On Strategies for Selection Games Related to Countable Dimension

Christopher Caruvana[a], Steven Clontz[b]

[a]School of Sciences, Indiana University Kokomo, 2300 S. Washington Street, Kokomo, IN 46902, USA
[b]College of Arts and Sciences, University of South Alabama, 411 N University Blvd North, Mobile, AL 36688, USA

Abstract. Two selection games from the literature, $G_c(O,O)$ and $G_1(O_{zd},O)$, are known to characterize countable dimension among certain spaces. This paper studies their perfect- and limited-information strategies, and investigates issues related to non-equivalent characterizations of zero-dimensionality for spaces that are not both separable and metrizable. To relate results on zero-dimensional and finite-dimensional spaces, a generalization of Telgársky's proof that the point-open and finite-open games are equivalent is demonstrated.

1. Introduction

In the field of topological dimension theory, there are three standard notions of dimension: the small inductive dimension, the large inductive dimension, and the covering dimension. Though some spaces, like the disjoint union of $[0,1]^n$ over positive integers n, are "weakly" infinite-dimensional, the Hilbert cube $[0,1]^\omega$ is "strongly" infinite-dimensional [8, 1.8.20]. Hurewicz, in [12], introduced the notion of countable-dimensional spaces as a first step to characterizing infinite-dimensional spaces.

Definition 1.1. A separable metrizable space is said to have **countable dimension** if it is the countable union of zero-dimensional subspaces.

Since a countable-dimensional space can be written as a countable union of zero-dimensional subspaces, we will be focusing our attention on these subspaces. We use the following terminology to disambiguate between two common characterizations of zero dimension from the literature.

Definition 1.2. A space is said to be **zero-ind** if it has a basis of clopen sets.

Definition 1.3. A space is said to be **zero-dim** or **0dim** if every open cover admits a refining open cover consisting of pairwise disjoint open sets.

Proposition 1.4. *Every T_1 **zero-dim** space X is **zero-ind**.*

Proof. Let U be an open neighborhood of x and consider the open cover $\{U, X \setminus \{x\}\}$. A pairwise disjoint open refinement covering X includes a clopen subset of U containing x. □

The "ind" above abbreviates "inductive" as this is the usual definition of zero inductive dimension; we use just "dim" for covering dimension to follow historical precedent. There is also a large inductive ("Ind") dimension defined in terms of closed subsets rather than points; in the context of normal spaces, covering dimension zero and this large inductive dimension zero coincide [8, 1.6.11]; we will not consider it further. In the further restricted context of separable metrizable spaces, all three notions of dimension coincide [8, 1.7.7]. However, in the general context of metrizable spaces, the small inductive dimension can differ from the covering dimension, the first example of such a space coming from [21]. The natural question then is how different can the two notions be? This general question of dimension spread has enjoyed steady progress [13, 14, 17–19].

We will be interested in the following generalization of σ-compactness.

Definition 1.5. Let $\mathcal{A}$ be a collection (resp. property) of subsets of a space X. Then X is said to be σ-$\mathcal{A}$ provided there exists a countable collection $\{A_n : n < \omega\}$ of sets in $\mathcal{A}$ (resp. sets satisfying $\mathcal{A}$) where $X = \bigcup_{n<\omega} A_n$.

Question 1.6. *When is σ-zero-ind equivalent to σ-zero-dim?*

Any answer to the following question is also an answer to the former.

Question 1.7. *When does zero-ind imply σ-zero-dim?*

As noted earlier, zero-ind and zero-dim (and therefore σ-zero-ind and σ-zero-dim) are equivalent for separable metrizable spaces. This assumption may be weakened by considering the following property.

Definition 1.8. A space is said to be **strongly paracompact** if every open cover of the space has a star-finite open refinement that covers the space, that is, an open refinement covering the space such that each member of the refinement meets only finitely-many other members.

From the definition, one observes that every strongly paracompact space is paracompact (every open cover has a locally-finite refinement that's also a cover). It may also be shown [9, Corollary 5.3.11] that every Lindelöf T_3 space, such as a separable metrizable space, is strongly paracompact.

Theorem 1.9. ([8, 3.1.30]) *Let X be strongly paracompact and T_2. Then X is zero-ind if and only if it is zero-dim.*

Due to this, the discussion of "zero dimension" and "countable dimension" is unambiguous in its usual context of separable metrizable spaces, but it seems more care must be taken if strong paracompactness is not guaranteed, even if the space is metrizable. As such, we prefer to refer to subspaces as zero-dim or zero-ind as appropriate throughout this paper, and only use zero/countable-dimensional when it is guaranteed these concepts are equivalent.

2. Relative covering dimension

It's important to note that zero-dim and zero-ind are both properties of topological spaces, and thus a subset of a space must be considered using its subspace topology. Consider then the following variation of zero covering dimension for a subset which does not consider the subspace topology. (According to [16], for paracompact spaces this definition coincides with a definition of relative dimension given in [24].)

