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A Characterization of 4-χ_ρ-(Vertex-)Critical Graphs

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Abstract. Given a graph G, a function $c : V(G) \longrightarrow \{1, \ldots, k\}$ with the property that for every $u \neq v$, $c(u) = c(v) = i$ implies that the distance between u and v is greater than i, is called a k-packing coloring of G. The smallest integer k for which there exists a k-packing coloring of G is called the packing chromatic number of G, and is denoted by $\chi_\rho(G)$. Packing chromatic vertex-critical graphs are the graphs G for which $\chi_\rho(G - x) < \chi_\rho(G)$ holds for every vertex x of G. A graph G is called a packing chromatic critical graph if for every proper subgraph H of G, $\chi_\rho(H) < \chi_\rho(G)$. Both of the mentioned variations of critical graphs with respect to the packing chromatic number have already been studied [6, 23]. All packing chromatic (vertex-)critical graphs G with $\chi_\rho(G) = 3$ were characterized, while there were known only partial results for graphs G with $\chi_\rho(G) = 4$. In this paper, we provide characterizations of all packing chromatic vertex-critical graphs G with $\chi_\rho(G) = 4$ and all packing chromatic critical graphs G with $\chi_\rho(G) = 4$.

1. Introduction

There are many variations of graph coloring and one of them is packing coloring. Given a graph G and a positive integer i, a *k-packing coloring* is a mapping $c : V(G) \longrightarrow \{1, 2, \ldots, k\}$ with the following property: if $c(u) = c(v) = i$, then $d(u, v) > i$ for any $u, v \in V(G)$, $u \neq v$, and $i \in \{1, 2, \ldots, k\}$ (note that $d(u, v)$ is the usual shortest-path distance between u and v). The *packing chromatic number* of G, denoted by $\chi_\rho(G)$, is the smallest integer k such that there exists a k-packing coloring of G.

The packing chromatic number was introduced by Goddard, S. M. Hedetniemi, S. T. Hedetniemi, Harris and Rall [20] in 2008 under the name broadcast chromatic number. The current name was given in the second paper on the topic by Brešar, Klavžar and Rall [10]. The concept has a very wide spectrum of potential applications, such as frequency assignments [20] or applications in resource placements and biological diversity [10].

Packing coloring has attracted many authors. This is reflected in the (probably non-exhaustive) list of papers on this topic that were published only in the last two years [2, 3, 7, 9, 12, 15, 18, 19, 22, 23, 26, 31] (see also a survey [8]). One of the main areas of investigation has been to determine the boundedness or the exact values of the packing chromatic numbers of several classes of (finite and infinite) graphs [4, 5, 10–14, 16, 17, 24–27, 29, 30]. Among them, a lot of attention was given to the question of boundedness of the invariant in the class of (sub)cubic graphs, which was already posed in the seminal paper. Finally, it was

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answered in the negative by Balogh, Kostochka and Liu [1] (see also an explicit construction in [5]). Also, the problem of boundedness or the exact values of the packing chromatic numbers for the infinite grids was studied by many authors [10, 14, 16, 17, 24, 27, 28].

It is clear that the packing chromatic number is hereditary: a graph cannot have smaller packing chromatic number than its subgraphs. In particular, if we delete a vertex v from a given graph G, then for the obtained graph we have: $\chi_\rho(G - v) \leq \chi_\rho(G)$. Klavžar and Rall [23] investigated the class of graphs G with the property that $\chi_\rho(G - v) < \chi_\rho(G)$ holds for every $v \in V(G)$. Such graphs are called *packing chromatic vertex-critical graph*, or shorter *χ_ρ-vertex-critical graph*. In the case when G is χ_ρ-vertex-critical and $\chi_\rho(G) = k$, we also say that G is *k-χ_ρ-vertex-critical*. Among other results, the mentioned authors proved that the only 3-χ_ρ-vertex-critical graphs are C_3, P_4 and C_4. In addition, they provided partial characterizations of 4-χ_ρ-vertex-critical graphs and considered χ_ρ-veretx-critical trees. Later, Brešar and Ferme [6] studied a different (basic) version of critical graphs for the packing chromatic number. Namely, they considered the class of graphs G satisfying the following property: $\chi_\rho(H) < \chi_\rho(G)$ for each proper subgraph H of a graph G. Such graphs are called *packing chromatic critical graph*, or shorter *χ_ρ-critical graph*. If G is χ_ρ-critical and $\chi_\rho(G) = k$, we can say that G is *k-χ_ρ-critical*. The mentioned authors characterized χ_ρ-critical graphs with diameter 2, χ_ρ-critical block graphs with diameter 3 and proved that the only 3-χ_ρ-critical graphs are C_3 and P_4. The authors also considered χ_ρ-critical trees. In both of the mentioned papers, a partial characterization of 4-χ_ρ-(vertex)-critical graphs is given. In this paper, we present a general characterization of such graphs.

The paper is organized as follows. In the next section, we establish the notation and define the concepts used throughout the paper. We present the known family of graphs G with $\chi_\rho(G) = 3$, which will help us to characterize 4-χ_ρ-vertex-critical graphs. In addition, we prove some lemmas, which will be very useful for the proofs in the sequel of this paper. In Section 3, we recall some partial characterizations of 4-χ_ρ-vertex-critical graphs. Then, we present all 4-χ_ρ-vertex-critical graphs and prove the characterization. Based on this result, in Section 4, we provide a complete characterization of 4-χ_ρ-critical graphs. We end the paper with some remarks.

2. Notation and preliminaries

Let G be a graph (unless stated otherwise, the term *graph* refers to a simple graph). We denote its vertex set by $V(G)$ and its set of edges by $E(G)$. The *(open) neighborhood* of an arbitrary vertex $v \in V(G)$, $N_G(v)$, is the set of all vertices adjacent to v. The *closed neighborhood* of v is $N_G(v) \cup \{v\}$ and is denoted by $N_G[v]$. The number of elements in $N_G(v)$, $|N_G(v)|$, is called the *degree* of v and is denoted by $\deg_G(v)$. In the case when $\deg_G(v) = 1$, we say that v is a *leaf* or a *pendant vertex*. If $\deg(v) > 1$, then v is called a *non-pendant vertex*. An *isolated vertex* is a vertex v with $\deg_G(v) = 0$. Further, the *distance* between two vertices $u, v \in V(G)$, denoted by $d_G(u, v)$, is the length of a shortest u-v-path in G. Note that the subscript in some of the above notations may be omitted if the graph G is clear from the context.

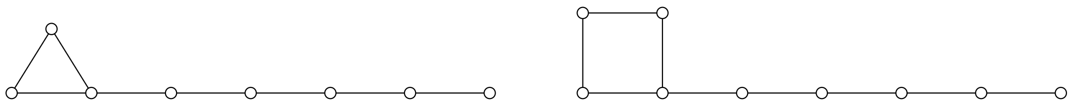
A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of a graph G is called a *proper subgraph* of G if $V(H)$ is a proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$. If H is a subgraph of G and $V(G) = V(H)$, then we say that H is a *spanning subgraph* of G. Further, a subgraph H of a graph G is an *induced subgraph* of G if for each pair of the vertices $a, b \in V(H)$ holds the following: if $ab \in E(G)$, then $ab \in E(H)$. If $A \subset V(G)$, then $G[A]$ denotes an induced subgraph of G with the vertex set A. We will also use the notation $G - v$, which is shorter for $G[V(G) \setminus \{v\}]$. Next, if $e = xy \in E(G)$, then $G - e$ (or $G - xy$) denotes the subgraph of G with $V(G - e) = V(G)$ and $E(G - e) = E(G) \setminus \{e\}$.

Graphs G and H are *isomorphic*, $G \cong H$, if there exists a bijection $h : V(G) \longrightarrow V(H)$ such that $uv \in E(G) \Leftrightarrow h(u)h(v) \in E(H)$ for any $u, v \in V(G)$.

Recall that the packing chromatic number is hereditary, which means that for any subgraph H of G, $\chi_\rho(H) \leq \chi_\rho(G)$. This property will be used several times in the sequel of this paper. Further, recall that $\chi_\rho(K_n) = n$ for every complete graph K_n, $n \geq 1$. Further, let C_n be a cycle of order $n \geq 3$. Then, $\chi_\rho(C_n) = 3$ if $n = 3$ or n is divisible by 4. Otherwise, $\chi_\rho(C_n) = 4$. For any path P_n we have: $\chi_\rho(P_1) = 1$, $\chi_\rho(P_2) = \chi_\rho(P_3) = 2$ and $\chi_\rho(P_n) = 3$ if $n \geq 4$.

Now, we prove two lemmas, which will be very useful in Sections 3 and 4.

Let n be a positive integer. The graph X_n is formed from the disjoint union of one copy of K_3 and one copy of P_n by joining a vertex of K_3 and a leaf (or an isolated vertex) of P_n (see Fig. 1a). Similarly, Y_n is the graph obtained from the disjoint union of one copy of C_4 and one copy of P_n by joining a vertex of C_4 and a leaf (or an isolated vertex) of P_n (see Fig. 1b).



(a) X_5 (b) Y_5

Figure 1: Graphs X_5 and Y_5

Lemma 2.1. *If $n \geq 1$, then $\chi_\rho(X_n) = 3$ and $\chi_\rho(Y_n) = 3$.*

Proof. Let $n \geq 1$ be an arbitrary positive integer. Recall that $\chi_\rho(K_3) = 3$ and $\chi_\rho(C_4) = 3$. Since X_n contains a subgraph isomorphic to K_3 and Y_n contains a subgraph isomorphic to C_4, the hereditary property of the packing chromatic number implies that $\chi_\rho(X_n) \geq 3$ and $\chi_\rho(Y_n) \geq 3$.

On the other hand, there exist 3-packing colorings of X_n and Y_n, defined as follows. Color the vertices belonging to K_3 in X_n with different colors from $\{1, 2, 3\}$ such that the vertex of degree 3 receives color 3. Similarly, color the vertices belonging to C_4 in Y_n with different colors from $\{1, 2, 3\}$ such that the vertex of degree 3 receives color 3. Next, in both cases, color the vertices belonging to the path one after another (starting with the vertex, which is adjacent to the vertex of degree 3) using the following pattern of colors: $1, 2, 1, 3$. Since the described colorings are 3-packing colorings of X_n, respectively Y_n, we derive that $\chi_\rho(X_n) = 3$ and $\chi_\rho(Y_n) = 3$. □

Lemma 2.2. *If T is the graph in Fig. 2, then $\chi_\rho(T) = 3$.*

Proof. Let T be the graph in Fig. 2. Since T has at least one edge, is a connected graph and is not a star, $\chi_\rho(T) \geq 3$ [20]. Clearly, there exists a 3-packing coloring c' of T defined as follows: $c'(y') = 3$, $c'(a) = c'(b) = c'(c) = 1$ and $c'(d) = c'(e) = 2$ (see Fig. 2 for the notation of the vertices). Thus, $\chi_\rho(T) = 3$ (see also [20]). □

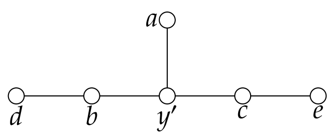


Figure 2: Graph T

We continue with the characterization of graphs G with $\chi_\rho(G) = 3$, which was proven by Goddard et al. [20].

Let v be an arbitrary vertex of a given graph G. A *T-add to a vertex v* is formed as follows. First, we add to G a vertex denoted by w_v and an independent set (of an arbitrary size), which is denoted by X_v. Then we add to G the edge vw_v and (some of the) edges, which join the vertices from $\{v, w_v\}$ with the vertices from X_v. An example of a T-add to $v \in V(G)$ is shown in Fig. 3.

Proposition 2.3. *[20] Let G be a graph. Then, $\chi_\rho(G) = 3$ if and only if G can be formed by taking some bipartite multigraph H with bipartition (U_1, U_3), subdividing every edge exactly once, adding leaves to some vertices in $U_1 \cup U_3$, and then performing a single T-add to some vertices in U_3 (i.e. we attach at most one T-add to each vertex from U_3).*

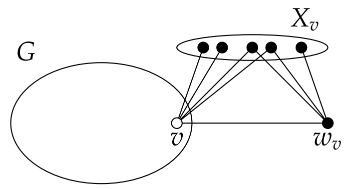


Figure 3: An example of a T-add to a vertex v of a given graph G

In this paper, we denote the family of graphs with packing chromatic number 3 by $\mathcal{G}_3$.

Let $G \in \mathcal{G}_3$ be an arbitrary connected graph. We denote the subsets of $V(G)$ as follows. If G is obtained from a bipartite multigraph H with bipartition (U_1, U_3) by subdividing every edge of H exactly once, adding leaves to some vertices in $U_1 \cup U_3$, and performing a single T-add to some vertices in U_3, then:

- $V_1 = U_1$;
- $V_3 = U_3$;
- V_2 is the set of all vertices obtained by subdivision;
- V_0 is the set of all leaves added to the vertices in V_1;
- V_4 is the set of all leaves adjacent to the vertices from V_3;
- V_5 is the set of all vertices from T-adds, which belong to a subgraph of G isomorphic to K_3 and have degree 2 (therefore, V_5 is the set of all vertices, which belong to a subgraph of G isomorphic to K_3 and have degree 2);
- V_6 is the set of all vertices from T-adds, which either: a) have degree at least 3, or b) have degree 2 and do not belong to a subgraph of G isomorphic to K_3 (therefore, V_6 is the set of all vertices adjacent to the vertices from V_3, which have degree at least 3, or have degree 2, but do not belong to a subgraph of G isomorphic to K_3 and are not obtained by subdivision);
- V_7 is the set of all vertices from T-adds, which are leaves and are adjacent to the vertices from V_6 (therefore, V_7 consists of all leaves adjacent to the vertices from V_6).

An example of the described labeling of subsets of $V(G)$, $G \in \mathcal{G}_3$, is shown in Fig. 4.

We observe that $V(G)$ is partitioned into sets $V_0, V_1, \ldots, V_7$. Further, since $V_1 \cup V_3$ induces a subgraph of G isomorphic to a bipartite multigraph in which each edge is subdivided exactly once, $d(v_1, v_1') = 4k_1$, $k_1 \geq 1$, and $d(v_3, v_3') = 4k_3$, $k_3 \geq 1$, for any $v_1, v_1' \in V_1$, $v_1 \neq v_1'$, and $v_3, v_3' \in V_3$, $v_3 \neq v_3'$. Next, suppose that $a \in V(G)$ belongs to a subgraph of G isomorphic to K_3. This implies that $a \in V_3 \cup V_5 \cup V_6$. More precisely, if $\deg(a) = 2$, then $a \in V_5$. If $\deg(a) \geq 3$, and each vertex from $N(a)$ is either a leaf or belongs to a triangle, then $a \in V_6$. Otherwise, $a \in V_3$. In addition, we observe that each vertex from V_6 has at most one neighbour which is not a leaf and does not belong to a triangle.

We end this section with two lemmas, which will be very useful in the sequel of this paper.

