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Further Hermite-Hadamard Type Inequalities Involving Operator h-Convex Functions

Mustapha Raïssouli[a], Lahcen Tarik[b], Mohamed Chergui[c], Bouazza El Wahbi[b]

[a]*Department of Mathematics, Faculty of Science, Moulay Ismail University, Meknes, Morocco.*
[b]*Department of Mathematics, Science Faculty, LAGA-Lab Ibn Tofail University, Kenitra, Morocco.*
[c]*Department of Mathematics, CRMEF-Rabat-Sale-Kenitra, EREAM Team, LaREAMI-Lab, Kenitra, Morocco.*

Abstract. In this paper, we are interested to establish inequalities of Hermite-Hadamard type involving operator h-convex functions. We provide some generalizations to operators of some inequalities with real arguments recently pointed out. Several applications to some weighted operator means are presented as well.

1. Introduction and basic notions

Let $(H, \langle,\rangle)$ be a complex Hilbert space. As usual, the notation $\mathcal{B}(H)$ refers to the C^*-algebra of all bounded linear operators acting on H. An operator $A \in \mathcal{B}(H)$ is called positive, and we write $A \geq 0$, if A is self-adjoint and $\langle Ax, x\rangle \geq 0$ for all $x \in H$. We denote by $\mathcal{B}^+(H)$ the closed cone of all positive operators in $\mathcal{B}(H)$ and by $\mathcal{B}^{+*}(H)$ the open cone of all positive invertible operators in $\mathcal{B}(H)$. The concept of positive operators induces a partial order defined on the subspace of self-adjoint operators by: For $A, B \in \mathcal{B}(H)$ both self-adjoint, we write $A \leq B$ for meaning that $B - A \in \mathcal{B}^+(H)$. Henceforth, whenever we consider a positive operator or an operator inequality, it will be assumed that the involved operators are self-adjoint.

Let f be a real valued function defined on a nonempty interval $I \subset \mathbb{R}$. We say that f is operator convex (resp. operator concave) on I if the inequality

$$f\Big((1-\lambda)A + \lambda B\Big) \leq (\geq)(1-\lambda)f(A) + \lambda f(B), \tag{1}$$

holds for all self-adjoint operators $A, B \in \mathcal{B}(H)$ with spectrums in I. Here, $f(A)$ is defined via the techniques of functional calculus.

For the class of operator convex functions, we have the Hermite-Hadamard inequalities

$$f\left(\frac{A+B}{2}\right) \leq \int_0^1 f\Big((1-t)A + tB\Big)\, dt \leq \frac{f(A)+f(B)}{2}, \tag{2}$$

which hold for all self-adjoint operators $A, B \in \mathcal{B}(H)$ with spectrums in I. If f is operator concave on I then the inequalities in (2) are reversed.

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Communicated by Mohammad Sal Moslehian
Email addresses: raissouli.mustapha@gmail.com (Mustapha Raïssouli), lahcen.tarik@uit.ac.ma (Lahcen Tarik), chergui_m@yahoo.fr (Mohamed Chergui), bouazza.elwahbi@uit.ac.ma (Bouazza El Wahbi)

We will now focus on the h-convexity property introduced by Varošanec in [14]. Let $I, J \subseteq \mathbb{R}$ with $(0,1) \subseteq J$, $h : J \to \mathbb{R}$ and $f : I \to \mathbb{R}$ be two real nonnegative functions with h is not identically equal to 0. By analogy with (1), we say that f is operator h-convex (resp. operator h-concave) on I if, [2]

$$f\Big((1-\lambda)A + \lambda B\Big) \leq (\geq) h(1-\lambda)f(A) + h(\lambda)f(B) \tag{3}$$

holds for all self-adjoint operators $A, B \in \mathcal{B}(H)$ with spectrums in I.

We mention the following remark which may be of interest for the reader.

Remark 1.1. *If we put $A = B$ in (3) we get*

$$\Big(h(1-\lambda) + h(\lambda) - 1\Big)f(A) \geq 0$$

for all $\lambda \in [0,1]$ and all self-adjoint operator $A \in \mathcal{B}(H)$ with spectra in I. So, the fact that f is nonnegative is imposed for saying if f is operator h-convex. However, this condition is not needed for defining the concept of operator convexity. In another word, (3) extends (1) only when f is assumed to be nonnegative.

Sarikaya [13] generalized the Hermite-Hadamard inequalities for real h-convex functions as follows: Let $f : I \to \mathbb{R}$ be h-convex function on I, $a, b \in I$ with $a < b$, and $f \in L^1([a,b])$. Then

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \Big(f(a) + f(b)\Big)\int_0^1 h(t)\, dt. \tag{4}$$

Later, Darvish et al. pointed out in [2] the operator version of (4) as follows

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{A+B}{2}\right) \leq \int_0^1 f\Big(tA + (1-t)B\Big)\, dt \leq \Big(f(A) + f(B)\Big)\int_0^1 h(t)dt. \tag{5}$$

If f is h-concave (resp. operator h-concave) then the inequalities in (4) and in (5) are reversed. If $h(t) = t, 0 \leq t \leq 1$, then (5) coincides with (2)

In recent decades, some refinements, reverses, extensions and generalizations for the Hermite-Hadamard type inequalities have been investigated in the literature, see [1–6, 11, 13] and the related references cited therein.

We also need to recall some operator means. Let $A, B \in \mathcal{B}^{+*}(H)$ and $\lambda \in [0,1]$. The following expressions

$$A\nabla_\lambda B := (1-\lambda)A + \lambda B,\ A\sharp_\lambda B := A^{1/2}\Big(A^{-1/2}BA^{-1/2}\Big)^\lambda A^{1/2},\ A!_\lambda B := \Big((1-\lambda)A^{-1} + \lambda B^{-1}\Big)^{-1}$$

are known in the literature as the λ-weighted arithmetic mean, λ-weighted geometric mean and λ-weighted harmonic mean of A and B, respectively. When $\lambda = 1/2$, they are simply denoted by $A\nabla B, A\sharp B$ and $A!B$, respectively. The following operator inequalities hold, [8]

$$A!_\lambda B \leq A\sharp_\lambda B \leq A\nabla_\lambda B.$$

We also define the weighted power mean as follows:

$$\mathcal{P}_{\lambda,s}(A,B) := \begin{cases} A^{1/2}\Big((1-\lambda)I_H + \lambda\Big(A^{-1/2}BA^{-1/2}\Big)^s\Big)^{1/s} A^{1/2} & \text{if } 0 < |s| \leq 1, \\ A^{1/2}\Big(A^{-1/2}BA^{-1/2}\Big)^\lambda A^{1/2} & \text{if } s = 0, \end{cases} \tag{6}$$

where I_H denotes the identity operator of H, and the Heinz mean as

$$HZ_\lambda(A,B) := \frac{A\sharp_\lambda B + A\sharp_{1-\lambda} B}{2}. \tag{7}$$