Definition 2.1. A subset $Y \subseteq X$ is **relatively zero-dim to X**, **zero-dim$_X$**, or **$\mathbf{0dim}_X$** if for every cover of Y by sets open in X, there exists a pairwise disjoint refinement covering Y by sets open in X.

We proceed by first demonstrating that this is a sufficient condition for a subset to be zero-dim.

Proposition 2.2. *For any $Y \subseteq X$, if Y is zero-dim$_X$, then Y is zero-dim.*

Proof. Assume Y is zero-dim$_X$. Let $\mathcal{U}$ be a cover of Y by open subsets of Y. Let $\mathcal{U}'$ be a collection of open subsets of X such that for each $U \in \mathcal{U}$ there exists $U' \in \mathcal{U}'$ such that $U = U' \cap Y$. Let $\mathcal{V}'$ be a pairwise-disjoint open refinement of $\mathcal{U}'$ covering Y. Let $\mathcal{V} = \{V' \cap Y : V' \in \mathcal{V}'\}$. It follows that $\mathcal{V}$ is a pairwise disjoint refinement of $\mathcal{U}$ of sets open in Y that covers Y, so Y is zero-dim. $\quad\square$

However, zero-dim$_X$ is not equivalent, in general, to the usual covering dimension of a subspace.

Definition 2.3. Let $N = \mathbb{R} \times [0, \infty)$ have the **tangent disc topology**, where points in $\mathbb{R} \times (0, \infty)$ have their usual Euclidean neighborhoods, and points in $\langle x, 0 \rangle \in \mathbb{R} \times \{0\}$ have neighborhoods of the form $U_{x,\epsilon} = \{\langle x, 0 \rangle\} \cup B_{\epsilon}(\langle x, \epsilon \rangle)$ for $\epsilon > 0$ (where $B_r(P)$ is the open ball of radius r centered at P).

Definition 2.4. For $B \subseteq \mathbb{R}$, let $N(B) = (B \times \{0\}) \cup (\mathbb{R} \times (0, \infty))$ be the **bubble space** of B, with the subspace topology inherited from $N = N(\mathbb{R})$.

Example 2.5. If $B \subseteq \mathbb{R}$ is uncountable, then $N(B)$ is a $T_{3.5}$ space with a subset $B \times \{0\}$ which is zero-dim but not σ-zero-dim$_X$.

Proof. We first note that $B \times \{0\}$ is discrete, and any discrete space is zero-dim.
We proceed by showing that if $B \times \{0\}$ is σ-zero-dim$_X$, then B is countable. Consider the open cover $\{U_{z,1} : z \in \mathbb{R}\}$ of any subset of B. For any pairwise disjoint open refinement of this cover, only countably-many points $\langle z, 0 \rangle$ from the subset can be covered, as any open subset of $U_{z,1}$ containing $\langle z, 0 \rangle$ must contain a distinct point in the countable set $\mathbb{Q}^2$. $\quad\square$

Note that if $|B| = \aleph_1$ and $MA + \neg CH$ holds, then $N(B)$ is also T_4 [4, 10], so the properties zero-dim$_X$ and zero-dim are consistently distinct for normal spaces. Nonetheless, these notions do in fact coincide when considering metrizable spaces.

Theorem 2.6. *Let X be metrizable. Then for any $Y \subseteq X$, Y is zero-dim$_X$ if and only if Y is zero-dim.*

Proof. Assume Y is zero-dim. Let $\mathcal{U} = \{U_{\alpha} : \alpha < \kappa\}$ be a cover of Y by open subsets of X. Then $\mathcal{U}' = \{U'_{\alpha} : \alpha < \kappa\}$ where $U'_{\alpha} = U_{\alpha} \cap Y$ for each $\alpha < \kappa$ is a cover of Y by open subsets of Y. Let $\mathcal{V}' = \{V'_{\alpha,\beta} : \alpha < \kappa, \beta < \lambda_{\alpha}\}$ where $V'_{\alpha,\beta} \subseteq U'_{\alpha}$ be a pairwise-disjoint refinement of $\mathcal{U}'$ by sets open in Y covering Y.
By [15, II.XI.21.2] and the metrizability of X, there exists a pairwise disjoint collection $\mathcal{V} = \{V_{\alpha,\beta} : \alpha < \kappa, \beta < \lambda_{\alpha}\}$ of sets open in X such that $V'_{\alpha,\beta} = V_{\alpha,\beta} \cap Y$. Let $W_{\alpha,\beta} = V_{\alpha,\beta} \cap U_{\alpha}$, so $\mathcal{W} = \{W_{\alpha,\beta} : \alpha < \kappa, \beta < \lambda_{\alpha}\}$ is a pairwise disjoint refinement of $\mathcal{U}$ by sets open in X. Let $y \in Y$, and choose $\alpha < \kappa, \beta < \lambda_{\alpha}$ such that $y \in V'_{\alpha,\beta}$. Then both $y \in V'_{\alpha,\beta} \subseteq U'_{\alpha} \subseteq U_{\alpha}$ and $y \in V'_{\alpha,\beta} \subseteq V_{\alpha,\beta}$, so $y \in W_{\alpha,\beta}$. Thus $\mathcal{W}$ covers Y, and Y is zero-dim$_X$. $\quad\square$

The use of [15, II.XI.21.2] is no coincidence, as it was the technique utilized in [3] to obtain game-theoretic characterizations of countable dimension among strongly paracompact metrizable spaces. Put another way, topological games which deal with open covers of the entire space more naturally characterize spaces in terms of the relative dimension of their subsets, but this distinction is lost when only studying metrizable spaces. This is explored further in the following section.

