) \quad \leq \quad & (q + r)p(x, u) + sp(u, T(u)) \\ & +t[p(x, u) + p(u, T(u)) - p(u, u) + p(u, u)], \end{aligned}$$
leading to
$$p(u, T(u)) \leq \frac{(q + r + t)}{1 - (s + t)} p(x, u).$$
By subtracting $p(x, T(x))$ to both sides of the inequality above we obtain
$$p(u, T(u)) - p(x, T(x)) \leq \frac{(q + r + s + 2t) - 1}{1 - (s + t)} p(x, u),$$
and this entails $p(u, T(u)) - p(x, T(x)) \leq \dfrac{\lambda - 1}{1 - \mu} p(x, u)$ and, thus,
$$p(u, T(u)) + \frac{1 - \lambda}{1 - \mu} p(x, u) \leq p(x, T(x)) \leq p(x, T(x)) + \frac{1 - \lambda}{1 - \mu} p(x, x).$$
This is $[H_p]$ with $\alpha = \dfrac{1 - \lambda}{1 - \mu} \in (0, 1)$.

6. In a similar vein we obtain from $[H_6]$ the following cases:
   - Case $m < e$.

$$\begin{aligned} p(u,T(u)) &\leq (a+b)p(x,u)+cp(u,T(u))+mp(x,T(u))+ep(u,u)) \\ &\leq (a+b)p(x,u)+cp(u,T(u)) \\ &\qquad +m(p(x,u)+p(u,T(u))-p(u,u))+ep(x,u) \\ &\leq (a+b+m+e)p(x,u)+(c+m)p(u,T(u)) \\ &\leq (a+b+m+e)p(x,u)+(c+e)p(u,T(u)), \end{aligned}$$

and, thus,

$$p(u,T(u))+\alpha p(x,u) \leq p(x,T(x))+\alpha p(x,u).$$

This is $[H_p]$ with $\alpha = \dfrac{1-(a+b+c+m+2e)}{1-(c+e))} \in (0,1)$.

   - Case $d \geq e$.

$$\begin{aligned} p(u,T(u)) &\leq (a+b)p(x,u)+cp(u,T(u))+mp(x,T(u))+ep(u,u)) \\ &\leq (a+b)p(x,u)+cp(u,T(u)) \\ &\qquad +m(p(x,u)+p(u,T(u))-p(u,u))+ep(x,u) \\ &\leq (a+b)p(x,u)+cp(u,T(u))+m(p(x,u)+p(u,T(u))) \\ &\qquad -ep(u,u))+ep(x,u) \\ &\leq (a+b+m)p(x,u)+(c+m)p(u,T(u)), \end{aligned}$$

and, thus,

$$p(u,T(u))+\alpha p(x,u) \leq p(x,T(x))+\alpha p(x,u).$$

Since $a+b+c+2m \geq a+b+c+m+e < 1$, then $1-(a+b+c+2d) \leq 1-(a+b+c+m+e)$ and thus, $[H_p]$ holds with $\alpha = \dfrac{1-(a+b+c+2m)}{1-(c+m))} \in (0,1)$.

□

5. Fixed point results

Let $T : X \to X$ be a proper and continuous operator defined on a partial metric space (X,p) and let us denote the fixed point set of T by

$$\mathcal{F}_T = \{x \in X : x = T(x)\} = \{x \in X : p(x,T(x)) = p(x,x) = p(T(x),T(x))\}.$$

Also, for a given set $A \subset X$, we define $p(x,A) := \inf\{p(x,y),\ y \in A\}$.

Theorem 5.1. *Let $T : X \to X$ a continuous and proper function defined on a complete partial metric space (X,p). Suppose that:*

a) there exists $\alpha \in (0,1)$ such that, for all $x \in X$ with $x \neq T(x)$ and for some $u \in X \setminus \{x\}$, $[H_p]$ holds;

b) the application $x \mapsto p(x,T(x))$ is lower semi-continuous.

