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Exponential Stability in Mean Square of Neutral Stochastic Pantograph Integro-Differential Equations

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Abstract. In this paper, we show two new results on the existence and uniqueness of the solution of Neutral Stochastic Pantograph Integro-Differential Equations (NSPIDE) and the exponential stability in mean square using the one-sided Growth Condition. One example is exhibited to show the interest of our results.

1. Introduction

Stochastic differential equations (SDE) are an important tools to model much phenomena such as economics, biology ... In the last decades, SDE takes much more attention (see [2] and [10]). Many researchers have investigated the theory of stability for a SDE with delay (SDDE) (see [4], [7], [10], [13], [16], [18] and [20]).

If we consider the approximation error for the parameters and the perturbed noise term of the system, it would be better to approximate the parameters as a point estimator plus an error. Using the central limit theorem, the error may be characterized by a random variable with normal distribution, which gives the pantograph form of the SDDE.

The stochastic pantograph differential equation (SPDE) is an important extension of SDDE (see [1], [3], [5], [6] and [11]). The SPDE appears in many different areas on pure and applied mathematics such as dynamical systems, number theory and electrodynamics (see [1], [3], [5], [6] and [11]). In the last decades, as one of the most important class of neutral SDE, the neutral SPDE (NSPDE) (see [8], [9], [15] and [19]).

Stability of NSPDE has attracted much more attention (see [9] and [12]). Many researchers have investigated the almost sure exponential and polynomial stability of NSPDE (see [1], [5],[6], [8], [11] and [12]-[19]). To the best of our knowledge, there is no existing result on the existence, the uniqueness and the exponential stability in mean square of NSPIDE. By applying the Gronwall inequality, the stochastic analysis techniques and the one-sided Growth Condition, we investigate the existence and uniqueness

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of solution of NSPIDE and exhibit a new sufficient conditions ensuring the exponential stability in mean square. Our paper generalize the results obtained in [17] and [19].

Now, we will summarize the main contributions of our paper:
(1) Under assumptions $\mathcal{A}_5$-$\mathcal{A}_7$, the theory of the exponential stability in mean square cannot be applied as done in other papers like [17] and [19].
(2) In [17], the class of nonlinear hybrid stochastic pantograph equations is more restrictive than the ones in this paper due to the neutral and the Integro-Differential terms .
(3) Different from the work in [19], considering the influence of the neutral term in the stability analysis makes our results more general.

The framework of the paper is as follows: In Section 2, we give some Preliminaries and definitions. In Section 3, we study the existence and uniqueness of the solution of NSPIDE. In section 4, we investigate the exponential stability in mean square of NSPIDE. Finally, in Section 5, we exhibit an example to prove our main results.

2. Preliminaries and definitions

Let $\{\Omega, \mathcal{F}, (\mathcal{F}_\varrho)_{\varrho\geq\varrho_0}, \mathbb{P}\}$ be a complete probability space with a filtration satisfying the usual conditions and $W(\varrho)$ is an m-dimensional Brownian motion defined on the probability space.
Let $\mathcal{L}^2([c,e],\mathbb{R}^d)$ be the family of $\mathbb{R}^d$-valued $\mathcal{F}_\varrho$-adapted processes $\{k(\varrho)\}_{c\leq\varrho\leq e}$ such that $\int_c^e |k(\varrho)|^2 d\varrho < \infty$ a.s. and $\mathcal{M}^2([c,e],\mathbb{R}^d)$ the family of processes $\{k(\varrho)\}_{c\leq\varrho\leq e}$ in $\mathcal{L}^2([c,e],\mathbb{R}^d)$ such that $\mathbb{E}\int_c^e |k(\varrho)|^2 d\varrho < \infty$. Let $\varrho \geq \varrho_0 > 0$ and $C([q\varrho,\varrho];\mathbb{R}^d)$ denote the family of functions μ from $[q\varrho,\varrho]$ to $\mathbb{R}^d$ that are continuous and equipped with the norm $\|\mu\| = \sup_{q\varrho\leq s\leq\varrho}|\mu(s)|$ and $|\eta| = \sqrt{\eta^T\eta}$ for any $\eta \in \mathbb{R}^d$. Let $L^2_{\mathcal{F}_\varrho}([q\varrho,\varrho];\mathbb{R}^d)$, $\varrho \geq \varrho_0 > 0$, denote the family of all $\mathcal{F}_\varrho$-measurable, $C([q\varrho,\varrho];\mathbb{R}^d)$-valued random variables $\mu = \{\mu(\theta) : q\varrho \leq \theta \leq \varrho\}$ such that $\mathbb{E}\|\mu\|^2 < \infty$.

Consider the following NSPIDE: for $\varrho_0 \leq \varrho \leq \Lambda$,

$$d\left[\eta(\varrho) - D(\varrho,\eta(q\varrho))\right] = f_1\left(\varrho,\eta(\varrho),\eta(q\varrho),\int_{q\varrho}^{\varrho} g(t,\eta(t))dt\right)d\varrho + f_2\left(\varrho,\eta(\varrho),\eta(q\varrho),\int_{q\varrho}^{\varrho} g(t,\eta(t))dt\right)dW(\varrho), \quad (1)$$

with the initial condition

$$\{\eta(\varrho), q\varrho_0 \leq \varrho \leq \varrho_0, \varrho_0 \in (0,a]\} = \chi \in L^2_{\mathcal{F}_{\varrho_0}}([q\varrho_0,\varrho_0];\mathbb{R}^d), \quad (2)$$

where $q \in (0,1)$ and $a > 0$. We assume that

$$f_1 : [\varrho_0,\Lambda]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d \longrightarrow \mathbb{R}^d, \quad f_2 : [\varrho_0,\Lambda]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d \longrightarrow \mathbb{R}^{d\times m},$$

$$g : [\varrho_0,\Lambda]\times\mathbb{R}^d \longrightarrow \mathbb{R}^d, \quad D : [\varrho_0,\Lambda]\times\mathbb{R}^d \longrightarrow \mathbb{R}^d.$$

Using the Itô's stochastic techniques and integrating the equation (1) on both sides from ϱ_0 to ϱ, we have

$$\eta(\varrho) - D(\varrho,\eta(q\varrho)) = \eta(\varrho_0) - D(\varrho_0,\eta(q\varrho_0)) + \int_{\varrho_0}^{\varrho} f_1\left(s,\eta(s),\eta(qs),\int_{qs}^{s} g(t,\eta(t))dt\right)ds$$

$$+ \int_{\varrho_0}^{\varrho} f_2\left(s,\eta(s),\eta(qs),\int_{qs}^{s} g(t,\eta(t))dt\right)dW(s), \quad \varrho_0 \leq \varrho \leq \Lambda. \quad (3)$$

We assume that for all $\varrho \in [\varrho_0,\Lambda]$ and $q\varrho \leq s \leq \varrho$,

$$f_1(\varrho,0,0,0) = 0, \quad f_2(\varrho,0,0,0) = 0, \quad g(s,0) = 0, \quad D(\varrho,0) = 0. \quad (4)$$

We will impose some assumptions which guarantee the existence and uniqueness of a solution, denoted by $\eta(\varrho;\varrho_0;\chi)$, for equation (1).
$\mathcal{A}_1$: Assume that there is a constant $L > 0$ such that

$$|f_1(\varrho,\eta,z,v) - f_1(\varrho,\overline{\eta},\overline{z},\overline{v})|^2 \vee |f_2(\varrho,\eta,z,v) - f_2(\varrho,\overline{\eta},\overline{z},\overline{v})|^2$$

$$\leq L\left(|\eta-\overline{\eta}|^2 + |z-\overline{z}|^2 + |v-\overline{v}|^2\right), \tag{5}$$

for all $\varrho \in [\varrho_0,\Lambda]$ and $\eta, \overline{\eta}, z, \overline{z}, v, \overline{v} \in \mathbb{R}^d$, where the notation $c \vee e$ define the maximum of c and e.
$\mathcal{A}_2$: Assume that there is a constant $L_1 > 0$ such that for all $(\varrho,\eta,z,v) \in [\varrho_0,\Lambda]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d$

$$|f_1(\varrho,\eta,z,v)|^2 \vee |f_2(\varrho,\eta,z,v)|^2 \leq L_1\left(1 + |\eta|^2 + |z|^2 + |v|^2\right). \tag{6}$$

$\mathcal{A}_3$: Assume that there is a constant $\zeta \in (0,1)$ such that

$$|D(\varrho,z) - D(\varrho,\overline{z})| \leq \zeta|z-\overline{z}|, \tag{7}$$

for all $(\varrho,z,\overline{z}) \in [\varrho_0,\Lambda]\times\mathbb{R}^d\times\mathbb{R}^d$.
$\mathcal{A}_4$: Assume that there is a constant $L_2 \geq 0$ such that

$$|g(\varrho,\eta) - g(\varrho,\overline{\eta})| \leq L_2|\eta-\overline{\eta}|, \tag{8}$$

for all $(\varrho,\eta,\overline{\eta}) \in [\varrho_0,\Lambda]\times\mathbb{R}^d\times\mathbb{R}^d$.
We cite now a technical lemma which is very useful for our results.

