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Domain of q-Cesàro Matrix in Hahn Sequence Space h_d and the Space bv of the Sequences of Bounded Variation

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Abstract. Let $h_d = \left\{f = (f_k) \in \omega : \sum_k d_k|f_k - f_{k+1}| < \infty\right\} \cap c_0$, where $d = (d_k)$ is an unbounded and monotonic increasing sequence of positive reals. We study the matrix domains $h_d(C^q) = (h_d)_{C^q}$ and $bv(C^q) = (bv)_{C^q}$, where C^q is the q-Cesàro matrix, $0 < q < 1$. Apart from the inclusion relations and Schauder basis, we compute α-, β- and γ-duals of the spaces $h_d(C^q)$ and $bv(C^q)$. We state and prove theorems concerning characterization of matrix classes from the spaces $h_d(C^q)$ and $bv(C^q)$ to any one of the space ℓ_∞, c, c_0 or ℓ_1. Finally, we obtain certain identities concerning characterization of compact operators using Hausdorff measure of non-compactness in the space $h_d(C^q)$.

1. Notations, introduction and preliminaries

The following notations/symbols are used throughout the texts:

$$\begin{aligned}
\mathbb{N} &:= \{1, 2, 3, \cdots\}, \\
\omega &:= \text{The set of all real or complex valued sequences}, \\
bs_d &:= \left\{f = (f_k) \in \omega : \sup_n \frac{1}{d_n}\left|\sum_{k=1}^{\infty} f_k\right| < \infty\right\}, \\
[k]_q &:= \begin{cases} \dfrac{1-q^k}{1-q}, & k > 1, \\ 1, & k = 1, \\ 0, & k = 0, \end{cases}
\end{aligned}$$

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$B(X)$	:=	Unit ball in X,
σ	:=	Set of all finite sequences that ends in zeros,
$\mathcal{N}$	:=	Family of subsets of $\mathbb{N}$,
$\mathcal{N}_r$	:=	Sub-collection of $\mathcal{N}$ with elements that are greater than r.

We emphasize that $[k]_q = k$, when $q \to 1$. $[k]_q$ is well known as q-numbers. We refer to [6] for detailed study in q-numbers and q-theory. In the above notations, and in what follows summation and supremum over the set $\mathbb{N}$ are simply denoted by $\sum_k$ and $\sup_k$, respectively, and $d = (d_k)$ is an unbounded and monotonic increasing sequence of positive reals. Also any sequence with zero or negative subscript is considered to be naught i.e., $f_k = 0$ for $k \leq 0$ for any $f = (f_k) \in \omega$. Further the sets ℓ_∞, c, c_0, ℓ_1, bs and cs are standard notations and have the usual meanings.

Let $H = (h_{nk})_{n,k=0}^{\infty}$ be an infinite matrix over the complex field. Let $H_n = \left(h_{nk}\right)_{k=0}^{\infty}$ and $Hf = (Hf)_n$, where $(Hf)_n = \sum_k h_{nk} f_k$ for any $f \in \omega$, provided that the infinite sum exists. The sequence Hf is also known as H-transform of the sequence f. Let $X, Y \subset \omega$. Then, (X, Y) denotes the family of all matrices that map X into Y. That is $H \in (X, Y)$ if and only if $Hf \in Y$ for all $f \in X$. The set $X_H = \left\{f \in \omega : Hf \in X\right\}$ is called the domain of H in X. It is known that if X is a BK space with norm $\|(\cdot)\|_X$ and H is a triangular matrix, then X_H is also a BK space with norm $\|(\cdot)\|_{X_H} = \|H(\cdot)\|_X$. The readers may refer the papers [1, 2, 7, 12, 28] and the book [22] which are great sources for the theory of sequence spaces.

A sequence $\left(u_n\right)_{n=0}^{\infty}$ in a normed linear space $X \subset \omega$ is called a Schauder basis for the space X if for each $f \in X$, there corresponds a unique sequence of scalars, say $\left(b_k\right)$, such that $f = \sum_k b_k u_k$ for all $k \in \mathbb{N}$.
For $X \subset \omega$, the sets

$$X^\alpha := \left\{a = (a_k) \in \omega : af = (a_k f_k) \in \ell_1 \text{ for all } f = (f_k) \in X\right\};$$

$$X^\beta := \left\{a = (a_k) \in \omega : af = (a_k f_k) \in cs \text{ for all } f = (f_k) \in X\right\};$$

$$X^\gamma := \left\{a = (a_k) \in \omega : af = (a_k f_k) \in bs \text{ for all } f = (f_k) \in X\right\};$$

are called α-, β- and γ-dual of the space X.
Let $0 < q < 1$. Then, q-Cesàro matrix $C^q = (c_{nk}^q)$ (cf. [28]), $n, k \in \mathbb{N}$, is defined by

$$c_{nk}^q = \begin{cases} \dfrac{q^{k-1}}{[n]_q}, & 1 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

Indeed C^q is a triangular matrix, and so has a unique inverse $\left(C^q\right)^{-1} = (c_{nk}^{-q})$ defined by

$$c_{nk}^{-q} = \begin{cases} (-1)^{n-k}\dfrac{[k]_q}{q^{n-1}}, & n-1 \leq k \leq n, \\ 0, & \text{otherwise}, \end{cases}$$

We emphasize that the matrix C^q reduces to Cesàro matrix C_1 of first order as q tends to 1. Domain of q-Cesàro matrix in the spaces ℓ_∞, c, c_0 and ℓ_p $(1 \leq p < \infty)$ are examined by Yaying et al. [28] and Demiriz and Şahin [2].
The sequence space bv defined by

$$bv = \left\{f = (f_k) \in \omega : \sum_{k=1}^{\infty} |f_k - f_{k+1}| < \infty\right\}, \tag{1}$$

which is the forward difference operator Δ domain on the sequence space ℓ_1, where $\Delta f_k = f_k - f_{k+1}$, for all $k \in \mathbb{N}$. The space $bv_0 = bv \cap c_0$ and the inclusions $\ell_1 \subset bv_0 \subset bv \subset c$ strictly hold. Moreover, the sequence space bv is a BK space with the norm

$$\|f\|_{bv} = \sum_{k=1}^{\infty} |f_k - f_{k+1}| \text{ for all } f = (f_k) \in bv.$$