3. Perfect- and limited-information strategies

Our goal is to study countable dimension in the context of the following two games.

Definition 3.1. Let O collect the open covers of a space. The game $G_c(O, O)$ is played by ONE and TWO. During each round $n < \omega$, ONE chooses some $\mathcal{U}_n \in O$, and then TWO chooses some pairwise disjoint open refinement $\mathcal{V}_n$ of $\mathcal{U}_n$. TWO wins this game if $\bigcup\{\mathcal{V}_n : n < \omega\}$ covers X, and ONE wins otherwise.

Definition 3.2. Let $\mathcal{T}$ be the topology of a space, and $O_{\mathcal{A}} = \{\mathcal{U} \subseteq \mathcal{T} : \forall A \in \mathcal{A}\, \exists U \in \mathcal{U}(A \subseteq U)\}$. Then the game $G_1(O_{\mathcal{A}}, O)$ is played by ONE and TWO. During each round $n < \omega$, ONE chooses some $\mathcal{U}_n \in O_{\mathcal{A}}$, and then TWO chooses some open set $V_n \in \mathcal{U}_n$. TWO wins this game if $\{\mathcal{V}_n : n < \omega\}$ covers X, and ONE wins otherwise.

We will be particularly interested when $\mathcal{A}$ collects the "zero-dimensional" subsets of a space, and write e.g. $O_{0\dim_X}$.
We also will consider a natural variation of this game.

Definition 3.3. Let $\mathcal{T}$ be the topology of a space, $B_A = \{U \in \mathcal{T} : A \subseteq U\}$, and $\mathcal{N}_{\mathcal{A}} = \{B_A : A \in \mathcal{A}\}$. Then the game $G_1(\mathcal{N}_{\mathcal{A}}, \neg O)$ is played by ONE and TWO. During each round $n < \omega$, ONE chooses some $B_{A_n} \in \mathcal{N}_{\mathcal{A}}$, and then TWO chooses some open set $V_n \in B_{A_n}$. TWO wins this game if $\{\mathcal{V}_n : n < \omega\}$ fails to cover X, and ONE wins otherwise.

These are both examples of **selection games** $G_1(\mathcal{A}, \mathcal{B})$ (ONE chooses $A_n \in \mathcal{A}$, TWO chooses $b_n \in A_n$, TWO wins if $\{b_n : n < \omega\} \in \mathcal{B}$) used to characterize many topological properties [7].

Definition 3.4. Let G be a game where players choose from the set M. Then $\tau : M^{<\omega} \to M$ defines a **(perfect-information) strategy** for either player, where $\tau(\langle m_0, \ldots, m_N \rangle) \in M$ is the move selected by the strategy in response to the opponent choosing $m_i \in M$ during round $0 \leqslant i \leqslant N$. If player P has a winning strategy that defeats every play of the opponent for G, then we write $P \uparrow G$.

Likewise, $\tau : M \times \omega \to M$ defines a **Markov strategy** that makes its choice $\tau(m, N)$ based on only the most recent move $m \in M$ of the opponent and the current round number $N < \omega$, and $\tau : \omega \to M$ defines a **predetermined strategy** that ignores the moves of the opponent and makes a choice $\tau(N)$ based only on the value of the current round $N < \omega$. Then $P \uparrow_{\mathrm{mark}} G$ (resp. $P \uparrow_{\mathrm{pre}} G$) means the player P has a winning Markov (resp. predetermined) strategy that defeats every play of the opponent for G.

So for example, $ONE \not\uparrow_{\mathrm{pre}} G_1(O, O)$ characterizes the Rothberger covering property (commonly expressed as the **selection principle** $S_1(O, O)$) [22]. Likewise, $TWO \uparrow_{\mathrm{mark}} G_1(O, O)$ characterizes the countability of a T_1 space (see e.g. [6]); we will soon demonstrate a generalization of this result.
We now formalize the relationship between $G_1(O_{\mathcal{A}}, O)$ and $G_1(\mathcal{N}_{\mathcal{A}}, \neg O)$.

Definition 3.5. Let G, H be games with players ONE and TWO. Then we say G and H are **equivalent** provided:

- $ONE \uparrow G$ if and only if $ONE \uparrow H$.