Lemma 2.1. *Let $a_1, a_2 \geq 0$ and $0 < \varepsilon < 1$. Then*

$$(a_1+a_2)^2 \leq \frac{a_1^2}{\varepsilon} + \frac{a_2^2}{1-\varepsilon}.$$

Proof. see [10]. □

3. The existence and uniqueness of the solution

This section is devoted to study the existence and uniqueness of the solution for Equation (1). For this purpose, we provide the following Lemma which will give an estimation that could be used to prove theorem 3.2.

Lemma 3.1. *Let assumptions $\mathcal{A}_2 - \mathcal{A}_4$ hold. Let $\eta(\varrho)$ be a solution of equation (1) with the initial condition (2), then*

$$\begin{aligned}
&\mathbb{E}\left(\sup_{q\varrho_0\leq s\leq\Lambda}|\eta^i(s)|^2\right)\\
\leq\ &\left[1+\frac{\left(2+\Lambda^2L_2^2\right)}{(1-\sqrt{\zeta})^2}\left((1-\sqrt{\zeta})^2+\sqrt{\zeta}+\frac{3}{1+\sqrt{\zeta}}\left(1+L_1\left(\Lambda-\varrho_0+4\right)\left(\Lambda-\varrho_0\right)(1+\Lambda^2L_2^2)\right)\right)\mathbb{E}\|\chi\|^2\right]\\
\times\ &\exp\left[\frac{3L_1\left(\Lambda-\varrho_0+4\right)\left(\Lambda-\varrho_0\right)\left(2+\Lambda^2L_2^2\right)}{(1-\sqrt{\zeta})^2(1+\sqrt{\zeta})}\right].
\end{aligned} \tag{9}$$

In particular, $\eta(\varrho) \in \mathcal{M}^2\left([q\varrho_0,\Lambda],\mathbb{R}^d\right)$.

Proof. For any integer $i \geq 1$, let σ_i be the stopping time define by:

$$\sigma_i = \Lambda \wedge \inf\{\varrho \in [\varrho_0, \Lambda]; |\eta(\varrho)| \geq i\}.$$

It is clear that $\sigma_i \uparrow \Lambda$ a.s. Let $\eta^i(\varrho) = \eta(\varrho \wedge \sigma_i)$, for $\varrho \in [q\varrho_0, \Lambda]$. Thus, for $\varrho_0 \leq \varrho \leq \Lambda$, $\eta^i(\varrho) = D(\varrho, \eta^i(q\varrho)) - D(\varrho_0, \chi) + I^i(\varrho)$, where

$$\begin{aligned} I^i(\varrho) &= \eta(\varrho_0) + \int_{\varrho_0}^{\varrho} f_1\left(s, \eta^i(s), \eta^i(qs), \int_{qs}^{s} g(t, \eta^i(t))dt\right) 1_{[\varrho_0, \sigma_i]}(s)ds \\ &+ \int_{\varrho_0}^{\varrho} f_2\left(s, \eta^i(s), \eta^i(qs), \int_{qs}^{s} g(t, \eta^i(t))dt\right) 1_{[\varrho_0, \sigma_i]}(s)dW(s). \end{aligned} \tag{10}$$

By lemma 2.1 and condition (7), we can derive that

$$\begin{aligned} |\eta^i(\varrho)|^2 &\leq \frac{1}{\zeta}|D(\varrho, \eta^i(q\varrho)) - D(\varrho_0, \chi)|^2 + \frac{1}{1-\zeta}|I^i(\varrho)|^2, \\ &\leq \frac{1}{\zeta\sqrt{\zeta}}|D(\varrho, \eta^i(q\varrho))|^2 + \frac{1}{\zeta(1-\sqrt{\zeta})}|D(\varrho_0, \chi)|^2 + \frac{1}{1-\zeta}|I^i(\varrho)|^2, \\ &\leq \sqrt{\zeta}|\eta^i(q\varrho)|^2 + \frac{\zeta}{1-\sqrt{\zeta}}\|\chi\|^2 + \frac{1}{1-\zeta}|I^i(\varrho)|^2. \end{aligned} \tag{11}$$

Thus,

$$\begin{aligned} \mathbb{E}\left(\sup_{\varrho_0 \leq s \leq \varrho} |\eta^i(s)|^2\right) &\leq \sqrt{\zeta}\mathbb{E}\left(\sup_{\varrho_0 \leq s \leq \varrho} |\eta^i(qs)|^2\right) + \frac{\zeta}{1-\sqrt{\zeta}}\mathbb{E}\|\chi\|^2 + \frac{1}{1-\zeta}\mathbb{E}\left(\sup_{\varrho_0 \leq s \leq \varrho} |I^i(s)|^2\right) \\ &\leq \sqrt{\zeta}\mathbb{E}\left(\sup_{q\varrho_0 \leq s \leq \varrho} |\eta^i(s)|^2\right) + \frac{\zeta}{1-\sqrt{\zeta}}\mathbb{E}\|\chi\|^2 + \frac{1}{1-\zeta}\mathbb{E}\left(\sup_{\varrho_0 \leq s \leq \varrho} |I^i(s)|^2\right) \\ &= \sqrt{\zeta}\mathbb{E}\left(\sup_{\varrho_0 \leq s \leq \varrho} |\eta^i(s)|^2\right) + \sqrt{\zeta}\mathbb{E}\|\chi\|^2 + \frac{\zeta}{1-\sqrt{\zeta}}\mathbb{E}\|\chi\|^2 + \frac{1}{1-\zeta}\mathbb{E}\left(\sup_{\varrho_0 \leq s \leq \varrho} |I^i(s)|^2\right) \\ &= \frac{\sqrt{\zeta}}{1-\sqrt{\zeta}}\mathbb{E}\|\chi\|^2 + \sqrt{\zeta}\mathbb{E}\left(\sup_{\varrho_0 \leq s \leq \varrho} |\eta^i(s)|^2\right) + \frac{1}{1-\zeta}\mathbb{E}\left(\sup_{\varrho_0 \leq s \leq \varrho} |I^i(s)|^2\right). \end{aligned} \tag{12}$$

Hence,

$$\mathbb{E}\left(\sup_{\varrho_0 \leq s \leq \varrho} |\eta^i(s)|^2\right) \leq \frac{\sqrt{\zeta}}{(1-\sqrt{\zeta})^2}\mathbb{E}\|\chi\|^2 + \frac{1}{(1-\sqrt{\zeta})^2(1+\sqrt{\zeta})}\mathbb{E}\left(\sup_{\varrho_0 \leq s \leq \varrho} |I^i(s)|^2\right). \tag{13}$$

Moreover, by Hölder inequality, condition (6) and Doob's martingale inequality, we have

$$\begin{aligned} \mathbb{E}\left(\sup_{\varrho_0 \leq s \leq \varrho} \left|I^i(s)\right|^2\right) &\leq 3\mathbb{E}\|\chi\|^2 + 3\mathbb{E}\left|\int_{\varrho_0}^{\varrho} f_1\left(s, \eta^i(s), \eta^i(qs), \int_{qs}^{s} g(t, \eta^i(t))dt\right) 1_{[\varrho_0, \sigma_i]}(s)ds\right|^2 \\ &+ 3\mathbb{E}\left|\int_{\varrho_0}^{\varrho} f_2\left(s, \eta^i(s), \eta^i(qs), \int_{qs}^{s} g(t, \eta^i(t))dt\right) 1_{[\varrho_0, \sigma_i]}(s)dW(s)\right|^2 \\ &\leq 3\mathbb{E}\|\chi\|^2 + 3(\Lambda - \varrho_0)\,\mathbb{E}\int_{\varrho_0}^{\varrho} \left|f_1\left(s, \eta^i(s), \eta^i(qs), \int_{qs}^{s} g(t, \eta^i(t))dt\right)\right|^2 ds \\ &+ 12\mathbb{E}\int_{\varrho_0}^{\varrho} \left|f_2\left(s, \eta^i(s), \eta^i(qs), \int_{qs}^{s} g(t, \eta^i(t))dt\right)\right|^2 ds \end{aligned}$$

$$\leq 3\mathbb{E}\|\chi\|^2 + 3L_1\,(\Lambda - \varrho_0)\,\mathbb{E}\int_{\varrho_0}^{\varrho}\left(1 + |\eta^i(s)|^2 + |\eta^i(qs)|^2 + |\int_{qs}^{s} g(t,\eta^i(t))dt|^2\right)ds$$
$$+ 12L_1\mathbb{E}\int_{\varrho_0}^{\varrho}\left(1 + |\eta^i(s)|^2 + |\eta^i(qs)|^2 + |\int_{qs}^{s} g(t,\eta^i(t))dt|^2\right)ds. \quad (14)$$

By condition (8), we can derive that

$$|\int_{qs}^{s} g(t,\eta^i(t))dt|^2 \leq \Lambda\int_{qs}^{s}|g(t,\eta^i(t))|^2dt \leq \Lambda L_2^2\int_{qs}^{s}|\eta^i(t)|^2dt$$
$$\leq \Lambda L_2^2\int_{qs}^{s}\sup_{qs\leq r\leq s}|\eta^i(r)|^2dr \leq \Lambda^2L_2^2\sup_{qs\leq r\leq s}|\eta^i(r)|^2. \quad (15)$$