The sequence space h defined by

$$h := \Big\{ f = (f_k) \in \omega : \sum_k k|f_k - f_{k+1}| < \infty \Big\} \cap c_0$$

is called Hahn sequence space, named after its introducer H. Hahn [4]. He obtained that h is a BK space with norm

$$\|f\| = \sum_k k|f_k - f_{k+1}| + \sup_k |f_k| \text{ for all } f = (f_k) \in h.$$

Extending the studies of Hahn, Rao [16] proved that the space h is a BK space with AK with respect to the norm

$$\|f\|_h = \sum_k k|f_k - f_{k+1}| \text{ for all } f = (f_k) \in h.$$

A generalized Hahn sequence space h^d was introduced by Goes [3] for $d = (d_k) \in \omega$ with $d_k \neq 0$ for all k, defined by

$$h^d := \Big\{ f = (f_k) \in \omega : \sum_k |d_k||f_k - f_{k+1}| < \infty \Big\} \cap c_0.$$

Quiet recently a scientific study of a more generalized Hahn sequence space is carried out by Malkowsky et al. [11] as follows:

$$h_d := \Big\{ f = (f_k) \in \omega : \sum_k d_k|f_k - f_{k+1}| < \infty \Big\} \cap c_0.$$

The authors proved that h_d is a BK space with AK with respect to the norm

$$\|f\|_{h_d} = \sum_k d_k|f_k - f_{k+1}| \text{ for all } f = (f_k) \in h_d,$$

where $d = (d_k)$ is an unbounded and monotonic increasing sequence of positive reals. Besides, the authors stated and proved various significant results concerning characterization of matrix transformations between h_d and classical BK spaces, and characterization of compact operators on the space h_d using Hausdorff measure of non-compactness. We refer to [10, 15, 17, 18, 21, 27] and the survey paper [8] for more studies and results related to Hahn sequence space.

The main objective of this paper is to extend the studies related to Hahn sequence space. In doing so, we introduce matrix domains $h_d\big(C^q\big) = (h_d)_{C^q}$ and $bv\big(C^q\big) = (bv)_{C^q}$, obtain Schauder basis, α-, β- and γ-duals of these new spaces. Besides results related to matrix transformation from $h_d(C^q)$ and $bv\big(C^q\big)$ to any one of the space ℓ_∞, c, c_0 or ℓ_1 are obtained. In the final section, certain identities concerning characterization of compact operators on the space $h_d(C^q)$ using Hausdorff measure of non-compactness are determined. Since the matrix C^q is more generalized than C, we believe that the results so obtained in this paper strengthen the results of Kirişci [7].

2. Sequence spaces $h_d(C^q)$ and $bv(C^q)$

Let us define a sequence $g = (g_k)$ whose k^{th} term is given by $g_k = \left(C^q f\right)_k$. This means that the sequence g is the C^q-transform of the sequence f, and

$$g_n = \sum_{k=1}^{n} \frac{q^{k-1}}{[n]_q} f_k. \tag{2}$$

The sequence spaces $h_d(C^q)$ and $bv(C^q)$ are defined by

$$\begin{aligned} h_d(C^q) &:= \left\{f = (f_k) \in \omega : g = C^q f \in h_d\right\}, \\ bv(C^q) &:= \left\{f = (f_k) \in \omega : g = C^q f \in bv\right\}. \end{aligned}$$

It is trivial that $h_d(C^q)$ and $bv(C^q)$ can be expressed as $h_d(C^q) = (h_d)_{C^q}$ and $bv(C^q) = (bv)_{C^q}$. In other words, $h_d(C^q)$ and $bv(C^q)$ are domain of the matrix C^q in h_d and bv, respectively. We emphasize that the sequence space $h_d(C^q)$ reduces to the sequence space $h(C_1)$, when q tends to 1, as studied by Kirişci [7].
The equality (2) can also be redefined in terms of the sequence $g = (g_k)$ by

$$f_n = \sum_{k=n-1}^{n} (-1)^{n-k} \frac{[k]_q}{q^{n-1}} g_k, \ n \in \mathbb{N} \text{ with } f_1 = g_1. \tag{3}$$

Theorem 2.1. *We have the following statements:*

(a) $h_d(C^q)$ *is a BK-space with respect to the norm*

$$\|f\|_{h_d(C^q)} = \left\|C^q f\right\|_{h_d} = \sum_n d_n \left| \sum_{k=1}^{n} \frac{q^{k-1}}{[n]_q} f_k - \sum_{k=1}^{n+1} \frac{q^{k-1}}{[n+1]_q} f_k \right|.$$

(b) $bv(C^q)$ *is a BK space with respect to the norm*

$$\|f\|_{bv(C^q)} = \left\|C^q f\right\|_{bv} = \sum_n \left| \sum_{k=1}^{n} \frac{q^{k-1}}{[n]_q} f_k - \sum_{k=1}^{n+1} \frac{q^{k-1}}{[n+1]_q} f_k \right|.$$

Proof. This is easy to verify and hence details are omitted. □

Theorem 2.2. $h_d(C^q)$ *and* $bv(C^q)$ *are linearly isomorphic to* h_d *and* bv*, respectively.*

Proof. We know that C^q is a triangle and so is invertible. The result immediately follows from the fact that the mapping T defined by

$$\begin{array}{rccl} T : & X(C^q) & \longrightarrow & X \\ & f & \longmapsto & Tf = g = C^q f \end{array}$$

is a linear bijection and norm preserving, where X denotes either of the space h_d or bv. Hence $h_d(C^q) \cong h_d$ and $bv(C^q) \cong bv$. □

Theorem 2.3. *The following inclusions strictly hold:*

(1) $h_d \subset h_d(C^q)$,
(2) $h_d(C^q) \subset \ell_1(C^q)$,
(3) $h_d(C^q) \subset bv(C^q)$,

where $\ell_1(C^q) = (\ell_1)_{C^q}$ *is studied in [28].*

Proof. (1) It is evident that h_d is contained in $h_d(C^q)$. To examine the strictness, we consider $d = (d_k) = (k)_{k=0}^{\infty}$ and the sequence $r = (r_k) = \left(\frac{q[k+1]_q - [k]_q}{q}\right)$ defined for all $k \in \mathbb{N}$. Then

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty}\left([k+1]_q - \frac{[k]_q}{q}\right) = \frac{1}{1-q} - \frac{1}{q(1-q)} = \frac{-1}{q} \neq 0. \tag{4}$$

Thus r is not a sequence in h_d. On the other hand, $C^q r := s = (s_k) = \left(q^k\right)$ is a sequence in h_d. This follows from the following illustration:

Since $q^k \to 0$ as $k \to \infty$, it is enough to show that $\sum_k d_k|s_k - s_{k+1}| < \infty$. We have

$$\begin{aligned}
\sum_k k|q_k - q_{k+1}| &= |q-q^2| + 2|q^2-q^3| + 3|q^3-q^4| + \cdots \\
&= q(1-q) + 2q^2(1-q) + 3q^3(1-q) + \cdots \\
&= q(1-q)(1+2q+3q^2+\cdots) \\
&= q(1-q)\cdot\frac{1}{(1-q)^2} \\
&= \frac{q}{1-q} < \infty
\end{aligned}$$

as desired.