- $TWO \uparrow G$ if and only if $TWO \uparrow H$.

- $ONE \uparrow_{\mathrm{pre}} G$ if and only if $ONE \uparrow_{\mathrm{pre}} H$.

- $TWO \uparrow_{\mathrm{mark}} G$ if and only if $TWO \uparrow_{\mathrm{mark}} H$.

And we say G and H are **dual** provided:

- $ONE \uparrow G$ if and only if $TWO \uparrow H$.

- $TWO \uparrow G$ if and only if $ONE \uparrow H$.

- $ONE \uparrow_{\mathrm{pre}} G$ if and only if $TWO \uparrow_{\mathrm{mark}} H$.

- $TWO \uparrow_{\mathrm{mark}} G$ if and only if $ONE \uparrow_{\mathrm{pre}} H$.

The following may be proven from the techniques of [5].

Proposition 3.6. $G_1(O_{\mathcal{A}}, O)$ and $G_1(\mathcal{N}_{\mathcal{A}}, \neg O)$ are dual for any $\mathcal{A}$.

Proof. Let $\mathbf{C}(X)$ collect the choice functions $f : X \to \bigcup X$ such that $f(x) \in x$ for all $x \in X$. To see that $\mathcal{N}_{\mathcal{A}}$ is a reflection of $O_{\mathcal{A}}$ we may verify:

- ran $f \in O_{\mathcal{A}}$ for all $f \in \mathbf{C}(\mathcal{N}_{\mathcal{A}})$, and

- for each $\mathcal{U} \in O_{\mathcal{A}}$ there exists $f_A \in \mathbf{C}(\mathcal{N}_{\mathcal{A}})$ such that ran $f_{\mathcal{U}} \subseteq \mathcal{U}$.

Therefore the games are dual. $\quad\square$

Note when $\mathcal{A} = [X]^1$ collects the singletons of X, then $G_1(O_{\mathcal{A}}, O)$ is the Rothberger game and $G_1(\mathcal{N}_{\mathcal{A}}, \neg O)$ is the well-known point-open game. These were shown to be dual for perfect-information strategies by Galvin in [11].

The following result shows that $TWO \uparrow_{\mathrm{mark}} G_c(O, O)$ provides a very natural characterization of "countable relative dimension" for an arbitrary topological space.

Theorem 3.7. *$TWO \uparrow_{\mathrm{mark}} G_c(O, O)$ if and only if X is σ-zero-dim$_X$.*

Proof. Let τ be a winning Markov strategy for TWO. Let

$$X_n = \bigcap_{\mathcal{U} \in O} \bigcup \tau(\mathcal{U}, n)$$

Let $\mathcal{U}$ cover X, then $\tau(\mathcal{U}, n)$ is a pairwise disjoint refinement of $\mathcal{U}$ and covers X_n. Therefore X_n is zero-dim$_X$.

Then consider $x \in X$. If $x \notin X_n$ for all $n < \omega$, choose $\mathcal{U}_n \in O$ with $x \notin \bigcup \tau(\mathcal{U}_n, n)$. Then $\mathcal{U}_n$ is a successful counterattack to the winning strategy τ, a contradiction. Therefore $x \in X_n$ for some $n < \omega$, and $X = \bigcup_{n<\omega} X_n$. Thus X is σ-zero-dim$_X$.

Now assume $X = \bigcup_{n<\omega} X_n$ with X_n zero-dim$_X$. Let $\tau(\mathcal{U}, n)$ be a pairwise disjoint open refinement of $\mathcal{U}$ covering X_n. It follows that τ is a winning Markov strategy for TWO. $\quad\square$

Likewise, Markov strategies for TWO in $G_1(O_{\mathcal{A}}, O)$ also may characterize spaces of countable dimension (or spaces which are countable unions of anything you'd like).

Lemma 3.8. *Let X be T_1. Then $TWO \uparrow_{\mathrm{mark}} G_1(O_{\mathcal{A}}, O)$ if and only if X is σ-$\mathcal{A}$.*

Proof. Let τ be a winning Markov strategy for TWO. Let

$$X_n = \bigcap_{\mathcal{U} \in O_{\mathcal{A}}} \tau(\mathcal{U}, n)$$

Suppose $X_n \not\subseteq A$ for all $A \in \mathcal{A}$. Pick $x_A \in X_n \setminus A$ for each $A \in \mathcal{A}$. If $X \in \mathcal{A}$ we're done, so assume $X \notin \mathcal{A}$. Then $\mathcal{U} = \{X \setminus \{x_A\} : A \in \mathcal{A}\} \in O_{\mathcal{A}}$. But then $\tau(\mathcal{U}, n) = X \setminus \{x_A\}$ for some $A \in \mathcal{A}$. Thus $X_n \not\subseteq \tau(\mathcal{U}, n)$, contradiction. Thus we have $X_n \subseteq A_n \in \mathcal{A}$ for $n < \omega$.

