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On Some Set Properties

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Abstract. Motivated by the Arhangel'skii "s-Lindelöf cardinal function" definition, Kočinac, Konca, Bonanzinga, Maesano and Singh defined and studied the set covering properties. In this paper, we study some set versions of compact-type properties and give partial solution to a problem posed by Singh in [11].

1. Introduction

In [1], Arhangel'skii defined a cardinal function sL from *s-Lindelöf* space X such that $sL(X) = \omega$. A space X is said to be *s-Lindelöf* if for each subset A of X and each open cover $\mathcal{U}$ of $\overline{A}$ by sets open in X, there is a countable set $\mathcal{V} \subset \mathcal{U}$ such that $A \subset \overline{\bigcup \mathcal{V}}$. Motivated by this definition, Kočinac and Konca [8, 10] introduced new types of selective covering properties called set-covering properties. Later on, Kočinac, Konca and Singh [9, 10, 12] studied set star covering properties viz. set star-Menger, set star Hurewicz etc. using the star operator. Singh in [11] studied set starcompact and set strongly starcompact properties. Bonanzinga and Maesano in [3] studied the set $\mathcal{K}$-starcompact spaces and other set covering properties. Recently a number of papers on set covering properties of Menger, Hurewicz, starcompact etc. have been published (see also [3]).

In this paper we study set starcompactness, set strongly starcompactness (see Definition 1.5), set strongly 1-starcompactness (see Definition 2.1), set $\mathcal{K}$-starcompactness (see Definition 2.9) and investigate their relationship with other set covering properties. Note that recently Bonanzinga and Maesano showed that set strongly 1-starcompactness is equivalent to strongly 1-starcompactness [3, Proposition 1.3] and that set $\mathcal{K}$-starcompactness is equivalent to $\mathcal{K}$-starcompactness [3, Proposition 1.2]. Also we give a partial solution to the following question posed by Singh [11]:

Problem 1.1. ([11]) *Is the product of a set starcompact and a compact space a set starcompact space?*

Throughout this paper we use standard topological terminology and notations as in [5]. By "a space" we mean "a topological space", $\mathbb{N}$ denotes the set of natural numbers and an open cover $\mathcal{U}$ of a subset A of X means elements of $\mathcal{U}$ are open in X and $A \subset \cup\mathcal{U} = \cup\{U : U \in \mathcal{U}\}$. The cardinality of a set A is denoted by

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$|A|$. Let ω denote the first infinite ordinal, ω_1 the first uncountable ordinal, $\mathfrak{c}$ the cardinality of the set of real numbers. As usual a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. A cardinal is often viewed as a space with the usual order topology. If A is a subset of a space X and $\mathcal{U}$ is a collection of subsets of X, then the star of A with respect to $\mathcal{U}$ is the set $St(A, \mathcal{U}) = \cup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. We usually write $St(x, \mathcal{U}) = St(\{x\}, \mathcal{U})$. If $\mathcal{P}$ is a collection of subsets of X, we write $\bigcup \mathcal{P} := \cup\{P : P \in \mathcal{P}\}$.

Recall that a space X is Menger if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there is a sequence $\{\mathcal{V}_n : n \in \omega\}$ such that for each $n \in \omega$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{\cup\mathcal{V}_n : n \in \omega\}$ is an open cover of X. The following set star versions of the Menger property were introduced by Kočinac et al. in [9]. (Recently, relative versions of "selective" star-Menger properties were considered in [2].)

Definition 1.2. ([9]) A space X is said to have:

1. set star-Menger property if for each nonempty subset A of X and each sequence $(\mathcal{U}_n : n \in \omega)$ of collection of open sets in X such that $\overline{A} \subset \cup\mathcal{U}_n$, there is a sequence $(\mathcal{V}_n : n \in \omega)$ such that for each $n \in \omega$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $A \subset \bigcup St(\cup\mathcal{V}_n, \mathcal{U}_n)$.
2. set strongly star-Menger property if for each nonempty subset A of X and each sequence $(\mathcal{U}_n : n \in \omega)$ of collection of open sets in X such that $\overline{A} \subset \cup\mathcal{U}_n$, there is a sequence $(F_n : n \in \omega)$ of finite subsets of $\overline{A}$ such that $A \subset \bigcup St(F_n, \mathcal{U}_n)$.