Lemma 2.4. *Let G be the graph obtained by attaching a vertex to two adjacent vertices of C_n, $n \geq 4$. Then, G contains a subgraph isomorphic to C_n, $n \geq 5$, $n \not\equiv 0 \pmod 4$.*

Proof. Let G be formed by attaching a vertex v to two adjacent vertices of C_n, $n \geq 4$, with $V(C_n) = \{x_1, x_2, \ldots, x_n\}$ and $E(C_n) = \{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1\}$. Further, let $x_1v, x_2v \in E(G)$. Clearly, if $n \not\equiv 0 \pmod 4$, then we are done. Otherwise, G contains a subgraph isomorphic to a cycle C_{n+1} ($V(C_{n+1}) = \{v, x_1, x_2, \ldots, x_n\}$ and $E(C_{n+1}) = \{x_1v, vx_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1\}$). Since $n + 1 \geq 5$ and $n + 1 \not\equiv 0 \pmod 4$, our claim holds. $\square$

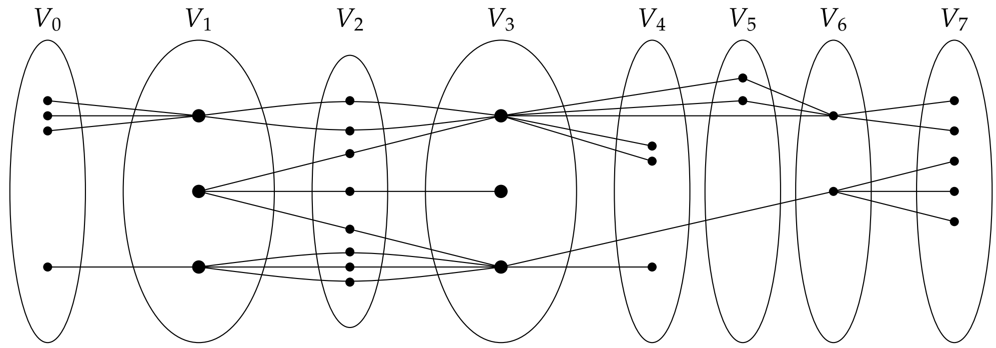


Figure 4: The sets $V_0, V_1, \ldots, V_7$

Lemma 2.5. *Let G be a graph, which does not contain a subgraph isomorphic to C_n, $n \geq 4$, $n \not\equiv 0 \pmod 4$. Next, let $u \in V(G)$ be an arbitrary vertex and let $N_i = \{v \in V(G) : d(u,v) = i\}$ for any $i \geq 1$. Further, let i be an arbitrary positive integer and let a, b be any two vertices from N_i. Then, $ab \notin E(G)$ if ab is not an edge of a subgraph of G isomorphic to K_3.*

Proof. Let G be a graph, which does not contain a subgraph isomorphic to C_n, $n \geq 4$, $n \not\equiv 0 \pmod 4$, $u \in V(G)$ and $N_i = \{v \in V(G) : d(u,v) = i\}$ for any $i \geq 1$. Next, let i be an arbitrary positive integer and let a, b be any two vertices from N_i. Suppose to the contrary that $ab \in E(G)$ and ab is not an edge of a subgraph of G isomorphic to K_3. Denote by P a shortest a-u-path and by Q a shortest b-u-path . Let $w \in V(P) \cap V(Q)$ such that the distance between w and u is the largest possible. Then, the vertices from P between w and a, and the vertices from Q between w and b form a cycle of odd order. Since ab is not an edge of a triangle, we infer that G contains a cycle of order $n > 3$, $n \not\equiv 0 \pmod 4$. This contradicts our assumption. □

3. 4-χ_ρ-vertex-critical graphs

In this section, we study 4-χ_ρ-vertex-critical graphs. First, we recall some known partial results for these graphs. We follow with the main result of this paper: a complete characterization of 4-χ_ρ-vertex-critical graphs.

The first known partial characterization of 4-χ_ρ-vertex-critical graphs considers all graphs that contain a cycle C_n, where $n \geq 5$ and is not divisible by 4. Note that for such cycles, we have $\chi_\rho(C_n) = 4$ and $\chi_\rho(C_n - u) = 3$ for any $u \in V(C_n)$, which implies that these cycles themselves are 4-χ_ρ-vertex-critical. However, if G has $\chi_\rho(G) = 4$ and G contains a non-spanning subgraph isomorphic to a cycle C_n, where $n \geq 5$ is not divisible by 4, then G is clearly not 4-χ_ρ-vertex-critical.

Denote by $\mathcal{C}_5$ the family of graphs, which are shown in Fig. 5, by $\mathcal{C}_6$ the family of graphs, which are shown in Fig. 6, and let $\mathcal{C} = \{C_n : n \geq 5, n \not\equiv 0 \pmod 4$. The following theorem provides a complete characterization of 4-χ_ρ-vertex-critical graphs that contain a cycle $C_n \in \mathcal{C}$.

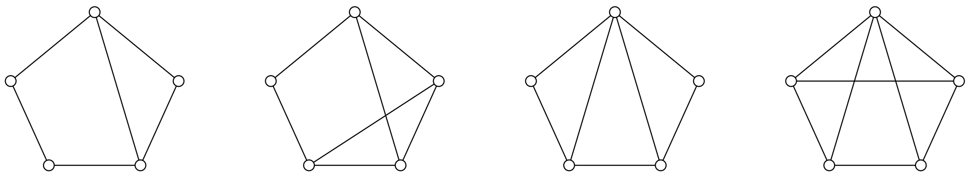


Figure 5: The graphs from $\mathcal{C}_5$

Theorem 3.1. [23] *If G is a graph that contains a cycle C_n, $n \geq 5$, $n \not\equiv 0 \pmod 4$, then G is 4-χ_ρ-vertex-critical if and only if one of the following holds.*

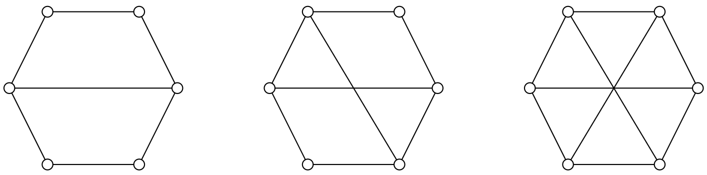


Figure 6: The graphs from $\mathcal{C}_6$

- $G \in \mathcal{C}_5$;
- $G \in \mathcal{C}_6$;
- $G \in \mathcal{C}$.

We continue by presenting yet another partial characterization of 4-χ_ρ-vertex-critical graphs. Let $\mathcal{D}$ be the class of graphs that contain exactly one cycle and have an arbitrary number of leaves attached to each of the vertices of the cycle. Next, recall that the *net graph* is obtained by attaching a single leaf to each vertex of K_3.

Theorem 3.2. *[23] A graph $G \in \mathcal{D}$ is a 4-χ_ρ-vertex-critical graph if and only if G is one of the following graphs:*

- *$G \in \mathcal{C}$;*
- *G is the net graph;*
- *G is obtained by attaching a single leaf to two adjacent vertices of C_4;*
- *G is obtained by attaching a single leaf to two vertices at distance 3 on C_8.*

In the sequel of this section, we provide a complete characterization of 4-χ_ρ-vertex-critical graphs. Recall that each χ_ρ-vertex-critical graph is connected [23].

First, we prove that the graphs shown in Fig. 7 are 4-χ_ρ-vertex-critical.

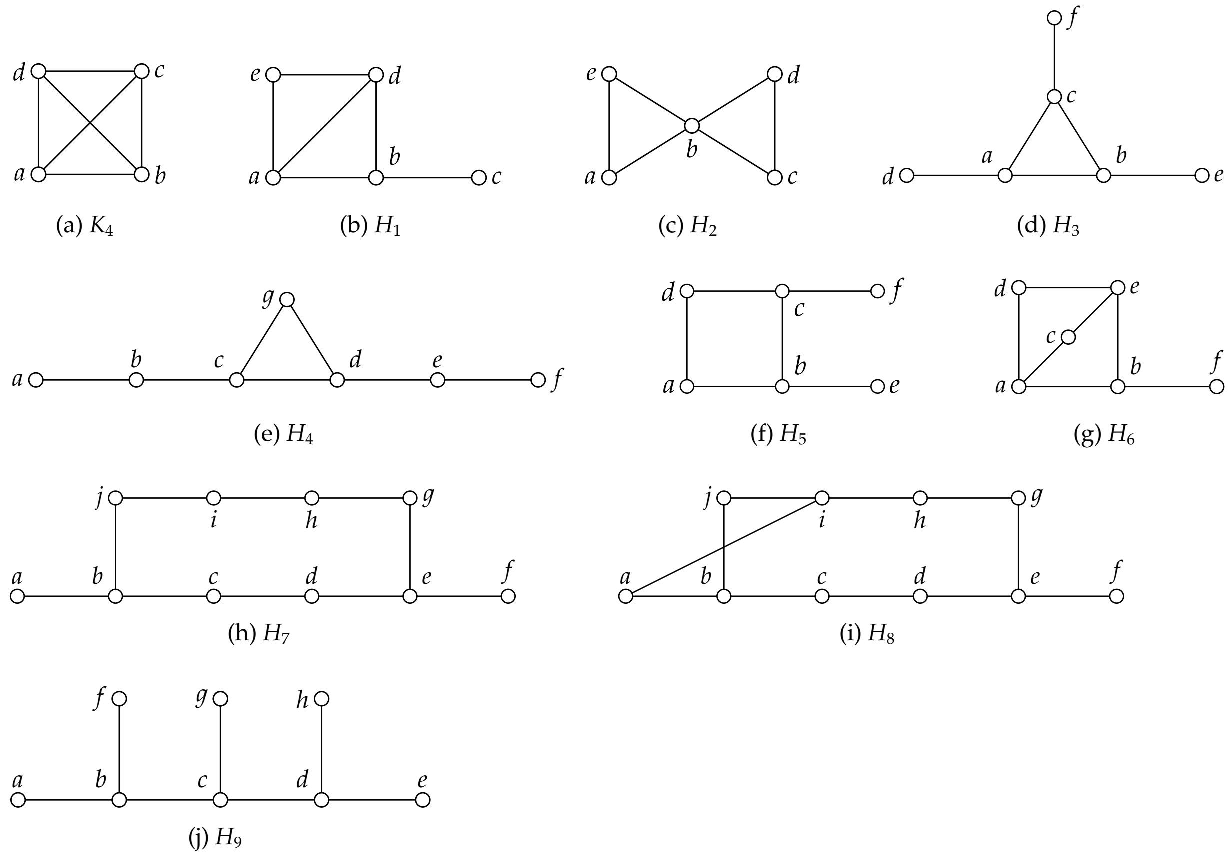
Theorem 3.3. *The graphs $K_4, H_1, H_2, H_3, H_4, H_5, H_6, H_7, H_8$ and H_9 shown in Fig. 7 are 4-χ_ρ-vertex-critical.*

Proof. Theorem 3.2 implies that H_3, H_5 and H_7 are 4-χ_ρ-vertex-critical. In addition, from [23] follows that H_1 is a 4-χ_ρ-vertex-critical graph. Clearly, also K_4 is 4-χ_ρ-vertex-critical.

Next, we prove that H_2 (see Fig. 7c) is 4-χ_ρ-vertex-critical. Since each packing coloring of H_2 assigns three distinct colors to the vertices a, b, e and (at most) one of these colors can be used for the vertices c and d, we have $\chi_\rho(H_2) \geq 4$. On the other hand, there exists a 4-packing coloring of H_2. Indeed, let a and c receive color 1, and let the other vertices of H_2 receive different colors from $\{2, 3, 4\}$. Thus, $\chi_\rho(H_2) = 4$. We observe that for any $u \in \{a, b, c, d, e\}$, $H_2 - u$ has exactly 4 vertices, but is not isomorphic to K_4. Hence, $\chi_\rho(H_2 - u) \leq 3$ and H_2 is a 4-χ_ρ-vertex-critical graph.

Further, consider the graph H_4 (see Fig. 7e). Suppose that there exists a 3-packing coloring c_4 of H_4. Then, $\{c_4(c), c_4(d)\} = \{2, 3\}$. Without loss of generality we may assume that $c_4(c) = 3$, $c_4(d) = 2$. Then, $c_4(e) = 1$ and there is no available color for f, a contradiction to our assumption. Thus, $\chi_\rho(H_4) \geq 4$. Now, color the vertices a, b, c, d, e, f one after another using the colors $1, 2, 1, 3, 1, 2$ and let g receive a color 4. This coloring is a 4-packing coloring of H_4, which implies that $\chi_\rho(H_4) = 4$. Next, let $u \in \{a, b, c, d, e, f, g\}$. If $u \in \{c, d, g\}$, then $H_4 - u$ is a path or a union of paths. Consequently, $\chi_\rho(H_4 - u) \leq 3$. If $u \in \{a, b\}$, then color the vertices g and e with color 1, c and f with color 2, d with color 3 and the other vertex with color 1. This coloring is a 3-packing coloring of $H_4 - u$ and hence, $\chi_\rho(H_4 - u) \leq 3$. In the case when $u \in \{e, f\}$, analogously we prove that $\chi_\rho(H_4 - u) \leq 3$. Therefore, H_4 is 4-χ_ρ-vertex-critical.

Next, we claim that H_6 (see Fig. 7g) is a 4-χ_ρ-vertex-critical graph. If there exists a 3-packing coloring c_6 of H_6, then $c_6(a), c_6(b), c_6(e) \neq 1$ and a, b, e receive pairwise distinct colors by c_6. This implies that c_6 uses at



(a) K_4 (b) H_1 (c) H_2 (d) H_3 (e) H_4 (f) H_5 (g) H_6 (h) H_7 (i) H_8 (j) H_9

Figure 7: 4-χ_ρ-vertex-critical graphs

least 4 colors, a contradiction to c_6 being a 3-packing coloring of H_6. Thus, $\chi_\rho(H_6) \geq 4$. Further, by letting $c'_6(b) = c'_6(c) = c'_6(d) = 1$, and by assigning different colors from $\{2,3,4\}$ to other vertices of H_6, we infer that c'_6 is a packing coloring of H_6 using 4 colors. Thus, $\chi_\rho(H_6) = 4$. Now, let $u \in \{a,b,c,d,e,f\}$. If $u \in \{a,e,f\}$, then c'_6 restricted to $H_6 - u$ (with a substitution of a color 4 by a color 2 or 3, if necessary) is a 3-packing coloring of $H_6 - u$. Hence, $\chi_\rho(H_6 - u) \leq 3$. If $u = b$, then all vertices of $H_6 - u$ can be colored with 3 colors, since $H_6 - u$ is the disjoint union of an isolated vertex and a cycle C_4. In the case when $u = c$, $u = d$, respectively, $H_6 - u \cong Y_1$ and Lemma 2.1 implies that $\chi_\rho(H_6 - u) = 3$. Thus, H_6 is 4-χ_ρ-vertex-critical.

Now, we prove that H_8 (see Fig. 7i) is 4-χ_ρ-vertex-critical. Since H_8 contains a subgraph isomorphic to H_7 and $\chi_\rho(H_7) = 4$, the hereditary property of the packing chromatic number implies that $\chi_\rho(H_8) \geq 4$. In order to show that $\chi_\rho(H_8) = 4$, we form a 4-packing coloring c_8 of H_8. Let $c_8(d) = c_8(i) = 2$, $c_8(b) = c_8(g) = 3$, $c_8(f) = 4$ and let the remaining vertices receive color 1. Since this is a 4-packing coloring of H_8, $\chi_\rho(H_8) = 4$. Next, let $u \in \{a,b,c,d,e,f,g,h,i,j\}$ and let G' be a subgraph of H_8 induced by $V(H_8) \setminus \{u\}$. Note that, if $u = f$, then c_8 restricted to G' is a 3-packing coloring of G', thus $\chi_\rho(G') \leq 3$. Further, suppose that $u = e$. Let $c'_8(x) = c_8(x)$ for any $V(G') \setminus \{f\}$ and let $c'_8(f) = 1$. Clearly, c'_8 is a 3-packing coloring of G', hence $\chi_\rho(G') \leq 3$. If $u \in \{a,j\}$, then G' is isomorphic to a subgraph of H_7 induced by $V(H_7) \setminus \{a\}$. Since $\chi_\rho(H_7 - a) \leq 3$, we have $\chi_\rho(G') \leq 3$. Similarly, if $u \in \{b,i\}$, then G' is isomorphic to a subgraph of H_7 induced by $V(H_7) \setminus \{i\}$. Since $\chi_\rho(H_7 - i) \leq 3$, we have $\chi_\rho(G') \leq 3$. Now, let $u \in \{c,d\}$. In this case, color the vertices e and i with color 2, b and g with color 3 and the others with color 1. Since such coloring is a 3-packing coloring of G', $\chi_\rho(G') \leq 3$. Analogously we prove that $\chi_\rho(G') \leq 3$ if $u \in \{h,g\}$. Therefore, H_8 is 4-χ_ρ-vertex-critical.