Then consider $x \in X$. If $x \notin X_n$ for all $n < \omega$, choose $\mathcal{U}_n \in O$ with $x \notin \tau(\mathcal{U}_n, n)$. Then $\mathcal{U}_n$ is a successful counterattack to the winning strategy τ, a contradiction. Therefore $x \in X_n$ for some $n < \omega$, and $X = \bigcup_{n<\omega} X_n = \bigcup_{n<\omega} A_n$. Thus X is σ-zero-dim$_X$.

Now assume $X = \bigcup_{n<\omega} X_n$ with $X_n \in \mathcal{A}$. Let $\tau(\mathcal{U}, n)$ be a member of $\mathcal{U}$ that contains X_n. It follows that τ is a winning Markov strategy for TWO. $\quad\square$

Corollary 3.9. *Let X be T_1. Then the following are equivalent.*

- *X is σ-zero-dim$_X$.*

- *$TWO \uparrow_{\mathrm{mark}} G_c(O, O)$*

- *$TWO \uparrow_{\mathrm{mark}} G_1(O_{0\,\dim_X}, O)$*

- *ONE* $\uparrow_{\mathrm{pre}} G_1(\mathcal{N}_{0\,\dim_X}, \neg O)$

We now turn to perfect-information strategies in these games.

Proposition 3.10. *If $\mathcal{A}$ has the property that for each $A \in \mathcal{A}$ there exists a G_δ $A^\star$ with $A \subseteq A^\star \in \mathcal{A}$, then ONE $\uparrow G_1(\mathcal{N}_\mathcal{A}, \neg O)$ if and only if ONE $\uparrow_{\mathrm{pre}} G_1(\mathcal{N}_\mathcal{A}, \neg O)$ if and only if X is σ-$\mathcal{A}$.*

Proof. Of course, if $X = \bigcup\{A_n : n \in \omega\}$ where $A_n \in \mathcal{A}$ for each $n \in \omega$, then One has a winning predetermined strategy. So we need only show that ONE having a winning strategy witnesses that X is σ-$\mathcal{A}$.

Suppose τ is winning for ONE; let τ' yield corresponding members of $\mathcal{A}$. Then $\tau'(\langle\rangle) \in \mathcal{A}$, so choose $U_{\langle n \rangle}$ open with $\tau'(\langle n \rangle) \subseteq X_{\langle\rangle} = \bigcap\{U_{\langle n \rangle} : n < \omega\} \in \mathcal{A}$. Now if $U_{s \upharpoonright n}$ is defined for $s \in \omega^{n+1}$ and $0 < n \leq |s|$, note $\tau'(\langle U_{s \upharpoonright 1}, \dots, U_s \rangle) \in \mathcal{A}$, so choose $U_{s^\frown \langle n \rangle}$ open with $\tau'(\langle U_{s \upharpoonright 1}, \dots, U_s \rangle) \subseteq X_s = \bigcap\{U_{s^\frown \langle n \rangle} : n < \omega\} \in \mathcal{A}$.

We now show that $X = \bigcup_{s \in \omega^{<\omega}} X_s$. If not, pick $x \notin X_s$ for all $s \in \omega^{<\omega}$. We then may define $f \in \omega^\omega$ such that $x \notin U_{f \upharpoonright n+1}$ for all $n < \omega$. Finally, note that the counterattack $\langle U_{f \upharpoonright 1}, U_{f \upharpoonright 2}, \dots \rangle$ defeats τ, a contradiction. $\quad\square$

From duality we may obtain the following existing result.

Corollary 3.11. ([2, Theorem 6]) *If $\mathcal{A}$ has the property that for each $A \in \mathcal{A}$ there exists a G_δ $A^\star$ with $A \subseteq A^\star \in \mathcal{A}$, then TWO $\uparrow G_1(O_\mathcal{A}, O)$ if and only if X is σ-$\mathcal{A}$.*

Putting Corollary 3.11 with Corollary 3.9, we see the following.

Corollary 3.12. *For T_1 spaces, if $\mathcal{A}$ has the property that for each $A \in \mathcal{A}$ there exists a G_δ $A^\star$ with $A \subseteq A^\star \in \mathcal{A}$, then the following are all equivalent.*

- *X is σ-$\mathcal{A}$.*
- *TWO $\uparrow G_1(O_\mathcal{A}, O)$.*
- *TWO $\uparrow_{\mathrm{mark}} G_1(O_\mathcal{A}, O)$.*
- *ONE $\uparrow G_1(\mathcal{N}_\mathcal{A}, \neg O)$.*
- *ONE $\uparrow_{\mathrm{pre}} G_1(\mathcal{N}_\mathcal{A}, \neg O)$.*

And in particular:

Corollary 3.13. *If every zero-$\dim_X$ subset of X is contained in a G_δ zero-$\dim_X$ subset, then the following are all equivalent.*

- *X is σ-zero-$\dim_X$.*
- *TWO $\uparrow_{\mathrm{mark}} G_c(O, O)$.*
- *TWO $\uparrow G_1(O_{0\,\dim_X}, O)$.*
- *TWO $\uparrow_{\mathrm{mark}} G_1(O_{0\,\dim_X}, O)$.*
- *ONE $\uparrow G_1(\mathcal{N}_{0\,\dim_X}, \neg O)$.*
- *ONE $\uparrow_{\mathrm{pre}} G_1(\mathcal{N}_{0\,\dim_X}, \neg O)$.*

It would seem natural for the following conjecture to hold, but its validity is currently an open question.