(2) Let us take $d_k = 2^k$ for every $k \in \mathbb{N}$ and define the sequence $g = (g_k)$ by

$$g_k = \begin{cases} 0, & k \neq 2^v, \\ \frac{1}{k}, & k = 2^v. \end{cases}$$

Then, $g \in \ell_1 \setminus h_d$. That is to say that the inclusion $h_d \subset \ell_1$ strictly holds. Now define the sequence $f = (f_k)$ by

$$f_k = \sum_{v=k-1}^{k} (-1)^{k-v}\frac{[v]_q}{q^{k-1}}g_v \text{ for each } k \in \mathbb{N} \text{ with } f_1 = g_1.$$

Then, one obtains $C^q f = g \in \ell_1 \setminus h_d$ which in turn leads us to the fact that $f \in \ell_1(C^q) \setminus h_d(C^q)$.

(3) It is easy to see that the inclusion $h_d \subset bv$ strictly holds which implies that the inclusion $h_d(C^q) \subset bv(C^q)$ holds. Now, let us take the sequence $e = \left((1)^k\right)$. Then, $C^q e = e \in bv \setminus h_d$ and so, $e \in bv(C^q) \setminus h_d(C^q)$ as required. □

Now we determine the Schauder basis of the space $bv\left(C^q\right)$. We recall Theorem 2.5 of Kirişçi [9] and Theorem 3.1 of Başar and Altay [1] wherein the authors obtained the Schauder bases of the spaces h_p and bv_p $(1 \leq p < \infty)$. Let $X \subset \omega$ and H be an infinite triangular matrix. Then, the matrix domain X_H has a Schauder basis if and only if X has basis. As a direct consequence of this fact, we conclude that the inverse image of the basis of each of the spaces h_d and bv form the Schauder basis of the spaces $h_d(C^q)$ and $bv(C^q)$. This fact allows us to present the following results:

Theorem 2.4. *Define the sequences $b^{(k)} = \left(b_n^{(k)}\right)$ and $\tilde{b}^{(k)} = \left(\tilde{b}_n^{(k)}\right)$, $k \in \mathbb{N}$, by*

$$b_n^{(k)} = \begin{cases} \frac{1}{d_k}, & k \geq n, \\ -\frac{[k]_q}{q^k d_k}, & k = n+1, \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad \tilde{b}_n^{(k)} = \begin{cases} 1, & k < n, \\ \frac{[n]_q}{q^{n-1}}, & k = n, \\ 0, & k > n. \end{cases}$$

Let $g_k = (C^q f)_k$ for each $k \in \mathbb{N}$. Then

(a) $b^{(k)}$ *is a Schauder basis for the space* $h_d(C^q)$, *and every* $f \in h_d(C^q)$ *is uniquely represented by* $f = \sum_k g_k b^{(k)}$.
(b) $\tilde{b}^{(k)}$ *is a Schauder basis for the space* $bv(C^q)$, *and every* $f \in bv(C^q)$ *is uniquely represented by* $f = \sum_k g_k \tilde{b}^{(k)}$.

Corollary 2.5. *The sequence spaces* $h_d(C^q)$ *and* $bv(C^q)$ *are separable.*

3. Alpha-, Beta- and Gamma-duals

In this section, we compute the α-, β- and γ-duals of the sequence spaces $h_d(C^q)$ and $bv(C^q)$. For our study, we need the following lemmas. Throughout $\mathcal{N}$ denotes the family of all finite subsets of $\mathbb{N}$. We assume that $H = (h_{nk})$ is an infinite matrix over the complex field.

Lemma 3.1. *[11] The following statements hold:*

(i) $H = (h_{nk}) \in (h_d, \ell_1)$ *if and only if*

$$\sup_m \frac{1}{d_m} \sum_n \left| \sum_{k=1}^{m} h_{nk} \right| < \infty.$$

(ii) $H = (h_{nk}) \in (h_d, \ell_\infty)$ *if and only if*

$$\sup_{n,m} \frac{1}{d_m} \left| \sum_{k=1}^{m} h_{nk} \right| < \infty. \tag{5}$$

(iii) $H = (h_{nk}) \in (h_d, c)$ *if and only if (5) holds, and*

$$\exists \alpha_k \in \mathbb{C} \ni \lim_{n\to\infty} h_{nk} = \alpha_k \ \textit{for each} \ k \in \mathbb{N}. \tag{6}$$

Lemma 3.2. *[5] The following statements hold:*

(i) $H = (h_{nk}) \in (bv, \ell_1)$ *if and only if*

$$\sup_k \sum_n \left| \sum_{j=k}^{\infty} h_{nj} \right| < \infty. \tag{7}$$

(ii) $H = (h_{nk}) \in (bv, \ell_\infty)$ *if and only if*

$$\sup_{n,k} \left| \sum_{j=k}^{\infty} h_{nj} \right| < \infty. \tag{8}$$

(iii) $H = (h_{nk}) \in (bv, c)$ *if and only if (8) holds, and*

$$\exists \beta_k \in \mathbb{C} \ni \lim_{n\to\infty} \sum_{j=k}^{\infty} h_{nj} = \beta_k \ \textit{for each} \ k \in \mathbb{N}. \tag{9}$$

Theorem 3.3. *Consider the following sets*

$$D_1 := \left\{ t = (t_k) \in \omega : \sup_m \frac{1}{d_m} \sum_n \left| \sum_{k=1}^{m} (-1)^{n-k} \frac{[k]_q}{q^{n-1}} t_n \right| < \infty \right\},$$
$$D_2 := \left\{ t = (t_k) \in \omega : \sup_m \sum_n \left| \sum_{k=m}^{\infty} (-1)^{n-k} \frac{[k]_q}{q^{n-1}} t_n \right| < \infty \right\}.$$

Then, $[h_d(C^q)]^\alpha = D_1$ *and* $[bv(C^q)]^\alpha = D_2$.