As pointed out in [7], we set $A\sharp_\nu B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2}$, for any $\nu \in \mathbb{R}$, as extension of the weighted geometric mean. With this, Liao and Wu established the following mixed mean operator inequalities, [10]

$$A\sharp(A\nabla B) \geq \int_0^1 A\sharp\,(A\nabla_t B)\ dt \geq A\nabla(A\sharp B) \tag{8}$$

and, for $s \in [-1, 0] \cup [1, 2]$

$$A\sharp_s\,(A\nabla_\lambda B) \leq A\nabla_\lambda\,(A\sharp_s B)\,,\quad A\sharp_s\,(A!_\lambda B) \geq A!_\lambda\,(A\sharp_s B)\,. \tag{9}$$

The remainder of this paper will be organized as follows. In Section 2, we present some new Hermite-Hadamard type inequalities for operator h-convex functions. Section 3 is devoted to illustrate our theoretical results by some concrete examples. In section 4, applications to operator mean inequalities are provided.

2. Main results

The current section will be devoted to establish some Hermite-Hadamard type inequalities involving operator h-convex (resp. h-concave) functions and so allow us to generalize some other inequalities existing in the literature.

Before stating our first result, we notice the following. As already pointed out before, the λ-weighted arithmetic mean of A and B, $A\nabla_\lambda B := (1-\lambda)A + \lambda B$, is usually defined for $\lambda \in [0, 1]$ and $A, B \in \mathcal{B}^{+*}(H)$. However, for the sake of simplicity we will use the same notation, namely $C\nabla_\lambda D = (1-\lambda)C + \lambda D$ for $\lambda \in [0, 1]$ and any operators $C, D \in \mathcal{B}(H)$.

Now, we are in a position to state the following main result.

Theorem 2.1. *Let $f : I \to \mathbb{R}$ be an operator h-convex function on I and h be an integrable function on* $[0, 1]$. *For all self-adjoint operators $A, B \in \mathcal{B}(H)$ with spectrums in I and $0 \leq p < q \leq 1$, we have*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(A\nabla_{\frac{p+q}{2}} B\right) \leq \frac{1}{q-p}\int_p^q f(A\nabla_t B)dt \leq \frac{1}{q-p}\int_p^q \Big(h(1-x)\ f(A) + h(x)\ f(B)\Big)dx. \tag{10}$$

If f is operator h-concave on I then the inequalities in (10) *are reversed.*

Proof. For $0 \leq p < q \leq 1$ we can set $p = \lambda - \alpha$ and $q = \lambda + \alpha$ with $\lambda = \frac{p+q}{2} \in (0, 1)$ and $0 < \alpha = \frac{q-p}{2} \leq \min\{1-\lambda, \lambda\}$. With this, for all $t \in [0, 1]$ we can easily check that

$$1 - \lambda \pm \alpha \mp 2\alpha t \in [0, 1],\ \lambda \mp \alpha \pm 2\alpha t \in [0, 1],\ (1 - \lambda \pm \alpha \mp 2\alpha t) + (\lambda \mp \alpha \pm 2\alpha t) = 1.$$

It follows that, $Sp\,(A\nabla_{1-\lambda\mp\alpha\pm2\alpha t}B) \subset I$ and $Sp\,(A\nabla_{1-\lambda\pm\alpha\mp2\alpha t}B) \subset I$. Otherwise, the obvious scalar identity

$$\lambda = \frac{(\lambda - \alpha + 2\alpha t) + (\lambda + \alpha - 2\alpha t)}{2},$$

implies the following operator equality

$$A\nabla_\lambda B = \Big[A\nabla_{\lambda-\alpha+2\alpha t}B]\nabla[A\nabla_{\lambda+\alpha-2\alpha t}B\Big].$$

So, applying two times (3), we obtain

$$\begin{aligned} f\,((1-\lambda)A + \lambda B) &\leq h\left(\frac{1}{2}\right) f\,(A\nabla_{\lambda-\alpha+2\alpha t}B) + h\left(\frac{1}{2}\right) f\,(A\nabla_{\lambda+\alpha-2\alpha t}B) \\ &\leq h\left(\frac{1}{2}\right)\Big[h(1-\lambda+\alpha-2\alpha t)f(A) + h(\lambda-\alpha+2\alpha t)f(B)\Big] + \\ &\qquad h\left(\frac{1}{2}\right)\Big[h(1-\lambda-\alpha+2\alpha t)f(A) + h(\lambda+\alpha-2\alpha t)f(B)\Big]. \end{aligned}$$

Integrating side by side these latter inequalities with respect to $t \in [0,1]$, we get

$$\int_0^1 f\left((1-\lambda)A+\lambda B\right)dt \le h\Big(\frac{1}{2}\Big)\Big(\int_0^1 f\left(A\nabla_{\lambda-\alpha+2\alpha t}B\right)dt + \int_0^1 f\left(A\nabla_{\lambda+\alpha-2\alpha t}B\right)\, dt\Big)$$
$$\le h\Big(\frac{1}{2}\Big)\Big[\int_0^1 \Big(h(1-\lambda+\alpha-2\alpha t)+h(1-\lambda-\alpha+2\alpha t)\Big)dt\Big] f(A)+$$
$$h\Big(\frac{1}{2}\Big)\Big[\int_0^1 \Big(h(\lambda-\alpha+2\alpha t)+h(\lambda+\alpha-2\alpha t)\Big)dt\Big] f(B). \quad (11)$$

Making simple change of variables, $u=\lambda-\alpha+2\alpha t/\ u=\lambda+\alpha-2\alpha t$, we get the following formulas

$$\int_0^1 f\left(A\nabla_{\lambda-\alpha+2\alpha t}B\right)dt = \int_0^1 f\left(A\nabla_{\lambda+\alpha-2\alpha t}B\right)dt = \frac{1}{2\alpha}\int_{\lambda-\alpha}^{\lambda+\alpha} f\left(A\nabla_u B\right)du,$$

$$\int_0^1 h(1-\lambda+\alpha-2\alpha t)dt = \int_0^1 h(1-\lambda-\alpha+2\alpha t)dt = \frac{1}{2\alpha}\int_{\lambda-\alpha}^{\lambda+\alpha} h(1-u)du,$$

$$\int_0^1 h(\lambda-\alpha+2\alpha t)dt = \int_0^1 h(\lambda+\alpha-2\alpha t)dt = \frac{1}{2\alpha}\int_{\lambda-\alpha}^{\lambda+\alpha} h(u)du.$$