Finally, consider the graph H_9 (see Fig. 7j). Suppose that there exists a 3-packing coloring c_9 of H_9. Clearly, $c_9(b), c_9(c), c_9(d) \neq 1$, which implies that there is no available color for one vertex from $\{b,c,d\}$, a contradiction to c_9 being a 3-packing coloring of H_9. This means that $\chi_\rho(H_9) \geq 4$. Next, we present a 4-packing coloring c_9' of H_9. Color the vertices a, e, f, g, h with color 1 and let $c_9'(b) = 2$, $c_9'(c) = 3$, $c_9'(d) = 4$. Clearly, c_9' is a 4-packing coloring of H_9, thus $\chi_\rho(H_9) = 4$. Further, let $u \in \{a, b, c, d, e, f, g, h\}$ and let G' be a subgraph of H_9 induced by $V(H_9) \setminus \{u\}$. If $u \in \{b, c, d\}$, then c_9' restricted to G' is a 3-packing coloring of G'. Hence, $\chi_\rho(G') \leq 3$. If $u = a$, then G' can be colored using 3 colors as follows. Let b, e, g, h receive color 1, f and d color 2 and c color 3. Since this is a 3-packing coloring of G', $\chi_\rho(G') \leq 3$. If $u \in \{e, f, h\}$, the proof is analogous. Finally, let $u = g$. In this case, color the vertices a, c, e, f, h with color 1, and the remaining two vertices with colors 2 and 3. Clearly, this is a 3-packing coloring of G', which means that $\chi_\rho(G') \leq 3$. This completes the proof. □

Next, we present five infinite families of 4-χ_ρ-vertex-critical graphs. Let ab be an edge of a given graph G. The *subdivision of ab k-times* is obtained by removing the edge ab from G, and adding k new vertices, $a_1, a_2, \ldots, a_k$, and $k+1$ new edges, $aa_1, a_1a_2, \ldots, a_kb$, to G.

The family $\mathcal{F}_1$ contains all graphs that can be constructed from the disjoint union of two copies of K_3 as follows. First, we make an edge joining both copies of K_3 and then, subdivide this edge $(4k)$-, $(4k+1)$- or $(4k+2)$-times, where $k \geq 0$. This family of graphs is shown in Fig. 8a.

Further, let A be a copy of the complete graph K_3 and let B be a copy of the path P_3. The family $\mathcal{F}_2$ (see Fig. 8b) contains all graphs that can be formed from the disjoint union of A and B by first joining a vertex $u \in V(A)$ with some non-pendant vertex $v \in V(B)$, and then subdividing the edge uv $(4k+2)$-times, where k is an arbitrary non-negative integer. In addition, also the graph with no subdivided edge uv belongs to $\mathcal{F}_2$.

The family $\mathcal{F}_3$ contains all graphs that can be obtained from the disjoint union of graphs A, where $A \in \{P_4, C_4\}$, and B, where B is isomorphic to K_3, by first joining a non-pendant vertex $u \in V(A)$ with some vertex $v \in V(B)$, and then subdividing the edge uv $(4k)$-times, where k is an arbitrary positive integer (see Fig. 8c).

The graphs from $\mathcal{F}_4$ are formed from the disjoint union of two copies of a path P_3, denoted by A and B, and one copy of a path P_2, denoted by C, in the following way. First, we join a non-pendant vertex $u \in V(A)$ with a vertex $v \in V(C)$, and a vertex $v \in V(C)$ with a non-pendant vertex $z \in V(B)$. Then, we subdivide the edge uv $(4k+2)$-times and the edge vz $(4k')$-times, where k and k' are two arbitrary non-negative integers (see Fig. 8d). The vertices obtained by subdivision (if exist) are labeled by $y_1, y_2, \ldots, y_l$ and $w_1, w_2, \ldots, w_{l'}$ as in Fig. 8d. In addition, a graph from $\mathcal{F}_4$ can also contain one edge from $\{v_1y_{l-1}, v_1w_2\}$.

Finally, the family $\mathcal{F}_5$ contains all graphs, which can be obtained from the disjoint union of graphs A and B, where $A, B \in \{P_4, C_4\}$, by first joining some non-pendant vertex $u \in V(A)$ with some non-pendant vertex $v \in V(B)$, and then subdividing the edge uv $(2k)$-times, where k is an arbitrary non-negative integer (see Fig. 8e).

Theorem 3.4. *Let $G \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5$. Then, G is a 4-χ_ρ-vertex-critical graph.*

Proof. Let $G \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5$ be an arbitrary graph with the vertices labeled as in Fig. 8. We claim that G is 4-χ_ρ-vertex-critical.

First, let $G \in \mathcal{F}_1$. Suppose that $\chi_\rho(G) = 3$ (clearly, $\chi_\rho(G) \geq 3$, since G contains a subgraph isomorphic to K_3). Then, $G \in \mathcal{G}_3$, which implies that $V(G)$ can be partitioned into the sets $V_0, V_1, \ldots, V_7$, defined in Section 2. Since $\deg(u) = \deg(v) = 3$, u, v belong to a subgraph of G isomorphic to K_3 and have neighbors, which are neither leaves nor belong to some triangle, we infer that $u, v \in V_3$. The fact that $u \in V_3$ implies that $x_1 \in V_2, x_2 \in V_1, x_3 \in V_2, x_4 \in V_3, \ldots$ Recall that any two vertices from V_3 are at distance $4a, a \geq 1$. Therefore, if $l \leq 2$, we have a contradiction, since $d(u, v) \leq 3$ and $u, v \in V_3$. Thus, $l \geq 3$, but then there exists $c \in \{x_{l-2}, x_{l-1}, x_l\}$, which belongs to V_3. Since $d(c, v) \leq 3$ and $c, v \in V_3$, a contradiction. Hence, $G \notin \mathcal{G}_3$. In order to prove that $\chi_\rho(G) \leq 4$, we form a 4-packing coloring c_1 of G. Let $c_1(u) = 3$, $c_1(u_1) = c_1(v_1) = 1$, $c_1(u_2) = 2$, $c_1(v) = 4$. Further, color the vertices $x_1, x_2, \ldots, x_l$ one after another using the following pattern of colors: 1, 2, 1, 3. Finally, let $c_1(v_2) = 2$ if $l = 4k$ or $l = 4k+1$ for any $k \geq 0$, and if $l = 4k+2, k \geq 0$, then let $c_1(v_2) = 3$. Since c_1 is a 4-packing coloring of G, $\chi_\rho(G) = 4$. Now, let $a \in V(G)$ be an arbitrary vertex. Note that $G - a$ is either $X_n, n \geq 2$, or the disjoint union of two graphs from $\{K_2, K_3\} \cup \{X_m : m \geq 1\}$. Using Lemma 2.1 and the facts that $\chi_\rho(K_2) = 2$, $\chi_\rho(K_3) = 3$, we infer that $\chi_\rho(G - a) \leq 3$. Thus, G is 4-χ_ρ-vertex-critical.

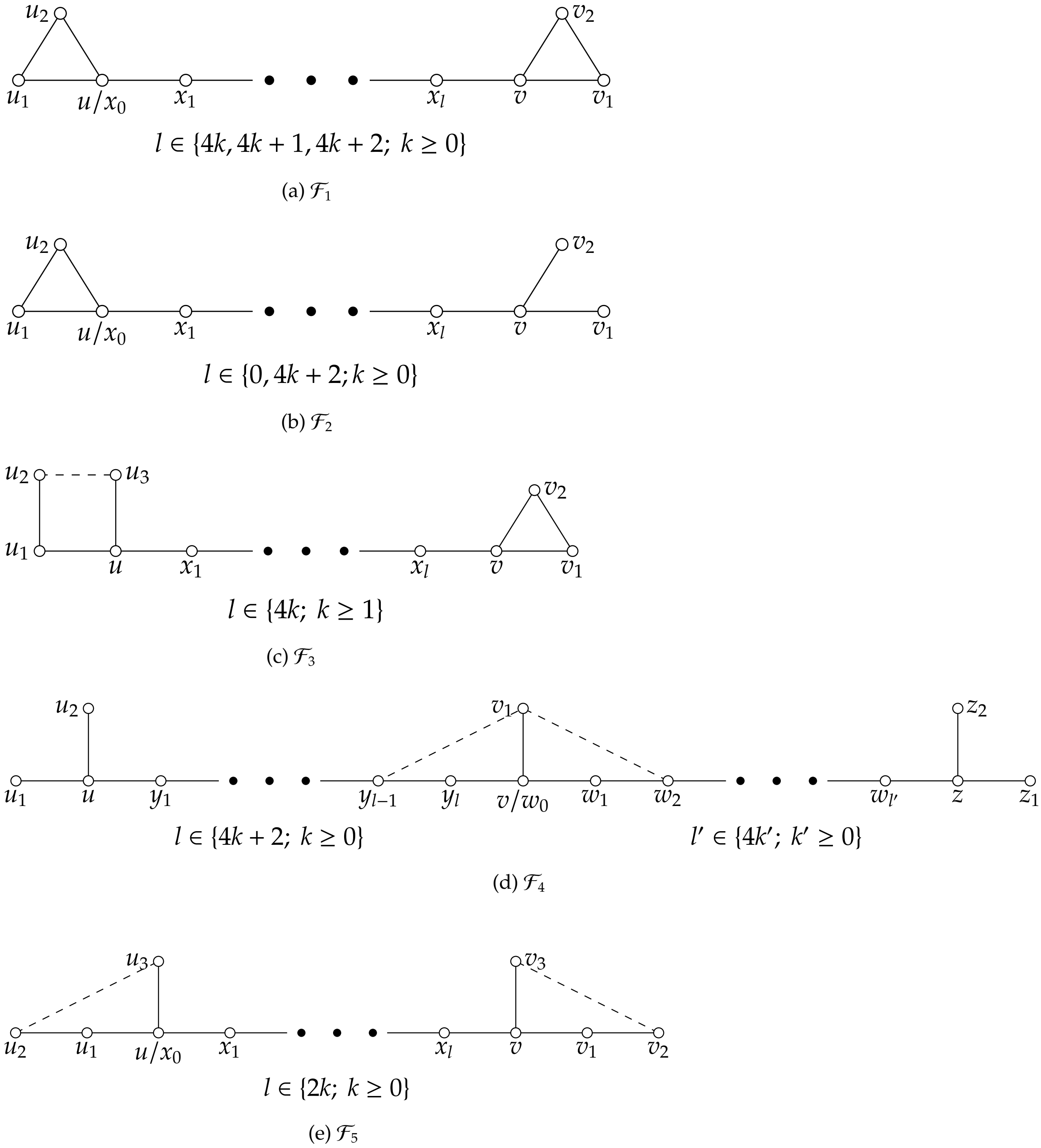


Figure 8: Families of 4-χ_ρ-vertex-critical graphs

Next, suppose that $G \in \mathcal{F}_2$. We claim that $\chi_\rho(G) = 4$. Note that G is isomorphic to a subgraph of some graph H from $\mathcal{F}_1$. We have already proved that $\chi_\rho(H) = 4$, which implies that $\chi_\rho(G) \leq 4$. Therefore, we need to prove that $\chi_\rho(G) \geq 4$. Suppose to the contrary that $\chi_\rho(G) = 3$ (clearly, $\chi_\rho(G) \geq 3$, since G contains a subgraph isomorphic to K_3). Then, $G \in \mathcal{G}_3$, which means that $V(G)$ can be partitioned into the sets $V_0, V_1, \ldots, V_7$ defined above. Since $\deg(u) = 3$, u belongs to a subgraph of G isomorphic to K_3 and has neighbors, which are neither leaves nor belong to some triangle, we derive that $u \in V_3$. If $l \neq 0$, then

$x_1 \in V_2, x_2 \in V_1, x_3 \in V_2, x_4 \in V_3, \ldots, x_l \in V_1$ and consequently, $v \in V_0 \cup V_2$, a contradiction since v has degree 3. If $l = 0$, then $u_1, u_2 \in V_5$, which implies that $v \in V_2$, again a contradiction since $\deg(v) = 3$. Therefore, in both cases, $G \notin \mathcal{G}_3$, which implies that $\chi_\rho(G) = 4$. Now, we need to prove that $\chi_\rho(G - a) \le 3$ holds for any $a \in V(G)$. Let $a \in V(G)$ be an arbitrary vertex. We observe that $G - a$ is a subgraph of a graph $H - b$, where $H \in \mathcal{F}_1$ and $b \in V(H)$. We have already proved $\chi_\rho(H - b) \le 3$, which implies that $\chi_\rho(G - a) \le 3$. Thus, G is a 4-χ_ρ-vertex-critical graph.

Now, let $G \in \mathcal{F}_3$. We claim that $\chi_\rho(G) = 4$. It is clear that $\chi_\rho(G) \ge 3$, since G contains a subgraph isomorphic to K_3. Suppose that $\chi_\rho(G) = 3$, which implies that $G \in \mathcal{G}_3$ and hence, $V(G)$ can be partitioned into the sets $V_0, V_1, \ldots, V_7$ defined above. Since $\deg(v) = 3$, v belongs to a subgraph of G isomorphic to K_3 and has a neighbor, which is neither leaf nor belongs to some triangle, we conclude that $v \in V_3$. This implies that $x_l \in V_2, x_{l-1} \in V_1, x_{l-2} \in V_2, x_{l-3} \in V_3, \ldots, x_1 \in V_3$. Then, $u \in V_6$ since $\deg(u) = 3$ and u does not belong to any triangle. Further, since u_1, u_3 do not belong to any subgraph of G isomorphic to K_3, they belong to V_7, a contradiction, since u_1 is not a leaf (recall that V_7 contains only the leaves). Thus, $\chi_\rho(G) \ge 4$. Next, note that $G - u_3$ is isomorphic to X_n, $n \ge 1$. Lemma 2.1 implies that there exists a 3-packing coloring c_3' of $G - u_3$. Let $c_3(a) = c_3'(a)$ for any $a \in V(G) \setminus \{u_3\}$ and let $c_3(u_3) = 4$. Clearly, c_3 is a 4-packing coloring of G, thus $\chi_\rho(G) = 4$. Now, we prove that $\chi_\rho(G - a) \le 3$ holds for any $a \in V(G)$. If $a = u_3$, then $G - a$ is isomorphic to X_n, $n \ge 1$. Next, if $a \in \{v_1, v_2\}$, then $G - a$ is isomorphic to (a subgraph of) $Y_n, n \ge 1$. In the case when $a \in V(G) \setminus \{u_2, u_3, v_1, v_2\}$, $G - a$ is isomorphic to the disjoint union of (a subgraph of) X_n, $n \ge 1$, and (a subgraph of) Y_n, $n \ge 1$. In each case, Lemma 2.1 implies that $\chi_\rho(G - a) \le 3$. If $a = u_2$, then we form a 3-packing coloring c_3'' of $G - u_2$ as follows. Let $c_3''(u_1) = c_3''(u_3) = c_3''(v_1) = 1$, $c_3''(u) = c_3''(v_2) = 2$, $c_3''(v) = 3$ and color the vertices $x_1, x_2, \ldots, x_l$ one after another using the following pattern of colors: $3, 1, 2, 1$. Since c_3'' is a 3-packing coloring of $G - u_2$, $\chi_\rho(G - u_2) \le 3$. Thus, G is 4-χ_ρ-vertex-critical.