From the definitions, it is clear that every set strongly star-Menger space is set star-Menger space. Further the following theorem shows that the class of set strongly star-Menger spaces is sufficiently large.

Theorem 1.3. ([9]) *Every Menger space is set strongly star-Menger.*

Proof. Let X be a Menger space. Let A be any nonempty subset of X and $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of collections of open sets in X such that $\overline{A} \subset \cup\mathcal{U}_n$. Since closed subsets of Menger spaces are Menger, then $\overline{A}$ is Menger. Therefore there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\overline{A} \subset \bigcup_{n\in\mathbb{N}}(\cup\mathcal{V}_n)$. Choose $x_V \in \overline{A} \cap V$ for each $V \in \mathcal{V}_n$. For each $n \in \mathbb{N}$, let $F_n = \{x_V : V \in \mathcal{V}_n\}$. Then each F_n is a finite subset of $\overline{A}$ and $A \subset \overline{A} \subset \cup_{n\in\mathbb{N}} St(F_n, \mathcal{U}_n)$. $\square$

However the converse of the above theorem does not hold. Kočinac et al. [9] gave an example of a Tychonoff set strongly star-Menger space which is not Menger space.

Definition 1.4. ([4]) A space X is said to be:

1. 1-starcompact if for each open cover $\mathcal{U}$ of X, there is a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $X = St(\cup\mathcal{V}, \mathcal{U})$.
2. strongly 1-starcompact if for each open cover $\mathcal{U}$ of X, there is a finite subset F of X such that $X = St(F, \mathcal{U})$.

Obviously every strongly 1-starcompact space is 1-starcompact space.

Definition 1.5. ([9, 11]) A space X is said to be:

1. set starcompact if for each nonempty subset A of X and each collection $\mathcal{U}$ of open sets in X such that $\overline{A} \subset \bigcup\mathcal{U}$, there is a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $A = St(\cup\mathcal{V}, \mathcal{U}) \bigcap A$.
2. set strongly star compact if for each nonempty subset A of X and for each collection $\mathcal{U}$ of open sets of X such that $\overline{A} \subset \bigcup\mathcal{U}$, there is a finite subset F of $\overline{A}$ such that $A = St(F, \mathcal{U}) \bigcap A$.

From the definitions, it is clear that every set strongly starcompact space is strongly 1-starcompact space and set starcompact space. And every set starcomapct space is 1-starcomapct.

2. On set starcompact and set strongly starcompact spaces

The following set version of strongly 1-starcompactness is in fact equivalent to strongly 1-starcompactness (see [3, Proposition 1.3], where different terminology is used).

Definition 2.1. A space X is said to be set strongly 1-starcompact if for each nonempty subset A of X and for each collection $\mathcal{U}$ of open sets of X such that $\overline{A} \subset \bigcup \mathcal{U}$, there is a finite subset F of X such that $A = St(F, \mathcal{U}) \bigcap A$.

Clearly every set strongly starcompact space is set strongly 1-starcompact (equivalently, strongly 1-starcompact).

We have the following diagram from the definitions which gives relationships between some of the star covering properties.

$$\begin{tikzcd}
 & \text{set strongly 1-starcompact} \arrow[r] & \text{1-starcompact} \\
 & \text{set strongly starcompact} \arrow[r] \arrow[u] \arrow[d] & \text{set starcompact} \arrow[u] \arrow[d] \\
\text{Menger} \arrow[r] & \text{set strongly star-Menger} \arrow[r] & \text{set star-Menger}
\end{tikzcd}$$

However converse of the above implications may not be true as shown in the following examples. We also refer the reader to [9, 11] to see the current state of knowledge about these relationships.

Example 2.2. There exists a T_1 set strongly 1-starcompact (equivalently, strongly 1-starcompact) space which is not set strongly starcompact.