Conjecture 3.14. *TWO $\uparrow G_c(O, O)$ may be added to Corollary 3.13.*

Furthermore, this would align with existing results on metrizable spaces, given the following lemma.

Lemma 3.15. ([8, 4.1.19]) *Let X be metrizable. Then every zero-dim subspace is contained in a G_δ zero-dim subspace.*

Recall from earlier that zero-dim and zero-dim$_X$ are equivalent in the context of metrizable spaces. So it follows that the equivalences in Corollary 3.13 are guaranteed for metrizable spaces.

Likewise, and zero-dim and zero-ind are equivalent for T_2 strongly paracompact spaces, so for the following theorem we relax our notation to allow the use of "zero dimensional" ("zd" for short) and "countable dimensional".

Theorem 3.16. *Let X be strongly paracompact and metrizable. Then the following are equivalent.*

- *X is countable-dimensional.*

- *TWO $\uparrow G_c(O,O)$.*

- *TWO $\uparrow_{\text{mark}} G_c(O,O)$.*

- *TWO $\uparrow G_1(O_{zd},O)$.*

- *TWO $\uparrow_{\text{mark}} G_1(O_{zd},O)$.*

- *ONE $\uparrow G_1(N_{zd}, \neg O)$.*

- *ONE $\uparrow_{\text{pre}} G_1(N_{zd}, \neg O)$.*

Proof. All equivalences except *TWO $\uparrow G_c(O,O)$* are obtained from Corollary 3.13 and the previous lemma. This missing equivalence is obtained from [3], whose proof assumes zero-ind is equivalent to zero-dim, which is guaranteed by strong paracompactness. □

4. Generalizing Telgársky's single-open vs. finite-open result

We begin with the following useful lemma, which generalizes the classic result [23, 4.3] of Telgársky on the equivalence of point-open and finite-open games.

Lemma 4.1. *Let $\mathcal{B}$ be the closure of $\mathcal{A}$ under finite unions. Then the games $G_1(N_\mathcal{A}, \neg O)$ and $G_1(N_\mathcal{B}, \neg O)$ are equivalent.*

Proof. Note that since $\mathcal{A} \subseteq \mathcal{B}$, $N_\mathcal{A} \subseteq N_\mathcal{B}$. Therefore we immediately have the following implications:

1. *ONE $\uparrow_{\text{pre}} G_1(N_\mathcal{A}, \neg O)$* implies *ONE $\uparrow_{\text{pre}} G_1(N_\mathcal{B}, \neg O)$*
2. *ONE $\uparrow G_1(N_\mathcal{A}, \neg O)$* implies *ONE $\uparrow G_1(N_\mathcal{B}, \neg O)$*
3. *TWO $\uparrow G_1(N_\mathcal{B}, \neg O)$* implies *TWO $\uparrow G_1(N_\mathcal{A}, \neg O)$*
4. *TWO $\uparrow_{\text{mark}} G_1(N_\mathcal{B}, \neg O)$* implies *TWO $\uparrow_{\text{mark}} G_1(N_\mathcal{A}, \neg O)$*

For each $B \in \mathcal{B}$, let $N_B < \omega$ and $A(B,n) \in \mathcal{A}$ with $B = \bigcup_{n \leqslant N_B} A(B,n)$. For convenience, let $A(B,n) = A(B,0)$ for $n > N_B$, so $B = \bigcup_{n < \omega} A(B,n)$ (we will not be concerned with the case $B = \emptyset$ since ONE's moves are always improved by choosing larger sets). We will assume strategies for and plays by ONE choose an element of $\mathcal{A}$ or $\mathcal{B}$ directly each round, rather than $N_\mathcal{A}$ or $N_\mathcal{B}$.

Converse 1. Suppose the sequence $\langle B_0, B_1, \dots \rangle$ witnesses *ONE $\uparrow_{\text{pre}} G_1(N_\mathcal{B}, \neg O)$*. Consider the predetermined strategy

$$\langle A(B_0,0), \dots, A(B_0, N_{B_0}), A(B_1, 0), \dots, A(B_1, N_{B_1}), \dots \rangle$$

by ONE in $G_1(N_\mathcal{A}, \neg O)$. Any response by TWO is of the form

$$\langle U(0,0), \dots, U(0, N_{B_0}), U(1,0), \dots, U(1, N_{B_1}), \dots \rangle$$

where $A(B_i, j) \subseteq U(i, j)$. Let $U_i = \bigcup_{j \leqslant N_{B_i}} U(i, j)$; then $B_i \subseteq \bigcup_{j \leqslant N_{B_i}} U(i, j) = U_i$. Then $\langle U_0, U_1, \ldots \rangle$ is an unsuccessful response by TWO against the winning predetermined strategy $\langle B_0, B_1, \ldots \rangle$. Thus $\{U_i : i < \omega\}$ is not a cover, and it follows that $\{U(i, j) : i < \omega, j \leqslant N_{B_i}\}$ is not a cover as well. Thus the above predetermined strategy for ONE in $G_1(\mathcal{N}_{\mathcal{B}}, \neg O)$ is winning.