Further, let $G \in \mathcal{F}_4$. First, we prove that $\chi_\rho(G) = 4$. Since G contains a subgraph isomorphic to P_4, $\chi_\rho(G) \ge 3$. Suppose that $\chi_\rho(G) = 3$, which means that $G \in \mathcal{G}_3$. Then, we can partition $V(G)$ into the sets $V_0, V_1, \ldots, V_7$, described above. Since $\deg(v) = 3$ and v has two neighbors, which are neither leaves nor belong to some triangle, $v \in V_1 \cup V_3$. Then, $y_l \in V_2, y_{l-1} \in V_1 \cup V_3, y_{l-2} \in V_2, y_{l-3} \in V_1 \cup V_3, \ldots, y_1 \in V_1 \cup V_3$. Further, since $\deg(u) = 3$, $u \in V_6$ and it follows that $v \in V_1$. Hence, $w_1 \in V_2$, $w_2 \in V_3$, $w_3 \in V_2$, $w_4 \in V_1, \ldots, w_{l'} \in V_1$. This means that $z \in V_0 \cup V_2$, a contradiction since $\deg(z) = 3$. Thus, $G \notin \mathcal{G}_3$, which implies that $\chi_\rho(G) \ge 4$. Now, let $c_4 : V(G) \longrightarrow \{1, 2, 3, 4\}$ and let $c_4(u_1) = c_4(u_2) = c_4(v_1) = c_4(z_1) = c_4(z_2) = 1$, $c_4(u) = c_4(v) = 2$, and $c_4(z) = 4$. Further, color the vertices $y_1, y_2, \ldots, y_l$ one after another using the pattern of colors $3, 1, 2, 1$, and the vertices $w_1, w_2, \ldots, w_{l'}$ one after another using the pattern of colors $1, 3, 1, 2$. Clearly, c_4 is a 4-packing coloring of G. Thus, $\chi_\rho(G) = 4$. Next, we claim that $\chi_\rho(G - a) \le 3$ holds for any $a \in V(G)$. Denote by G' a subgraph of G induced by $V(G) \setminus \{z_2\}$. Let $c_4'(u_1) = c_4'(u_2) = c_4'(v_1) = c_4'(z) = 1$, $c_4'(u) = 2$, $c_4'(z_1) = 3$. Then, color the vertices $y_1, y_2, \ldots, y_l, v, w_1, w_2, \ldots, w_{l'}$ one after another using the following pattern of colors: $3, 1, 2, 1$. Clearly, c_4' is a 3-packing coloring of G', thus $\chi_\rho(G - z_2) \le 3$. Analogously we prove that $\chi_\rho(G - z_1) \le 3$. Let G'' be a subgraph of G induced by $V(G) \setminus \{u_2\}$. In order to prove that $\chi_\rho(G'') \le 3$, we form a 3-packing coloring c_4'' of G''. Let $c_4''(u) = c_4''(v_1) = c_4''(z_1) = c_4''(z_2) = 1$, $c_4''(z) = 2$, $c_4''(u_1) = 3$. Then, color the vertices $y_1, y_2, \ldots, y_l, v, w_1, w_2, \ldots, w_{l'}$ one after another using the following pattern of colors: $2, 1, 3, 1$. Clearly, c_4'' is a 3-packing coloring of G'', thus $\chi_\rho(G - u_2) \le 3$. Analogously we prove that $\chi_\rho(G - u_1) \le 3$. If $a \in \{u, y_1, y_2, \ldots, y_{l-1}, w_2, \ldots, w_{l'}, z, v\}$, then $G - a$ is the disjoint union of a subgraph of G' and a subgraph of G''. Thus, the above considerations imply that $\chi_\rho(G - a) \le 3$. In the case when $a = v_1$, color the vertices of $G - v_1$ as follows. Let u_1, u_2, z_1, z_2 receive color 1, u and z color 3, and let the vertices $y_1, y_2, \ldots, y_l, v, w_1, w_2, \ldots, w_{l'}$ be colored one after another using the pattern $1, 2, 1, 3$. The described coloring is a 3-packing coloring of $G - v_1$, thus $\chi_\rho(G - v_1) \le 3$. Finally, if $a \in \{y_l, w_1\}$, then $G - a$ is isomorphic to $G - v_1$, or to the disjoint union of a subgraph of G' and a subgraph of G''. Hence, $\chi_\rho(G - a) \le 3$ and we conclude that G is 4-χ_ρ-vertex-critical.

It remains to consider the case when $G \in \mathcal{F}_5$. First, we prove that $\chi_\rho(G) = 4$. It is clear that $\chi_\rho(G) \ge 3$ since G contains a subgraph isomorphic to P_4. Suppose that $G \in \mathcal{G}_3$. Then, we make a partition of $V(G)$ into the sets $V_0, V_1, \ldots, V_7$, which are described above. Since $\deg(u), \deg(v) \ge 3$ and $d(u, v) = 2k + 1$, it follows that u or v belongs to V_6. But this is a contradiction, since u (respectively, v) has at least two neighbors, which are neither leaves nor belong to a triangle. Thus, $\chi_\rho(G) \ge 4$. Let $c_5 : V(G) \longrightarrow \{1, 2, 3, 4\}$ and let $c_5(u_1) = c_5(u_3) = c_5(v_1) = c_5(v_3) = 1$, $c_5(u_2) = c_5(v_2) = 2$, $c_5(u) = 3$ and $c_5(v) = 4$. Further, color the vertices

$x_1, x_2, \dots, x_l$ one after another using the pattern of colors: $1, 2, 1, 3$. Clearly, c_5 is a 4-packing coloring of G and thus, $\chi_\rho(G) = 4$. Next, we prove that $\chi_\rho(G - a) \le 3$ holds for any $a \in V(G)$. If $a \in V(G) \setminus \{u_2, v_2\}$, then $G - a$ is isomorphic to (a subgraph of) Y_n, $n \ge 1$, or it is the disjoint union of graphs isomorphic to (subgraphs of) Y_n, $n \ge 1$. In each case, Lemma 2.1 implies that $\chi_\rho(G - a) \le 3$. If $a = v_2$ and $l = 4k', k' \ge 0$, then we form a 3-packing coloring c_5' of $G - v_2$ as follows: $c_5'(a) = c_5(a)$ for any $a \in V(G) \setminus \{v, v_2\}$ and $c_5'(v) = 2$. In the case when $a = v_2$ and $l = 4k' + 2, k' \ge 0$, let $c_5''(u_1) = c_5''(u_3) = c_5''(v_1) = c_5''(v_3) = 1$, $c_5''(u_2) = 3$, $c_5''(u) = c_5''(v) = 2$, and color the vertices $x_1, x_2, \dots, x_l$ one after another using the pattern of colors: $1, 3, 1, 2$. These colorings imply that $\chi_\rho(G - v_2) \le 3$. Analogously one can prove that $\chi_\rho(G - u_2) \le 3$. Hence, G is a 4-χ_ρ-vertex-critical graph. This completes the proof. □

The following two lemmas will help us to prove a complete characterization of 4-χ_ρ-vertex-critical graphs.

Lemma 3.5. *Let H be obtained by attaching a single leaf to two vertices of C_n, $n \ge 4$, which are at distance $2k + 1$, $k \ge 0$. Then, $\chi_\rho(H) \ge 4$.*

Proof. Let H be obtained by attaching a single leaf to two vertices of C_n, $n \ge 4$, which are at distance $2k + 1$, $k \ge 0$. Clearly, if $n \ne 4a, a \ge 1$, then H contains a subgraph H' isomorphic to a graph from $\mathcal{C}$. Since $\chi_\rho(H') = 4$, the hereditary property of the packing chromatic number implies that $\chi_\rho(H) \ge 4$. Therefore, we only need to consider the case when $n = 4a, a \ge 1$. If $a \ge 2$, then H contains a subgraph isomorphic to a graph from $\mathcal{F}_5$. In this case, the hereditary property of the packing chromatic number and Theorem 3.4 imply that $\chi_\rho(H) \ge 4$. If $a = 1$, then H is isomorphic to H_5 and by Theorem 3.3, $\chi_\rho(H) = 4$. □

Lemma 3.6. *Let $\mathcal{X} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5 \cup \mathcal{C}_5 \cup \mathcal{C}_6 \cup \mathcal{C} \cup \{K_4, H_1, H_2, H_3, H_4, H_5, H_6, H_7, H_8, H_9\}$ and let $H \in \mathcal{X}$. If a graph $G \notin \mathcal{X}$ and it contains a subgraph isomorphic to H, then G is not 4-χ_ρ-vertex-critical.*

Proof. Let $H \in \mathcal{X} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5 \cup \mathcal{C}_5 \cup \mathcal{C}_6 \cup \mathcal{C} \cup \{K_4, H_1, H_2, H_3, H_4, H_5, H_6, H_7, H_8, H_9\}$ and let $G \notin \mathcal{X}$ be an arbitrary graph containing a subgraph isomorphic to H. Clearly, since $G \notin \mathcal{X}$, but $H \in \mathcal{X}$, $G \ncong H$.

First, suppose that $V(G) \ne V(H)$, which means that there exists a vertex $a \in V(G) \setminus V(H)$. Using the fact that $G - a$ contains a subgraph isomorphic to H, the hereditary property of the packing chromatic number and Theorems 3.1, 3.3, 3.4, we derive that $\chi_\rho(G - a) \ge 4$. It follows that G is not a 4-χ_ρ-vertex-critical graph.

Now, let $V(G) = V(H)$. If $H \in \mathcal{C}_5 \cup \mathcal{C}_6 \cup \mathcal{C}$, then Theorem 3.1 implies that G is not 4-χ_ρ-vertex-critical (recall that $G \notin \mathcal{C}_5 \cup \mathcal{C}_6 \cup \mathcal{C}$). Further, if $H \cong K_4$, then $G \cong H$, a contradiction to the fact that $G \ncong H$.

Clearly, since $G \ncong H$ and $V(G) = V(H)$, $E(H) \ne E(G)$. Then, there exists $e' \in E(G) \setminus E(H)$.

Therefore, if $H \cong H_1$ (see Fig. 7b), then at least one of the edges be, ca, ce, cd belongs to $E(G)$. This implies that G contains a subgraph isomorphic to a graph from $\mathcal{C}$ or it contains a subgraph isomorphic to K_4. The above considerations imply that G is not a 4-χ_ρ-vertex-critical graph. If $H \cong H_2$ (see Fig. 7c), then G contains a subgraph isomorphic to a graph from $\mathcal{C}$ and by the above reasoning, it is not 4-χ_ρ-vertex-critical. In the case when $H \cong H_3$ or $H \cong H_6$ (see Fig. 7d, 7g), G contains a subgraph isomorphic to a graph from $\mathcal{C}$ or a subgraph isomorphic to H_1. In both cases, the above considerations imply that G is not 4-χ_ρ-vertex-critical. Now, let $H \cong H_4$ (see Fig. 7e). In this case, G contains a subgraph isomorphic to a graph from $\mathcal{C} \cup \{H_1, H_2\}$. Again, the above considerations imply that G is not 4-χ_ρ-vertex-critical. If $H \cong H_5$ (see Fig. 7f), then G contains a subgraph isomorphic to a graph from $\mathcal{C} \cup \{H_1\}$ or $G \cong H_6$. In the first case, the above considerations imply that G is not 4-χ_ρ-vertex-critical, and in the second case we have a contradiction to $G \notin \mathcal{X}$. Next, let $H \cong H_7$ (see Fig. 7h). If $e' \in E(G) \setminus \{ad, ai, cf, fh\}$, then G contains a subgraph isomorphic to a graph from $\mathcal{C}$ and the above consideration implies that G is not 4-χ_ρ-vertex-critical. Further, if ad or cf belongs to $E(G)$, then G contains a non-spanning subgraph isomorphic to a graph from $\mathcal{F}_5$ (the vertices from $V(G)$ corresponding to the vertices of degree 3 in a graph from $\mathcal{F}_5$ are d, e or b, c). By Theorem 3.4, G is not 4-χ_ρ-vertex-critical. Next, if at least one of the edges from $\{ai, fh\}$ belongs to $E(G)$, then $G \cong H_8$ or G contains a non-spanning subgraph isomorphic to a graph from $\mathcal{F}_5$. In the first case, we have a contradiction to our assumption, and in the second case we derive that G is not 4-χ_ρ-vertex-critical. If $H \cong H_8$ (see Fig. 7i), then we analogously prove that G is not 4-χ_ρ-vertex-critical (note that H_7 is a subgraph of H_8). Futher, consider the case when $H \cong H_9$ (see Fig. 7j). It is easy to observe that G contains a non-spanning subgraph isomorphic to a graph from $\{H_3, H_5, C_5\} \cup \mathcal{F}_2$. Hence, using the fact that $\chi_\rho(C_5) = 4$ and Theorems 3.3, 3.4, we deduce that G is not 4-χ_ρ-vertex-critical.

Next, suppose that $H \in \mathcal{F}_1$ (see Fig. 8a). If $e' = u_1a$, $a \in \{v, v_1, v_2\} \cup \{x_1, x_2, \ldots, x_l\}$, then G contains a (non-spanning) subgraph isomorphic to H_1 or a (non-spanning) subgraph isomorphic to a graph from $\mathcal{C}$ (G contains a subgraph obtained by attaching a vertex to two adjacent vertices of a cycle of order at least 4, hence, by Lemma 2.4, G also contains a subgraph isomorphic to a graph from $\mathcal{C}$). In both cases, the above consideration implies that G is not a 4-χ_ρ-vertex-critical graph, if $\deg(u_1) \geq 3$. Analogously we prove that G is not 4-χ_ρ-vertex-critical, if $\deg(u_2), \deg(v_1), \deg(v_2) \geq 3$. Now, let $e' = ab$, $a, b \in \{u, v\} \cup \{x_1, x_2, \ldots, x_l\}$. Then, G contains a (non-spanning) subgraph isomorphic to H_2, a subgraph isomorphic to H_4 or a non-spanning subgraph obtained by attaching a single leaf to two adjacent vertices of a cycle of order at least 4. In each case, the above reasoning and Lemma 3.5 imply that $\chi_\rho(G - v_2) \geq 4$. Consequently, G is not 4-χ_ρ-vertex-critical.

Now, let $H \in \mathcal{F}_2$ (see Fig. 8b). If $e' = u_1a$ (or $e' = u_2a$), $a \in \{v, v_1, v_2\} \cup \{x_1, x_2, \ldots, x_l\}$, then analogously as in the case when $H \in \mathcal{F}_1$, we prove that G is not 4-χ_ρ-vertex-critical, if $\deg(u_1) \geq 3$ or $\deg(u_2) \geq 3$. If $e' = ab$, $a, b \in \{u\} \cup \{x_1, x_2, \ldots, x_l\}$, then again, analogously as in the case when $H \in \mathcal{F}_1$, we prove that G is not 4-χ_ρ-vertex-critical. Next, $e' = uv$ implies that G contains a non-spanning subgraph obtained by attaching a single leaf to two adjacent vertices of $C_n, n \geq 4$. Hence, Lemma 3.5 implies that G is not 4-χ_ρ-vertex-critical. If $e' = uv_1$ or $e' = uv_2$, then G contains a subgraph isomorphic to a graph from $\{H_2\} \cup \mathcal{C}$, hence the above reasoning yields that G is not 4-χ_ρ-vertex-critical. Now, let $e' = vx_i$, $i \in \{1, 2, \ldots, l\}$. Then, G contains a non-spanning subgraph isomorphic to a graph from $\mathcal{F}_1$ or a non-spanning subgraph obtained by attaching a single leaf to two adjacent vertices of $C_n, n \geq 4$. Hence, Theorem 3.4 and Lemma 3.5 imply that G is not a 4-χ_ρ-vertex-critical graph. If $e' = v_1v_2$, then $G \in \mathcal{F}_1$ or it contains a subgraph isomorphic to a graph from $\mathcal{F}_1$. We have a contradiction to our assumption or the above consideration yields that G is not 4-χ_ρ-vertex-critical. Finally, let $e' = v_1x_i$ or $e' = v_2x_i$, $i \in \{1, 2, \ldots, l\}$. Then, G contains a non-spanning subgraph isomorphic to a graph from $\mathcal{C} \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$. The above consideration and Theorem 3.4 imply that G is not 4-χ_ρ-vertex-critical.