Proof. Let $X = A \cup B$, where $A = \{a_\alpha : \alpha < \mathfrak{c}\}$ is any set with $|A| = \mathfrak{c}$ and B is any finite set with $|B| \geq 3$ such that any element of B is not in A. Topologize X as follows: for each $a_\alpha \in A$ and each finite subset F of B, $\{a_\alpha\} \cup (B \setminus F)$ is a basic open neighborhood of a_α and each element of B is isolated.

First we show that X is set strongly 1-starcompact. Let C be any nonempty subset of X and $\mathcal{U}$ be any open cover of $\overline{C}$. If $C \subset A$. Then for each $a_\alpha \in C$, there exists $U_\alpha \in \mathcal{U}$ such that $a_\alpha \in U_\alpha$. Then for each $a_\alpha \in C$, we can find a finite set F_α such that $\{a_\alpha\} \cup (B \setminus F_\alpha) \subseteq U_\alpha$. Then $C \subset St(B, \mathcal{U})$ and B is a finite subset of X. Now if $C \subset B$. Since B is finite, we have nothing to prove. Hence X is set strongly 1-starcompact.

In [11], it is showed that X is not set strongly starcompact. □

It is not difficult to see that every countably compact space is set strongly 1-starcompact (equivalently, strongly 1-starcompact). In [3, Proposition 2.2] the authors show that for Hausdorff spaces set strongly starcompactness, strongly 1-starcompactness and countable compactness are equivalent properties. Such an equivalence is not true for T_1 spaces (see [4, Example 2.3.7]). (Note that in [3] there is a misprint in the statement of the definition of "relatively* SSC": the authors write that the set "F is a finite subset of A" instead of "F is a finite subset of $\overline{A}$".)

Example 2.3. There exists a Tychonoff set strongly star-Menger space (hence set star-Menger space) which is not set strongly 1-starcompact (equivalently, strongly 1-starcompact).

Proof. Let

$$X = ([0, \omega] \times [0, \omega]) \setminus \{\langle \omega, \omega \rangle\}$$

be the subspace of the product space $[0, \omega] \times [0, \omega]$. Then X is a Tychonoff space.

Since X is countable, it is a Menger space. Since every Menger space is set strongly star-Menger, we have that X is set strongly star-Menger.

Next we show that X is not set strongly 1-starcompact. Let $A = \{\langle n, \omega\rangle : n \in \omega\}$ and for each $n \in \omega$, let

$$U_n = \{n\} \times [n, \omega].$$

Consider the open cover $\mathcal{U} = \{U_n : n \in \omega\}$ of $A = \overline{A}$. Let F be any finite subset of X. Then there exists $n_1 \in \omega$ such that $U_n \cap F = \emptyset$ for each $n > n_1$. Let $a = \langle m, \omega\rangle$ with $m > n_1$. But U_m is the only element of $\mathcal{U}$ containing $\langle m, \omega\rangle$. Thus $a = \langle m, \omega\rangle \notin St(F, \mathcal{U})$ which shows that X is not set strongly 1-starcompact. This completes the proof. □

Since set strongly 1-starcompactness and strongly 1-starcompact properties are equivalent, we have the following two examples:

Example 2.4. ([4, Example 2.3.8]) There exists a Hausdorff 1-starcompact space which is not set strongly 1-starcompact.

Example 2.5. ([4, Example 2.3.7]) There exists a T_1 set strongly 1-starcompact space which is not countably compact.

Recall that a space X is metacompact if each open cover of X has a point finite refinement.

Theorem 2.6. ([4]) *Strongly 1-starcompact metacompact spaces are compact.*

Since compact ⇒ countably compact ⇒ set strongly starcompact ⇒ set strongly 1-starcompact ⇔ strongly 1-starcompact, thus Theorem 2.6 leads to the following theorem.

Theorem 2.7. *If X is a metacompact space, then the following are equivalent:*

1. *X is compact;*
2. *X is countably compact;*
3. *X is set strongly starcompact;*
4. *X is set strongly 1-starcompact.*

Ikenaga [6] gave the concept of $\mathcal{K}$-starcompact spaces and later Song [13, 14] concentrated on the basic topological properties of this space.