Substituting these latter formulas in (11) it holds

$$f\Big(A\nabla_\lambda B\Big) := f\Big((1-\lambda)A+\lambda B\Big) \le \frac{h\left(\frac{1}{2}\right)}{\alpha}\int_{\lambda-\alpha}^{\lambda+\alpha} f\left(A\nabla_t B\right)dt$$
$$\le \frac{h\left(\frac{1}{2}\right)}{\alpha}\Big[f(A)\int_{\lambda-\alpha}^{\lambda+\alpha} h(1-x)dx + f(B)\int_{\lambda-\alpha}^{\lambda+\alpha} h(x)dx\Big],$$

hence (10) after simple algebraic manipulations. The proof is complete. □

Theorem 2.1 provides a generalization for (5), in the sense that taking $p=0,\ q=1$ in (10) leads to (5). Furthermore, Theorem 2.1 has many consequences. For instance, we mention the following corollary which states a generalization of (2).

Corollary 2.2. *Let $f: I \to \mathbb{R}$ be operator convex on I. For all self-adjoint operators $A, B \in \mathcal{B}(H)$ with spectrums in I, the following inequalities*

$$f(A\nabla_{\frac{p+q}{2}}B) \le \frac{1}{q-p}\int_p^q f(A\nabla_t B)dt \le f(A)\nabla_{\frac{p+q}{2}} f(B), \quad (12)$$

hold for any $0 \le p < q \le 1$. If f is operator concave on I then (12) are reversed.

Proof. Assume that f is with positive values. Since f is operator convex on I, so f is operator h-convex on I, with $h(x)=x, x\in[0,1]$. Whence (12), since $h(1/2)=1/2$ and

$$\int_p^q h(1-x)dx = (q-p)\Big(1-\frac{p+q}{2}\Big), \text{ and } \int_p^q h(x)dx = (q-p)\frac{p+q}{2}.$$

Otherwise, the same proof as for Theorem 2.1 works for f operator convex not necessarily with positive values. □

The following remark may be of interest for the reader.

Remark 2.3. *Let $f(x) = 1/x,\ x > 0$, which is operator convex on $(0, \infty)$. Applying* (12) *we get*

$$\Big(A\nabla_{\frac{p+q}{2}}B\Big)^{-1} \le \frac{1}{q-p}\int_p^q \Big(A\nabla_t B\Big)^{-1}dt \le A^{-1}\nabla_{\frac{p+q}{2}}B^{-1},$$

which was obtained in [12, Corollary 3.8] under a functional point of view.

The left inequalities in (10) and in (12) can be generalized as recited in what follows.

Corollary 2.4. *Let f, h and A, B be as in Theorem 2.1. Let $n \ge 0$ be an integer and $0 \le p := x_0 < x_1 < ... < x_n < x_{n+1} := q \le 1$. Then we have*

$$\int_p^q f\Big(A\nabla_t B\Big)\,dt \ge \frac{1}{2h(1/2)}\sum_{i=0}^{n}(x_{i+1} - x_i)f\Big(A\nabla_{\frac{x_i+x_{i+1}}{2}}B\Big). \tag{13}$$

In particular, if f is operator convex on I then

$$\int_p^q f\Big(A\nabla_t B\Big)\,dt \ge \sum_{i=0}^{n}(x_{i+1} - x_i)f\Big(A\nabla_{\frac{x_i+x_{i+1}}{2}}B\Big). \tag{14}$$

If f is operator h-concave, resp. operator concave, then (13) *and* (14) *are reversed.*

Proof. We apply successively the left inequality of (10) in the subintervals $[x_i, x_{i+1}]$ of $[p, q]$. So, for any fixed $i = 0, 1, ..., n + 1$, we have

$$\int_{x_i}^{x_{i+1}} f\Big(A\nabla_t B\Big)\,dt \ge \frac{1}{2h(1/2)}(x_{i+1} - x_i)f\Big(A\nabla_{\frac{x_i+x_{i+1}}{2}}B\Big).$$

Summing side by side these latter inequalities from $i = 0$ to $i = n$, with the fact that $x_0 = p$ and $x_{n+1} = q$, we get (13). □

Remark 2.5. *(i) For $n = 0$,* (13) *and* (14) *coincide with the left inequality of* (10) *and* (12)*, respectively.*
(ii) If we apply the same way, as in the previous corollary, to the right inequality in (10) *we can easily see that its upper bound is still unchanged.*
(iii) The right expression in (13)*, as in* (14)*, increases with respect to $n \ge 0$. This implies two facts. Firstly,* (13) *and* (14) *refine* (10) *and* (12)*, respectively. Secondly, an upper bound for the integral $\int_p^q f\Big(A\nabla_t B\Big)\,dt$ similar to the right expression in* (14) *does not provide any new information.*

The following example illustrates and explains more the previous discussions.

Example 2.6. *If in Corollary 2.4 we choose an equidistant subdivision of $[p, q]$, namely $x_{i+1} - x_i = \frac{q-p}{n+1}$, then* (13) *becomes, after elementary computations*

$$\frac{1}{q-p}\int_p^q f\Big(A\nabla_t B\Big)dt \ge \frac{1}{2h(1/2)}\frac{1}{n+1}\sum_{i=0}^{n} f\Big(A\nabla_{r_{i,n}^{p,q}}B\Big) \ge \frac{1}{2h(1/2)}f\Big(A\nabla_{\frac{p+q}{2}}B\Big),$$

where we set

$$r_{i,n}^{p,q} := p\nabla_{s_{i,n}}q, \text{ with } s_{i,n} := \frac{2i+1}{2(n+1)}.$$

The details are simple and therefore omitted here.

Another example of application for Corollary 2.4 is contained in the following result.