Consequently,

$$\mathbb{E}\left(\sup_{\varrho_0\leq s\leq\varrho}\left|I^i(s)\right|^2\right) \leq 3\mathbb{E}\|\chi\|^2 + 3L_1\,(\Lambda - \varrho_0)\,\mathbb{E}\int_{\varrho_0}^{\varrho}\left(1 + |\eta^i(s)|^2 + |\eta^i(qs)|^2 + \Lambda^2L_2^2\sup_{qs\leq r\leq s}|\eta^i(r)|^2\right)ds$$
$$+ 12L_1\mathbb{E}\int_{\varrho_0}^{\varrho}\left(1 + |\eta^i(s)|^2 + |\eta^i(qs)|^2 + \Lambda^2L_2^2\sup_{qs\leq r\leq s}|\eta^i(r)|^2\right)ds$$

Therefore,

$$\mathbb{E}\left(\sup_{\varrho_0\leq s\leq\varrho}\left|I^i(s)\right|^2\right) \leq 3\mathbb{E}\|\chi\|^2 + 3L_1\,(\Lambda - \varrho_0 + 4)\,\mathbb{E}\int_{\varrho_0}^{\varrho}\left(1 + |\eta^i(s)|^2 + |\eta^i(qs)|^2 + \Lambda^2L_2^2\sup_{qs\leq r\leq s}|\eta^i(r)|^2\right)ds. \quad (16)$$

Then, taking the supremum on the right hand side of (16), we have

$$\mathbb{E}\left(\sup_{\varrho_0\leq s\leq\varrho}\left|I^i(s)\right|^2\right)$$
$$\leq 3\mathbb{E}\|\chi\|^2 + 3L_1\,(\Lambda - \varrho_0 + 4)\int_{\varrho_0}^{\varrho}\left(1 + \mathbb{E}\sup_{\varrho_0\leq r\leq s}|\eta^i(r)|^2 + \mathbb{E}\sup_{\varrho_0\leq r\leq s}|\eta^i(qr)|^2 + \Lambda^2L_2^2\mathbb{E}\sup_{q\varrho_0\leq r\leq s}|\eta^i(r)|^2\right)ds$$
$$\leq 3\left(1 + L_1\,(\Lambda - \varrho_0 + 4)\,(\Lambda - \varrho_0)(1 + \Lambda^2L_2^2)\right)\mathbb{E}\|\chi\|^2$$
$$+ 3L_1\,(\Lambda - \varrho_0 + 4)\int_{\varrho_0}^{\varrho}\left(1 + \left(2 + \Lambda^2L_2^2\right)\mathbb{E}\sup_{\varrho_0\leq r\leq s}|\eta^i(r)|^2\right)ds. \quad (17)$$

Substituting (17) into (13), it yields that

$$\mathbb{E}\left(\sup_{q\varrho_0\leq s\leq\varrho}|\eta^i(s)|^2\right)$$
$$\leq \frac{1}{(1-\sqrt{\zeta})^2}\left((1-\sqrt{\zeta})^2 + \sqrt{\zeta} + \frac{3}{1+\sqrt{\zeta}}\left(1 + L_1\,(\Lambda - \varrho_0 + 4)\,(\Lambda - \varrho_0)(1 + \Lambda^2L_2^2)\right)\right)\mathbb{E}\|\chi\|^2$$
$$+ \frac{1}{(1-\sqrt{\zeta})^2(1+\sqrt{\zeta})}3L_1\,(\Lambda - \varrho_0 + 4)\int_{\varrho_0}^{\varrho}\left(1 + \left(2 + \Lambda^2L_2^2\right)\mathbb{E}\sup_{q\varrho_0\leq r\leq s}|\eta^i(r)|^2\right)ds. \quad (18)$$

Therefore,

$$\begin{aligned}
&1+\left(2+\Lambda^2L_2^2\right)\mathbb{E}\left(\sup_{q\varrho_0\le s\le\varrho}|\eta^i(s)|^2\right)\\
\le\;& 1+\frac{\left(2+\Lambda^2L_2^2\right)}{(1-\sqrt{\zeta})^2}\left((1-\sqrt{\zeta})^2+\sqrt{\zeta}+\frac{3}{1+\sqrt{\zeta}}\left(1+L_1\left(\Lambda-\varrho_0+4\right)\left(\Lambda-\varrho_0\right)(1+\Lambda^2L_2^2)\right)\right)\mathbb{E}\|\chi\|^2\\
+\;& \frac{\left(2+\Lambda^2L_2^2\right)}{(1-\sqrt{\zeta})^2(1+\sqrt{\zeta})}3L_1\left(\Lambda-\varrho_0+4\right)\int_{\varrho_0}^{\varrho}\left(1+\left(2+\Lambda^2L_2^2\right)\mathbb{E}\sup_{q\varrho_0\le r\le s}|\eta^i(r)|^2\right)ds. \qquad (19)
\end{aligned}$$

Applying Gronwall inequality, we have

$$\begin{aligned}
&1+\left(2+\Lambda^2L_2^2\right)\mathbb{E}\left(\sup_{q\varrho_0\le s\le\Lambda}|\eta^i(s)|^2\right)\\
\le\;& \left[1+\frac{\left(2+\Lambda^2L_2^2\right)}{(1-\sqrt{\zeta})^2}\left((1-\sqrt{\zeta})^2+\sqrt{\zeta}+\frac{3}{1+\sqrt{\zeta}}\left(1+L_1\left(\Lambda-\varrho_0+4\right)\left(\Lambda-\varrho_0\right)(1+\Lambda^2L_2^2)\right)\right)\mathbb{E}\|\chi\|^2\right]\\
\times\;& \exp\left[\frac{3L_1\left(\Lambda-\varrho_0+4\right)\left(\Lambda-\varrho_0\right)\left(2+\Lambda^2L_2^2\right)}{(1-\sqrt{\zeta})^2(1+\sqrt{\zeta})}\right]. \qquad (20)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\left(\sup_{q\varrho_0\le s\le\sigma_i}|\eta^i(s)|^2\right)\\
\le\;& \left[1+\frac{\left(2+\Lambda^2L_2^2\right)}{(1-\sqrt{\zeta})^2}\left((1-\sqrt{\zeta})^2+\sqrt{\zeta}+\frac{3}{1+\sqrt{\zeta}}\left(1+L_1\left(\Lambda-\varrho_0+4\right)\left(\Lambda-\varrho_0\right)(1+\Lambda^2L_2^2)\right)\right)\mathbb{E}\|\chi\|^2\right]\\
\times\;& \exp\left[\frac{3L_1\left(\Lambda-\varrho_0+4\right)\left(\Lambda-\varrho_0\right)\left(2+\Lambda^2L_2^2\right)}{(1-\sqrt{\zeta})^2(1+\sqrt{\zeta})}\right]. \qquad (21)
\end{aligned}$$

Finally, (9) holds by letting $i\to\infty$. □

Theorem 3.2. *Let assumptions $\mathcal{A}_1-\mathcal{A}_3$ hold. Then, there is a unique solution $\eta(\varrho)$ of equation (1) such that $\eta(\varrho)\in\mathcal{M}^2([q\varrho_0,\Lambda];\mathbb{R}^d)$.*

Proof. Uniqueness: Let $\eta(\varrho)$ and $\overline{\eta}(\varrho)$ be two solutions of equation (1) such that $\eta(\varrho)=\overline{\eta}(\varrho)$, for $\varrho\in[q\varrho_0,\varrho_0]$. By lemma 3.1, $\eta(\varrho)$ and $\overline{\eta}(\varrho)$ belong to $\mathcal{M}^2([q\varrho_0,\Lambda];\mathbb{R}^d)$. Note that,

$$\eta(\varrho)-\overline{\eta}(\varrho)=D(\varrho,\eta(q\varrho))-D(\varrho,\overline{\eta}(q\varrho))+I(\varrho),$$

where

$$\begin{aligned}
I(\varrho)\;=\;& \int_{\varrho_0}^{\varrho}\left(f_1\left(s,\eta(s),\eta(qs),\int_{qs}^{s}g(t,\eta(t))dt\right)-f_1\left(s,\overline{\eta}(s),\overline{\eta}(qs),\int_{qs}^{s}g(t,\overline{\eta}(t))dt\right)\right)ds\\
+\;& \int_{\varrho_0}^{\varrho}\left(f_2\left(s,\eta(s),\eta(qs),\int_{qs}^{s}g(t,\eta(t))dt\right)-f_2\left(s,\overline{\eta}(s),\overline{\eta}(qs),\int_{qs}^{s}g(t,\overline{\eta}(t))dt\right)\right)dW(s).
\end{aligned}$$

By condition (7) and Lemma 2.1, we have

$$|\eta(\varrho)-\overline{\eta}(\varrho)|^2\le\zeta|\eta(q\varrho)-\overline{\eta}(q\varrho)|^2+\frac{1}{1-\zeta}|I(\varrho)|^2.$$