Definition 2.8. A space X is said to be $\mathcal{K}$-starcompact if for every open cover $\mathcal{U}$ of the space X, there exists a compact subset K of X such that $St(K, \mathcal{U}) = X$.

In [3] Bonanzinga and Maesano introduced the following set version of $\mathcal{K}$-starcompactness and showed that it is equivalent to $\mathcal{K}$-starcompactness [3, Proposition 1.2].

Definition 2.9. A space X is said to be set $\mathcal{K}$-starcompact if for each nonempty subset A of X and for each collection $\mathcal{U}$ of open sets of X such that $\overline{A} \subset \bigcup \mathcal{U}$, there is a compact subset K of X such that $A \subset St(K, \mathcal{U})$.

Note that a Hausdorff $\mathcal{K}$-starcompact (equivalently, set $\mathcal{K}$-starcompact) space need not be set starcompact [3, Example 3.3] and a set starcompact (hence 1-starcompact) space need not be $\mathcal{K}$-starcompact (equivalently, set $\mathcal{K}$-starcompact) space.

3. Topological properties

In this section, we study the topological properties of the considered set covering properties. Recall that the set strongly 1-starcompactness (equiv., strongly 1-starcompactness) and set $\mathcal{K}$-starcompactness (equiv., $\mathcal{K}$-starcompactness) are not hereditary properties. Next we show that these properties are preserved by clopen subspaces.

Theorem 3.1. *A clopen subspace of a set strongly 1-starcompact (equiv., strongly 1-starcompact) space is also set strongly 1-starcompact (equiv., strongly 1-starcompact) space.*

Proof. Let X be set strongly 1-starcompact space and $A \subset X$ be a clopen subset of X. Let B be any subset of A and $\mathcal{U}$ be any collection of open sets in (A, τ_A) such that $Cl_A(B) \subset \bigcup \mathcal{U}$. Since A is open, then $\mathcal{U}$ is a collection of open sets in X and since A is closed, we have $Cl_A(B) = Cl_X(B)$. Then by set strongly 1-starcompactness of X, there exists a finite subset F of X such that $B \subset St(F, \mathcal{U})$ and since $\mathcal{U}$ is collection of open sets in A, we have $St(F, \mathcal{U}) = St(G, \mathcal{U})$, where $G = F \cap A$. Hence A is also set strongly 1-starcompact. □

Theorem 3.2. *A clopen subspace of a set $\mathcal{K}$-starcompact (equiv., $\mathcal{K}$-starcompact) space is also set $\mathcal{K}$-starcompact (equiv., $\mathcal{K}$-starcompact) space.*

Now we consider the Alexandroff duplicate $A(X) = X \times \{0, 1\}$ of a space X. The basic neighborhoods of a point $\langle x, 0\rangle \in X \times \{0\}$ is of the form $(U \times \{0\}) \bigcup (U \times \{1\} \setminus \{\langle x, 1\rangle\})$, where U is any neighborhood of $x \in X$ and each point $\langle x, 1\rangle \in X \times \{1\}$ is isolated point.

Example 3.3. There exists a Tychonoff set strongly 1-starcompact (equiv., strongly 1-starcompact) space X such that $A(X)$ is not set strongly 1-starcompact (equiv., strongly 1-starcompact).

Proof. Let X be the same space X as in the proof of Example 2.2. Thus X is set strongly 1-starcompact. Let $A = \{\langle \alpha, 1\rangle : \alpha < \mathfrak{c}\}$. Then A is a clopen subset of $A(X)$ with $|A| = \mathfrak{c}$ and each point of A is isolated which clearly shows that A is not set strongly 1-starcompact. Hence $A(X)$ is not set strongly 1-starcompact since every clopen subset of set strongly 1-starcompact space is set strongly 1-starcompact. □

Recall that the extent of a topological space X, $e(X)$ is the supremum of cardinalities of all closed discrete subsets of X. Singh showed that any space whose Alexandroff duplicate space is set starcompact has finite extent. We show some similar result for set strongly 1-starcompact space and set $\mathcal{K}$-starcompact spaces.