Theorem 2.7. *Let f and h be as in Theorem 2.1 and let $A, B \in \mathcal{B}(H)$ be self-adjoint with spectrums in I. For any $0 \leq p < q \leq 1$ and $r \in [0,1]$, we have the following operator inequalities*

$$\frac{1}{2h(1/2)}\Big\{f\Big(A\nabla_{r_1}B\Big)\nabla_{1-r}f\Big(A\nabla_{r_2}B\Big)\Big\} \leq \frac{1}{q-p}\int_p^q f\Big(A\nabla_t B\Big)\,dt$$
$$\leq \Big\{f(A\nabla_p B) + f(A\nabla_q B)\Big\}\int_0^1 h(x)\,dx \leq \Big\{\Big(h(1-p)+h(1-q)\Big)f(A) + \Big(h(p)+h(q)\Big)f(B)\Big\}\int_0^1 h(x)dx, \tag{15}$$

where we set

$$r_1 := r_1(p,q) := p\nabla_{\frac{r}{2}}q, \quad r_2 := r_2(p,q) := p\nabla_{\frac{1+r}{2}}q. \tag{16}$$

If f is operator h-concave on I then (15) *are reversed.*

Proof. If we take $n = 1$ in (13) with $x_0 = p < x_1 = (1-r)p + rq < x_2 = q$, $r \in [0,1]$ we get the left inequality of (15) after simple algebraic operations. To show the second inequality in (15), we use the change of variable $t = (1-s)p + sq$, $s \in [0,1]$. With this, it is easy to check that

$$A\nabla_t B := (1-t)A + tB = (1-s)A\nabla_p B + sA\nabla_q B.$$

It follows that

$$\frac{1}{q-p}\int_p^q f\Big(A\nabla_t B\Big)dt = \int_0^1 f\Big((1-s)A\nabla_p B + sA\nabla_q B\Big)ds,$$

which, with (10) and the fact that $\int_0^1 h(1-x)dx = \int_0^1 h(x)$, yields the middle inequality of (15). We then deduce the last inequality by using (3). □

As for Corollary 2.2, the following result is immediate from Theorem 15.

Corollary 2.8. *Let $f : I \to \mathbb{R}$ be operator convex on I. For all self-adjoint operators $A, B \in \mathcal{B}(H)$ with spectrums in I, the following inequalities*

$$f\Big(A\nabla_{r_1}B\Big)\nabla_{1-r}f\Big(A\nabla_{r_2}B\Big) \leq \frac{1}{q-p}\int_p^q f\Big(A\nabla_t B\Big)\,dt \leq f(A\nabla_p B)\nabla f(A\nabla_q B) \leq f(A)\nabla_{\frac{p+q}{2}}f(B), \tag{17}$$

hold for any $0 \leq p < q \leq 1$ and $r \in [0,1]$, where r_1 and r_2 are given by (16). *If f is operator concave on I then* (17) *are reversed.*

If we choose $p = 0, q = 1$ and $r \in \{0,1\}$ in Theorem 15 and in Corollary 2.8 we get (5) and (2), respectively. Otherwise, taking $r = 0$ or $r = 1$ in the left inequalities of (15) and (17) we get the left ones in (10) and (12), respectively. More precisely, we have the following result which provides refinements for the left inequalities of (10) and (12).

Corollary 2.9. *With the same hypotheses as in Theorem 2.1, we have*

$$\frac{1}{q-p}\int_p^q f\Big(A\nabla_t B\Big)\,dt \geq \frac{1}{2h(1/2)}\sup_{r\in[0,1]}\Big\{f\Big(A\nabla_{r_1}B\Big)\nabla_{1-r}f\Big(A\nabla_{r_2}B\Big)\Big\} \geq \frac{1}{2h(1/2)}f\Big(A\nabla_{\frac{p+q}{2}}B\Big), \tag{18}$$

with reversed inequalities and "$\inf$" instead of "$\sup$", if f is operator h-concave on I. Similar inequalities can be stated for (17).

Theorem 2.10. *Let $f : I \to \mathbb{R}$ be operator h-convex on I and $A, B \in \mathcal{B}(H)$ be self-adjoint with $Sp(A) \cup Sp(B) \subseteq I$. Let a and b be two real numbers such that $0 \le b \le 1 \le a \le 2$ and $a + b \ge 2$. Then the following inequalities*

$$\begin{aligned}
&\frac{1}{(a+b)h\left(\frac{1}{2}\right)} f\left(\frac{(a-m)A+(b+m)B}{a+b}\right) \le \frac{a+b-1}{b-m(a+b-1)} \int_{\beta_{a,b,m}}^{\frac{b}{a+b-1}} f(A\nabla_t B)dt \\
&\le f(A\nabla_m B) \int_{\frac{a+b-2}{a+b}}^{1} h(1-x)dx + \left[h\left(\frac{a-1}{a+b-1}\right) f(A) + h\left(\frac{b}{a+b-1}\right) f(B)\right] \int_{\frac{a+b-2}{a+b}}^{1} h(x)dx \\
&\le \left[h(1-m) \int_{\frac{a+b-2}{a+b}}^{1} h(1-x)dx + h\left(\frac{a-1}{a+b-1}\right) \int_{\frac{a+b-2}{a+b}}^{1} h(x)\right] f(A) \\
&\qquad + \left[h(m) \int_{\frac{a+b-2}{a+b}}^{1} h(1-x)dx + h\left(\frac{b}{a+b-1}\right) \int_{\frac{a+b-2}{a+b}}^{1} h(x)dx\right] f(B) \qquad (19)
\end{aligned}$$

hold for any $m \in [0, 1]$ with $m \neq \dfrac{b}{a+b-1}$, where we set

$$\beta_{a,b,m} := \frac{2m}{a+b} + \frac{b(a+b-2)}{(a+b)(a+b-1)}.$$

Proof. Let us consider

$$C := A\nabla_m B \text{ and } D := \frac{a-1}{a+b-1}A + \frac{b}{a+b-1}B.$$

It is easy to check that

$$\frac{(a-m)A+(b+m)B}{a+b} = \frac{1}{a+b}C + \frac{(a+b-1)}{a+b}D. \qquad (20)$$

So, the condition $Sp(A) \cup Sp(B) \subseteq I$ ensures that $Sp(C) \cup Sp(D) \subseteq I$. Applying Theorem 2.1 for the operators C and D with $\lambda = \dfrac{a+b-1}{a+b}$ and $\alpha = 1 - \lambda = \dfrac{1}{a+b} \le \lambda$, we deduce

$$\begin{aligned}
&\frac{1}{(a+b)h\left(\frac{1}{2}\right)} f\left(\frac{1}{a+b}C + \frac{a+b-1}{a+b}D\right) \le \int_{\frac{a+b-2}{a+b}}^{1} f(C\nabla_t D)dt \\
&\qquad \le f(C) \int_{\frac{a+b-2}{a+b}}^{1} h(1-x)dx + f(D) \int_{\frac{a+b-2}{a+b}}^{1} h(x)dx. \qquad (21)
\end{aligned}$$