Hence,

$$\begin{aligned}\mathbb{E}\left(\sup_{\varrho_0\le s\le\varrho}|\eta(s)-\overline{\eta}(s)|^2\right) &\le \zeta\mathbb{E}\left(\sup_{\varrho_0\le s\le\varrho}|\eta(qs)-\overline{\eta}(qs)|^2\right)+\frac{1}{1-\zeta}\mathbb{E}\left(\sup_{\varrho_0\le s\le\varrho}|I(s)|^2\right)\\ &\le \zeta\mathbb{E}\left(\sup_{\varrho_0\le s\le\varrho}|\eta(s)-\overline{\eta}(s)|^2\right)+\frac{1}{1-\zeta}\mathbb{E}\left(\sup_{\varrho_0\le s\le\varrho}|I(s)|^2\right), \qquad (22)\end{aligned}$$

which yields that

$$\mathbb{E}\left(\sup_{\varrho_0\le s\le\varrho}|\eta(s)-\overline{\eta}(s)|^2\right)\le\frac{1}{(1-\zeta)^2}\mathbb{E}\left(\sup_{\varrho_0\le s\le\varrho}|I(s)|^2\right). \qquad (23)$$

Moreover, we can easily prove that

$$\mathbb{E}\left(\sup_{\varrho_0\le s\le\varrho}|I(s)|^2\right)\le 2L\left(\Lambda-\varrho_0+4\right)\left(2+\Lambda^2L_2^2\right)\int_{\varrho_0}^{\varrho}\mathbb{E}\left(\sup_{\varrho_0\le r\le s}|\eta(r)-\overline{\eta}(r)|^2\right)ds. \qquad (24)$$

Therefore substituting (24) into (23), we have

$$\mathbb{E}\left(\sup_{\varrho_0\le s\le\varrho}|\eta(s)-\overline{\eta}(s)|^2\right)\le\frac{2L\left(\Lambda-\varrho_0+4\right)\left(2+\Lambda^2L_2^2\right)}{(1-\zeta)^2}\int_{\varrho_0}^{\varrho}\mathbb{E}\left(\sup_{\varrho_0\le r\le s}|\eta(r)-\overline{\eta}(r)|^2\right)ds. \qquad (25)$$

By Gronwall inequality, we obtain

$$\mathbb{E}\left(\sup_{\varrho_0\le s\le\Lambda}|\eta(s)-\overline{\eta}(s)|^2\right)=0,$$

which implies that $\eta(\varrho)=\overline{\eta}(\varrho)$ for $\varrho_0\le\varrho\le\Lambda$, therefore for all $q\varrho_0\le\varrho\le\Lambda$, almost surely.
Existence: **Step 1**:
We suppose that $\Lambda-\varrho_0$ is sufficiently small such that

$$\theta=\zeta+\frac{2L\left(\Lambda-\varrho_0+4\right)\left(\Lambda-\varrho_0\right)}{(1-\zeta)}<1. \qquad (26)$$

Let $\eta^0(\varrho_0)=\eta^0(q\varrho_0)=\chi$ and $\eta^0(\varrho)=\chi(0)$ for $\varrho_0\le\varrho\le\Lambda$. For each $i=1,2,3,....$, set $\eta^i(\varrho)=\chi$, for $q\varrho_0\le\varrho\le\varrho_0$, and define, by the Picard iterations,

$$\eta^i(\varrho)=D(\varrho,\eta^{i-1}(q\varrho))-D(\varrho_0,\chi)+J^{i-1}(\varrho), \qquad (27)$$

where, for $\varrho\in[\varrho_0,\Lambda]$,

$$\begin{aligned}J^{i-1}(\varrho) &= \chi(0)+\int_{\varrho_0}^{\varrho}f_1\left(s,\eta^{i-1}(s),\eta^{i-1}(qs),\int_{qs}^{s}g(t,\eta^{i-1}(t))dt\right)ds\\ &\quad+\int_{\varrho_0}^{\varrho}f_2\left(s,\eta^{i-1}(s),\eta^{i-1}(qs),\int_{qs}^{s}g(t,\eta^{i-1}(t))dt\right)dW(s). \qquad (28)\end{aligned}$$

It is easy to see that $\eta^0(\cdot)\in\mathcal{M}^2([q\varrho_0,\Lambda],\mathbb{R}^d)$. On the other hand, we can prove by induction that $\eta^i(\cdot)\in\mathcal{M}^2([q\varrho_0,\Lambda],\mathbb{R}^d)$. By (27), we have:

$$\begin{aligned}|\eta^i(\varrho)|^2 &\le \frac{1}{\zeta}|D(\varrho,\eta^{i-1}(q\varrho))-D(\varrho_0,\chi)|^2+\frac{1}{1-\zeta}|J^{i-1}(\varrho)|^2\\ &\le \frac{1}{\zeta\sqrt{\zeta}}|D(\varrho,\eta^{i-1}(q\varrho))|^2+\frac{1}{\zeta(1-\sqrt{\zeta})}|D(\varrho_0,\chi)|^2+\frac{1}{1-\zeta}|J^{i-1}(\varrho)|^2\\ &\le \sqrt{\zeta}|\eta^{i-1}(q\varrho)|^2+\frac{\zeta}{1-\sqrt{\zeta}}\|\chi\|^2+\frac{1}{1-\zeta}|J^{i-1}(\varrho)|^2. \qquad (29)\end{aligned}$$

Taking the expectation on both sides, we get

$$\mathbb{E}\left(\sup_{\varrho_0\le s\le\varrho}|\eta^i(s)|^2\right)\le\sqrt{\zeta}\mathbb{E}\left(\sup_{\varrho_0\le s\le\varrho}|\eta^{i-1}(qs)|^2\right)+\frac{\zeta}{1-\sqrt{\zeta}}\mathbb{E}\|\chi\|^2+\frac{1}{1-\zeta}\mathbb{E}\left(\sup_{\varrho_0\le s\le\varrho}|J^{i-1}(s)|^2\right). \tag{30}$$

For any $n\ge 1$, we can derive from (30) that

$$\begin{aligned}\max_{1\le i\le n}\mathbb{E}\left(\sup_{\varrho_0\le s\le\varrho}|\eta^i(s)|^2\right) &\le \frac{\sqrt{\zeta}}{1-\sqrt{\zeta}}\mathbb{E}\|\chi\|^2+\sqrt{\zeta}\max_{1\le i\le n}\mathbb{E}\left(\sup_{\varrho_0\le s\le\varrho}|\eta^{i-1}(s)|^2\right)\\ &\quad+\frac{1}{1-\zeta}\max_{1\le i\le n}\mathbb{E}\left(\sup_{\varrho_0\le s\le\varrho}|J^{i-1}(s)|^2\right).\end{aligned} \tag{31}$$

Proceeding as (17), we obtain

$$\begin{aligned}\mathbb{E}\left(\sup_{\varrho_0\le s\le\varrho}|J^i(s)|^2\right) &\le 3\left(1+L_1(\Lambda-\varrho_0+4)(\Lambda-\varrho_0)(1+\Lambda^2L_2^2)\right)\mathbb{E}\|\chi\|^2\\ &\quad+3L_1(\Lambda-\varrho_0+4)\int_{\varrho_0}^{\varrho}\left(1+\left(2+\Lambda^2L_2^2\right)\mathbb{E}\sup_{\varrho_0\le r\le s}|\eta^i(r)|^2\right)ds.\end{aligned} \tag{32}$$

On the other hand, we have

$$\begin{aligned}\max_{1\le i\le n}\mathbb{E}\left(\sup_{\varrho_0\le s\le\varrho}|\eta^{i-1}(s)|^2\right) &= \max\{\mathbb{E}|\chi(0)|^2,\mathbb{E}(\sup_{\varrho_0\le s\le\varrho}|\eta^1(s)|^2),..,\mathbb{E}(\sup_{\varrho_0\le s\le\varrho}|\eta^{n-1}(s)|^2)\}\\ &\le \max\{\mathbb{E}\|\chi\|^2,\mathbb{E}(\sup_{\varrho_0\le s\le\varrho}|\eta^1(s)|^2),..,\mathbb{E}(\sup_{\varrho_0\le s\le\varrho}|\eta^{n-1}(s)|^2),\mathbb{E}(\sup_{\varrho_0\le s\le\varrho}|\eta^n(s)|^2)\}\\ &= \max\{\mathbb{E}\|\chi\|^2,\max_{1\le i\le n}\mathbb{E}(\sup_{\varrho_0\le s\le\varrho}|\eta^i(s)|^2)\}\\ &\le \mathbb{E}\|\chi\|^2+\max_{1\le i\le n}\mathbb{E}\left(\sup_{\varrho_0\le s\le\varrho}|\eta^i(s)|^2\right).\end{aligned} \tag{33}$$