Theorem 3.4. *1If X is a T_1-space and $A(X)$ is a set strongly 1-starcompact (equiv., strongly 1-starcompact) space. Then $e(X) < \omega$.*

Proof. Suppose that $e(X) \geq \omega$. Then there exists a discrete closed subset B of X such that $|B| \geq \omega$. Hence $B \times \{1\}$ is a clopen subset of $A(X)$ and every point of $B \times \{1\}$ is an isolated point. Clearly $B \times \{1\}$ is not set strongly 1-starcompact. By Theorem 3.1, every clopen subset of a set strongly 1-starcompact space is set strongly 1-starcompact, hence $A(X)$ is not set strongly 1-starcompact. □

Theorem 3.5. *If X is a T_1-space and $A(X)$ is a set $\mathcal{K}$-starcompact (equiv., $\mathcal{K}$-starcompact) space, then $e(X) < \omega$.*

Theorem 3.6. *Let X be a space such that the Alexandroff duplicate $A(X)$ of X is set strongly 1-starcompact (equiv., strongly 1-starcompact). Then X is also set strongly 1-starcompact (equiv., strongly 1-starcompact).*

Proof. Let B be any nonempty subset of X and $\mathcal{U}$ be any open cover of $\overline{B}$. Let $C = B \times \{0\}$ and $A(\mathcal{U}) = \{U \times \{0, 1\} : U \in \mathcal{U}\}$. Then $A(\mathcal{U})$ is an open cover of $\overline{C}$. Since $A(X)$ is set strongly 1-starcompact, there is a finite subset F of $A(X)$ such that $C = B \times \{0\} \subset St(F, A(\mathcal{U}))$. As F is finite subset of $A(X)$, let $F = \{(x_i, 0) : 1 \leq i \leq m\} \cup \{(y_j, 1) : 1 \leq j \leq n\}$. Take $G = \{x_i : 1 \leq i \leq m\} \cup \{y_j : 1 \leq j \leq n\}$. Then clearly $B \subset St(G, \mathcal{U})$ and since G is finite subset of X, we can say that X is set strongly 1-starcompact. □

Lemma 3.7. ([15]) *For a T_1-space X, $e(X) = e(A(X))$.*

Next we give the partial solution of the open problem posed by Singh in ([11], Problem 3.8).

Theorem 3.8. *If X is a T_1 set starcompact space (resp., set strongly starcompact) and $e(X) < \omega$, then $A(X)$ is set starcompact space (resp., set strongly starcompact).*

Proof. We prove the result only for the set starcompact spaces. The proof for set strongly starcompact spaces is on similar lines.

Let B be any nonempty subset of $A(X)$. If $|\{x : \langle x, 0\rangle \in B\}| < \omega$, then by the hypothesis $A(X)$ is set starcompact. So, let $|\{x : \langle x, 0\rangle \in B\}| > \omega$ and $\mathcal{U}$ be any open cover of $\overline{B}$. Then $C = \{x \in X : \langle x, 0\rangle \in B\}$ is a nonempty subset of X. For each $x \in \overline{C}$, choose an open neighborhood $W_x = (V_x \times \{0, 1\}) \setminus \{\langle x, 1\rangle\}$ of $\langle x, 0\rangle$ satisfying that there exists some $U \in \mathcal{U}$ such that $W_x \subset U$, where V_x is an open subset of X containing x.