Simple computations lead to $C \nabla_t D = A \nabla_v B$, $t \in [0, 1]$, with $v = m + \left(\dfrac{b}{a+b-1} - m\right)t$, and simple integration by change of variable yields

$$\int_{\frac{a+b-2}{a+b}}^{1} f(C\nabla_t D)dt = \frac{a+b-1}{b-m(a+b-1)} \int_{\beta_{a,b,m}}^{\frac{b}{a+b-1}} f(A\nabla_v B)dv. \qquad (22)$$

On the other hand, since f is operator h-convex on I then we have

$$f(D) = f\left(\frac{a-1}{a+b-1}A + \frac{b}{a+b-1}B\right) \le h\left(\frac{a-1}{a+b-1}\right) f(A) + h\left(\frac{b}{a+b-1}\right) f(B) \qquad (23)$$

and

$$f(A\nabla_m B) \le h(1-m)f(A) + h(m)f(B). \qquad (24)$$

Combining (20), (21), (22), (23) and (24) we get (19). □

Remark 2.11. *Choosing in Theorem 2.10, $h(t) = t,\ t \in [0,1]$ and taking successively $(a,b) = (2,0)$ and $(a,b) = (1,1)$, $m \in (0,1)$, we obtain the following refinements of Hermite-Hadamard inequalities stated by Dragomir in [6] for operator convex functions,*

$$f\Big(\frac{A+B}{2}\Big) \le (1-m)f\left[\frac{(1-m)A+(1+m)B}{2}\right] + mf\left[\frac{(2-m)A+mB}{2}\right]$$
$$\le \int_0^1 f((1-t)A+tB)dt \le \frac{1}{2}[f((1-m)A+mB)+(1-m)f(B)+mf(A)] \le \frac{f(A)+f(B)}{2}. \quad (25)$$

3. Some examples

In this section, we shall be dealing with some examples illustrating the previous theoretical results. We then preserve the same notations as in the previous section.

Recall that, a function $f : I \longrightarrow \mathbb{R}$ is called operator monotone if $A \ge B$ implies $f(A) \ge f(B)$, for any self-adjoint operators $A, B \in \mathcal{B}(H)$ with spectrums in I. As useful examples of monotone/ operator convex (resp. operator concave) functions we recall the following.

Example 3.1. *(i) The function $x \longmapsto x^r$, $x \in (0,\infty)$, is operator convex for $r \in [-1,0] \cup [1,2]$, operator concave for $r \in [0,1]$, neither operator convex nor operator concave if $r \in (-\infty,-1) \cup (2,+\infty)$. It is operator monotone on $[0,\infty)$ for $r \in [0,1]$.*
(ii) The logarithm function $x \longmapsto \log x$ is operator monotone and operator concave on $(0,\infty)$ while the exponential function $x \longmapsto e^x$ is neither operator monotone nor operator convex.

We now may state the following result as example of application.

Proposition 3.2. *Let $A, B \in \mathcal{B}^{+*}(H)$. For any $s \in [-1,0] \cup [1,2]$ we have*

$$\Big(A\nabla B\Big)^s \le \sup_{r\in[0,1]} \Big\{\Big(A\nabla_{r_1} B\Big)^s \nabla_{1-r}\Big(A\nabla_{r_2} B\Big)^s\Big\} \le \frac{1}{q-p}\int_p^q \Big(A\nabla_t B\Big)^s dt \le \Big(A\nabla_p B\Big)^s \nabla \Big(A\nabla_q B\Big)^s,$$

with reversed inequalities if $s \in [0,1]$. Here r_1 and r_2 are as in (16).

Proof. Using Theorem 15 and Corollary 2.9, with Example 3.1,(i) we get the requested inequalities. □

In order to give more examples, we need to recall the following lemma [9], known in the literature as Hansen's theorem.

Lemma 3.3. *If f is a nonnegative operator monotone function on $[0,+\infty)$, then the following inequality*

$$C^* f(A) C \le f\Big(C^* A C\Big) \quad (26)$$

holds for every positive operator A and every contraction C, i.e. $C^*C \le I_H$.

Using this lemma, we state the following result which provides an example of an operator h-concave function.

Proposition 3.4. *The function $f(x) = x^s$, for $s \in [0,1]$, is operator h-concave on $(0,\infty)$, with $h(t) = \dfrac{t^s}{a^{1-s}}$ for any $a \ge 2$.*

Proof. Let $A, B \in \mathcal{B}(H)$ be two self-adjoint operators, $\lambda \in [0,1]$ and $a \geq 2$. We put,

$$C = \frac{1}{\sqrt{a}} I_H,\ U = a(1-\lambda)A \ \text{ and } \ V = a\lambda B.$$

We have $C^* = C$ and $2C^*C \leq I_H$, so $\begin{pmatrix} C & 0 \\ C & 0 \end{pmatrix}$ is a contraction. Since f is a nonnegative operator monotone function on $(0,\infty)$, by virtue of (26) we get

$$\begin{aligned} \begin{pmatrix} C\big(f(U)+f(V)\big)C & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} C & C \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f(U) & 0 \\ 0 & f(V) \end{pmatrix} \begin{pmatrix} C & 0 \\ C & 0 \end{pmatrix} \\ &= \begin{pmatrix} C & 0 \\ C & 0 \end{pmatrix}^* f\left(\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}\right) \begin{pmatrix} C & 0 \\ C & 0 \end{pmatrix} \\ &\leq f\left(\begin{pmatrix} C & 0 \\ C & 0 \end{pmatrix}^* \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} C & 0 \\ C & 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} f\left(C(U+V)C\right) & 0 \\ 0 & f(0) \end{pmatrix}. \end{aligned}$$

Thus, we have

$$f\big(C(U+V)C\big) \geq C\big(f(U)+f(V)\big)C,$$

which means that

$$\big((1-\lambda)A + \lambda B\big)^s \geq \frac{1}{a}\big(a(1-\lambda)A\big)^s + \frac{1}{a}(a\lambda B)^s$$

or, equivalently,

$$\big((1-\lambda)A + \lambda B\big)^s \geq \frac{(1-\lambda)^s}{a^{1-s}}A^s + \frac{\lambda^s}{a^{1-s}}B^s. \tag{27}$$

Hence, f is operator h-concave on $(0,\infty)$ with $h(t) = \dfrac{t^s}{a^{1-s}}$. □

Now, a pertinent question arises from the above: Is Proposition 3.4 stronger than the operator concavity of $x \longmapsto x^s,\ s \in [0,1]$? In what follows we discuss the answer to this question. Indeed, for $s \in [0,1]$ the function $x \longmapsto x^s$ is operator concave and so we have

$$\big((1-t)A + tB\big)^s \geq (1-t)A^s + tB^s. \tag{28}$$

With $h(t) = \dfrac{t^s}{a^{1-s}}$, Proposition 3.4 is equivalent to

$$\big((1-t)A + tB\big)^s \geq h(1-t)A^s + h(t)B^s. \tag{29}$$

It is then natural to compare (28) and (29). It is easy to check that $h(t) \leq t$ for $t \geq 1/a$ and, $h(t) \geq t$ for $t \leq 1/a$. This means that, neither (28) nor (29) is uniformly stronger than the other. Concluding, (29) is an improvement of (28) for $t \in [1/a, 1]$.