Proof. For all $n, k \in \mathbb{N}$, consider the matrix $A^q = (a^q_{nk})$ defined by

$$a^q_{nk} = \begin{cases} (-1)^{n-k}\dfrac{[k]_q}{q^{n-1}}t_n, & n-1 \le k \le n, \\ 0, & \text{otherwise}, \end{cases} \tag{10}$$

where $t = (t_k) \in \omega$. Then, we get the following equality:

$$t_n f_n = \sum_{k=n-1}^{n} (-1)^{n-k}\frac{[k]_q}{q^{n-1}} g_k t_n = (A^q g)_n$$

for each $n \in \mathbb{N}$, where $g = C^q f$. Thus $tf = (t_n f_n) \in \ell_1$ whenever $f \in h_d(C^q)$ if and only if $A^q g \in \ell_1$ whenever $g \in h_d$. This leads us to the fact that $t = (t_n) \in [h_d(C^q)]^\alpha$ if and only if $A^q \in (h_d, \ell_1)$. Thus, using Part (i) of Lemma 3.1, we obtain

$$\sup_m \frac{1}{d_m} \sum_n \left| \sum_{k=1}^{m} (-1)^{n-k} \frac{[k]_q}{q^{n-1}} t_n \right| < \infty.$$

Therefore $[h_d(C^q)]^\alpha = D_1$.
The α-dual of the space $bv(C^q)$ is obtained in the similar way by using Part (i) of Lemma 3.2 instead of Part (i) of Lemma 3.1 in the aforementioned statements. Hence, details are excluded to avoid repetition of the similar statements. □

Theorem 3.4. *Define the matrix $T = (t_{nk})$ by*

$$t_{nk} = [k]_q \left(\frac{t_k}{q^{k-1}} - \frac{t_{k+1}}{q^k} \right)$$

for all $n, k = 1, 2, 3, \cdots$. Then, we have the following statements:

(i) $t = (t_k) \in [h_d(C^q)]^\beta$ if and only if $T = (t_{nk}) \in (h_d, c)$ and

$$\left(\frac{[r]_q t_r}{q^{r-1}} \right) \in \ell_\infty. \tag{11}$$

(ii) $t = (t_k) \in [bv(C^q)]^\beta$ if and only if $T = (t_{nk}) \in (bv, c)$ and

$$\left(\frac{[r]_q t_r}{q^{r-1}} \right) \in c_0. \tag{12}$$

Proof. (i) Assume that $t = (t_k) \in [h_d(C^q)]^\beta$. Then, the sequence $tf = (t_k f_k) \in cs$. That is to say that the series $\sum_k t_k f_k$ is convergent for all $f = (f_k) \in h_d(C^q)$. Now, we consider the following equality obtained by the r^{th} partial sum of the series $\sum_k t_k f_k$ with (3)

$$\begin{aligned} \sum_{k=0}^{r} t_k f_k &= \sum_{k=0}^{r} t_k \left[\sum_{v=k-1}^{k} (-1)^{k-v} \frac{[v]_q}{q^{k-1}} g_v \right] \\ &= \sum_{k=0}^{r-1} [k]_q \left(\frac{t_k}{q^{k-1}} - \frac{t_{k+1}}{q^k} \right) g_k + \frac{[r]_q}{q^{r-1}} g_r t_r \end{aligned} \tag{13}$$

for all $n, r \in \mathbb{N}$. Bearing in mind the fact $h_d(C^q) \cong h_d$, we pass to limit, as $r \to \infty$, in (13). Then, since the series $\sum_k t_k f_k$ is convergent by the hypothesis, the series

$$\sum_k [k]_q \left(\frac{t_k}{q^{k-1}} - \frac{t_{k+1}}{q^k} \right) g_k$$

is also convergent and the term $[r]_q t_r g_r / q^{r-1}$ in the right hand side of (13) must tend to zero, as $r \to \infty$. Since $h_d \subset c_0$ this is achieved with $\left\{[r]_q t_r / q^{r-1}\right\} \in \ell_\infty$, we therefore have

$$\sum_k t_k f_k = \sum_k [k]_q \left(\frac{t_k}{q^{k-1}} - \frac{t_{k+1}}{q^k} \right) g_k = (Tg)_k \tag{14}$$

for all $k \in \mathbb{N}$. Hence, $T = (t_{nk}) \in (h_d, c)$. Thus, the conditions in (5) and (6) of Part (iii) of Lemma 3.1 are satisfied by the matrix T. Hence, the conditions are necessary.

Conversely, suppose that $T = (t_{nk}) \in (h_d, c)$ and the condition in (11) holds. Then, we again obtain the relation (14) by using (13). Therefore, since we have $T = (t_{nk}) \in (h_d, c)$ the series $\sum_k t_k f_k$ is convergent for all $f = (f_k) \in h_d(C^q)$. Hence, $t = (t_k) \in [h_d(C^q)]^\beta$, that is, the conditions are sufficient.

(ii) Part (ii) can easily be obtained using similar arguments as in the proof of Part (i), with $\left(\frac{[r]_q t_r}{q^{r-1}}\right) \in c_0$ instead of the space ℓ_∞, and the conditions in (8) and (9) of Part (iii) of Lemma 3.2 instead of the conditions in (5) and (6) of Part (iii) of Lemma 3.1. □

Theorem 3.5. *The following statements hold:*

(i) $t = (t_k) \in [h_d(C^q)]^\gamma$ if and only if $T = (t_{nk}) \in (h_d, \ell_\infty)$ and the condition (11) holds.

(ii) $t = (t_k) \in [bv(C^q)]^\gamma$ if and only if $T = (t_{nk}) \in (bv, \ell_\infty)$ and the condition (12) holds.

Proof. This can be obtained by the similar technique used in proving Parts (i) and (ii) of Theorem 3.4 with Part (ii) of Lemma 3.1 in the proof of (i) and Part (ii) of Lemma 3.2 in the proof of (ii) instead of Part (iii) of Lemma 3.1 and Part (iii) of Lemma 3.2, respectively. We omit details to avoid repetition of the similar statements. □

4. Matrix transformations

In this section, we characterize some classes of matrix transformations from the sequence spaces $h_d(C^q)$ and $bv(C^q)$ to any one of the space ℓ_∞, c, c_0 or ℓ_1.