Converse 2. Suppose τ is a strategy witnessing $ONE \uparrow G_1(\mathcal{N}_{\mathcal{B}}, \neg O)$. Let $B_0 = \tau(\langle \rangle)$ and $m_0 = N_{B_0}$. Then $\tau(\langle \rangle) = \bigcup_{i \leqslant m_0} A(B_0, i)$. Let $\tau'(\langle V_{0,0}, \ldots, V_{0,i-1} \rangle) = A(B_0, i)$ for $i \leqslant m_0$. Note then that $\bigcup_{i \leqslant m_0} \tau'(\langle V_{0,k}, \ldots, V_{0,i-1} \rangle) = B_0 = \tau(\langle \rangle)$.

Suppose $m_i < \omega$ is defined for $i \leqslant p < \omega$ and τ' has been defined for each initial segment of $\langle V_{0,0}, \cdots, V_{0,m_0}, \cdots, V_{p,0}, \cdots, V_{p,m_p-1} \rangle$ where

$$U_k = \bigcup_{i \leqslant m_k} V_{k,i} \supseteq \tau(\langle U_0, \cdots, U_{k-1} \rangle)$$

for $k < p$ and

$$\tau(\langle U_0, \cdots, U_{p-1} \rangle) = \bigcup_{i \leqslant m_p} \tau'(\langle V_{0,0}, \cdots, V_{0,m_0}, \cdots, V_{p-1,0}, \cdots, V_{p-1,i-1} \rangle).$$

Consider $\langle V_{0,0}, \cdots, V_{0,m_0}, \cdots, V_{p,0}, \cdots, V_{p,m_p} \rangle$, and let

$$U_p = \bigcup_{i \leqslant m_p} V_{p,i} \supseteq \tau(\langle U_0, \cdots, U_{p-1} \rangle).$$

Set $B_{p+1} = \tau(\langle U_0, \cdots, U_p \rangle)$ and $m_{p+1} = N_{B_{p+1}}$. Then $\tau(\langle U_0, \cdots, U_p \rangle) = \bigcup_{i \leqslant m_{p+1}} A(B_{p+1}, i)$. Let

$$\tau'(\langle V_{0,0}, \cdots, V_{0,m_0}, \cdots, V_{p,0}, \cdots, V_{p,m_p}, V_{p+1,0}, \cdots, V_{p+1,i-1} \rangle) = A(B_{p+1}, i)$$

for $i \leqslant m_{p+1}$. Note then that

$$\bigcup_{i \leqslant m_{p+1}} \tau'(\langle V_{0,0}, \cdots, V_{0,m_0}, \cdots, V_{p+1,0}, \cdots, V_{p+1,i-1} \rangle) = B_{p+1} = \tau(\langle U_0, \cdots, U_p \rangle).$$

Finally, by following the construction we observe that any counter-play against τ' produces a corresponding counter-play against τ with the same union. Thus since τ is winning, so is τ'.

Converse 3. Suppose τ is a strategy witnessing $TWO \uparrow G_1(\mathcal{N}_{\mathcal{A}}, \neg O)$. Let

$$\tau'(\langle B_0, \cdots, B_n \rangle) = \bigcup_{i \leqslant N_{B_n}} \tau(\langle A(B_0, 0), \cdots, A(B_0, N_{B_0}), \cdots, A(B_n, 0), \cdots, A(B_n, i) \rangle).$$

Then any counter-play $\langle B_0, B_1, \cdots \rangle$ against τ' corresponds to a counter-play

$$\langle A(B_0, 0), \cdots, A(B_0, N_{B_0}), A(B_1, 0), \cdots \rangle$$

against τ where both strategies cover the same subset of the space. Therefore since τ is winning, so is τ'.

Converse 4. Suppose τ is a strategy witnessing $TWO \uparrow_{\mathrm{mark}} G_1(\mathcal{N}_{\mathcal{A}}, \neg O)$. Let $\theta : \omega^2 \to \omega$ be a bijection. Then we define the Markov strategy τ' in $G_1(\mathcal{N}_{\mathcal{B}}, \neg O)$ by $\tau'(B, n) = \bigcup_{i < \omega} \tau(A(B, i), \theta(n, i))$. Then if ONE chooses B_n during round n against τ', consider when ONE chooses $A(B_n, i)$ during round $\theta(n, i)$ against τ. It follows that both plays result in TWO constructing covers of the same subspace, so since τ is winning, so is τ'. $\square$

By duality we have the following.