Using Proposition 3.4 we will now establish the following result which contains, in its turn, an example of application.

Proposition 3.5. *Let $A,B\in\mathcal{B}^{+*}(H)$, $0\le p<q\le 1$ and $a\ge 2$. For any $s\in[0,1]$ we have*

$$\left(\frac{a}{2}\right)^{1-s}\left(A\nabla_{\frac{p+q}{2}}B\right)^s\ge\left(\frac{a}{2}\right)^{1-s}\inf_{r\in[0,1]}\left\{\left(A\nabla_{r_1}B\right)^s\nabla_{1-r}\left(A\nabla_{r_2}B\right)^s\right\}\ge\frac{1}{q-p}\int_p^q\left(A\nabla_tB\right)^s dt,\tag{30}$$

where r_1 and r_2 are defined as in (16).

Proof. By Proposition 3.4, the function $f(x)=x^s, s\in[0,1]$ is operator h-concave on $(0,\infty)$ with $h(t)=a^{s-1}t^s$. Applying Theorem 15 and Corollary 2.9, with the fact that $h(1/2)=a^{s-1}2^{-s}$, we obtain (30). □

Finally, the following result gives another example of application.

Proposition 3.6. *Let a,b,m be as in Theorem 2.10 and $s\in[0,1]$. For all $A,B\in\mathcal{B}^{+*}(H)$, the following inequalities hold*

$$\begin{aligned}\left(\frac{2}{a+b}\right)^s\left(\frac{(a-m)A+(b+m)B}{a+b}\right)^s&\ge\frac{a+b-1}{b-m(a+b-1)}\int_{\beta_{a,b,m}}^{\frac{b}{a+b-1}}(A\nabla_tB)^s dt\\&\ge\frac{1}{(1+s)(a+b)^2}\left[2^{s+1}(A\nabla_mB)^s+\frac{\gamma_{a,b,s}}{(a+b)^{1-s}}\left((a-1)^sA^s+b^sB^s\right)\right]\\&\ge\frac{1}{(1+s)(a+b)^{3-s}}\left[\left(2^{s+1}(1-m)^s+(a-1)^s\gamma_{a,b,s}\right)A^s+\left(2^{s+1}m^s+b^s\gamma_{a,b,s}\right)B^s\right],\end{aligned}\tag{31}$$

where

$$\beta_{a,b,m}=\frac{2m}{a+b}+\frac{b(a+b-2)}{(a+b)(a+b-1)},\ \gamma_{a,b,s}=\frac{(a+b)^{1+s}-(a+b-2)^{1+s}}{(a+b-1)^s}.$$

In particular, for any $m\in[0,1)$ we have

$$\begin{aligned}\left(\frac{(1-m)A+(1+m)B}{2}\right)^s\ge\frac{1}{1-m}\int_m^1(A\nabla_tB)^s dt&\ge\frac{1}{4(1+s)}\left[2^{s+1}(A\nabla_mB)^s+2^{2s}B^s\right]\\&\ge\frac{1}{4^{1-s}(1+s)}\left[(1-m)^sA^s+(1+m^s)B^s\right].\end{aligned}\tag{32}$$

Proof. According to Proposition 3.4, the function $f(x)=x^s$ is operator h-concave on $(0,\infty)$ with $h(t)=\dfrac{t^s}{(a+b)^{1-s}}$. Furthermore, we have

$$\int_{\frac{a+b-2}{a+b}}^1 h(x)dx=\frac{(a+b)^{1+s}-(a+b-2)^{1+s}}{(s+1)(a+b)^2},\quad\int_{\frac{a+b-2}{a+b}}^1 h(1-x)dx=\frac{2^{1+s}}{(s+1)(a+b)^2}.$$

Substituting these latter expressions in (19), we obtain (31). Taking $a=b=1$ in (31) we get (32). □

4. Application to operator means

In this section, we will apply our main results in establishing some operator mean inequalities. Our first result in this context is recited as follows.

Theorem 4.1. *Let $A,B\in\mathcal{B}^{+*}(H)$, $\nu\in[0,1]$ and $0<\alpha\le\min\{1-\nu,\nu\}$. Then the following inequalities*

$$\mathcal{P}_{\nu,s}(A,B)\ge\frac{1}{2\alpha}\int_{\nu-\alpha}^{\nu+\alpha}\mathcal{P}_{t,s}(A,B)dt\ge A\nabla_\nu B,\tag{33}$$

$$\mathcal{P}_{\nu,-s}(A,B)\le 2\alpha\left(\int_{\nu-\alpha}^{\nu+\alpha}\left(\mathcal{P}_{t,-s}(A,B)\right)^{-1}dt\right)^{-1}\le A!_\nu B.\tag{34}$$

hold for any $s\in[1,2]$. If $s\in\left[\frac{1}{2},1\right]$, then (33) *and* (34) *are reversed.*

Proof. Consider the function $f(x) = x^{\frac{1}{s}},\ x \in (0, +\infty)$.