Consider the matrix Z^q whose $(n,k)^{\text{th}}$ term z^q_{nk} is given by

$$z^q_{nk} = [k]_q \left(\frac{h_{nk}}{q^{k-1}} - \frac{h_{n,k+1}}{q^k} \right) \text{ for all } n, k = 1, 2, 3, \cdots. \tag{15}$$

Theorem 4.1. *$H = (h_{nk}) \in (h_d(C^q), \ell_\infty)$ if and only if*

$$\sup_{n,m} \frac{1}{d_m} \left| \sum_{k=1}^{m} [k]_q \left(\frac{h_{n,k}}{q^{k-1}} - \frac{h_{n,k+1}}{q^k} \right) \right| < \infty, \tag{16}$$

$$\left(\frac{[r]_q h_{nr}}{q^{r-1}} \right)_{r \in \mathbb{N}} \in \ell_\infty \tag{17}$$

for all $m, n \in \mathbb{N}$.

Proof. Let $H \in (h_d(C^q), \ell_\infty)$. Then, Hf exists for all $f \in h_d(C^q)$, and belongs to the space ℓ_∞. Thus, $H_n \in [h_d(C^q)]^\beta$ which confirms the necessity of the conditions in (16) and (17).

Conversely, assume that the conditions in (16) and (17) hold. Let $f = (f_k) \in h_d(C^q)$. Then, $H_n \in [h_d(C^q)]^\beta$ for each $n \in \mathbb{N}$, and Hf exists. Therefore, we have the following equality:

$$\begin{aligned} \sum_{k=1}^{r} h_{nk} f_k &= \sum_{k=1}^{r} h_{nk} \sum_{v=k-1}^{k} (-1)^{k-v} \frac{[v]_q}{q^{k-1}} g_v \\ &= \sum_{k=1}^{r-1} [k]_q \left(\frac{h_{n,k}}{q^{k-1}} - \frac{h_{n,k+1}}{q^k} \right) g_k + \frac{[r]_q}{q^{r-1}} h_{nr} g_r \end{aligned} \tag{18}$$

for all $n, r \in \mathbb{N}$. In the light of the condition in (17) and passing to limits as $r \to \infty$ in (18), we deduce the following equality

$$\sum_k h_{nk} f_k = \sum_k z^q_{nk} g_k \tag{19}$$

for all $n, k = 1, 2, 3, \cdots$, where the matrix $Z^q = (z^q_{nk})$ is defined as in (15). Thus Z^q maps h_d into ℓ_∞. This implies that $Z^q g = Hf \in \ell_\infty$ as required. □

Theorem 4.2. *$H = (h_{nk}) \in (h_d(C^q), c)$ if and only if (16) and (17) hold, and there exists $\alpha_k \in \mathbb{C}$ such that*

$$\lim_{n\to\infty} [k]_q \left(\frac{h_{n,k}}{q^{k-1}} - \frac{h_{n,k+1}}{q^k} \right) = \alpha_k \text{ for each } k \in \mathbb{N}. \tag{20}$$

Proof. Assume that $H \in (h_d(C^q), c)$. Then Hf exists for all $f \in h_d(C^q)$, and belongs to the space c. Since the inclusion $c \subset \ell_\infty$ holds, the necessity of the conditions in (16) and (17) are straightforward. To prove the necessity of the condition in (20), we consider equality (19) for $f = C^{-q} e^{(k)}$. Then, we get that the sequence

$$Hf = H\Big(C^{-q} e^{(k)}\Big) = Z^q\Big(C^q\Big(C^{-q} e^{(k)}\Big)\Big) = Z^q e^{(k)} = Z^q_k = (z^q_{nk})_{n\in\mathbb{N}}$$

is convergent for each $k \in \mathbb{N}$. This proves the necessity of the condition in (20).
Conversely, we assume that the conditions in (16), (17) and (20) hold. Then, under these assumptions $H_n \in [h_d(C^q)]^\beta$ and so Hf exists. Thus we again get the equality (19). We observe that the conditions in (16) and (20) corresponds to the conditions in (5) and (6), respectively, with z^q_{nk} instead of h_{nk}. This implies that $Hf = Z^q g \in c$. Thus $H \in (h_d(C^q), c)$. □

Replacing the space c by the space c_0, the above theorem gives the following result:

Corollary 4.3. *$H = (h_{nk}) \in (h_d(C^q), c_0)$ if and only if (16) and (17) hold, and (20) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$.*

Theorem 4.4. *$H = (h_{nk}) \in (h_d(C^q), \ell_1)$ if and only if (17) holds, and*

$$\sup_m \frac{1}{d_m} \sum_n \left| \sum_{k=1}^m [k]_q \left(\frac{h_{n,k}}{q^{k-1}} - \frac{h_{n,k+1}}{q^k} \right) \right| < \infty, \tag{21}$$

for all $m \in \mathbb{N}$.

Proof. The proof is similar to the proof of Theorem 4.1 and so details are omitted. □

In the similar way, matrix transformation from $bv(C^q)$ to any one of the space ℓ_∞, c, c_0 or ℓ_1 can be characterized. Hence we present the results without proof.

Theorem 4.5. *$H = (h_{nk}) \in (bv(C^q), \ell_\infty)$ if and only if*

$$\sup_{m,n} \left| \sum_{k=m}^\infty [k]_q \left(\frac{h_{n,k}}{q^{k-1}} - \frac{h_{n,k+1}}{q^k} \right) \right| < \infty, \tag{22}$$

$$\left(\frac{[m]_q}{q^{m-1}} h_{nm} \right) \in c_0 \tag{23}$$

for all $m, n \in \mathbb{N}$.

Theorem 4.6. *$H = (h_{nk}) \in (bv(C^q), c)$ if and only if (20), (22) and (23) hold.*

Theorem 4.7. *$H = (h_{nk}) \in (bv(C^q), c_0)$ if and only if (22) and (23) hold, and (20) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$.*

Theorem 4.8. *$H = (h_{nk}) \in (bv(C^q), \ell_1)$ if and only if (23) holds, and*

$$\sup_m \sum_n \left| \sum_{k=m}^{\infty} [k]_q \left(\frac{h_{n,k}}{q^{k-1}} - \frac{h_{n,k+1}}{q^k} \right) \right| < \infty \tag{24}$$

for all $m \in \mathbb{N}$.