Next, let $H \in \mathcal{F}_3$ (see Fig. 8c). If $e' \in \{v_1a, v_2a, bc\}$, where $a \in \{u, u_1, u_2, u_3\} \cup \{x_1, x_2, \ldots, x_l\}$, $b, c \in \{u, v\} \cup \{x_1, x_2, \ldots, x_l\}$, then analogously as in the case when $H \in \mathcal{F}_1$, we prove that G is not 4-χ_ρ-vertex-critical. Further, if $e' = u_1v$ or $e' = u_3v$, then G contains a non-spanning subgraph isomorphic to a graph from $\mathcal{C}$, so it is not 4-χ_ρ-vertex-critical. Also in the case when $e' = vu_2$, G is not 4-χ_ρ-vertex-critical. Namely, G contains a non-spanning subgraph obtained by attaching a single leaf to two vertices of $C_n, n \geq 4$, which are at distance 3, and Lemma 3.5 implies the result. Next, we observe that $e' = uu_2$ or $e' = u_1u_3$ imply that G contains a non-spanning subgraph isomorphic to a graph from $\mathcal{F}_1$. Hence, by Theorem 3.4, G is not a 4-χ_ρ-vertex-critical graph. Now, suppose that $e' = u_1x_i$, $i \in \{1, 2, \ldots, l\}$. Then, G contains a non-spanning subgraph isomorphic to H_3 or a non-spanning subgraph obtained by attaching a single leaf to two adjacent vertices of a cycle of order at least 4. By Theorem 3.3 and Lemma 3.5, G is not 4-χ_ρ-vertex-critical. If $e' = u_2x_i$, $i \in \{1, 2, \ldots, l\}$, then G contains a non-spanning subgraph isomorphic to a graph from $\{H_5\} \cup \mathcal{C}$ or a non-spanning subgraph obtained by attaching a single leaf to two vertices of $C_n, n \geq 4$, which are at distance 3. Hence, by Theorems 3.1 and 3.3, and Lemma 3.5, G is not 4-χ_ρ-vertex-critical. Finally, if $e' = u_3x_i$, $i \in \{1, 2, \ldots, l\}$, then G contains a subgraph isomorphic to a graph from $\{H_4\} \cup \mathcal{C} \cup \mathcal{F}_2$. From the above consideration, we derive that G is not a 4-χ_ρ-vertex-critical graph.

Further, suppose that $H \in \mathcal{F}_4$ with the vertices labeled as in Fig. 8d. If $e' \in \{u_1y_1, u_2y_1, u_1u_2, v_1w_1, z_1w_{l'}, z_2w_{l'}\}$, then G contains a subgraph isomorphic to a graph from $\mathcal{F}_2$. By the previous consideration, G is not 4-χ_ρ-vertex-critical. Further, suppose that $e' \in \{u_1v, u_1v_1, u_1z, u_1z_1, u_1z_2, u_2v, u_2v_1, u_2z, u_2z_1, u_2z_2, uv_1, uz, uz_1, uz_2, vz, vz_1, vz_2, v_1z\}$. In this case, G contains a subgraph isomorphic to a cycle from $\mathcal{C}$. Thus, by the above reasoning, G is not 4-χ_ρ-vertex-critical. If $e' \in \{v_1z_1, v_1z_2, uv\} \cup \{uw_{i'} : 1 \leq i' \leq l'\}$, then G contains a non-spanning subgraph obtained by attaching a single leaf to two vertices of $C_n, n \geq 4$, which are at distance $2k+1, k \geq 0$. Lemma 3.5 implies that G is not a 4-χ_ρ-vertex-critical graph. Next, let $e' = ab$, $a, b \in \{v, y_i, w_{i'} : 1 \leq i \leq l, 1 \leq i' \leq l'\}$. In this case, G contains a non-spanning subgraph isomorphic to H_4 or a non-spanning subgraph obtained by attaching a single leaf to two adjacent vertices of $C_n, n \geq 4$. Using Theorem 3.3 and Lemma 3.5, we derive that G is not 4-χ_ρ-vertex-critical. Further, suppose that $e' \in \{u_1y_i, u_2y_i, z_1w_{i'}, z_2w_{i'}, u_1w_{j'}, u_2w_{j'}, z_1y_j, z_2y_j : 2 \leq i \leq l, 1 \leq i' \leq l'-1, 1 \leq j' \leq l', 1 \leq j \leq l\}$. Then, G contains a non-spanning subgraph isomorphic to a graph from $\mathcal{F}_5 \cup \mathcal{C}$ or a non-spanning subgraph obtained by attaching a single leaf to two adjacent vertices of $C_n, n \geq 4$ (if $l' = 0$). Hence, by the above consideration, Theorem 3.4 and Lemma 3.5, we infer that G is not 4-χ_ρ-vertex-critical. Now, let $e' = z_1z_2$, $e' = uy_i$ or $e' = zw_{i'}$, where $1 \leq i \leq l$, $1 \leq i' \leq l'$. This implies that G contains a subgraph isomorphic to a graph from $\mathcal{F}_2 \cup \mathcal{F}_3$ or a non-spanning subgraph obtained by attaching a single leaf to two adjacent vertices of $C_n, n \geq 4$. Therefore, the above reasoning and Lemma 3.5 imply that G is not 4-χ_ρ-vertex-critical. If $e' = zy_i$, $1 \leq i \leq l$, then G contains a non-spanning subgraph isomorphic to a graph H_3 or a non-spanning subgraph obtained by attaching a single leaf to two adjacent vertices of $C_n, n \geq 4$. In both cases, we derive that G is not 4-χ_ρ-vertex-critical. Finally, let $e' = v_1y_i$ or $e = v_1w_{i'}$, $i \in \{1, 2, \dots, l-2, l\}$, $i' \in \{3, 4, \dots, l'\}$. Then, G contains a non-spanning subgraph isomorphic to a graph from $\{H_4\} \cup \mathcal{C} \cup \mathcal{F}_4$. The above consideration implies that G is not 4-χ_ρ-vertex-critical.

Finally, let $H \in \mathcal{F}_5$ (see Fig. 8e). If $e' = ab, a, b \in \{u, u_1, v, v_1\} \cup \{x_1, x_2, \dots, x_l\}$, then G contains a non-spanning subgraph isomorphic to a graph from $\{H_3, H_4\}$, or a non-spanning subgraph obtained by attaching a single leaf to two adjacent vertices of $C_n, n \geq 4$. By Theorem 3.3 and Lemma 3.5, G is not 4-χ_ρ-vertex-critical. If $e' \in \{uu_2, u_1u_3, vv_2, v_1v_3\}$, then G contains a subgraph isomorphic to a graph from $\mathcal{F}_2 \cup \mathcal{F}_3$. By the previous consideration, G is not 4-χ_ρ-vertex-critical. Now, suppose that $e' = u_3x_i$ or $e' = v_3x_i$, where $i \in \{1, 2, \dots, l\}$. In this case, G contains a non-spanning subgraph isomorphic to a graph from $\{H_4\} \cup \mathcal{C} \cup \mathcal{F}_5$. Then, Theorems 3.1, 3.3 and 3.4 imply that G is not 4-χ_ρ-vertex-critical. If $e' \in E(G) \setminus \{ab, uu_2, u_1u_3, vv_2, v_1v_3, u_3x_i, v_3x_i : 1 \leq i \leq l \ \wedge a, b \in \{u, u_1, v, v_1\} \cup \{x_1, x_2, \dots, x_l\}\}$, then G contains a subgraph isomorphic to H_7 (note that $G \not\cong H_7$), a non-spanning subgraph isomorphic to a graph from $\mathcal{C} \cup \mathcal{F}_1$ or a non-spanning subgraph obtained by attaching a single leaf to two vertices of $C_n, n \geq 4$, which are at distance $2k+1, k \geq 0$. Using the above consideration, Theorems 3.1, 3.4 and Lemma 3.5, we derive that G is not a 4-χ_ρ-vertex-critical graph. This completes the proof. □

Theorem 3.7. *A graph G is 4-χ_ρ-vertex-critical if and only if $G \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5 \cup \mathcal{C}_5 \cup \mathcal{C}_6 \cup \mathcal{C} \cup \{K_4, H_1, H_2, \dots, H_8, H_9\}$.*

Proof. Let $\mathcal{X} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5 \cup \mathcal{C}_5 \cup \mathcal{C}_6 \cup \mathcal{C} \cup \{K_4, H_1, H_2, \dots, H_8, H_9\}$. If $G \in \mathcal{X}$, then Theorems 3.1, 3.3 and 3.4 imply that G is 4-χ_ρ-vertex-critical.

To prove the converse implication, let G be a 4-χ_ρ-vertex-critical graph. The assumption that $G \notin \mathcal{X}$ will lead us to a contradiction. Note that G is connected, and by Lemma 3.6, G does not contain a subgraph isomorphic to a graph from $\mathcal{X}$.

In the sequel of this proof, we distinguish two cases.

Case 1. G contains a subgraph isomorphic to T, which is shown in Fig. 2.

Case 1.1. T is a spanning subgraph of G ($V(T) = V(G)$).

Let the vertices of T be denoted as is shown in Fig. 2. Since G is 4-χ_ρ-vertex-critical, Lemma 2.2 implies $G \not\cong T$. Then, G contains at least one of the edges from $\{ab, ac, ad, ae, bc, be, cd, de, dy', ey'\}$. Using the facts that $G \notin \mathcal{X}$ and it does not contain a subgraph isomorphic to a graph from $\mathcal{X}$, we infer that $bc, be, cd, de \notin E(G)$. Next, it is easy to observe that $\chi_\rho(T+f) = 3$ for any $f \in \{ab, ac, ad, ae, dy', ey'\}$. This means that G contains at least two edges from $\{ab, ac, ad, ae, dy', ey'\}$. First, consider the case when exactly two of the mentioned edges belong to $E(G)$. If $\{ab, dy'\} \subset E(G)$, $\{ac, ey'\} \subset E(G)$, $\{ad, dy'\} \subset E(G)$, $\{ad, ey'\} \subset E(G)$, $\{ae, dy'\} \subset E(G)$ or $\{ae, ey'\} \subset E(G)$, then $\chi_\rho(G) = 3$, a contradiction to G being a 4-χ_ρ-vertex-critical graph. Otherwise, G contains a subgraph isomorphic to H_1, H_2 or to a graph from $\mathcal{C}$. This is a contradiction, since G does not contain a subgraph isomorphic to a graph from $\mathcal{X}$. Hence, G contains at least three edges from $\{ab, ac, ad, ae, dy', ey'\}$. By reasoning from the case when G contains exactly two edges from this set, we derive a contradiction (since G does not contain a subgraph isomorphic to a graph from $\mathcal{X}$) for any triple of the chosen edges.

Case 1.2. T is not a spanning subgraph of G ($V(T) \neq V(G)$).

Let $y \in V(G)$ be a vertex belonging to a subgraph of G isomorphic to T, and corresponding to the vertex $y' \in V(T)$. Further, let $N_i = \{u : d(u, y) = i\}$ for any $i \geq 1$ and let $N_0 = \{y\}$. In the sequel of this proof, we present some properties of G, which ensure that $G \in \mathcal{G}_3$ (and this will contradict to our assumption).

First, consider the vertices from $\bigcup_{j\equiv 1,3 \pmod 4} N_j$ with degree at least 3. Let $A = \{x : \deg(x) \geq 3, x \in N_j, j \equiv 1,3 \pmod 4\}$ and let $x \in A$ ($x \in N_j$) be an arbitrary vertex.

We claim that x has exactly one neighbor from N_{j-1}. Suppose to the contrary that x is adjacent to distinct vertices $x_1, x_2 \in N_{j-1}$ (note that in this case $j \geq 3$). Then, there exist $x'_1, x'_2 \in N_{j-2}$ such that $x_1x'_1, x_2x'_2 \in E(G)$. If $x'_1 = x'_2$, then G contains a subgraph isomorphic to a graph from $\mathcal{F}_5 \cup C$ (the vertices corresponding to the vertices of degree 3 in a graph from $\mathcal{F}_5$ are y and x'_1), a contradiction to our assumption. Thus, $x'_1 \neq x'_2$. If $j \geq 5$, we again derive that G contains a subgraph isomorphic to a graph from $\mathcal{F}_5$ (in this case, the vertices corresponding to the vertices of degree 3 in a graph from $\mathcal{F}_5$ are y and x), which contradicts our assumption. Otherwise, $x'_1y, x'_2y \in E(G)$, which implies that G contains a subgraph isomorphic to a graph from C, again a contradiction to our assumption. Therefore, x has exactly one neighbor from N_{j-1}.

Now, suppose that there exist $x_1, x_2, x_3 \in V(G)$ such that $xx_1, xx_2, xx_3 \in E(G)$, $x_1 \in N_{j-1}$, $x_2 \in N_j$ and $x_3 \in N_j \cup N_{j+1}$. We prove that all vertices from $N_j \cap N(x)$ are adjacent exactly to x and x_1 (have degree 2). Since $x_2 \in N_j$, it has a neighbor from N_{j-1}. If $x_2x_1 \notin E(G)$, then there exists $x'_2 \in N_{j-1}$ such that $x_2x'_2 \in E(G)$. But then, G contains a subgraph isomorphic to a graph from $\mathcal{F}_5$ (in this case the vertices corresponding to the vertices of degree 3 in a graph from $\mathcal{F}_5$ are y or x_1, and x) or a subgraph isomorphic to a graph from C. In both cases, we have a contradiction to our assumption. Consequently, every vertex from N_j adjacent to x is also adjacent to x_1. Moreover, each such vertex x_2 has degree 2. Indeed, if $x_2x_3 \in E(G)$, then G contains a subgraph isomorphic to a graph from $\{H_1, K_4\}$, a contradiction to G being 4-χ_ρ-vertex-critical. If there exists $x''_2 \in V(G)$ such that $x_2x''_2 \in E(G)$ ($x''_2 \neq x_3$), then G contains a subgraph isomorphic to $\{H_1, H_3\}$, again a contradiction to G being 4-χ_ρ-vertex-critical. Hence, each vertex from $N_j \cap N(x)$ has degree 2; it is adjacent to x and x_1.

Next, we claim that each vertex from $N_{j+1} \cap N(x)$ is a leaf. Let $x_1, x_2, x_3 \in V(G)$ such that $xx_1, xx_2, xx_3 \in E(G)$, $x_1 \in N_{j-1}$, $x_2 \in N_j \cup N_{j+1}$ and $x_3 \in N_{j+1}$. If $x_2x_3 \in E(G)$, then G contains a subgraph isomorphic to a graph from $\mathcal{F}_2 \cup \mathcal{F}_3$, a contradiction to our assumption. Further, if there exists $x'_3 \in V(G)$ such that $x_3x'_3 \in E(G)$ ($x'_3 \neq x_2$), then G contains a subgraph isomorphic to a graph from $\{H_3, H_4, H_5, C_5\} \cup \mathcal{F}_5$, again a contradiction to our assumption. Hence, each vertex from $N_{j+1} \cap N(x)$ is a leaf.

In conclusion, each $x \in \bigcup_{j\equiv 1,3 \pmod 4} N_j$ either has degree at most 2 or it belongs to the set A (has degree at least 3). In the second case, $x \in N_j$ and its neighbors satisfy the following properties.

Property 1: Vertex x has exactly one neighbor from N_{j-1}, say x_1.

Property 2: All vertices from $N_j \cap N(x)$ are adjacent exactly to x and x_1 (have degree 2).

Property 3: Each vertex from $N_{j+1} \cap N(x)$ is a leaf.

We observe that these properties imply that $N[x] \setminus \{x_1\}$ induces a T-add attached to x_1. Therefore, each vertex from A belongs to a T-add.