Then, $\mathcal{F}_T \neq \emptyset$ and, for all $x \in X$,

$$p(x,\mathcal{F}_T) \leq \alpha^{-1}p(x,T(x))+p(x,x).$$

Proof. Under these hypotheses the assumptions required by Lemma 3.7 are satisfied and, we get the existence of $u \in X$, for all $x \in X$, such that $p(u,u) = p(x,x)$,

$$p(u,T(u)) + \alpha p(x,u) \leq p(x,T(x)) + \alpha p(x,x), \tag{4}$$

and

$$p(u,T(u)) = \alpha p(u,u). \tag{5}$$

Since $0 < \alpha < 1$ and $p(u,T(u)) \geq p(u,u)$ the equality (5) can only be true if and only if

$$p(u,T(u)) = p(u,u) = 0 \Leftrightarrow u = T(u).$$

This entails that $\mathcal{F}_T \neq \emptyset$. Moreover, inequality (4) leads to

$$p(x,\mathcal{F}_T) \leq p(x,u) \leq \alpha^{-1} p(x,T(x)) + p(x,x).$$

This conclusion ends the proof. □

Corollary 5.2. *Let $T : X \to X$ an operator defined in a complete partial metric space (X,p). If the application $x \mapsto p(x,T(x))$ is lower semicontinuous and if one of assumptions $[H_i]$ with $i \in \{1,\dots,6\}$ is satisfied, then T has a unique fixed point $\bar{x}$ and*

$$p(x,\bar{x}) \leq \alpha^{-1} p(x,T(x)) + p(x,x) \qquad \text{for all } x \in X.$$

with

$$\alpha = \begin{cases} 1-a & \text{in the case } [H_1] \\ \dfrac{1-(a+b)}{1-b)} & \text{in the case } [H_2] \\ \dfrac{1-(a+b+c)}{1-b} & \text{in the case } [H_3] \\ \dfrac{1-2k}{1-k} & \text{in the case } [H_4] \\ \dfrac{1-\lambda}{1-\mu} & \text{in the case } [H_5] \\ \left.\begin{array}{ll} \dfrac{1-(a+b+c+2d)}{1-(c+d))} & \text{if } d < e \\ \dfrac{1-(a+b+c+d+2e)}{1-(c+e))} & \text{if } d \geq e \end{array}\right\} & \text{in the case } [H_6]. \end{cases}$$

Proof. Under the conditions stated above, Lemma 4.1 entails that the assumptions required by Theorem 5.1 hold and, therefore, the existence of the fixed point follows. It remains to show the uniqueness.

Assume that there are two points $x_1 \neq x_2$ such that $T(x_1) = x_1$ and $T(x_2) = x_2$.

1) If $[H_1]$ holds, then, by writing $[H_1]$ for x_1 and x_2, we obtain

$$p(x_1,x_2) \leq a p(x_1,x_2).$$

Since $a < 1$, we obtain $p(x_1,x_2) = 0$, and, then, $p(x_1,x_2) = p(x_1,x_1) = p(x_2,x_2) = 0$. Thus, $x_1 = x_2$.

2) If $[H_2]$ holds, then, writing $[H_2]$ for x_1 and x_2, we obtain

$$p(x_1, x_2) \le ap(x_1, x_2) + ap(x_1, x_2) = 2ap(x_1, x_2).$$

This leads to $x_1 = x_2$ since $2a < 1$.

3) If $[H_3]$ holds, then, by writing $[H_3]$ for x_1 and x_2, we obtain

$$p(x_1, x_2) \le ap(x_1, x_1) + bp(x_2, x_2).$$

However, since $a + b < 1$, we get

$$ap(x_1, x_2) + bp(x_1, x_2) < p(x_1, x_2) \le ap(x_1, x_1) + bp(x_2, x_2),$$

yielding $p(x_1, x_2) < p(x_1, x_1)$ and $p(x_1, x_2) < p(x_2, x_2)$. This contradiction leads to $p(x_1, x_2) = p(x_1, x_1) = p(x_2, x_2) = 0$;, thus $x_1 = x_2$.