Substituting (32) and (33) into (31), we obtain

$$\begin{aligned}&\max_{1\le i\le n}\mathbb{E}\left(\sup_{\varrho_0\le s\le\varrho}|\eta^i(s)|^2\right)\\ \le\ & \frac{1}{(1-\sqrt{\zeta})^2}\left(2\sqrt{\zeta}-\zeta+\frac{3\left(1+L_1(\Lambda-\varrho_0+4)(\Lambda-\varrho_0)(1+\Lambda^2L_2^2)\right)}{(1+\sqrt{\zeta})}\right)\mathbb{E}\|\chi\|^2\\ +\ & \frac{3L_1(\Lambda-\varrho_0+4)}{(1+\sqrt{\zeta})(1-\sqrt{\zeta})^2}\int_{\varrho_0}^{\varrho}\left(1+\left(2+\Lambda^2L_2^2\right)\max_{1\le i\le n}\mathbb{E}\sup_{\varrho_0\le r\le s}|\eta^{i-1}(r)|^2\right)ds\\ \le\ & \frac{1}{(1-\sqrt{\zeta})^2}\left(2\sqrt{\zeta}-\zeta+\frac{3\left(1+L_1(\Lambda-\varrho_0+4)(\Lambda-\varrho_0)(1+\Lambda^2L_2^2)\right)}{(1+\sqrt{\zeta})}\right)\mathbb{E}\|\chi\|^2\\ +\ & \frac{3L_1(\Lambda-\varrho_0+4)}{(1+\sqrt{\zeta})(1-\sqrt{\zeta})^2}\int_{\varrho_0}^{\varrho}\left(1+\left(2+\Lambda^2L_2^2\right)\left(\mathbb{E}\|\chi\|^2+\max_{1\le i\le n}\mathbb{E}\sup_{\varrho_0\le r\le s}|\eta^i(s)|^2\right)\right)ds\end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{(1-\sqrt{\zeta})^2}\left(2\sqrt{\zeta}-\zeta+\frac{3\left(1+L_1(\Lambda-\varrho_0+4)(\Lambda-\varrho_0)(1+\Lambda^2L_2^2)\right)}{(1+\sqrt{\zeta})}\right)\mathbb{E}\|\chi\|^2 \\ &+ \frac{3L_1(\Lambda-\varrho_0+4)(\Lambda-\varrho_0)}{(1+\sqrt{\zeta})(1-\sqrt{\zeta})^2}\left(1+\left(2+\Lambda^2L_2^2\right)\mathbb{E}\|\chi\|^2\right) \\ &+ \frac{3L_1(\Lambda-\varrho_0+4)}{(1+\sqrt{\zeta})(1-\sqrt{\zeta})^2}\left(2+\Lambda^2L_2^2\right)\int_{\varrho_0}^{\varrho}\max_{1\leq i\leq n}\mathbb{E}\sup_{\varrho_0\leq r\leq s}|\eta^i(r)|^2ds \end{aligned} \quad (34)$$

By Gronwall inequality, we have:

$$\max_{1\leq i\leq n}\mathbb{E}\left(\sup_{\varrho_0\leq\varrho\leq\Lambda}|\eta^i(\varrho)|^2\right)\leq C_1\exp\left(C_2(\Lambda-\varrho_0)\right), \quad (35)$$

where

$$\begin{aligned} C_1 &= \frac{1}{(1-\sqrt{\zeta})^2}\left(2\sqrt{\zeta}-\zeta+\frac{3\left(1+L_1(\Lambda-\varrho_0+4)(\Lambda-\varrho_0)(1+\Lambda^2L_2^2)\right)}{(1+\sqrt{\zeta})}\right)\mathbb{E}\|\chi\|^2 \\ &+\frac{3L_1(\Lambda-\varrho_0+4)(\Lambda-\varrho_0)}{(1+\sqrt{\zeta})(1-\sqrt{\zeta})^2}\left(1+\left(2+\Lambda^2L_2^2\right)\mathbb{E}\|\chi\|^2\right) \end{aligned} \quad (36)$$

and

$$C_2=\frac{3L_1(\Lambda-\varrho_0+4)}{(1+\sqrt{\zeta})(1-\sqrt{\zeta})^2}\left(2+\Lambda^2L_2^2\right). \quad (37)$$

Since n is arbitrary, we must obtain, $\forall i=0,1,2,3,..$

$$\mathbb{E}\left(\sup_{q\varrho_0\leq\varrho\leq\Lambda}|\eta^i(\varrho)|^2\right)\leq C_1\exp\left(C_2(\Lambda-\varrho_0)\right)+\mathbb{E}\|\chi\|^2, \quad (38)$$

which implies that $\eta^i(\varrho)\in\mathcal{M}^2([q\varrho_0,\Lambda];\mathbb{R}^d)$.
Noting that for $\varrho_0\leq\varrho\leq\Lambda$, we obtain

$$\eta^1(\varrho)-\eta^0(\varrho)=\eta^1(\varrho)-\chi(0)=D(\varrho,\eta^0(q\varrho))-D(\varrho_0,\chi)+I_1(\varrho),$$

where

$$I_1(\varrho)=\int_{\varrho_0}^{\varrho}f_1\left(s,\eta^0(s),\eta^0(qs),\int_{qs}^{s}g(t,\eta^0(t))dt\right)ds+\int_{\varrho_0}^{\varrho}f_2\left(s,\eta^0(s),\eta^0(qs),\int_{qs}^{s}g(t,\eta^0(t))dt\right)dW(s).$$

Proceeding as in the techniques of the uniqueness, one has:

$$\begin{aligned} \mathbb{E}\left(\sup_{\varrho_0\leq\varrho\leq\Lambda}|\eta^1(\varrho)-\eta^0(\varrho)|^2\right) &\leq \zeta\mathbb{E}\left(\sup_{\varrho_0\leq\varrho\leq\Lambda}|\eta^0(q\varrho)-\chi(0)|^2\right)+\frac{1}{1-\zeta}\mathbb{E}\left(\sup_{\varrho_0\leq\varrho\leq\Lambda}|I_1(\varrho)|^2\right) \\ &\leq 4\zeta\mathbb{E}\|\chi\|^2+\frac{2L_1}{1-\zeta}(\Lambda-\varrho_0+4)(\Lambda-\varrho_0)\left[1+(2+\Lambda^2L_2^2)\mathbb{E}\|\chi\|^2\right] \\ &= M. \end{aligned} \quad (39)$$

For $n\geq 1$ and $\varrho_0\leq\varrho\leq\Lambda$,

$$\eta^{i+1}(\varrho)-\eta^i(\varrho)=D(\varrho,\eta^i(q\varrho))-D(\varrho,\eta^{i-1}(q\varrho))+K^i(\varrho),$$

where

$$
\begin{aligned}
K^i(\varrho) &= \int_{\varrho_0}^{\varrho}\left(f_1\left(s,\eta^i(s),\eta^i(qs),\int_{qs}^{s} g(t,\eta^i(t))dt\right)-f_1\left(s,\eta^{i-1}(s),\eta^{i-1}(qs),\int_{qs}^{s} g(t,\eta^{i-1}(t))dt\right)\right)ds\\
&+ \int_{\varrho_0}^{\varrho}\left(f_2\left(s,\eta^i(s),\eta^i(qs),\int_{qs}^{s} g(t,\eta^i(t))dt\right)-f_2\left(s,\eta^{i-1}(s),\eta^{i-1}(qs),\int_{qs}^{s} g(t,\eta^{i-1}(t))dt\right)\right)dW(s).
\end{aligned}
$$

Proceeding as in the techniques of the uniqueness, one can derive that:

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{\varrho_0\le\varrho\le\Lambda}|\eta^{i+1}(\varrho)-\eta^i(\varrho)|^2\right)\\
\le\ & \zeta\mathbb{E}\left(\sup_{\varrho_0\le\varrho\le\Lambda}|\eta^i(q\varrho)-\eta^{i-1}(q\varrho)|^2\right)+\frac{1}{1-\zeta}\mathbb{E}\left(\sup_{\varrho_0\le\varrho\le\Lambda}|K^i(\varrho)|^2\right)\\
\le\ & \zeta\mathbb{E}\left(\sup_{\varrho_0\le\varrho\le\Lambda}|\eta^i(\varrho)-\eta^{i-1}(\varrho)|^2\right)+\frac{2L\left(\Lambda-\varrho_0+4\right)\left(2+\Lambda^2L_2^2\right)}{1-\zeta}\int_{\varrho_0}^{\Lambda}\mathbb{E}\left(\sup_{\varrho_0\le r\le\varrho}|\eta^i(r)-\eta^{i-1}(r)|^2\right)d\varrho\\
\le\ & \left(\zeta+\frac{2L\left(\Lambda-\varrho_0+4\right)\left(2+\Lambda^2L_2^2\right)(\Lambda-\varrho_0)}{1-\zeta}\right)\mathbb{E}\left(\sup_{\varrho_0\le\varrho\le\Lambda}|\eta^i(\varrho)-\eta^{i-1}(\varrho)|^2\right)\\
=\ & \theta\mathbb{E}\left(\sup_{\varrho_0\le\varrho\le\Lambda}|\eta^i(\varrho)-\eta^{i-1}(\varrho)|^2\right)\\
\le\ & \theta^i\mathbb{E}\left(\sup_{\varrho_0\le\varrho\le\Lambda}|\eta^1(\varrho)-\eta^0(\varrho)|^2\right)\\
\le\ & M\theta^i \qquad (40)
\end{aligned}
$$

Now, we prove that $\{\eta^i(\varrho)\}$ converges to $\eta(\varrho)$ in the sense of L^2 and with probability 1 on $\mathcal{M}^2\left([q\varrho_0,\Lambda];\mathbb{R}^d\right)$. On the other hand, $\eta(\varrho)$ is the solution of equation (1). By Chebyshev theorem, we have

$$
\mathbb{P}\left(\sup_{\varrho_0\le\varrho\le\Lambda}|\eta^{i+1}(\varrho)-\eta^i(\varrho)|^2>\frac{1}{2^i}\right)\le M(2\theta)^i. \qquad (41)
$$

Since $\sum_{i\ge0}M(2\theta)^i<\infty$, the Borel-Cantelli lemma implies that, for almost all $\overline{\omega}\in\Omega$, there is $i_0=i_0(\overline{\omega})\in\mathbb{N}^*$ such that

$$
\sup_{\varrho_0\le\varrho\le\Lambda}|\eta^{i+1}(\varrho)-\eta^i(\varrho)|^2\le\frac{1}{2^i}, i\ge i_0,
$$

which yields that, with probability 1, the partial sums

$$
\eta^0(\varrho)+\sum_{n=0}^{i-1}\left(\eta^{n+1}(\varrho)-\eta^n(\varrho)\right)=\eta^i(\varrho), \qquad (42)
$$

converges uniformly in $\varrho\in[\varrho_0,\Lambda]$. Let $\eta(\varrho)$ be the limit of (42). It is clear that $\eta(\varrho)$ is continuous and $\mathbb{F}_\varrho$-adapted. Moreover, we can see by (40) that for every ϱ, $\{\eta^i(\varrho)\}_{i\ge1}$ is a Cauchy sequence in $\mathcal{M}^2([q\varrho_0,\Lambda];\mathbb{R}^d)$ as well.