Next, we prove that $A \subseteq \bigcup_{j\equiv 1 \pmod 4} N_j$ or $A \subseteq \bigcup_{j\equiv 3 \pmod 4} N_j$. Clearly, if $|A| \leq 1$, then the statement holds. Hence, consider the case when $|A| \geq 2$. Suppose to the contrary that there exist $x_1, x_2 \in A$ such that $x_1 \in N_{j_1}$, where $j_1 \equiv 1 \pmod 4$, and $x_2 \in N_{j_2}$, where $j_2 \equiv 3 \pmod 4$. Let P_1 be a shortest x_1-y-path, P_2 a shortest x_2-y-path and $w \in P_1 \cap P_2$ such that the distance between y and w is the largest possible. Note that $w \in N_j, j \equiv 0,2 \pmod 4$ and consequently, $w \neq x_1, x_2$. Namely, if $w \in N_j, j \equiv 1,3 \pmod 4$, then $w \in A$ (note that $\deg(w) \geq 3$) and thus, by Property 3, each vertex from $N(w) \cap N_{j+1}$ is a leaf. This is implies that x_1, x_2 are both adjacent to w (or one of them is w) and both belong to N_{j+1} (or one to N_j), a contradiction, since $x_1 \in N_{j_1}, j_1 \equiv 1 \pmod 4$ and $x_2 \in N_{j_2}, j_2 \equiv 3 \pmod 4$. Therefore, without loss of generality we may assume that $w \in N_j, j \equiv 0 \pmod 4$. Then, $d(w, x_1) \equiv 1 \pmod 4$ and $d(w, x_2) \equiv 3 \pmod 4$. This implies that G contains a subgraph isomorphic to a graph from $\mathcal{F}_4$ (the vertices corresponding to some of the vertices of degree 3 in a graph from $\mathcal{F}_4$ are x_1, x_2 and w), a contradiction to our assumption. Thus, $A \subseteq \bigcup_{j\equiv 1 \pmod 4} N_j$ or $A \subseteq \bigcup_{j\equiv 3 \pmod 4} N_j$.

Without loss of generality we may assume that $A \subseteq \bigcup_{j\equiv 3 \pmod 4} N_j$.

We summarize our findings as follows. Since $A \subseteq \bigcup_{j\equiv 3 \pmod 4} N_j$, each vertex from $\bigcup_{j\equiv 1 \pmod 4} N_j$ has degree at most 2. Next, it is possible that G contains T-adds, which are attached to the vertices from $\bigcup_{j\equiv 2 \pmod 4} N_j$. In particular, it is possible that G contains subgraphs isomorphic to K_3, which satisfy the following property: one vertex of each such triangle belongs to N_j and the other two to N_{j+1} for some $j \equiv 2 \pmod 4$ (we have proved that $j + 1 \equiv 3 \pmod 4$).

Now, we check whether G contains a subgraph isomorphic to K_3 of a different type (i.e., triangle, which does not satisfy the property that one of its vertices belongs to N_j and the other two to N_{j+1} for some $j \equiv 2 \pmod 4$).

Let $z, z_1, z_2 \in V(G)$ such that $zz_1, zz_2, z_1z_2 \in E(G)$. We can assume that $z, z_1, z_2 \in N_j \cup N_{j+1}$, $j \geq 0$. Moreover, let $z \in N_j$ and $z_1, z_2 \in N_j \cup N_{j+1}$, $j \geq 0$. Further, let $z' \in V(G)$ such that $zz' \in E(G)$ and $z' \in N_{j-1}$ (or $z' \in N_{j+1}$ if $j = 0$). If $j \equiv 1, 3 \pmod 4$, then $z \in A$ and consequently, it satisfies the Properties 1, 2 and 3. This implies that z_1 (respectively, z_2) is a leaf, or it has degree 2 and is adjacent to z and z'. In both cases, we have a contradiction, since z_1, z_2 do not satisfy the written properties. Therefore, $j \equiv 0, 2 \pmod 4$. Now, we prove that $z_1, z_2 \in N_{j+1}$. Suppose to the contrary that $z_1 \in N_j$. This implies that $j \geq 2$ ($j \neq 0$, because $|N_0| = 1$). Note that z_1 has a neighbor from N_{j-1}. If $z'z_1 \in E(G)$, then G contains a subgraph isomorphic to H_1, a contradiction to G being 4-χ_ρ-vertex-critical. Therefore, there exists $z_1' \in N_{j-1}$ such that $z_1z_1' \in E(G)$. Since $z', z_1' \in N_{j-1}$, there exist $z'', z_1'' \in N_{j-2}$ such that $z_1'z_1'', z'z'' \in E(G)$. If $z_1'' = z''$, then G contains a graph from $\mathcal{C}$ (namely C_5) as a non-spanning subgraph, which contradicts G being 4-χ_ρ-vertex-critical. Otherwise, G contains a subgraph isomorphic to H_4, a contradiction to our assumption. Thus, $z_1, z_2 \in N_{j+1}$.

Therefore, each subgraph of G isomorphic to K_3 has exactly one vertex from N_j and the other two vertices from N_{j+1} for some $j \equiv 0, 2 \pmod 4$.

Let $B = \{z : z \in N_j, j \equiv 0, 2 \pmod 4, z \text{ belongs to a subgraph of } G \text{ isomorphic to } K_3\}$.

We prove that $B \subseteq \bigcup_{j \equiv 0 \pmod 4} N_j$ or $B \subseteq \bigcup_{j \equiv 2 \pmod 4} N_j$. Let $z, z' \in B$ and suppose that $z \in N_{j_1}, j_1 \equiv 0 \pmod 4$ and $z' \in N_{j_2}, j_2 \equiv 2 \pmod 4$. Let P_1' be a shortest z-y-path, P_2' a shortest z'-y-path and $w' \in P_1' \cap P_2'$ such that the distance between y and w' is the largest possible. Note that $w' \in N_j, j \equiv 0, 2 \pmod 4$. Namely, if $w' \in N_j, j \equiv 1, 3 \pmod 4$, then G contains a subgraph isomorphic to a graph from $\mathcal{F}_5$ (the vertices corresponding to the vertices of degree 3 in a graph from $\mathcal{F}_5$ are y and w') or a subgraph isomorphic to H_5. In both cases, we have a contradiction to our assumption. Thus, $w' \in N_j, j \equiv 0, 2 \pmod 4$. But then, G contains a subgraph from $\mathcal{F}_1$ and Lemma 3.6 again implies a contradiction to G being 4-χ_ρ-vertex-critical. Thus, $B \subseteq \bigcup_{j \equiv 0 \pmod 4} N_j$ or $B \subseteq \bigcup_{j \equiv 2 \pmod 4} N_j$.

Further, we prove that $B \subseteq \bigcup_{j \equiv 2 \pmod 4} N_j$. First, consider the case when $A \neq \emptyset$. Let $x \in A$ and let $z \in B$. Then, $x \in N_{j_1}, j_1 \equiv 3 \pmod 4$ and $z \in N_{j_2}, j_2 \equiv 0, 2 \pmod 4$. Suppose that $j_2 \equiv 0 \pmod 4$. Let Q_1 be a shortest x-y-path, Q_2 a shortest z-y-path and $v \in Q_1 \cap Q_2$ such that the distance between y and v is the largest possible. Analogously as for w', also in this case we prove that $v \in N_j, j \equiv 0, 2 \pmod 4$ and without loss of generality assume that $v \in N_j, j \equiv 0 \pmod 4$. This implies that G contains a subgraph isomorphic to a graph from $\mathcal{F}_2$, a contradiction to our assumption. Hence, $z \in N_{j_2}, j_2 \equiv 2 \pmod 4$ and consequently, $B \subseteq \bigcup_{j \equiv 2 \pmod 4} N_j$. Note that, if $|A| = 0$, then without loss of generality we may assume that $B \subseteq \bigcup_{j \equiv 2 \pmod 4} N_j$.

In conclusion, each subgraph of G isomorphic to K_3 satisfies the above written property: one of its vertices, denoted by z, belongs to N_j and the other two, denoted by x_1, x_2, to N_{j+1}, where $j \equiv 2 \pmod 4$. Moreover, if x_1 (respectively, x_2) belongs to A (i.e., has degree at least 3), then it satisfies the Properties 1,2 and 3. Therefore, the vertices from $(N[x_1] \cup N[x_2]) \setminus \{z\}$ induce a T-add to a vertex z.

Now, we make a partition of $V(G)$ into the sets $V_0', V_1', \ldots, V_7'$ as follows.

$V_0' = \{u \in N_j : j \equiv 1 \pmod 4, \deg(u) = 1\}$,

$V_1' = \{u \in N_j : (j \equiv 0 \pmod 4) \wedge (\exists u' \in N_{j-1}, uu' \in E(G), \deg(u') = 2)\} \cup \{y\}$,

$V_2' = \{u \in N_j : j \equiv 1, 3 \pmod 4, \deg(u) = 2, u \text{ does not belong to a subgraph of } G \text{ isomorphic to } K_3\}$,

$V_3' = \{u \in N_j : j \equiv 2 \pmod 4\}$ (note that $B \subseteq V_3'$),

$V_4' = \{u \in N_j : j \equiv 3 \pmod 4, \deg(u) = 1\}$,

$V_5' = \{u \in N_j : j \equiv 3 \pmod 4, \deg(u) = 2, u \text{ belongs to a subgraph of } G \text{ isomorphic to } K_3\}$,

$V_6' = \{u \in N_j : j \equiv 3 \pmod 4, \deg(u) \geq 3\} = A$,

$V_7' = \{u \in N_j : (j \equiv 0 \pmod 4) \wedge (\exists u' \in N_{j-1}, uu' \in E(G), \deg(u') \geq 3)\}$.

First, we claim that each vertex from $V(G)$ belongs to exactly one set of $V_0', V_1', \ldots, V_7'$. Let $u \in V(G)$ be an arbitrary vertex. Suppose that $u \in N_j, j \equiv 0 \pmod 4$. If $u = y$, then $u \in V_1'$. Otherwise, u has a neighbor $u' \in N_{j-1}$. Clearly, u' is not a leaf. If $\deg(u') \geq 3$, then $u' \in A$ and using Property 3, we infer that u is a leaf. Consequently, there does not exist $u'' \in N_{j-1}, uu'' \in E(G)$ with $\deg(u'') = 2$, which implies that u is only in V_7' (clearly, $u' \neq y$). If $\deg(u') = 2$, then analogously we derive that there does not exist $u''' \in N_{j-1}, uu''' \in E(G)$ with $\deg(u''') \geq 3$. This means that u is only in V_1'. Therefore, u belongs either to V_1' or to V_7' and clearly, $u \notin V_0' \cup V_2' \cup V_3' \cup V_4' \cup V_5' \cup V_6'$. Next, let $u \in N_j, j \equiv 1 \pmod 4$. If $\deg(u) = 1$, then $u \in V_0'$. Further, suppose that $\deg(u) = 2$. Note that u does not belong to a subgraph of G isomorphic to K_3, since each subgraph of G isomorphic to K_3 has one vertex in $B \subseteq \bigcup_{j\equiv 2 \pmod 4} N_j$ and the other two in $\bigcup_{j\equiv 3 \pmod 4} N_j$. This implies that $u \in V_2'$. Next, suppose that $\deg(u) \geq 3$. Then $u \in A$ and consequently, $u \in \bigcup_{j\equiv 3 \pmod 4} N_j$, a contradiction since $u \in N_j, j \equiv 1 \pmod 4$. Therefore, u belongs to exactly one of the sets V_0', V_2', and $u \notin V_1' \cup V_3' \cup V_4' \cup V_5' \cup V_6' \cup V_7'$. Next, using the definitions of the sets $V_0', V_1', \ldots, V_7'$, we derive that each $u \in N_j, j \equiv 2 \pmod 4$ belongs only to the set V_3'. Finally, let $u \in N_j, j \equiv 3 \pmod 4$. If u is a leaf, then $u \in V_4'$. If $\deg(u) \geq 3$, then $u \in V_6'$. Further, let $\deg(u) = 2$. If u belongs to a subgraph of G isomorphic to K_3, then $u \in V_5'$, and otherwise, $u \in V_2'$. Therefore, u belongs to exactly one of the sets V_2', V_4', V_5', V_6' and $u \notin V_0' \cup V_1' \cup V_3' \cup V_7'$. These observations imply that $(V_0', V_1', \ldots, V_7')$ is a partition of $V(G)$.

As we mentioned above, our goal is to prove that $G \in \mathcal{G}_3$. Hence, we show that G is formed by taking a bipartite multigraph with bipartition (V_1', V_3'), subdividing every edge exactly once, adding leaves to some vertices in $V_1' \cup V_3'$, and then performing a single T-add to some vertices in V_3'.

First, we prove that $V_1' \cup V_3'$ induces a bipartite multigraph in G, in which each edge is subdivided exactly once. Clearly, each vertex from $V_1' \cup V_3'$ belongs to exactly one of the sets V_1', V_3'. Therefore, we only need to prove that $d(v_1, v_1') = 4k_1$, $k_1 \geq 1$, and $d(v_3, v_3') = 4k_3$, $k_3 \geq 1$, for any $v_1, v_1' \in V_1'$ and any $v_3, v_3' \in V_3'$. Let v_1, v_1' be two distinct vertices from V_1'. Further, let P be a shortest v_1-v_1'-path, Q_1 a shortest v_1-y-path, Q_1' a shortest v_1'-y-path and $w \in Q_1 \cap Q_1'$ such that the distance between y and w is the largest possible. Recall that $v_1, v_1' \in \bigcup_{j\equiv 0 \pmod 4} N_j$. If $w \in N_j, j \equiv 1,3 \pmod 4$, then $w \in A$, because $\deg(w) \geq 3$. Using Property 3, we derive that $wv_1, wv_1' \in E(G)$ (v_1, v_1' are leaves) and consequently, $v_1, v_1' \in V_7'$. Since $(V_0, V_1', \ldots, V_7')$ is a partition of $V(G)$, $v_1, v_1' \notin V_1'$, a contradiction. Therefore, $w \in N_j, j \equiv 0,2 \pmod 4$. Clearly, if $w \in P$, then $d(v_1, v_1') = 4k_1$ for some $k_1 \geq 1$. Next, consider the case when $w \notin P$. If $P \cap (Q_1 \cup Q_1') = \{v_1, v_1'\}$, then $d(v_1, v_1') = 4k_1$ for some $k_1 \geq 1$. Namely, otherwise G contains a subgraph isomorphic to a graph from $\mathcal{C}$, a contradiction to our assumption. Now, let $P \cap (Q_1 \cup Q_1') \neq \{v_1, v_1'\}$. Without loss of generality we may assume that $|P \cap Q_1| \geq 2$ and let $w_1 \in P \cap Q_1$, $w_1 \neq v_1$, such that the distance between v_1 and w_1 is the largest possible. Further, let $w_1' \in P \cap Q_1'$ (w_1' can be v_1', but $w_1 \neq w_1'$) such that the distance between v_1' and w_1' is the largest possible. Note that $d(v_1, w_1) \neq 1$. Namely, if v_1 and w_1 are adjacent, then $v_1 \in V_7'$ (note that $\deg(w_1) \geq 3$), a contradiction. Analogously, if $v_1' \neq w_1'$, then $d(v_1', w_1') \neq 1$. Moreover, $w_1 \in N_j, j \equiv 0,2 \pmod 4$. Indeed, $w_1 \in N_j, j \equiv 1,3 \pmod 4$ implies that $w_1 \in A$ and consequently, $w_1v_1 \in E(G)$, which is not true. Analogously we derive that $w_1' \in N_j, j \equiv 0,2 \pmod 4$. Next, we observe that the vertices from Q_1 between w_1 and w, the vertices from Q_1' between w_1' and w, and the vertices from P between w_1 and w_1' form a cycle C_n in G, $n \geq 3$. Note that $w_1w \notin E(G)$ since $w_1 \in N_{j_1}, j_1 \equiv 0,2 \pmod 4$ and $w \in N_{j_2}, j_2 \equiv 0,2 \pmod 4$, but $j_1 \neq j_2$ (clearly, $w \notin P$, $w_1 \in P$, hence $w_1 \neq w$). Thus, C_n is not a triangle. Therefore, $n \geq 4$ and moreover, $n = 4a, a \geq 1$, since G does not contain a subgraph from $\mathcal{C}$. This implies that $d(w_1, w_1') = 4k$, $k \geq 1$, if $w_1 \in N_{j_1}, w_1' \in N_{j_2}, j_1, j_2 \equiv 0 \pmod 4$ or $w_1 \in N_{j_1}, w_1' \in N_{j_2}, j_1, j_2 \equiv 2 \pmod 4$, and otherwise $d(w_1, w_1') = 4l + 2, l \geq 0$. In the first case, $d(v_1, w_1) + d(v_1' + w_1') = 4k', k' \geq 1$. Thus, $d(v_1, v_1') = 4k_1, k_1 \geq 1$. In the second case, $d(v_1, w_1) + d(v_1' + w_1') = 4l' + 2, l' \geq 0$, and hence, $d(v_1, v_1') = 4l_1, l_1 \geq 1$. Analogously, we prove that $d(v_3, v_3') = 4k_3$ for some $k_3 \geq 1$. Therefore, $V_1' \cup V_3'$ induces a subdivided bipartite multigraph in G.