4) If $[H_4]$ holds, then, by writing $[H_4]$ for x_1 and x_2, we obtain

$$p(x_1, x_2) \le ap(x_1, x_1) + bp(x_2, x_2) + cp(x_1, x_2).$$

Since $a + b + c < 1$ we get $ap(x_1, x_2) + bp(x_1, x_2) + cp(x_1, x_2) < p(x_1, x_2)$ and $p(x_1, x_2) \le ap(x_1, x_1) + bp(x_2, x_2) + cp(x_1, x_2)$. Thus $p(x_1, x_2) < p(x_1, x_1)$ and $p(x_1, x_2) < p(x_2, x_2)$. This contradiction leads to $p(x_1, x_2) = p(x_1, x_1) = p(x_2, x_2) = 0$, thus, $x_1 = x_2$.

5) If $[H_5]$ holds, then, by writing $[H_5]$ for x_1 and x_2, we obtain (again, we omit again the arguments of the functions q, r, s, t)

$$p(x_1, x_2) \quad \le \quad qp(x_1, x_2) + rp(x_1, x_1) + sp(x_2, x_2) + t[p(x_1, x_2) + p(x_2, x_1)].$$

Since $q+r+s+2t < 1$, we get $(q+2t)p(x_1, x_2)+rp(x_1, x_2)+sp(x_1, x_2) < p(x_1, x_2)$; thus $p(x_1, x_2) < p(x_1, x_1)$ and $p(x_1, x_2) < p(x_2, x_2)$. This contradiction leads to the conclusion that $p(x_1, x_2) = p(x_1, x_1) = p(x_2, x_2) = 0$; thus $x_1 = x_2$.

6) If $[H_6]$ holds, then by writing $[H_6]$ for x_1, and x_2, we obtain

$$p(x_1, x_2) \quad \le \quad ap(x_1, x_2) + bp(x_1, x_1) + cp(x_2, x_2) + mp(x_1, x_2) + ep(x_2, x_1).$$

Thus

$$p(x_1, x_2) \le (a + m + e)p(x_1, x_2) + bp(x_1, x_1) + cp(x_2, x_2).$$

Assume $d \ge e$. Since $a + b + c + m + e < 1$, we get $(a + m + e)p(x_1, x_2) + bp(x_1, x_2) + cp(x_1, x_2) < p(x_1, x_2)$ and $p(x_1, x_2) \le (a + m + e)p(x_1, x_2) + bp(x_1, x_1) + cp(x_2, x_2)$. Thus, on the one hand, $p(x_1, x_2) < p(x_1, x_1)$ and, on the other hand, $p(x_1, x_2) < p(x_2, x_2)$. This contradiction leads to the conclusion that $p(x_1, x_2) = p(x_1, x_1) = p(x_2, x_2) = 0$, and, thus, $x_1 = x_2$.

Assume that $d < e$. Since $a + b + c + 2e < 1$, we get $(a + 2e)p(x_1, x_2) + bp(x_1, x_2) + cp(x_1, x_2) < p(x_1, x_2)$ and $p(x_1, x_2) \le (a + 2e)p(x_1, x_2) + bp(x_1, x_1) + cp(x_2, x_2)$. Thus, $p(x_1, x_2) < p(x_1, x_1)$ and $p(x_1, x_2) < p(x_2, x_2)$. This contradiction leads to $p(x_1, x_2) = p(x_1, x_1) = p(x_2, x_2) = 0$ and, thus, $x_1 = x_2$.

Since $\mathcal{F}_T = \{\bar{x}\}$ and for all $x \in X$,

$$p(x, \mathcal{F}_T) = p(x, \bar{x}) \le \alpha^{-1} p(x, T(x)) + p(x, x),$$

the proof is complete. □

Definition 5.3. *(From [1]). A self mapping T of PMS (X, p) is called a Caristi mapping on X if there is a function $f : X \to [0, +\infty)$ which is lower semicontinous for (X, p^s) and satisfies:*

$$p(x, T(x)) \leq p(x, x) + f(x) - f(T(x))$$

for all $x \in X$.

From this definition and taking $y = Tx$ in Lemma 3.6, we get the following theorem.