We need the following Lemmas due to Başar and Altay [1] and Malkowsky et al. [11] for our next results:

Lemma 4.9. *[1, Lemma 5.3] Assume that X and Y are any two sequence spaces, H is an infinite matrix, and M is a triangle. Then, $H \in (X, Y_M) \Leftrightarrow MH \in (X, Y)$.*

Lemma 4.10. *[11]*

(1) $H = (h_{nk}) \in (\ell_\infty, h_d)$ if and only if

$$\sup_N \sum_k \left| \sum_{n \in N} d_n \Big(h_{nk} - h_{n+1,k} \Big) \right| < \infty, \tag{25}$$

$$\lim_{n \to \infty} \sum_k |h_{nk}| = 0. \tag{26}$$

(2) $H = (h_{nk}) \in (c, h_d)$ if and only if (6) holds, and

$$\sup_K \sum_n \left| \sum_{k \in K} d_n \Big(h_{nk} - h_{n+1,k} \Big) \right| < \infty, \tag{27}$$

$$\left(\sum_k h_{nk} \right)_{n=1}^{\infty} \in h_d. \tag{28}$$

(3) $H = (h_{nk}) \in (c_0, h_d)$ if and only if (6) and (27) holds.

(4) $H = (h_{nk}) \in (\ell_1, h_d)$ if and only if (6) holds, and

$$\sup_k \sum_n \left| d_n \Big(h_{nk} - h_{n+1,k} \Big) \right| < \infty. \tag{29}$$

Lemma 4.11. *[20]*

(1) $H = (h_{nk}) \in (\ell_\infty, bv) = (c, bv) = (c_0, bv)$ if and only if

$$\sup_{N,K} \left| \sum_{n \in N} \sum_{k \in K} h_{nk} - h_{n-1,k} \right| < \infty, \tag{30}$$

(2) $H = (h_{nk}) \in (\ell_1, bv)$ if and only if

$$\sup_k \sum_n \left| h_{nk} - h_{n-1,k} \right| < \infty. \tag{31}$$

As one of the immediate consequence of Lemma 4.9, we characterize the matrix class $(X, h_d(C^q))$ by using Lemma 4.10, where X is any one of the space ℓ_∞, c, c_0 or ℓ_1.
Define the matrix $\tilde{C}^q = (\tilde{c}^q_{nk})$ for all $n, k \in \mathbb{N}$ by

$$\tilde{c}^q_{nk} = \sum_{v=1}^{n} \frac{q^{v-1}}{[n]_q} h_{vk}.$$

Theorem 4.12. *We have the following statements:*

(1) $H \in (\ell_\infty, h_d(C^q))$ if and only if (30) and (26) hold with $\tilde{c}^q_{nk}$ instead of h_{nk}.

(2) $H \in (c, h_d(C^q))$ *if and only if (6), (27) and (28) hold with* $\tilde{c}^q_{nk}$ *instead of* h_{nk}.

(3) $H \in (c_0, h_d(C^q))$ *if and only if (6) and (27) hold with* $\tilde{c}^q_{nk}$ *instead of* h_{nk}.

(4) $H \in (\ell_1, h_d(C^q))$ *if and only if (6) and (29) hold with* $\tilde{c}^q_{nk}$ *instead of* h_{nk}.

Proof. (1) By using the definition of the matrix $\tilde{C}^q = (\tilde{c}^q_{nk})$ and Lemma 4.9, it is evident that $H \in (\ell_\infty, h_d(C^q))$ if and only if $\tilde{C}^q \in (\ell_\infty, h_d)$. The desired conclusion follows immediately by using Lemma 4.10.
The remainder of the theorem can be shown using similar arguments as in the proofs of each of the Parts 2, 3 and 4 of Lemma 4.10. □

Theorem 4.13. *We have the following statements:*

(1.) $H \in (\ell_\infty, bv(C^q)) = (c, bv(C^q)) = (c_0, bv(C^q))$ *if and only if (30) holds with* $\tilde{c}^q_{nk}$ *instead of* h_{nk}.

(2.) $H \in (\ell_1, bv(C^q))$ *if and only if (31) holds with* $\tilde{c}^q_{nk}$ *instead of* h_{nk}.

Proof. It is similar to the proof of Theorem 4.12. □

Another important application of Lemma 4.9 is to characterize matrix classes from the sequence space $X \in \left\{h_d(C^q), bv(C^q)\right\}$ to some of the well-known sequence spaces in the literature.
Define the matrix $S = (s_{nk})$ by $s_{nk} = 1$ if $1 \leq k \leq n$ and 0, otherwise. Then, considering M as the summability matrix S and the Cesàro matrix C_1 in Lemma 4.9, we define matrices whose $(n,k)^{\text{th}}$ entries for all $n, k \in \mathbb{N}$ are given by

$$\tilde{s}_{nk} = \sum_{v=1}^{n} h_{vk} \text{ and } c^1_{nk} = \sum_{v=1}^{n} \frac{1}{n} h_{vk}.$$

Corollary 4.14. *We have the following statements:*

(1) $H \in (h_d(C^q), bs)$ *if and only if (16) and (17) hold with* $\tilde{s}_{nk}$ *instead of* h_{nk}.

(2) $H \in (h_d(C^q), cs)$ *if and only if (16), (17) and (20) hold with* $\tilde{s}_{nk}$ *instead of* h_{nk}.

(3) $H \in (h_d(C^q), cs_0)$ *if and only if (16), (17) and (20) hold with* $\tilde{s}_{nk}$ *instead of* h_{nk} *and* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$.

(4) $H \in (bv(C^q), bs)$ *if and only if (22) and (23) hold with* $\tilde{s}_{nk}$ *instead of* h_{nk}.

(5) $H \in (bv(C^q), cs)$ *if and only if (20), (22) and (23) hold with* $\tilde{s}_{nk}$ *instead of* h_{nk}.

(6) $H \in (bv(C^q), cs_0)$ *if and only if (22) and (23) hold, and (20) holds with* $\tilde{s}_{nk}$ *instead of* h_{nk} *and* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$.

Corollary 4.15. *Let* X_∞, X_c, X_0 *and* X_1 *be the Cesàro sequence spaces studied in [14, 19]. Then*

(1) $H \in (h_d(C^q), X_\infty)$ *if and only if (16) and (17) hold with* c^1_{nk} *instead of* h_{nk}.

(2) $H \in (h_d(C^q), X_c)$ *if and only if (16), (17) and (20) hold with* c^1_{nk} *instead of* h_{nk}.