Now, we prove that V_2' is the set of all vertices obtained by subdivision of bipartite multigraph with bipartition (V_1', V_3'). We have to show that each $v_2 \in V_2'$ has exactly two neighbors: one from V_1' and the other from V_3'. Let $v_2 \in V_2'$ be an arbitrary vertex. Then, $v_2 \in N_j, j \equiv 1,3 \pmod 4$. First, consider the case when $j \equiv 1 \pmod 4$. Clearly, $v_2v \in E(G)$ for some $v \in N_{j-1}$ ($j - 1 \equiv 0 \pmod 4$). This implies that $v \in V_1' \cup V_7'$. It is easy to observe that each vertex from V_7' has a neighbor from $V_6' = A$. By Property 3, each vertex from V_7' is a leaf. This implies that $v \in V_1'$. Therefore, $N(v_2) \cap N_{j-1} \subseteq V_1'$. Since the vertices from V_1' are pairwise at distance $4a$, $a \geq 1$, we derive that $|N(v_2) \cap N_{j-1}| = 1$ (v_2 has exactly one neighbor from N_{j-1}, more precisely, from V_1'). Suppose that $v_2u \in E(G)$, $u \in N_j$. Then, $u \in V_0' \cup V_2'$. By Lemma 2.5, v_2 belongs to a subgraph of G isomorphic to K_3, a contradiction to the definition of V_2'. Since $\deg(v_2) = 2$, v_2 has one neighbor from N_{j+1}, namely from V_3' (note that $j + 1 \equiv 2 \pmod 4$). In conclusion, v_2 has one neighbor from V_1' and the other from V_3'. Next, suppose that $j \equiv 3 \pmod 4$. Clearly, $v_2v \in E(G)$ for some $v \in N_{j-1}$. Since $j - 1 \equiv 2 \pmod 4$), we infer that $N(v_2) \cap N_{j-1} \subseteq V_3'$. Again, since the vertices from V_3' are pairwise at distance $4a$, $a \geq 1$, we derive that v_2 has exactly one neighbor from N_{j-1}. Suppose that $v_2u \in E(G)$, $u \in N_j$. Again, by Lemma 2.5, v_2 belongs to a subgraph of G isomorphic to K_3, a contradiction to the definition of V_2'. Since $\deg(v_2) = 2$, $v_2z \in E(G)$ for some $z \in N_{j+1}$. Note that $z \in V_1' \cup V_7'$. The fact that $\deg(v_2) = 2$ implies that $z \in V_1'$. In conclusion, v_2 has one neighbor from V'_1 and the other from V'_3.

Further, let $v_1 \in V'_1$ ($v_1 \in N_j$, $j \equiv 0 \pmod 4$) be an arbitrary vertex. We need to prove that $N(v_1)$ can contain only leaves and the vertices from V'_2. First, consider the case when $v_1 \neq y$. Since $v_1 \in N_j$, $j \equiv 0 \pmod 4$, we have: $N(v_1) \subseteq V'_0 \cup V'_1 \cup V'_2 \cup V'_4 \cup V'_5 \cup V'_6 \cup V'_7$. Clearly, v_1 does not have a neighbor from V'_1 since $d(v_1, v'_1) = 4k_1, k_1 \geq 1$, for each $v'_1 \in V'_1$, $v'_1 \neq v_1$. If there exists $v_7 \in V'_7$ such that $v_1v_7 \in E(G)$, then $\deg(v_7) \geq 2$ (v_7 has a neighbor v_1 and a neighbor from N_{j-1}), a contradiction to v_7 being a leaf (we have already argued that each vertex from V'_7 is a leaf). Thus, $N(v_1)$ does not contain the vertices from V'_7. Next, suppose that $v_1v_6 \in E(G)$ for some $v_6 \in V'_6$. By the definition of V'_1, v_1 is adjacent to $u' \in N_{j-1}$ such that $\deg(u') = 2$, which implies that $u' \notin V'_6$. Therefore, v_1 has at least two neighbors (u' and v_6). On the other hand, since $v_6 \in A$, Property 3 implies that v_1 is a leaf, a contradiction. Hence, $N(v_1)$ does not contain the vertices from V'_6. Clearly, v_1 is not adjacent to a vertex from V'_4 since each vertex from V'_4 (from N_{j-1}) is a leaf and hence does not have a neighbor from N_j. Further, recall that each subgraph of G isomorphic to K_3 has one vertex from N_{j_1}, $j_1 \equiv 2 \pmod 4$ and the other two from N_{j_2}, $j_2 \equiv 3 \pmod 4$. Next, each vertex from V'_5 belongs to a triangle and has degree 2. These facts imply that v_1 does not have a neighbor from V'_5. Thus, $N(v_1) \subseteq V'_0 \cup V'_2$. Obviously, $N(y)$ also contains only the vertices from $V'_0 \cup V'_2$. Since each vertex from V'_0 is a leaf, our claim holds.

It remains to prove that to each vertex $v_3 \in V'_3$ ($v_3 \in N_j$, $j \equiv 2 \pmod 4$) only a single T-add, leaves and the vertices from V'_2 can be attached. Since $v_3 \in N_j$, $j \equiv 2 \pmod 4$, we have: $N(v_3) \subseteq V'_0 \cup V'_2 \cup V'_3 \cup V'_4 \cup V'_5 \cup V'_6$. Clearly, v_3 does not have a neighbor from V'_3 since $d(v_3, v'_3) = 4k_3, k_3 \geq 1$, for each $v'_3 \in V'_3$, $v'_3 \neq v_3$. Note that v_3 is not adjacent to a vertex from V'_0, since each vertex from V'_0 (from N_{j-1}) is a leaf and hence does not have a neighbor from N_j. Next, suppose that $v_5 \in V'_5$ is adjacent to v_3. Note that v_5 belongs to a triangle and has degree 2. This implies that v_3 also belongs to this triangle (with vertices v_5, v_3, b; $b \in V'_5 \cup V'_6$) and thus, $v_3 \in B$. As we have argued above, the vertices from $(N[v_5] \cup N[b]) \setminus \{v_3\}$ induce a T-add attached to v_3. Now, we prove that only one T-add can be attached to v_3. Suppose that there exist $x_1, x_2 \in N_{j+1} \setminus \{v_5, b\}$ such that $\{v_3, x_1, x_2\}$ also induce a K_3. Then, G contains a subgraph isomorphic to H_2, a contradiction to G being 4-χ_ρ-vertex-critical. Next, suppose that there exists $x_1 \in (N_{j+1} \cap N(v_3)) \setminus \{b\}$ such that $\deg(x_1) \geq 3$ ($x_1 \in V'_6$). Then, G contains a subgraph isomorphic to a graph from $\{H_1\} \cup \mathcal{F}_2$, a contradiction. Therefore, if v_3 is adjacent to a vertex from V'_5, only one T-add can be attached to v_3, and perhaps some leaves and the vertices from V'_2 (note that each vertex from $V'_5 \cup V'_6$ belongs to a T-add). Thus, we can now assume that $N(v_3) \cap V'_5 = \emptyset$. Suppose that $v_3v_6 \in E(G)$ for some $v_6 \in V'_6$. Note that v_3 does not belong to a subgraph of G isomorphic to K_3, because $N(v_3) \cap V'_5 = \emptyset$ and G does not contains a subgraph isomorphic to H_1 or H_3. Therefore, since $v_6 \in A$, the Properties 1,2,3 imply that all vertices from $N(v_6) \setminus \{v_3\}$ are leaves. This means that $N[v_6] \setminus \{v_3\}$ induces a T-add attached to v_3. Now, suppose that two T-adds can be attached to v_3. Since $N(v_3) \cap V'_5 = \emptyset$, v_3 is adjacent to two vertices from V'_6. Since each vertex from V'_6 belongs to A and v_3 does not belong to any triangle, the Properties 1,2,3 imply that G contains a subgraph isomorphic to H_9, a contradiction to our assumption. Therefore, also in the case when $N(v_3) \cap V'_5 = \emptyset$ and v_3 has a neighbor from V'_6 we derive that only one T-add can be attached to v_3 (and perhaps some leaves and the vertices from V'_2). If $N(v_3) \cap (V'_5 \cup V'_6) = \emptyset$, then only the vertices from $V'_2 \cup V'_4$ can be adjacent to v_3. In this case, our claim clearly holds.

Thus, $G \in \mathcal{G}_3$, a contradiction to G being 4-χ_ρ-vertex-critical.

Case 2. G does not contain a subgraph isomorphic to T, which is shown in Fig. 2.

Case 2.1. G contains a subgraph isomorphic to C_n, $n \geq 3$.

By Lemma 3.6, $n = 3$ or $n = 4a, a \geq 1$.

First, suppose that $n = 4a, a \geq 1$. Since $\chi_\rho(G) = 4$, $G \ncong C_n$. Thus, if $a \geq 2$, G contains a subgraph isomorphic to T or a subgraph isomorphic to a graph from $\mathcal{C}$, a contradiction to our assumptions. Hence, $a = 1$. Let $C \cong C_4$ and let $V(C) = \{a, b, c, d\}$, $\{ab, bc, cd, da\} \subseteq E(G)$. Note that G is not isomorphic to K_4 and it does not contain a subgraph isomorphic to K_4. Hence, without loss of generality we may assume that $bd \notin E(G)$. Since $\chi_\rho(G) = 4$, G is not isomorphic to H_1 and G does not contain a subgraph isomorphic to H_1, without loss of generality we may also assume that there exists a vertex from $V(G) \setminus \{b, c, d\}$ adjacent to a. This implies that $\deg(b) = \deg(d) = 2$, because G does not contain a subgraph isomorphic to a graph from $\{H_5\} \cup \mathcal{C}$. If each vertex adjacent to a is adjacent to c and each vertex adjacent to c is adjacent to a, then $\chi_\rho(G) \leq 3$ (recall that G does not contain a subgraph isomorphic to H_1, H_5 or K_4), a contradiction to our assumption. Thus, there exists a vertex adjacent to a, which is not adjacent to c (or a vertex adjacent to c that is not adjacent to a). Next, note that if all vertices adjacent to a, which are not adjacent to c, are leaves, and also all vertices from $N(c) \setminus N[a]$ are leaves, then $\chi_\rho(G) = 3$, again a contradiction. Therefore, there exist $a', a'' \in V(G)$, such that $aa', a'a'' \in E(G)$ and $a'c \notin E(G)$ (note that $a'' \neq b, c, d$). But then, G contains a subgraph isomorphic to T, a contradiction. These findings imply that $n \neq 4$ (therefore, G does not contain a subgraph isomorphic to C_n, $n \geq 4$).

Now assume that G contains a triangle C and let $V(C) = \{a, b, c\}$, $E(C) = \{ab, bc, ca\}$. Since $\chi_\rho(G) = 4$, at least one vertex from $\{a, b, c\}$ has degree at least 3. Note that at least one vertex from $\{a, b, c\}$ has degree 2. Namely, otherwise, G contains a subgraph isomorphic to H_3 or to C_4, a contradiciton to our assumption. Therefore, without loss of generality we may assume that $\deg(c) = 2$.

First, suppose that $\deg(a), \deg(b) \geq 3$. This implies that there exist $a', b' \in V(G)$ such that $aa', bb' \in E(G)$. Note that $a' \neq b'$ ($a'b, b'a \notin E(G)$) since G does not contain a subgraph isomorphic to C_4. Moreover, each vertex from $(N(a) \cup N(b)) \setminus \{a, b, c\}$ has exactly one neighbor in $N(a) \cup N(b) \cup \{a, b, c\}$, because G does not contain a subgraph isomorphic to H_2 or C_4. Hence, since $\chi_\rho(G) = 4$, there exists $a'' \in V(G) \setminus (N(a) \cup N(b) \cup \{a, b, c\})$ (or $b'' \in V(G) \setminus (N(a) \cup N(b) \cup \{a, b, c\})$ such that $a'a'' \in E(G)$ (or $b'b'' \in E(G)$). Then, G contains a subgraph isomorphic to T, a contradiction.

Therefore, $\deg(a) \geq 3$ and $\deg(b) = 2$. Let $N_0 = \{a\}$ and let $N_i = \{u;\ d(u, a) = i\}$ for each $i = 1, 2, \ldots, k$. Note that $k \geq 2$. Namely, $\chi_\rho(G) = 4$ and G does not contain a subgraph isomorphic to H_2, which imply that there exists $a' \in V(G)$ such that $d(a, a') = 2$. Further, the fact that G does not contain a subgraph isomorphic to T implies that all vertices from $N_1 \cup N_2 \cup \ldots \cup N_{k-2}$ have degree at most 2 and in addition, $\deg(a) = 3$. Now, let $\deg(u_k) \geq 2$ for some $u_k \in N_k$. This implies that there exist $u_{k-1} \in N_{k-1}$ and $v \in (N_k \cup N_{k-1})$ such that $\{u_k, u_{k-1}, v\}$ induces a triangle. The fact that G does not contain a subgraph isomorphic to T implies that $\deg(u_{k-1}) = 3$, $\deg(v) = 2$, $\deg(u_k) = 2$. Since G does not contain a subgraph isomorphic to a graph from $\mathcal{F}_1$, $k - 2 = 4m + 3$, $m \geq 0$. But then, there exists a 3-packing coloring c' of G. Namely, let $c'(c) = c'(v) = 1$, $c'(b) = c'(u_k) = 2$, $c'(a) = c'(u_{k-1}) = 3$. Further, if $m \geq 1$, then let the vertices from a subgraph of G isomorphic to P_{k-2} be colored one after another using the following pattern of colors: $1, 2, 1, 3, \ldots, 1, 2, 1$. Otherwise, these vertices color with the following color pattern: $1, 2, 1$. Clearly, c' is a 3-packing coloring of G, a contradiction to G being 4-χ_ρ-vertex-critical. Hence, $\deg(u_k) = 1$ for every $u_k \in N_k$. Since G does not contain a subgraph isomorphic to T, all vertices from $N(u_{k-1})$ except one, are leaves. If $\deg(u_{k-1}) = 2$, then Lemma 2.1 implies that $G \in \mathcal{G}_3$, a contradiction to G being 4-χ_ρ-vertex-critical. Therefore, $\deg(u_{k-1}) \geq 3$. Since G does not contain a subgraph isomorphic to a graph from $\mathcal{F}_2$, we derive that $k - 2 \neq 0$ and $k - 2 \neq 4l + 2$ for any $l \geq 0$. In this case, there exists a 3-packing coloring c'' of G, defined as follows. Let $c''(c) = 1, c''(b) = 2$, $c''(a) = 3$. If $k - 2 = 4l + 1$, $l \geq 0$, or $k - 2 = 4l', l' \geq 1$, then let $c''(u_{k-1}) = 2$ and the leaves from $N(u_{k-1})$ receive color 1. If $l \geq 1$, then the other vertices color one after another using the following pattern of colors $1, 2, 1, 3, \ldots$. The same pattern is used for any $l' \geq 1$. If $l = 0$, then color the uncolored vertex by 1. If $k - 2 = 4l'' + 3$, $l'' \geq 0$, then let $c''(u_{k-1}) = 3$ and the leaves from $N(u_{k-1})$ receive color 1. Further, if $l'' \geq 1$, then color the other vertices one after another using the following pattern of colors $1, 2, 1, 3, \ldots$. Otherwise, color the remaining three vertices one after another by $1, 2, 1$. Clearly, in each case, the obtained coloring is a 3-packing coloring of G, a contradiction to G being 4-χ_ρ-vertex-critical.