Therefore, we get $\eta^i(\varrho)\to\eta(\varrho)$ in $\mathcal{M}^2([q\varrho_0,\Lambda];\mathbb{R}^d)$ when $i\to\infty$. Letting $i\to\infty$ in (38), we obtain

$$
\mathbb{E}\left(|\eta(\varrho)|^2\right)\le C_1\exp\left(C_2(\Lambda-\varrho_0)\right)+\mathbb{E}\|\chi\|^2,\quad \text{for}\quad q\varrho_0\le\varrho\le\Lambda. \qquad (43)
$$

Hence, using (43), we have

$$\begin{aligned}\mathbb{E}\int_{q\varrho_0}^{\Lambda}|\eta(\varrho)|^2d\varrho &= \mathbb{E}\int_{q\varrho_0}^{\varrho_0}|\eta(\varrho)|^2d\varrho+\mathbb{E}\int_{\varrho_0}^{\Lambda}|\eta(\varrho)|^2d\varrho\\ &\le \mathbb{E}\int_{q\varrho_0}^{\varrho_0}\|\chi\|^2d\varrho+\mathbb{E}\int_{\varrho_0}^{\Lambda}\Big(C_1\exp\left(C_2(\Lambda-\varrho_0)\right)+\mathbb{E}\|\chi\|^2\Big)d\varrho\\ &< \infty. \end{aligned} \tag{44}$$

Therefore, $\eta(\varrho)\in\mathcal{M}^2([q\varrho_0,\Lambda];\mathbb{R}^d)$. Now, we show that $\eta(\varrho)$ satisfies (3). Using condition (5), we have

$$\mathbb{E}|\int_{\varrho_0}^{\varrho}\left(f_1\left(s,\eta^i(s),\eta^i(qs),\int_{qs}^{s}g(t,\eta^i(t))dt\right)-f_1\left(s,\eta(s),\eta(qs),\int_{qs}^{s}g(t,\eta(t))dt\right)\right)ds|^2$$

$$+\mathbb{E}|\int_{\varrho_0}^{\varrho}\left(f_2\left(s,\eta^i(s),\eta^i(qs),\int_{qs}^{s}g(t,\eta^i(t))dt\right)-f_2\left(s,\eta(s),\eta(qs),\int_{qs}^{s}g(t,\eta(t))dt\right)\right)dW(s)|^2$$

$$\begin{aligned} &\le (\Lambda-\varrho_0)\,\mathbb{E}\int_{\varrho_0}^{\varrho}|f_1\left(s,\eta^i(s),\eta^i(qs),\int_{qs}^{s}g(t,\eta^i(t))dt\right)-f_1\left(s,\eta(s),\eta(qs),\int_{qs}^{s}g(t,\eta(t))dt\right)|^2ds\\ &+ \mathbb{E}\int_{\varrho_0}^{\varrho}|f_2\left(s,\eta^i(s),\eta^i(qs),\int_{qs}^{s}g(t,\eta^i(t))dt\right)-f_2\left(s,\eta(s),\eta(qs),\int_{qs}^{s}g(t,\eta(t))dt\right)|^2ds\\ &\le L\,(\Lambda-\varrho_0+1)\left(1+\frac{1}{q}+\Lambda^2L_2^2\right)\int_{\varrho_0}^{\varrho}\mathbb{E}\left(\sup_{\varrho_0\le r\le s}|\eta^i(r)-\eta(r)|^2\right)ds, \end{aligned} \tag{45}$$

which goes to 0 when $i\to\infty$. Therefore, we can let $i\to\infty$ in (27) to have that

$$\eta(\varrho)-D(\varrho,\eta(q\varrho))=\chi(0)-D(\varrho_0,\chi)+\int_{\varrho_0}^{\varrho}f_1\left(s,\eta(s),\eta(qs),\int_{qs}^{s}g(t,\eta(t))dt\right)ds$$

$$+\int_{\varrho_0}^{\varrho}f_2\left(s,\eta(s),\eta(qs),\int_{qs}^{s}g(t,\eta(t))dt\right)dW(s),\quad on\quad \varrho_0\le\varrho\le\Lambda.$$

Then, $\eta(\varrho)$ is a solution of (1).

Step 2: We will delete the condition (26). Set $\beta>0$ be sufficiently small for $\theta=\zeta+\frac{2L(\beta+4)}{1-\zeta}\beta<1$. By step 1, there exists a solution of (1) on $[q\varrho_0,\varrho_0+\beta]$. Now consider (1) on $[\varrho_0+\beta,\varrho_0+2\beta]$ with initial condition $\eta(\varrho_0+\beta)$. Using Step 1 again, there exists a solution of (1) on $[\varrho_0+\beta,\varrho_0+2\beta]$. Iterating this procedure, we can deduce that there exists a solution of (1) on the whole interval $[q\varrho_0,\Lambda]$, as desired. □

4. Exponential stability in mean square of the solution of Neutral Stochastic Pantograph Integro-Differential Equations

In this section we will present some sufficient conditions for the exponential stability in mean square of the solution of equation (1).

Definition 4.1. *The solution of system (1) is said to be exponentially stable in mean square if there exist $\xi, C>0$ such that*

$$E\Big(|\eta(\varrho,\varrho_0,\chi)|^2\Big)\le Ce^{-\xi(\varrho-\varrho_0)}E\Big(\|\chi\|^2\Big)$$

for all $\varrho\ge\varrho_0\ge0$ and $\chi\in L^2_{\mathcal{F}_{\varrho_0}}([q\varrho_0,\varrho_0];\mathbb{R}^d)$.

In order to show the exponential stability in mean square of equation (1), we shall impose the following assumptions.
$\mathcal{A}_5$: Assume that there exist a nonnegative constants L_2 and δ such that

$$|g(\varrho,\eta)-g(\varrho,\overline{\eta})| \leq L_2 e^{-\frac{\delta}{2}\varrho}|\eta-\overline{\eta}|, \tag{46}$$

for all $(\varrho,\eta,\overline{\eta}) \in [\varrho_0,\Lambda]\times\mathbb{R}^d\times\mathbb{R}^d$.
$\mathcal{A}_6$: Assume that there exist a constants $\alpha_2 \geq 0$, $\alpha_3 \geq 0$, $\alpha_1 \leq -\alpha_2-\alpha_3 L_2^2(1-q)$ and $\lambda \geq 0$ such that: for any $(\varrho,\eta,z,v) \in [\varrho_0,\Lambda]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d$

$$(\eta-D(\varrho,z))^T f_1(\varrho,\eta,z,v)+\frac{1}{2}|f_2(\varrho,\eta,z,v)|^2 \leq \alpha_1 e^{\lambda\varrho}|\eta|^2+\alpha_2 q|z|^2+\alpha_3|v|^2. \tag{47}$$

$\mathcal{A}_7$: Assume that there exist a constants $\zeta \in [0,1)$ and $\epsilon > 0$ such that

$$|D(\varrho,z)-D(\varrho,\overline{z})| \leq \zeta e^{-\frac{\epsilon}{2}\varrho}|z-\overline{z}|, \tag{48}$$

for all $(\varrho,z,\overline{z}) \in [\varrho_0,\Lambda]\times\mathbb{R}^d\times\mathbb{R}^d$.
We will assume that f_1 and f_2 are smooth enough to ensure the existence of a unique solution of equation (1).