Corollary 4.2. *Let $\mathcal{B}$ be the closure of $\mathcal{A}$ under finite unions. Then the games $G_1(O_{\mathcal{A}}, O)$ and $G_1(O_{\mathcal{B}}, O)$ are equivalent.*

5. Conclusions

In [1, Corollary 19], Babinkostova cites an example of Pol [20] where *ONE* $\not\uparrow_{\text{pre}}$ $G_c(O, O)$, but *ONE* $\uparrow_{\text{pre}}$ $G_1(O_{\mathcal{A}}, O)$, where $\mathcal{A}$ is the collection of "finite-dimensional" subsets of the space. Since this example is separable and metrizable, finite-dimensional here is equivalent to finite unions of zero-dimensional (per your favorite definition) subsets [8, 4.1.17]. Therefore despite the equivalence of *TWO* $\uparrow_{\text{mark}}$ $G_1(O_{0\,\dim_X}, O)$ and *TWO* $\uparrow_{\text{mark}}$ $G_c(O, O)$ among T_1 spaces, we have the following.

Corollary 5.1. *The games $G_1(O_{0\,\dim_X}, O)$ and $G_c(O, O)$ are not equivalent, even for separable metrizable spaces.*

A stronger version of Theorem 3.16 was asserted in [3] without assuming strong paracompactness, but its proof seems to nonetheless assume the equivalence of inductive and covering zero dimension. Thus we believe the following question remains open.

Question 5.2. *Is strong paracompactness required in Theorem 3.16?*

Acknowledgement

References

[1] L. Babinkostova, When does the Haver property imply selective screenability?, Topology Appl. 154 (2007) 1971–1979.
[2] L. Babinkostova, M. Scheepers, Selection principles and countable dimension, Note Mat. 27 (2007) 5–15.
[3] L. Babinkostova, M. Scheepers, Countable dimensionality, a game and the Haver property, Proc. Amer. Math. Soc. 1 (2021) 3939–3945.
[4] R.H. Bing. Metrization of topological spaces, Canad. J. Math. 3 (1951) 175–186.
[5] S. Clontz, Dual selection games, Topology Appl. 272 (2020) Art. 107056.
[6] S. Clontz, J. Holshouser, Limited information strategies and discrete selectivity, Topology Appl. 265 (2019) Art. 106815.
[7] R.R. Dias, M. Scheepers, Selective games on binary relations, Topology Appl. 192 (2015), 58–83. (Proceedings of Brazilian Conference on General Topology and Set Theory (STW-2013)—)
[8] R. Engelking, Dimension Theory, North-Holland Publishing Company, Amsterdam, 1978.
[9] R. Engelking, General Topology, Vol. 6 of Sigma Series in Pure Mathematics, Heldermann Verlag, Berlin, Second edition, 1989.
[10] W.G. Fleissner, A.W Miller, On Q sets, Proc. Amer. Math. Soc. 78 (1980) 280–284.
[11] F. Galvin, Indeterminacy of point-open games, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 26 (1978) 445–449.
[12] W. Hurewicz, Über unendlich-dimensionale Punktmengen, Proc. Akademie van Wetenschappen Amsterdam 31 (1928) 916–922.
[13] J. Kulesza, Metrizable spaces where the inductive dimensions disagree, Trans. Amer. Math. Soc. 318 (1990) 763–781.
[14] J. Kulesza, New properties of Mrowka's space $\nu\mu_0$, Proc. Amer. Math. Soc. 133 (2005) 899–904.
[15] K. Kuratowski, Topology, Vol. I, Elsevier, 2014.
[16] A.C. Megaritis, Relative dimension r-dim and finite spaces, Appl. Gen. Topol. 13 (2012) 91–102.
[17] S. Mrówka, Small inductive dimension of completions of metric spaces, Proc. Amer. Math. Soc. 125 (1997) 1545–1554.
[18] S. Mrówka, Small inductive dimension of completions of metric spaces II, Proc. Amer. Math. Soc. 128 (2000) 1247–1256.
[19] A. Ostaszewski, A note on the Prabir Roy space, Topology Appl. 35 (1990)) 95–107.
[20] E. Pol, Spaces whose nth power is weakly infinite-dimensional but whose (n+1)th power is not, Proc. Amer. Math. Soc. 117 (1993) 871–876.
[21] P. Roy, Failure of equivalence of dimension concepts for metric spaces, Bull. Amer. Math. Soc. 68 (1962) 609–613.
[22] M. Scheepers, Combinatorics of open covers I. Ramsey theory, Topology Appl. 69 (1996) 31–62.
[23] R. Telgársky, Spaces defined by topological games, Fund. Math. 88 (1975) 193–223.
[24] J. Valuyeva, On relative dimension concepts, Questions Answers Gen. Topology 15 (1997) 21–24.