• If $s \in [1, 2]$ then Example 3.1,(i) tells us that f is operator concave on $(0, +\infty)$. Hence, using (12) with $X = I_H$ and $Y = \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^s$ we get

$$\left((1-\nu)I_H + \nu\left(A^{-1/2}BA^{-1/2}\right)^s\right)^{1/s} \geq \frac{1}{2\alpha}\int_{\nu-\alpha}^{\nu+\alpha}\left((1-t)I_H + t\left(A^{-1/2}BA^{-1/2}\right)^s\right)^{1/s} dt$$
$$\geq (1-\nu)I_H + \nu A^{-1/2}BA^{-1/2}. \quad (35)$$

Multiplying (35) at left and at right by $A^{1/2}$, and utilizing (6), we obtain

$$\mathcal{P}_{\nu,s}(A,B) \geq \frac{1}{2\alpha}\int_{\nu-\alpha}^{\nu+\alpha}\mathcal{P}_{t,s}(A,B)dt \geq (1-\nu)A + \nu B = A\nabla_\nu B.$$

Thus, (33) is established. Now, let us replace in (33) A and B by A^{-1} and B^{-1} respectively. Then we get

$$\mathcal{P}_{\nu,s}(A^{-1},B^{-1}) \geq \frac{1}{2\alpha}\int_{\nu-\alpha}^{\nu+\alpha}\mathcal{P}_{t,s}(A^{-1},B^{-1})dt \geq A^{-1}\nabla_\nu B^{-1}. \quad (36)$$

Taking into account the following relations,

$$A^{-1}\nabla_\nu B^{-1} = (A!_\nu B)^{-1} \text{ and } \mathcal{P}_{\nu,s}(A^{-1},B^{-1}) = \left(\mathcal{P}_{\nu,-s}(A,B)\right)^{-1},$$

with the fact that $x \longmapsto 1/x$ is operator monotone decreasing on $(0, \infty)$, we get (34).

• If $s \in \left[\frac{1}{2}, 1\right]$ then, following Example 3.1,(i), f is operator convex on $(0, +\infty)$. Thus, by Corollary 2.2 the inequalities (35) should be reversed. □

The following theorem provides our second result of application.

Theorem 4.2. *Let $A, B \in \mathcal{B}^{+*}(H)$, $\nu \in [0, 1]$ and $0 < \alpha \leq \min\{1-\nu, \nu\}$. The following inequalities*

$$A\sharp_s(A\nabla_\nu B) \geq \frac{1}{2\alpha}\int_{\nu-\alpha}^{\nu+\alpha} A\sharp_s\,(A\nabla_t B)\,dt \geq A\nabla_\nu(A\sharp_s B) \quad (37)$$

$$A\sharp_s(A!_\nu B) \leq 2\alpha\left(\int_{\nu-\alpha}^{\nu+\alpha}\left(A\sharp_s\,(A!_t B)\right)^{-1} dt\right)^{-1} \leq A!_\nu(A\sharp_s B) \quad (38)$$

hold for any $s \in [0, 1]$. If $s \in [-1, 0] \cup [1, 2]$ then (37) and (38) are reversed.

Proof. Let us define the function g by $g(x) = x^s,\ x \in (0, \infty)$.

• For $s \in [0, 1]$, again by Example 3.1,(i), g is operator concave on $(0, +\infty)$. So, applying (12) with $X = I_H$ and $Y = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, we deduce

$$\left((1-\nu)I_H + \nu A^{-1/2}BA^{-1/2}\right)^s \geq \frac{1}{2\alpha}\int_{\nu-\alpha}^{\nu+\alpha}\left((1-t)I_H + tA^{-1/2}BA^{-1/2}\right)^s dt$$
$$\geq (1-\nu)I_H + \nu\left(A^{-1/2}BA^{-1/2}\right)^s, \quad (39)$$

which can be rewritten as follows

$$\left(A^{-1/2}\left((1-\nu)A + \nu B\right)A^{-1/2}\right)^s \geq \frac{1}{2\alpha}\int_{\nu-\alpha}^{\nu+\alpha}\left(A^{-1/2}\left((1-t)A + tB\right)A^{-1/2}\right)^s dt$$
$$\geq A^{-1/2}\left((1-\nu)A + \nu A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^s A^{1/2}\right)A^{-1/2}.$$

Multiplying left and right each term in these last inequalities by $A^{1/2}$, we obtain (37).

Replacing (37) A and B by A^{-1} and B^{-1}, respectively, and remarking that

$$A^{-1}\sharp_s\left(A^{-1}\nabla_\nu B^{-1}\right)=\left(A\sharp_s(A!_\nu B)\right)^{-1}\text{ and }A^{-1}\nabla_\nu\left(A^{-1}\sharp_s B^{-1}\right)=\left(A!_\nu(A\sharp_s B)\right)^{-1},$$

we get

$$\left(A\sharp_s(A!_\nu B)\right)^{-1}\geq\frac{1}{2\alpha}\int_{\nu-\alpha}^{\nu+\alpha}\left(A\sharp_s(A!_t B)\right)^{-1}dt\geq\left(A!_\nu(A\sharp_s B)\right)^{-1}\tag{40}$$

Since $x\longmapsto 1/x$ is operator monotone decreasing on $(0,\infty)$ we then obtain (38).

• For $s\in[-1,0]\cup[1,2]$, g is operator convex on $(0,\infty)$. So, (39) should be reversed and the proof is finished. □

Remark 4.3. *Theorem 4.1 and Theorem 4.2 give generalizations for* (8) *and* (9)*, respectively.*

As a deduction of Theorem 4.2, we have the following corollary.

Corollary 4.4. *Let $A,B\in\mathcal{B}^{+*}(H)$, $\nu\in[0,1]$, $0<\alpha\leq\min\{1-\nu,\nu\}$. For any $s\in[0,1]$ we have*

$$HZ_s(A,A\nabla_\nu B)\geq\frac{1}{2\alpha}\int_{\nu-\alpha}^{\nu+\alpha}HZ_s(A,A\nabla_t B)dt\geq A\nabla_\nu HZ_s(A,B),\tag{41}$$

with reversed inequalities if $s\in[-1,0]\cup[1,2]$.

Proof. Let $s\in[0,1]$. From (33) we deduce

$$A\sharp_s(A\nabla_\nu B)\geq\frac{1}{2\alpha}\int_{\nu-\alpha}^{\nu+\alpha}A\sharp_s(A\nabla_t B)\,dt\geq A\nabla_\nu(A\sharp_s B)$$

and

$$A\sharp_{1-s}(A\nabla_\nu B)\geq\frac{1}{2\alpha}\int_{\nu-\alpha}^{\nu+\alpha}A\sharp_{1-s}(A\nabla_t B)\,dt\geq A\nabla_\nu(A\sharp_{1-s} B).$$

Summing these latter inequalities side by side, we obtain

$$\begin{aligned}A\sharp_s(A\nabla_\nu B)+A\sharp_{1-s}(A\nabla_\nu B)&\geq\frac{1}{2\alpha}\int_{\nu-\alpha}^{\nu+\alpha}\Big[A\sharp_s(A\nabla_t B)+A\sharp_{1-s}(A\nabla_t B)\Big]dt\\&\geq A\nabla_\nu(A\sharp_s B)+A\nabla_\nu(A\sharp_{1-s}B)=A\nabla_\nu\Big(A\sharp_s B+A\sharp_{1-s}B\Big).\end{aligned}$$

This, with (7), yields (41). If $s\in[-1,0]\cup[1,2]$ the previous inequalities should be reversed and the proof is complete. □

Another main result of application may be stated as follows.