Let $\mathcal{W} = \{V_x : x \in \overline{C}\}$. Then $\mathcal{W}$ is an open cover of $\overline{C}$. Since X is set starcompact, there is a finite subset $\mathcal{V}$ of $\mathcal{W}$ such that $C \subset St(\cup\mathcal{V}, \mathcal{W})$. For each $V_x \in \mathcal{V}$, we have W_x neighborhood of $\langle x, 0\rangle$ such that $W_x \subset U$ for some $U \in \mathcal{U}$. Collect all such U and let $\mathcal{H} = \{U : W_x \subset U, V_x \in \mathcal{V}\}$. Then $\mathcal{H}$ is a finite subset of $\mathcal{U}$ and $C \times \{0\} \subset St(\cup\mathcal{H}, \mathcal{U})$. Let $A = [C \times \{0, 1\} \cap (A(X) \setminus St(\cup\mathcal{H}, \mathcal{U})] \cup \{\langle y, 1\rangle \in B : \langle y, 1\rangle \notin C \times \{1\}\}$. Then A is a discrete closed subset of $A(X)$. Since every subspace of T_1 space is T_1 space, by lemma 3.7 the set A is finite so we can enumerate A as $\{a_1, a_2, \ldots a_n\}$. Let $\mathcal{V}' = \mathcal{V} \cup \{V_{a_1}, V_{a_2}, \ldots, V_{a_n}\}$, where V_{a_i} is an element of $\mathcal{U}$ containing a_i for $i = 1, 2, \ldots, n$. Thus $\mathcal{V}'$ is a finite subset of $\mathcal{U}$ and $B \subset St(\cup\mathcal{V}', \mathcal{U})$. Hence $A(X)$ is set starcompact space. □

Recall that the continuous images of strongly 1-starcompact (equiv., set strongly 1-starcompact) and $\mathcal{K}$-starcompact (equiv., set $\mathcal{K}$-starcompact) spaces are strongly 1-starcompact (equiv., set strongly 1-starcompact) and $\mathcal{K}$-starcompact (equiv., set $\mathcal{K}$-starcompact) spaces, respectively.

Theorem 3.9. *Let $f : X \to Y$ be an open, closed and finite-to-one continuous mapping from a space X onto a set strongly 1-starcompact (equiv., strongly 1-starcompact) space Y. Then X is also a set strongly 1-starcompact (equiv., strongly 1-starcompact) space.*

Proof. Let $A \subset X$ be any nonempty set and $\mathcal{U}$ be a collection of open sets in X such that $\overline{A} \subset \bigcup\mathcal{U}$. Then $B = f(A)$ is a subset of Y. Let $y \in \overline{B}$, then $f^{-1}(y)$ is a finite subset of X, thus there is a finite subset $\mathcal{U}_y$ of $\mathcal{U}$ such that $f^{-1}(y) \subset \bigcup\mathcal{U}_y$ and $U \bigcap f^{-1}(y) \neq \emptyset$ for each $U \in \mathcal{U}_y$. Since f is closed, there exists an open neighborhood V_y of y in Y such that $f^{-1}(V_y) \subset \bigcup\{U : U \in \mathcal{U}_y\}$. Since f is open, we can assume that $V_y \subset \bigcap\{f(U) : U \in \mathcal{U}_y\}$. then $\mathcal{V} = \{V_y : y \in \overline{B}\}$ is an open cover of $\overline{B}$. Since Y is set strongly 1-starcompact, there exists a finite subset F of Y such that $B \subset St(F, \mathcal{V})$. Since f is finite to one mapping, therefore $f^{-1}(F)$ is also a finite subset of X.

We prove that $A \subset St(f^{-1}(F), \mathcal{U})$. Let $x \in A$. Then there exists some $y \in \overline{B}$ such that $f(x) \in V_y$ and $V_y \cap F \neq \emptyset$. Since $x \in f^{-1}(V_y) \subset \bigcup\{U : U \in \mathcal{U}_y\}$, we can choose $U \in \mathcal{U}_y$ with $x \in U$. then $V_y \subset f(U)$ which implies $U \cap f^{-1}(F) \neq \emptyset$. Hence $x \in St(f^{-1}(F), \mathcal{U})$. Therefore X is set strongly 1-starcompact. □

Recall that [9] a function $f : X \to Y$ is said to have the containment property if for each subset $A \subset X$ such that $f(A) = B$, then $f^{-1}(B) \subseteq \overline{A}$.