(3) $H \in (h_d(C^q), X_0)$ *if and only if (16), (17) and (20) hold with* c^1_{nk} *instead of* h_{nk} *and* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$.

(4) $H \in (h_d(C^q), X_1)$ *if and only if (17) and (21) hold with* c^1_{nk} *instead of* h_{nk}.

(5) $H \in (bv(C^q), X_\infty)$ *if and only if (22) and (23) hold with* c^1_{nk} *instead of* h_{nk}.

(6) $H \in (bv(C^q), X_c)$ *if and only if (20), (22) and (23) hold with* c^1_{nk} *instead of* h_{nk}.

(7) $H \in (bv(C^q), X_0)$ *if and only if (22), (23) and (20) hold with* c^1_{nk} *instead of* h_{nk} *and* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$.

(8) $H \in (bv(C^q), X_1)$ *if and only if (23) and (24) hold with* c^1_{nk} *instead of* h_{nk}.

5. Compactness of operators via Hmnc

Throughout this section, Hmnc is the abbreviation for Hausdorff measure of non-compactness. For $u = (u_k) \in \omega$, define

$$\|u\|_X^{\dagger} = \sup_{f \in B(X)} \left| \sum_k u_k f_k \right|,$$

where we assume that the series on the right hand side exists. It is observed that $u \in X^{\beta}$. Denote the family of all bounded linear operators from Banach spaces X to Y by $B(X, Y)$, which is a Banach space endowed with the norm $\|T\| = \sup_{f \in B(X)} \|Tf\|$.

Let $D(X)$ denote the domain of X. An operator T is called compact if $D(X) = X$ and the sequence $\left((Tf)_k\right)$ has a convergent subsequence in Y, for every bounded sequence $f = (f_k)$ in X.

Let $B(f_k, r_k)$ represent the unit ball with centre f_k and radius r_k, for $k = 1, 2, \ldots, n$. Then Hmnc of a bounded set Q in a metric space X is defined by

$$\chi(Q) = \inf\left\{\epsilon > 0 : Q \subset \bigcup_{k=1}^{n} B(f_k, r_k), f_k \in X, r_k < \epsilon \ (k = 1, 2, \ldots, n), n \in \mathbb{N}\right\}.$$

Hmnc of any operator $T : X \to Y$ is defined by $\|T\|_{\chi} = \chi(T(B(X)))$. The relation

$$\|T\|_{\chi} = 0 \Leftrightarrow T \text{ is compact}$$

plays a significant role to determine the compactness of an operator between BK-spaces. We refer to [11–13, 23–26] for studies concerning compactness via Hmnc.

In the rest of the paper the set of all finite sequences that ends in zeros is denoted by σ. We recall some known results in the literature that are necessary for our investigation:

Lemma 5.1. *[11] We have $h_d^{\beta} = bs_d$. Further, $\|f\|_{h_d}^{\dagger} = \|f\|_{bs_d}$.*

Lemma 5.2. *[22, Theorem 4.2.8] Assume that X and Y are any two BK sequence spaces. Then, $(X, Y) \subset B(X, Y)$, i.e. every $H \in (X, Y)$ defines a linear operator $T_H \in B(X, Y)$, where $T_H f = Hf$ for all $f \in X$.*

Lemma 5.3. *[12, Theorem 1.23] Consider $X \supset \sigma$ as a BK-space and $H \in (X, Y)$. Then*

$$\|T_H\| = \|H\|_{(X,Y)} = \sup_{n \in \mathbb{N}} \|H_n\|_X^{\dagger} < \infty.$$

Lemma 5.4. *[13, Theorem 3.7] Consider $X \supset \sigma$ as a BK-space. Then, we have the following statements:*

(a) Assume that $H \in (X, c_0)$. Then $\|T_H\|_{\chi} = \limsup_{n\to\infty} \|H_n\|_X^{\dagger}$ and T_H is compact if and only if $\|H_n\|_X^{\dagger} = 0$ $(n \to \infty)$.

(b) Assume that X has AK and $H \in (X, c)$. Then

$$\frac{1}{2} \limsup_{n\to\infty} \|H_n - h\|_X^{\dagger} \leq \|T_H\|_{\chi} \leq \limsup_{n\to\infty} \|H_n - h\|_X^{\dagger}$$

and T_H is compact if and only if $\|H_n - h\|_X^{\dagger} = 0$ $(n \to \infty)$, where $h = (h_k)$ with $h_k = \lim_{n\to\infty} h_{nk}$ for all $k \in \mathbb{N}$.

(c) Assume that $H \in (X, \ell_{\infty})$. Then, $0 \leq \|T_H\|_{\chi} \leq \limsup_{n\to\infty} \|H_n\|_X^{\dagger}$ and T_H is compact if $\|H_n\|_X^{\dagger} = 0$ $(n \to \infty)$.

Lemma 5.5. *[13, Theorem 3.11] Assume $X \supset \sigma$ to be a BK-space and $H \in (X, \ell_1)$. Then*

$$\lim_{r\to\infty} \sup_{N \in \mathcal{N}_r} \left\| \sum_{n \in N} H_n \right\|_X^{\dagger} \leq \|T_H\|_{\chi} \leq 4 \cdot \lim_{r\to\infty} \sup_{N \in \mathcal{N}_r} \left\| \sum_{n \in N} H_n \right\|_X^{\dagger}$$

and T_H is compact if and only if $\sup_{N \in \mathcal{N}_r} \left\| \sum_{n \in N} H_n \right\|_X^{\dagger} = 0$ $(r \to \infty)$.

We write the next lemma that follows from our previous results.

Lemma 5.6. *Let X be any sequence spacen and $H \in (h_d(C^q), X)$. Then, $Z^q = (z^q_{nk}) \in (h_d, X)$ and $Hf = Z^q g$ for all $f \in h_d(C^q)$, where the matrix $Z^q = (z^q_{nk})$ is defined as in (15).*

Proof. It is straightforward from Theorem 4.1. □

Theorem 5.7. *We have the following statements :*

(a) Assume that $H \in (h_d(C^q), c_0)$. Then

$$\|T_H\|_\chi = \limsup_{\substack{n\to\infty \\ m\in\mathbb{N}}} \left(\frac{1}{d_m} \left| \sum_{k=1}^{m} z^q_{nk} \right| \right).$$

(b) Assume that $H \in (h_d(C^q), c)$. Then

$$\frac{1}{2} \limsup_{\substack{n\to\infty \\ m\in\mathbb{N}}} \left(\frac{1}{d_m} \sum_{k=1}^{m} \left| z^q_{nk} - z_k \right| \right) \le \|T_H\|_\chi \le \limsup_{\substack{n\to\infty \\ m\in\mathbb{N}}} \left(\frac{1}{d_m} \sum_{k=1}^{m} \left| z^q_{nk} - z_k \right| \right),$$

where $z = (z_k)$ and $z_k = \lim_{n\to\infty} z^q_{nk}$ for each $k \in \mathbb{N}$.