Theorem 4.5. *Let $A,B\in\mathcal{B}^{+*}(H)$, $\nu\in[0,1]$ and $0<\alpha\leq\min\{1-\nu,\nu\}$. We have the following assertions:*
(i) If $s\in[-1,0]\cup[1,2]$ then

$$\begin{aligned}A\sharp_s(A\nabla B)&\leq[A\sharp_s(A\nabla_{\frac{1+\nu}{2}}B)]\nabla_\nu[A\sharp_s(A\nabla_{\frac{\nu}{2}}B)]\\&\leq\int_0^1 A\sharp_s(A\nabla_t B)dt\leq[A\sharp_s(A\nabla_\nu B)]\nabla[(A\sharp_s B)\nabla_\nu A]\leq A\nabla(A\sharp_s B)\end{aligned}\tag{42}$$

and

$$A\sharp_s(A!B) \geq [A\sharp_s(A!_{\frac{1+\nu}{2}}B)]!_\nu[A\sharp_s(A!_{\frac{\nu}{2}}B)]$$
$$\geq \left(\int_0^1 (A\sharp_s(A!_tB))^{-1}dt\right)^{-1} \geq [A\sharp_s(A!_\nu B)]![(A\sharp_s B)!_\nu A] \geq A!(A\sharp_s B) \quad (43)$$

(ii) If $s \in [0,1]$ then (42) *and* (43) *are reversed.*

Proof. Let $f(x) = x^s,\ x \in (0,+\infty)$ which is operator convex on $(0,+\infty)$ for $s \in [-1,0] \cup [1,2]$. Thus, applying (25) for the operators I_H and $A^{-1/2}BA^{-1/2}$, we get

$$\left(\frac{I_H + A^{-1/2}BA^{-1/2}}{2}\right)^s \leq (1-\nu)\left(\frac{(1-\nu)I_H + (1+\nu)A^{-1/2}BA^{-1/2}}{2}\right)^s$$
$$+\nu\left(\frac{(2-\nu)I_H + \nu A^{-1/2}BA^{-1/2}}{2}\right)^s \leq \int_0^1 \left((1-t)I_H + tA^{-1/2}BA^{-1/2}\right)^s dt$$
$$\leq \frac{1}{2}\left[\left((1-\nu)I_H + \nu A^{-\frac{1}{2}}BA^{-1/2}\right)^s + (1-\nu)\left(A^{-1/2}BA^{-1/2}\right)^s + \nu I_H\right] \leq \frac{I_H + \left(A^{-1/2}BA^{-1/2}\right)^s}{2}.$$

Multiplying at left and at right by $A^{1/2}$ all sides of this latter chain of inequalities, the required inequality (42) is deduced. Substituting A and B by A^{-1} and B^{-1} successively and taking the inverse in (42), we obtain (43). The remainder of the theorem is deduced from the fact that the previous inequalities should be reversed since f is operator convex on $(0,+\infty)$ for $s \in [0,1]$. □

Finally, we state the following result.

Theorem 4.6. *Let $A, B \in \mathcal{B}^{+*}(H)$, $\nu \in [0,1]$ and $0 < \alpha \leq \min\{1-\nu,\nu\}$. For any $s \in [0,1]$ and $a \geq 2$, we have*

$$\left(\frac{a}{2}\right)^{1-s} A\sharp_s(A\nabla_\nu B) \geq \frac{1}{2\alpha}\int_{\nu-\alpha}^{\nu+\alpha} A\sharp_s(A\nabla_t B)dt \geq \frac{2}{(s+1)a^{1-s}}\Big[A\sharp_s(A\nabla_{\nu-\alpha} B)\Big]\nabla\Big[A\sharp_s(A\nabla_{\nu+\alpha} B)\Big] \quad (44)$$

and

$$\left(\frac{2}{a}\right)^{1-s} A\sharp_s(A!_\nu B) \leqslant 2\alpha\left(\int_{\nu-\alpha}^{\nu+\alpha} (A\sharp_s(A!_tB))^{-1}dt\right)^{-1} \leqslant \frac{(s+1)a^{1-s}}{2}\Big[A\sharp_s(A!_{\nu-\alpha}B)\Big]!\Big[A\sharp_s(A!_{\nu+\alpha}B)\Big]. \quad (45)$$

Proof. Let $s \in [0,1]$ and $f(x) = x^s,\ x \in (0,+\infty)$. By Proposition 3.4, f is operator h-concave on $(0,+\infty)$, with $h(t) = \dfrac{t^s}{a^{1-s}}$. From (17) with (18), we have

$$\frac{1}{2h\left(\frac{1}{2}\right)} f(A\nabla_\lambda B) \leqslant \frac{1}{2\alpha}\int_{\lambda-\alpha}^{\lambda+\alpha} f(A\nabla_t B)dt \leqslant [f(A\nabla_{\lambda-\alpha} B) + f(A\nabla_{\lambda+\alpha} B)]\int_0^1 h(t)dt.$$

So,

$$\frac{a^{1-s}}{2\left(\frac{1}{2}\right)^s}\left((1-\nu)I_H + \nu A^{-1/2}BA^{-1/2}\right)^s \geq \frac{1}{2\alpha}\int_{\nu-\alpha}^{\nu+\alpha}\left((1-t)I_H + tA^{-1/2}BA^{-1/2}\right)^s dt$$
$$\geq \Big[\left((1-\nu+\alpha)I_H + (\nu-\alpha)A^{-1/2}BA^{-1/2}\right)^s + \left((1-\nu-\alpha)I_H + (\nu+\alpha)A^{-1/2}BA^{-1/2}\right)^s\Big]\int_0^1 \frac{t^s}{a^{1-s}}dt.$$

Multiplying both sides at left and at right by $A^{1/2}$, we obtain

$$\left(\frac{a}{2}\right)^{1-s} A\sharp_s(A\nabla_\nu B) \geq \frac{1}{2\alpha}\int_{\nu-\alpha}^{\nu+\alpha} A\sharp_s(A\nabla_t B)dt \geq \frac{1}{(s+1)a^{1-s}}\left(A\sharp_s(A\nabla_{\nu-\alpha} B) + A\sharp_s(A\nabla_{\nu+\alpha} B)\right).$$

Whence (44). Substituting, in a similar manner as previously, A and B by A^{-1} and B^{-1} respectively, we get (45). $\square$

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