Theorem 3.10. *If f is an open perfect map with containment property from a space X onto a set starcompact space Y, then X is set starcompact.*

Proof. Let A be a nonempty subset of X and $\mathcal{U}$ be any open cover of $\overline{A}$. Since f is closed, thus $f(\overline{A})$ is closed. Let $f(\overline{A}) = B$ for some subset B of Y and let $y \in B$. Since $f^{-1}(y)$ is compact and f has the containment property, there exists a finite subcollection $\mathcal{U}_y$ of $\mathcal{U}$ such that $f^{-1}(y) \subseteq \bigcup\mathcal{U}_y$ and $U \cap f^{-1}(y) \neq \emptyset$ for each $U \in \mathcal{U}_y$. Let $U_y = \bigcup\mathcal{U}_y$. Then $V_y = Y \setminus f(X \setminus U_y)$ is a neighborhood of y since f is closed. Then $\mathcal{V} = \{V_y : y \in B\}$ is an open cover of closed set B. Since Y is set starcompact, there exists a finite subset $\mathcal{V}'$ of $\mathcal{V}$ such that $B \subset St(\bigcup\mathcal{V}', \mathcal{V})$. Without loss of generality, we may assume that $\mathcal{V}' = \{V_{y_i} : 1 \leq i \leq n\}$. Let $\mathcal{W} = \bigcup_i \mathcal{U}_{y_i}$, then $\mathcal{W}$ is a finite subset of $\mathcal{U}$ and $f^{-1}(\bigcup\mathcal{V}') = \bigcup\mathcal{W}$. Since $f^{-1}(V_{y_i}) \subset \bigcup\{U : U \in \mathcal{U}_{y_i}\}$ for each i, we have $A \subset St(\bigcup \mathcal{W}, \mathcal{U})$. To show this, let $x \in A$. Then there exists $y \in B$ such that $f(x) \in V_y$ and $V_y \bigcap (\bigcup \mathcal{V}') \neq \emptyset$. Since $x \in f^{-1}(V_y) \subset \bigcup\{U : U \in \mathcal{U}_y\}$, we can choose $U \in \mathcal{U}_y$ with $x \in U$. As $V_y \subset f(U)$, we have $U \bigcap f^{-1}(\bigcup \mathcal{V}') \neq \emptyset$. Hence $x \in St(f^{-1}(\bigcup \mathcal{V}'), \mathcal{U})$ and therefore $x \in St(\bigcup \mathcal{W}, \mathcal{U})$ which shows that $A \subset St(\bigcup \mathcal{W}, \mathcal{U})$. Hence X is set starcompact space. □

Call the product space $X \times Y$ rectangular set starcompact if it satisfies the defining condition for the set starcompact property for subsets of $X \times Y$ of the form $A = B \times C$, where B and C are the nonempty subsets of X and Y, respectively. Now we prove that the product of set starcompact space and a compact Hausdorff space is rectangular set starcompact space.

Theorem 3.11. *The product of a set starcompact space X and a compact Hausdorff space Y is rectangular set starcompact.*

Proof. Let X be any set starcompact space and Y be any compact Hausdorff space. Let $A = B \times C$ be the subset of $X \times Y$ and $\mathcal{U}$ be any cover of $\overline{A} = \overline{B \times C} = \overline{B} \times \overline{C}$. Then there exists open covers $\mathcal{V}$ and $\mathcal{W}$ of $\overline{B}$ and $\overline{C}$ such that $\mathcal{U} = \mathcal{V} \times \mathcal{W}$. Since X is set starcompact, so there exists finite subset $\mathcal{V}'$ of $\mathcal{V}$ such that $B \subset St(\cup \mathcal{V}', \mathcal{V})$. Since Y is compact, so there exists finite subset $\mathcal{W}'$ of $\mathcal{W}$ such that $C \subset \cup \mathcal{W}'$. Put $\mathcal{U}' = \mathcal{V}' \times \mathcal{W}'$ which is a finite subset of $\mathcal{U}$ such that $A \subset St(\cup \mathcal{U}', \mathcal{U})$. Hence the space $X \times Y$ is rectangular set starcompact. □

The above theorem is a partial solution of the following open problem posed by Singh in [11].

Problem 3.12. ([11]) *Is the product of a set starcompact and a compact space a set starcompact space?*

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