(c) Assume that $H \in (h_d(C^q), \ell_\infty)$. Then

$$0 \le \|T_H\|_\chi \le \limsup_{\substack{n\to\infty \\ m\in\mathbb{N}}} \left(\frac{1}{d_m} \left| \sum_{k=1}^{m} z^q_{nk} \right| \right).$$

(d) Assume that $H \in (h_d(C^q), \ell_1)$. Then

$$\lim_{r\to\infty} \sup_{\substack{N\in\mathcal{N}_r \\ m\in\mathbb{N}}} \frac{1}{d_m} \left| \sum_{k=1}^{m} \left(\sum_{n\in N} z^q_{nk} \right) \right| \le \|T_H\|_\chi \le 4 \lim_{r\to\infty} \sup_{\substack{N\in\mathcal{N}_r \\ m\in\mathbb{N}}} \frac{1}{d_m} \left| \sum_{k=1}^{m} \left(\sum_{n\in N} z^q_{nk} \right) \right|.$$

Proof. (a) Assume that $H \in (h_d(C^q), c_0)$. We have

$$\|H_n\|^\dagger_{h_d(C^q)} = \left\| Z^q_n \right\|^\dagger_{h_d} = \left\| Z^q_n \right\|_{bs_d} = \sup_{m\in\mathbb{N}} \left(\frac{1}{d_m} \left| \sum_{k=1}^{m} z^q_{nk} \right| \right).$$

for each $n \in \mathbb{N}$. In the view of Part (a), Lemma 5.4, we conclude that

$$\|T_H\|_\chi = \limsup_{\substack{n\to\infty \\ m\in\mathbb{N}}} \left(\frac{1}{d_m} \left| \sum_{k=1}^{m} z^q_{nk} \right| \right).$$

(b) Note that

$$\left\| Z^q_n - z_k \right\|^\dagger_{h_d} = \left\| Z^q_n - z_k \right\|_{bs_d} = \sup_{m\in\mathbb{N}} \left(\frac{1}{d_m} \left| \sum_{k=1}^{m} z^q_{nk} - z_k \right| \right) \tag{32}$$

for each $n \in \mathbb{N}$. Assume that $H \in (h_d(C^q), c)$. Then, in the light of Lemma 5.6, we get that $Z^q \in (h_d, c)$. Keeping in mind that h_d is a BK space with AK, by Part (b) of Lemma 5.4 we deduce

$$\frac{1}{2} \limsup_{n\to\infty} \left\| Z^q_n - z \right\|^\dagger_{h_d} \le \|T_H\|_\chi \le \limsup_{n\to\infty} \left\| Z^q_n - z \right\|^\dagger_{h_d}$$

which on employing (32) yields the desired result.
(c) It is obtained in the similar way as in the proof of Part (a) except that we use Part (c) of Lemma 5.4

instead of Part (a) of Lemma 5.4.
(d) Observe that

$$\left\|\sum_{n\in\mathbb{N}} Z_n^q\right\|_{h_d}^{\dagger} = \left\|\sum_{n\in\mathbb{N}} Z_n^q\right\|_{bs_d} = \sup_{m\in\mathbb{N}}\left(\frac{1}{d_m}\sum_{k=1}^{m}\left|\sum_{n\in\mathbb{N}} z_{nk}^q\right|\right). \tag{33}$$

Assume that $H \in (h_d(C^q), \ell_1)$. Then, by applying Lemma 5.6, we get that $Z^q \in (h_d, \ell_1)$. By Lemma 5.5, we obtain

$$\lim_{r\to\infty}\sup_{N\in\mathcal{N}_r}\left\|\sum_{n\in N} Z_n^q\right\|_{h_d}^{\dagger} \le \|T_H\|_{\chi} \le 4\cdot\lim_{r\to\infty}\sup_{N\in\mathcal{N}_r}\left\|\sum_{n\in N} Z_n^q\right\|_{h_d}^{\dagger}$$

and by (33) these inequalities imply

$$\lim_{r\to\infty}\sup_{\substack{N\in\mathcal{N}_r\\ m\in\mathbb{N}}}\frac{1}{d_m}\left|\sum_{k=1}^{m}\left(\sum_{n\in N} z_{nk}^q\right)\right| \le \|T_H\|_{\chi} \le 4\lim_{r\to\infty}\sup_{\substack{N\in\mathcal{N}_r\\ m\in\mathbb{N}}}\frac{1}{d_m}\left|\sum_{k=1}^{m}\left(\sum_{n\in N} z_{nk}^q\right)\right|.$$

This completes the proof. □

As a direct consequence of the above theorem, we have the following result:

Corollary 5.8. *The following statements hold:*

(a) Assume that $H \in (h_d(C^q), c_0)$. Then, T_H is compact if and only if

$$\sup_{m\in\mathbb{N}}\left(\frac{1}{d_m}\left|\sum_{k=1}^{m} z_{nk}^q\right|\right) \to 0\ (n\to\infty).$$

(b) Assume that $H \in (h_d(C^q), c)$. Then, T_H is compact if and only if

$$\sup_{m\in\mathbb{N}}\left(\frac{1}{d_m}\left|\sum_{k=1}^{m} z_{nk}^q - z_k\right|\right) \to 0\ (n\to\infty).$$

(c) Assume that $H \in (h_d(C^q), \ell_\infty)$. Then, T_H is compact if

$$\sup_{m\in\mathbb{N}}\left(\frac{1}{d_m}\left|\sum_{k=1}^{m} z_{nk}^q\right|\right) \to 0\ (n\to\infty).$$

(d) Assume that $H \in (h_d(C^q), \ell_1)$. Then, T_H is compact if and only if

$$\sup_{\substack{N\in\mathcal{N}_r\\ m\in\mathbb{N}}}\frac{1}{d_m}\left|\sum_{k=1}^{m}\left(\sum_{n\in N} z_{nk}^q\right)\right| \to 0\ (r\to\infty).$$

Declaration

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