Theorem 4.2. *Let Assumptions $\mathcal{A}_5$-$\mathcal{A}_7$ hold. Then, the solution of equation (1) is exponentially stable in mean square.*

Proof. Let $\gamma \in (0,\epsilon)$ be arbitrary such that $\gamma \leq \left(\frac{q\lambda}{1-q}\right)\wedge(q\delta-1)\wedge\left(-\frac{\left(\alpha_1+\alpha_2+\alpha_3 L_2^2(1-q)\right)}{(1+\frac{\zeta^2}{q})}\right)$.
Define $u(\varrho)=\eta(\varrho)-D(\varrho,\eta(q\varrho))$ and $V(\varrho)=e^{\gamma(\varrho-\varrho_0)}|u(\varrho)|^2$. By Itô formula, we obtain

$$\begin{aligned} dV(\varrho) &= e^{\gamma(\varrho-\varrho_0)}\left[\gamma|u(\varrho)|^2+2u^T(\varrho)f_1\left(\varrho,\eta(\varrho),\eta(q\varrho),\int_{q\varrho}^{\varrho}g(t,\eta(t))dt\right)\right]d\varrho \\ &+ e^{\gamma(\varrho-\varrho_0)}|f_2\left(\varrho,\eta(\varrho),\eta(q\varrho),\int_{q\varrho}^{\varrho}g(t,\eta(t))dt\right)|^2 d\varrho \\ &+ 2e^{\gamma(\varrho-\varrho_0)}u^T(\varrho)f_2\left(\varrho,\eta(\varrho),\eta(q\varrho),\int_{q\varrho}^{\varrho}g(t,\eta(t))dt\right)dW(\varrho). \end{aligned} \tag{49}$$

Then,

$$\begin{aligned} e^{\gamma(\varrho-\varrho_0)}|u(\varrho)|^2 &= |u(\varrho_0)|^2+\gamma\int_{\varrho_0}^{\varrho}e^{\gamma(s-\varrho_0)}|u(s)|^2ds+2\int_{\varrho_0}^{\varrho}e^{\gamma(s-\varrho_0)}u^T(s)f_1\left(s,\eta(s),\eta(qs),\int_{qs}^{s}g(t,\eta(t))dt\right)ds \\ &+ \int_{\varrho_0}^{\varrho}e^{\gamma(s-\varrho_0)}|f_2\left(s,\eta(s),\eta(qs),\int_{qs}^{s}g(t,\eta(t))dt\right)|^2ds \\ &+ 2\int_{\varrho_0}^{\varrho}e^{\gamma(s-\varrho_0)}u^T(s)f_2\left(s,\eta(s),\eta(qs),\int_{qs}^{s}g(t,\eta(t))dt\right)dW(s). \end{aligned} \tag{50}$$

Taking the expectation on both sides of (50) and using Assumption $\mathcal{A}_6$, we can derive that

$$\begin{aligned} e^{\gamma(\varrho-\varrho_0)}\mathbb{E}|u(\varrho)|^2 &= \mathbb{E}|u(\varrho_0)|^2 + \gamma\mathbb{E}\int_{\varrho_0}^{\varrho} e^{\gamma(s-\varrho_0)}|u(s)|^2 ds \\ &+ 2\mathbb{E}\int_{\varrho_0}^{\varrho} e^{\gamma(s-\varrho_0)}u^T(s)f_1\left(s,\eta(s),\eta(qs),\int_{qs}^{s} g(t,\eta(t))dt\right)ds \\ &+ \mathbb{E}\int_{\varrho_0}^{\varrho} e^{\gamma(s-\varrho_0)}|f_2\left(s,\eta(s),\eta(qs),\int_{qs}^{s} g(t,\eta(t))dt\right)|^2 ds \\ &\le \mathbb{E}|u(\varrho_0)|^2 + \gamma\mathbb{E}\int_{\varrho_0}^{\varrho} e^{\gamma(s-\varrho_0)}|u(s)|^2 ds \\ &+ 2\mathbb{E}\int_{\varrho_0}^{\varrho} e^{\gamma(s-\varrho_0)}\left(\alpha_1 e^{\lambda s}|\eta(s)|^2 + \alpha_2 q|\eta(qs)|^2 + \alpha_3|\int_{qs}^{s} g(t,\eta(t))dt|^2\right)ds. \end{aligned} \tag{51}$$

Noting that by assumptions $\mathcal{A}_5$ and $\mathcal{A}_7$, we have

$$|u(\varrho)|^2 = |\eta(\varrho) - D(\varrho,\eta(q\varrho))|^2 \le 2|\eta(\varrho)|^2 + 2\zeta^2|\eta(q\varrho)|^2, \tag{52}$$

$$|u(\varrho_0)|^2 = |\eta(\varrho_0) - D(\varrho,\eta(q\varrho_0))|^2 \le 2|\eta(\varrho_0)|^2 + 2\zeta^2|\eta(q\varrho_0)|^2 \le 2\left(1+\zeta^2\right)\|\chi\|^2, \tag{53}$$

and

$$|\int_{qs}^{s} g(t,\eta(t))dt|^2 \le L_2^2 s(1-q)\int_{qs}^{s} e^{-\delta t}|\eta(t)|^2 dt. \tag{54}$$

Substituting (52)-(54) into (51), we get

$$\begin{aligned} e^{\gamma(\varrho-\varrho_0)}\mathbb{E}|u(\varrho)|^2 &\le 2\left(1+\zeta^2\right)\mathbb{E}\|\chi\|^2 + 2\gamma\mathbb{E}\int_{\varrho_0}^{\varrho} e^{\gamma s}|\eta(s)|^2 ds + 2\gamma\zeta^2\mathbb{E}\int_{\varrho_0}^{\varrho} e^{\gamma s}|\eta(qs)|^2 ds \\ &+ 2\alpha_1\mathbb{E}\int_{\varrho_0}^{\varrho} e^{(\gamma+\lambda)s}|\eta(s)|^2 ds + 2\alpha_2 q\mathbb{E}\int_{\varrho_0}^{\varrho} e^{\gamma s}|\eta(qs)|^2 ds \\ &+ 2\alpha_3 L_2^2(1-q)\mathbb{E}\int_{\varrho_0}^{\varrho}\left(\int_{qs}^{s} e^{-\delta t}|\eta(t)|^2 dt\right)se^{\gamma s}ds \\ &\le 2\left(1+\zeta^2\right)\mathbb{E}\|\chi\|^2 + 2\gamma\mathbb{E}\int_{\varrho_0}^{\varrho} e^{\gamma s}|\eta(s)|^2 ds + 2\gamma\frac{\zeta^2}{q}\mathbb{E}\int_{q\varrho_0}^{\varrho} e^{\frac{\gamma}{q}s}|\eta(s)|^2 ds \\ &+ 2\alpha_1\mathbb{E}\int_{\varrho_0}^{\varrho} e^{\frac{\gamma}{q}s}|\eta(s)|^2 ds + 2\alpha_2\mathbb{E}\int_{q\varrho_0}^{\varrho} e^{\frac{\gamma}{q}s}|\eta(s)|^2 ds \\ &+ 2\alpha_3 L_2^2(1-q)\mathbb{E}\int_{\varrho_0}^{\varrho}\left(\int_{qs}^{s}|\eta(t)|^2 dt\right)se^{(\gamma-q\delta)s}ds. \end{aligned} \tag{55}$$

Moreover, for any $0 \le \varrho_0 \le a$, we have

$$\begin{aligned} \mathbb{E}\int_{\varrho_0}^{\varrho}\left(\int_{qs}^{s}|\eta(t)|^2 dt\right)se^{(\gamma-q\delta)s}ds &\le \int_0^{\varrho} se^{(\gamma-q\delta)s}ds\,\mathbb{E}\left(\int_{q\varrho_0}^{\varrho} e^{\frac{\gamma}{q}s}|\eta(s)|^2 ds\right) \\ &\le \frac{1}{(\gamma-q\delta)^2}\mathbb{E}\left(\int_{q\varrho_0}^{\varrho} e^{\frac{\gamma}{q}s}|\eta(s)|^2 ds\right) \\ &\le \mathbb{E}\left(\int_{q\varrho_0}^{\varrho} e^{\frac{\gamma}{q}s}|\eta(s)|^2 ds\right) \\ &\le \left(\int_0^{a} e^{\frac{\gamma}{q}s}ds\right)\mathbb{E}\|\chi\|^2 + \mathbb{E}\left(\int_{\varrho_0}^{\varrho} e^{\frac{\gamma}{q}s}|\eta(s)|^2 ds\right) \end{aligned} \tag{56}$$

Hence,

$$
\begin{aligned}
& e^{\gamma(\varrho-\varrho_0)}\mathbb{E}|u(\varrho)|^2 \\
\leq\ & 2\left(1+\zeta^2\right)\mathbb{E}\|\chi\|^2 + 2\gamma\mathbb{E}\int_{\varrho_0}^{\varrho} e^{\frac{\gamma}{q}s}|\eta(s)|^2 ds + 2\gamma\frac{\zeta^2}{q}\left(\int_0^a e^{\frac{\gamma}{q}s}ds\right)\mathbb{E}\|\chi\|^2 \\
+\ & 2\gamma\frac{\zeta^2}{q}\mathbb{E}\int_{\varrho_0}^{\varrho} e^{\frac{\gamma}{q}s}|\eta(s)|^2 ds + 2\alpha_1\mathbb{E}\int_{\varrho_0}^{\varrho} e^{\frac{\gamma}{q}s}|\eta(s)|^2 ds \\
+\ & 2\alpha_2\left(\int_0^a e^{\frac{\gamma}{q}s}ds\right)\mathbb{E}\|\chi\|^2 + 2\alpha_2\mathbb{E}\int_{\varrho_0}^{\varrho} e^{\frac{\gamma}{q}s}|\eta(s)|^2 ds \\
+\ & 2\alpha_3 L_2^2(1-q)\left(\int_0^a e^{\frac{\gamma}{q}s}ds\right)\mathbb{E}\|\chi\|^2 + 2\alpha_3 L_2^2(1-q)\mathbb{E}\left(\int_{\varrho_0}^{\varrho} e^{\frac{\gamma}{q}s}|\eta(s)|^2 ds\right). \qquad (57)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& e^{\gamma(\varrho-\varrho_0)}\mathbb{E}|u(\varrho)|^2 \\
\leq\ & 2\left[\left(1+\zeta^2\right) + \gamma\frac{\zeta^2}{q}\left(\int_0^a e^{\frac{\gamma}{q}s}ds\right) + \alpha_2\left(\int_0^a e^{\frac{\gamma}{q}s}ds\right) + \alpha_3 L_2^2(1-q)\left(\int_0^a e^{\frac{\gamma}{q}s}ds\right)\right]\mathbb{E}\|\chi\|^2 \\
+\ & 2\left(\gamma(1+\frac{\zeta^2}{q}) + \alpha_1 + \alpha_2 + \alpha_3 L_2^2(1-q)\right)\mathbb{E}\int_{\varrho_0}^{\varrho} e^{\frac{\gamma}{q}s}|\eta(s)|^2 ds \\
\leq\ & C_3\mathbb{E}\|\chi\|^2. \qquad (58)
\end{aligned}
$$