Theorem 5.4. *(From [1]). Let (X, p) be a complete partial metric space. Then, every Caristi mapping on X has a fixed point.*

Example 5.5. *Let $X = [-1, 1]$ and $p(x, y) = |x - y| + a$ with $a > 0$, then (X, p) is complete partial metric space. Define a function $T : X \to X$ by $T(x) = \sqrt{|x|}$ and let $f : X \to [0, +\infty]$ be a lower semicontinous function for (X, p^s) defined by $f(x) = 1 - x$. Then we have, for all $x \in \mathbb{R}$,*

$$p(x, T(x)) = \sqrt{|x|} - x + a \leq p(x, x) + f(x) - f(T(x)) = \sqrt{|x|} - x + a.$$

Thus, $p(x, T(x)) \leq p(x, x) + f(x) - f(T(x))$.

Since, all the conditions of the Theorem 5.4 are satisfied result that T has a fixed point, i.e., $T(0) = 0$.

6. Application in coupled fixed point theory

In this section we apply our main fixed point results to the coupled fixed point problem. Coupled fixed point problems started with the work of Opoitsev, [26]-[28]. The topic started being strongly investigated with the article of D. Guo and V. Lakshmikantham, see [18], for the monotone iterations case and with T. Gnana Bhaskar and V. Lakshmikantham results, see [17], in the context of contraction-type methods.

Let us recall the coupled fixed point problem. If X is a nonempty set and $F : X \times X \to X$ is a given operator, find $(x, u) \in X \times X$ satisfying the system

$$\begin{cases} x = F(x, u) \\ u = F(u, x). \end{cases} \tag{6}$$

An important problem is to find the strong fixed points of F, i.e., the elements $x \in X$ with the property $x = F(x, x)$.

H. Aydi et al. introduced the η-partial metric space in [5]. (X, p) is said to be a η-partial metric if it is partial metric space and $X \times X$ is endowed with the partial metric η defined by $\eta((x, y), (u, v)) = p(x, u) + p(y, v)$, for all $(x, y), (u, v) \in X \times X$.

Let us define in this case the set of coupled fixed point ς_F as follows

$$\zeta_F = \{(x, u) \in X \times X : x = F(x, u), u = F(u, x)\}.$$

Let us give the main coupled fixed point theorem of this section.

Theorem 6.1. *Let (X, p) be a complete partial metric space and let $F : X \times X \to X$ a given operator. Suppose that:*

a) there exists $\alpha \in (0, 1)$ such that, for all $(x, u) \in X \times X$ with $(x, u) \neq (F(x, u), F(u, x))$, and for some $(y, v) \in X \times X \setminus \{(x, u)\}$, we have

$$\begin{cases} p(y, F(y, v)) + \alpha p(x, y) \leq p(x, F(x, u)) + \alpha p(x, x) \\ p(v, F(v, y)) + \alpha p(u, v) \leq p(u, F(u, x)) + \alpha p(u, u) \end{cases}, \text{ and}$$

b) the application $(x, u) \mapsto p(x, F(x, u))$ is lower semicontinuous.

Then $\zeta_F \neq \emptyset$.

Proof. Let the operator $T : Z \to Z$ where $Z = X \times X$ given by $T(z) = T(x,u) := (F(x,u), F(u,x))$ it is clearly that (Z, η) is a complete partial metric space and that T satisfies both assumptions of theorem 5.1.

The conclusion follows by applying Theorem 5.1, since it yields $\eta(z, T(z)) = \eta(z,z) = 0$. Thus,

$$\begin{cases} x^* = F(x^*, u^*) \\ u^* = F(u^*, x^*). \end{cases} \tag{7}$$

In other words (x^*, u^*) is a coupled fixed point of F. □

7. Conclusions

Sufficient conditions for the existence of fixed points were derived in the context of partial metric spaces by using metric methods in the context of a new extension of Ekeland's variational principle. We also showed how the developments of this article relates with various fixed points for several types of contraction mappings that have been considered in the literature. Finally, an application to existence of solution to the coupled fixed point problem in a strong sense illustrates the power of our results.

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