Case 2.2. G does not contain a subgraph isomorphic to C_n, $n \geq 3$.

Note that in this case, G is a tree. Clearly, if every vertex has degree at most 2, then G is a path and thus, it is not 4-χ_ρ-vertex-critical, a contradiction to our assumption. Hence, there exists $y \in V(G)$ such that $\deg(y) \geq 3$. Let $N_0 = \{y\}$ and let $N_i = \{u : d(u, y) = i\}$ for each $i = 1, 2, \ldots, k$. Recall that $\chi_\rho(G) = 4$, G is a tree and G does not contain a subgraph isomorphic to T. Thus, $k \geq 3$. Moreover, all vertices from $N_1 \cup N_2 \cup \ldots N_{k-2}$ have degree at most 2, all vertices from $N(y)$, except one, are leaves, and $|N_i| = 1$ for any $i \in \{2, 3, \ldots, k-1\}$. Further, $\deg(u_k) = 1$ for every $u_k \in N_k$. Therefore, G contains at most two vertices of degree at least 3: one is y and let the other be y' ($y' \in N_{k-1}$). Next, the fact that G is a tree implies that all vertices from $N(y') \setminus N_{k-2}$ are leaves. Now, we prove that G is 3-packing colorable. Let $c : V(G) \longrightarrow \{1, 2, 3\}$. First, suppose that $k - 2 = 4l$, $l \geq 1$, or $k - 2 = 4l' + 1, l' \geq 0$. In this case, let $c(y) = 3$, $c(y') = 2$ and all leaves from $N(y) \cup N(y')$ receive color 1. The remaining vertices (of P_{k-2}) color one after another using the following pattern of colors: $1, 2, 1, 3$ (note that if $l' = 0$, then the vertex receives color 1). Clearly, c is a 3-packing coloring of G, thus $\chi_\rho(G) \leq 3$, a contradiction to G being 4-χ_ρ-vertex-critical. Next, let $k - 2 = 4l'' + 2$, $l'' \geq 0$.

Then, let $c(y) = c(y') = 2$ and let all leaves from $N(y) \cup N(y')$ receive color 1. The remaining vertices (of P_{k-2}) are colored one after another using the pattern of colors: $1, 3, 1, 2$ (if $l'' = 0$, then use only colors $1, 3$). The described coloring c is a 3-packing coloring of G, a contradiction to G being 4-χ_ρ-vertex-critical. Finally, suppose that $k - 2 = 4l''' + 3$, $l''' \geq 0$. In this case, let $c(y) = c(y') = 3$ and all leaves from $N(y) \cup N(y')$ color with color 1. Next, the remaining vertices (of P_{k-2}) are colored one after another using the pattern of colors: $1, 2, 1, 3$ (if $l''' = 0$, then use only colors $1, 2, 1$). Again, this is a 3-packing coloring of G, a contradiction to G being 4-χ_ρ-vertex-critical. This concludes the proof. □

4. 4-χ_ρ-critical graphs

Recall that a graph G is a χ_ρ-critical graph, if for every proper subgraph H of G, $\chi_\rho(H) < \chi_\rho(G)$. If G is χ_ρ-critical and $\chi_\rho(G) = k$, then we say that G is k-χ_ρ-critical. In this section, we characterize 4-χ_ρ-critical graphs.

Brešar and Ferme [6] proved that every χ_ρ-critical graph is also χ_ρ-vertex-critical. In addition, in trees these two types of critical graphs coincide. The mentioned authors also provided two partial characterizations of 4-χ_ρ-critical graphs, which are given below.

Proposition 4.1. *[6] If G is a graph containing a cycle C_n, where $n \geq 5$ is an integer not divisible by 4, then G is a 4-χ_ρ-critical graph if and only if G is isomorphic to C_n.*

Recall that $\mathcal{D}$ is the class of graphs that contain exactly one cycle and have an arbitrary number of leaves attached to each of the vertices of the cycle.

Theorem 4.2. *[6] A graph $G \in \mathcal{D}$ is a 4-χ_ρ-critical graph if and only if G is one of the following graphs:*

- $G \cong C_n$, $n \geq 5$, $n \not\equiv 0 \pmod 4$;
- *G is the net graph;*
- *G is obtained by attaching a single leaf to two adjacent vertices of C_4.*

Let $\mathcal{F}_1'$ be the subfamily of graphs from $\mathcal{F}_1$ for which $l \notin \{0, 4k+2;\ k \geq 0\}$ (see Fig. 8a). Next, we denote by $\mathcal{F}_3'$ the subfamily of graphs from $\mathcal{F}_3$, which do not contain the edge u_2u_3 (see Fig. 8c). Further, $\mathcal{F}_4'$ consists of all graphs from $\mathcal{F}_4$, which do not contain the edges v_1y_{l-1} and v_1w_2 (see Fig. 8d). Finally, let $\mathcal{F}_5'$ be the subfamily of graphs from $\mathcal{F}_5$, which do not contain the edges u_2u_3 and v_2v_3 (see Fig. 8e). Note that each graph from $\mathcal{F}_4' \cup \mathcal{F}_5'$ is a tree.

Theorem 4.3. *A graph G is 4-χ_ρ-critical if and only if $G \in \mathcal{F}_1' \cup \mathcal{F}_2 \cup \mathcal{F}_3' \cup \mathcal{F}_4' \cup \mathcal{F}_5' \cup \mathcal{C} \cup \{K_4, H_1, H_2, H_3, H_4, H_5, H_9\}$.*

Proof. The fact that each χ_ρ-critical graph is also χ_ρ-vertex-critical and Theorem 3.7 imply that 4-χ_ρ-critical graphs can only be the graphs from $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5 \cup \mathcal{C}_5 \cup \mathcal{C}_6 \cup \mathcal{C} \cup \{K_4, H_1, H_2, \ldots, H_9\}$.

From Proposition 4.1, we derive that the graphs from $\mathcal{C}_5 \cup \mathcal{C}_6$ are not 4-χ_ρ-critical. By Theorem 4.2, the graphs from $\mathcal{C} \cup \{H_3, H_5\}$ are 4-χ_ρ-critical. In addition, Theorem 4.2 also shows that H_7 is not a 4-χ_ρ-critical graph. Consequently, H_8 is not 4-χ_ρ-critical, since it contains a proper subgraph isomorphic to H_7. Further, since each tree is a χ_ρ-critical graph if and only if it is χ_ρ-vertex-critical, we know that H_9 is 4-χ_ρ-critical. Clearly, K_4 is also a 4-χ_ρ-critical graph. Therefore, it remains to consider the graphs from $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5 \cup \{H_1, H_2, H_4, H_6\}$. Let $G \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5 \cup \{H_1, H_2, H_4, H_6\}$. By Theorem 3.7, $\chi_\rho(G) = 4$. Hence, we only need to check if $\chi_\rho(G - e) \leq 3$ for every edge e of G. Suppose that $e = xy$ and x is a leaf. We observe that $G - e$ is isomorphic to the disjoint union of $G - x$ and K_1. Since G is 4-χ_ρ-vertex-critical, $\chi_\rho(G - x) \leq 3$, which implies that $\chi_\rho(G - e) \leq 3$. Therefore, we only need to consider the edges of G, which connect the vertices of degree at least 2.

Let $G \cong H_1$ (see Fig. 7b). If $e \in \{ab, ad, bd\}$, then $G - e \in \{X_2, Y_1\}$. By Lemma 2.1, $\chi_\rho(G - e) = 3$. In the case when $e \in \{ae, de\}$, $G - e$ is isomorphic to $H_3 - d$. Since H_3 is 4-χ_ρ-vertex-critical, we have $\chi_\rho(G - e) \leq 3$. Thus,

H_1 is 4-χ_ρ-critical. Now, suppose that $G \cong H_2$ (see Fig. 7c). If $e \in \{ab, be, bc, bd\}$, then $G - e \cong X_2$ and Lemma 2.1 implies that $\chi_\rho(G - e) = 3$. Otherwise, color the leaves of $G - e$ with color 1 and the remaining vertices with colors 1, 2, 3 such that the vertex of degree 4 receives color 2. The described coloring is a 3-packing coloring of $G - e$, which implies that H_2 is 4-χ_ρ-critical. Next, let $G \cong H_4$ (see Fig. 7e). If $e = cd$, then $G - e$ is a path and hence, it is 3-packing colorable. Further, if $e \in \{bc, de\}$, then $G - e$ is the disjoint union of X_2 and K_2. By Lemma 2.1, $\chi_\rho(G - e) = 3$. Now, let $e = cg$. Then, a 3-packing coloring c' of $G - e$ can be formed as follows: $c'(a) = c'(c) = c'(e) = c'(g) = 1$, $c'(b) = c'(f) = 2$ and $c'(d) = 3$. This implies that $\chi_\rho(G - e) \leq 3$, if $e = cg$. Analogously we prove that $\chi_\rho(G - e) \leq 3$ holds for $e = dg$. Therefore, H_4 is 4-χ_ρ-critical. Further, we claim that H_6 (see Fig. 7g) is not a 4-χ_ρ-critical graph. Note that $H_6 - ad$ is isomorphic to H_5 and $\chi_\rho(H_5) = 4$. Hence, $\chi_\rho(H_6 - ad) = 4$ and the claim is true.

Now, let $G \in \mathcal{F}_1$ (see Fig. 8a). If $e \in E(G) \setminus \{u_1u_2, v_1v_2\}$, then $G - e$ is isomorphic to X_n, $n \geq 1$, or to the disjoint union of two graphs from $\{K_3, X_n : n \geq 1\}$. Using Lemma 2.1 and the fact that $\chi_\rho(K_3) = 3$, we derive that $\chi_\rho(G - e) = 3$. Next, let $e \in \{u_1u_2, v_1v_2\}$. If $l = 0$ or $l = 4k + 2$, $k \geq 0$, then $G - e$ is isomorphic to a graph from $\mathcal{F}_2$. Thus, $\chi_\rho(G - e) = 4$ and G is not 4-χ_ρ-critical. If $l = 4k, k \geq 1$, then $G - e$ is isomorphic to $H - u_2$, where $H \in \mathcal{F}_3$. By Theorem 3.4, $\chi_\rho(G - e) \leq 3$ and consequently, G is 4-χ_ρ-critical. Finally, let $l = 4k + 1$, $k \geq 0$. In this case, we can form a 3-packing coloring of $G - e$ as follows. Let the vertices of the triangle receive colors 1, 2, 3 such that the vertex of degree 3 receives a color 3. Further, color the leaves with color 1, and the remaining vertices (of P_{4k+2}) one after another with the following pattern of colors: 1, 2, 1, 3. This implies that G is 4-χ_ρ-critical. In conclusion, each graph from $\mathcal{F}'_1$ is 4-χ_ρ-critical.

Further, let $G \in \mathcal{F}_2$ (see Fig. 8b). If $e \in E(G) \setminus \{u_1u_2\}$, then $G - e$ is isomorphic to X_n, to a subgraph of X_n or to the disjoint union of such graphs. Hence, by Lemma 2.1, $\chi_\rho(G - e) \leq 3$. If $e = u_1u_2$, then $G - e$ is isomorphic to $H - \{u_2, v_2\}$, where $H \in \mathcal{F}_5$. Using Theorem 3.4, we infer that $\chi_\rho(G - e) \leq 3$. Hence, every $G \in \mathcal{F}_2$ is 4-χ_ρ-critical.

Now, consider a graph $G \in \mathcal{F}_3$ (see Fig. 8c). Clearly, if $u_2u_3 \in E(G)$, then $G - u_2u_3 \in \mathcal{F}_3$ and hence, G is not 4-χ_ρ-critical. Thus, we only need to consider the case when $G \in \mathcal{F}'_3$. If $e \in E(G) \setminus \{u_1u_2, v_1v_2\}$, then $G - e$ is isomorphic to (a subgraph of) X_n, to a subgraph of Y_n or to the disjoint union of such graphs. Using Lemma 2.1, we derive that $\chi_\rho(G - e) \leq 3$. If $e = v_1v_2$, then $G - e$ is isomorphic to $H - v_2$, where $H \in \mathcal{F}_5$. By Theorem 3.4, $\chi_\rho(G - e) \leq 3$. Note that $\chi_\rho(G - u_1u_2) \leq 3$, since u_2 is a leaf. Hence, every graph $G \in \mathcal{F}'_3$ is 4-χ_ρ-critical.

Next, let $G \in \mathcal{F}_4$ (see Fig. 8d). First, suppose that $v_1y_{l-1} \in E(G)$. Then $G - v_1y_{l-1} \in \mathcal{F}_4$ and by Theorem 3.4, $\chi_\rho(G - v_1y_{l-1}) = 4$. Hence, G is not 4-χ_ρ-critical. Analogously, we prove that G is not a 4-χ_ρ-critical graph, if $v_1w_2 \in E(G)$. If $v_1y_{l-1}, v_1w_2 \notin E(G)$, then G is a tree and consequently, it is 4-χ_ρ-critical. Therefore, each graph from $\mathcal{F}'_4$ is 4-χ_ρ-critical.

Finally, let $G \in \mathcal{F}_5$ (see Fig. 8e). If G contains at least one of the edges from $\{u_2u_3, v_2v_3\}$, then $G - e \in \mathcal{F}_5$ for $e \in \{u_2u_3, v_2v_3\}$. By Theorem 3.4, $\chi_\rho(G - e) = 4$ and thus, G is not 4-χ_ρ-critical. Otherwise, G is a tree and clearly, it is 4-χ_ρ-critical. Therefore, every graph from $\mathcal{F}'_5$ is 4-χ_ρ-critical. This concludes the proof. □

5. Concluding remarks and open problems

In this paper, we have characterized 4-χ_ρ-vertex-critical graphs and 4-χ_ρ-critical graphs.

It is easy to prove that every finite graph G with $\chi_\rho(G) = k$ contains a χ_ρ-critical subgraph H with $\chi_\rho(H) = k$. Therefore, every graph G with $\chi_\rho(G) = 4$ can be constructed by adding edges to a graph from $\mathcal{F}'_1 \cup \mathcal{F}_2 \cup \mathcal{F}'_3 \cup \mathcal{F}'_4 \cup \mathcal{F}'_5 \cup \mathcal{C}_5 \cup \mathcal{C}_6 \cup \mathcal{C} \cup \{K_4, H_1, H_2, \ldots, H_9\}$ (or it belongs to this set). In this way, the family of graphs with packing chromatic number 4 can be characterized.

Note that S-packing colorings generalize the notion of packing colorings. Recall that, if $S = (a_1, a_2, \ldots, a_k)$ is a non-decreasing sequence of positive integers, then an S-packing k-coloring of a graph G is a partition of $V(G)$ into sets $X_1, X_2, \ldots, X_k$ such that for each pair of distinct vertices in the set X_i, the distance between them is larger than a_i. The S-packing chromatic number of G, $\chi_S(G)$, is the smallest k such that G has an S-packing k-coloring [21]. Clearly, a k-packing coloring coincides with an S-packing k-coloring where $S = (1, 2, 3, \ldots, k)$. Holub, Jakovac and Klavžar [22] studied S-packing chromatic vertex-critical graphs. These are graphs G with the property that $\chi_S(G - u) < \chi_S(G)$ holds for every $u \in V(G)$. Among other results, the authors in [22] have partially characterized 4-χ_S-critical graphs when $s_1 > 1$. It would be natural to study also the 4-χ_S-critical graphs for different sequences S.

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