where $C_3 = 2\left[\left(1+\zeta^2\right) + \gamma\frac{\zeta^2}{q}\left(\int_0^a e^{\frac{\gamma}{q}s}ds\right) + \alpha_2\left(\int_0^a e^{\frac{\gamma}{q}s}ds\right) + \alpha_3 L_2^2(1-q)\left(\int_0^a e^{\frac{\gamma}{q}s}ds\right)\right]$.
On the other hand, using assumption $\mathcal{A}_7$, for any $\Lambda > 0$, we have

$$
\begin{aligned}
\sup_{\varrho_0\leq\varrho\leq\Lambda}\left(e^{\gamma(\varrho-\varrho_0)}\mathbb{E}\left|\eta(\varrho)\right|^2\right) &\leq 2\sup_{\varrho_0\leq\varrho\leq\Lambda}\left(e^{\gamma(\varrho-\varrho_0)}\mathbb{E}\left|\eta(\varrho)-D(\varrho,\eta(q\varrho))\right|^2\right) + 2\sup_{\varrho_0\leq\varrho\leq\Lambda}\left(e^{\gamma(\varrho-\varrho_0)}\mathbb{E}\left|D(\varrho,\eta(q\varrho))\right|^2\right) \\
&\leq 2C_3\mathbb{E}\|\chi\|^2 + 2\zeta^2\sup_{\varrho_0\leq\varrho\leq\Lambda}\left(e^{(\gamma-\epsilon)(\varrho-\varrho_0)}\mathbb{E}\left|\eta(q\varrho)\right|^2\right) \\
&\leq 2C_3\mathbb{E}\|\chi\|^2 + 2\zeta^2\sup_{q\varrho_0\leq\varrho\leq\Lambda}\left(e^{(\gamma-\epsilon)\frac{(\varrho-\varrho_0)}{q}}\mathbb{E}\left|\eta(\varrho)\right|^2\right) \\
&\leq 2\left(C_3 + \zeta^2 e^{\frac{a}{q}(\epsilon-\gamma)(1-q)}\right)\mathbb{E}\|\chi\|^2 + 2\zeta^2\sup_{\varrho_0\leq\varrho\leq\Lambda}\left(e^{\gamma(\varrho-\varrho_0)}\mathbb{E}\left|\eta(\varrho)\right|^2\right). \qquad (59)
\end{aligned}
$$

Thus, by choosing ζ such that $\zeta < 2^{-\frac{1}{2}}$, we have

$$\sup_{\varrho_0\leq\varrho\leq\Lambda}\left(e^{\gamma(\varrho-\varrho_0)}\mathbb{E}\left|\eta(\varrho)\right|^2\right) \leq \frac{1}{1-2\zeta^2}\times 2\left(C_3 + \zeta^2 e^{\frac{a}{q}(\epsilon-\gamma)(1-q)}\right)\mathbb{E}\|\chi\|^2.$$

Letting $\Lambda \to +\infty$, we have

$$\sup_{\varrho_0\leq\varrho\leq+\infty}\left(e^{\gamma(\varrho-\varrho_0)}\mathbb{E}\left|\eta(\varrho)\right|^2\right) \leq C_4\mathbb{E}\|\chi\|^2,$$

where $C_4 = \frac{1}{1-2\zeta^2}\times 2\left(C_3 + \zeta^2 e^{\frac{a}{q}(\epsilon-\gamma)(1-q)}\right)$. This implies that for all $\varrho \geq \varrho_0$

$$\mathbb{E}\left|\eta(\varrho)\right|^2 \leq C_4 e^{-\gamma(\varrho-\varrho_0)}\mathbb{E}\|\chi\|^2,$$

as desired. □

5. Example

In this section we illustrate our results with an example.

Example 5.1. *Consider the following NSPIDE:*

$$d\left[\eta(\varrho)-D(\varrho,\eta(\frac{1}{2}\varrho))\right]=f_1\left(\varrho,\eta(\varrho),\eta(\frac{1}{2}\varrho),\int_{\frac{1}{2}\varrho}^{\varrho}g(t,\eta(t))dt\right)d\varrho+f_2\left(\varrho,\eta(\varrho),\eta(\frac{1}{2}\varrho),\int_{\frac{1}{2}\varrho}^{\varrho}g(t,\eta(t))dt\right)dW(\varrho), \quad (60)$$

where the initial condition is $\chi(\varrho)=1$, *for* $\varrho\in[\frac{1}{2},1]$, *with* $\varrho_0=1$ *and* $W(t)$ *is a one dimensional Brownian motions. Let* $0\leq\lambda\leq\frac{1}{4}$ *and*

$$D(\varrho,\eta(\frac{1}{2}\varrho))=\frac{1}{2}e^{-\frac{1}{4}\varrho}\sin(\eta(\frac{1}{2}\varrho)),$$

$$f_1\left(\varrho,\eta(\varrho),\eta(\frac{1}{2}\varrho),\int_{\frac{1}{2}\varrho}^{\varrho}g(t,\eta(t))dt\right)=-24e^{\lambda\varrho}\eta(\varrho)+\frac{1}{2}\eta(\frac{1}{2}\varrho)+\frac{1}{10}\int_{\frac{1}{2}\varrho}^{\varrho}e^{-\frac{3}{2}s}\sin(\eta(s))ds,$$

$$f_2\left(\varrho,\eta(\varrho),\eta(\frac{1}{2}\varrho),\int_{\frac{1}{2}\varrho}^{\varrho}g(t,\eta(t))dt\right)=\frac{1}{2}\eta(\varrho)+\frac{1}{2}\eta(\frac{1}{2}\varrho)-\frac{1}{10}\int_{\frac{1}{2}\varrho}^{\varrho}e^{-\frac{3}{2}s}\sin(\eta(s))ds.$$

Then,

$$(\eta-D(\varrho,z))^T f_1(\varrho,\eta,z,v)+\frac{1}{2}|f_2(\varrho,\eta,z,v)|^2\leq-17.125e^{\lambda\varrho}|\eta|^2+0.5(14)|z|^2+0.75|v|^2. \quad (61)$$

Thus, $\alpha_1=-17.125$, $\alpha_2=14$, $\alpha_3=0.75$, $L_2=\frac{1}{5}$, $\delta=3$, $\gamma=\frac{1}{8}$ *and* $q=\zeta=\epsilon=\frac{1}{2}$.
Then, $\alpha_1=-17.125\leq-\alpha_2-\alpha_3L_2^2(1-q)=-14.015$.
Therefore, all the assumptions of theorem 4.2 holds. Consequently, system (60) is exponentially stable in mean square.

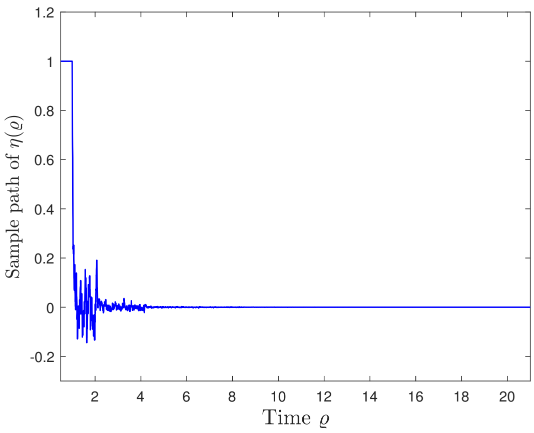

Figure 1: Trajectory's simulation of $\eta(\varrho)$ on the interval $[0.5,21]$ with initial condition $\chi(\varrho)=1$.

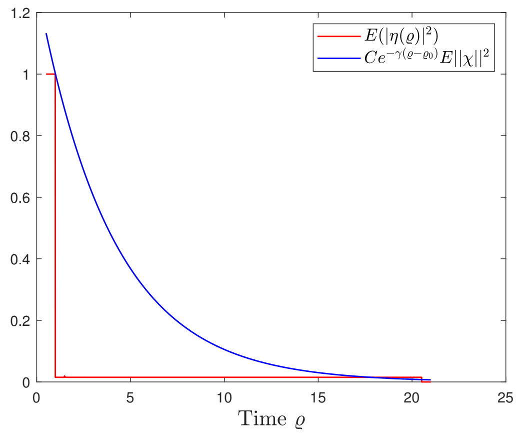

Figure 2: Exponential stability in mean square of system (60) on the interval $[0.5,21]$.

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