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Hyers-Ulam Stability of Fractional Integro-differential Equation with a Positive Constant Coefficient involving the Generalized Caputo Fractional Derivative

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Abstract. The purpose of this paper is to investigate the existence and uniqueness of a solution, and the continuous dependence on the input data of the solution of integro-differential equations with a positive constant coefficient involving fractional order derivative (FIDEs). In addition, we also provide the sufficient conditions for the Hyers-Ulam stability (HU-stability) and the Hyers-Ulam-Rassias stability (HUR-stability) of FIDEs. Finally, the HUR-stability of the well-known model of RLC circuit in the form of FIDEs is also surveyed.

1. Introduction

In this paper, we consider FIDEs with a positive constant coefficient under the form

$$\begin{cases} {}^{C}_{\psi}\mathbf{D}^{\beta}_{a^+}\xi(t) = \lambda\xi(t) + \mathbf{F}\Big(t, \xi(t), \int_a^t \mathbf{K}(t,s)\xi(s)ds\Big) \\ \xi(a) = \xi_a \in \mathbf{R}, \end{cases} \tag{1}$$

where $\beta \in (0,1), t \in [a,b]$; ${}^{C}_{\psi}\mathbf{D}^{\beta}_{a^+}\xi(t)$ denotes the concept of generalized Caputo-type fractional derivative with respect to another function ψ (it is also called the $\psi-$Caputo fractional derivative) defined as in Definition 2.1; the functions $\mathbf{F} : [a,b] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $\mathbf{K} : [a,b] \times [a,b] \to \mathbf{R}$ are continuous; λ is a positive constant.

Fractional differential equations (FDEs) and fractional integral equations arise naturally in various fields of physics and engineering such as electrical circuits, chemical sciences, biology, control theory, fitting of experimental data, biophysics, etc. An interesting account in the study of the fields of economics and engineering can be found in [1, 2, 6, 10, 13, 21, 22, 25, 28, 34, 35], and references therein. In recent years, the theory of fractional calculus and FDEs, as an important research branch, have been further developed

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and have attracted much attention. For the general theory and applications of such equations we refer the interested reader to see the monographs of Kilbas et al. [16], Miller et al. [18], Lakshmikantham et al. [17] and the references therein.

In [33], Suganya et al. studied the existence of a mild solution of FIDEs with the nonlocal condition involving a strongly continuous fractional cosine family of the form

$$ {}_{t}^{C}\mathbf{D}_{0^+}^{\alpha}\xi(t) = \mathcal{A}\xi(t) + f\Big(t, \xi(t), \int_a^t e(t,s,\xi(s))ds\Big), \quad \forall t \in [0,T], $$

with nonlocal condition

$$ \xi(0) + g(x) = \xi_0, \quad x'(0) + h(x) = \xi_1 $$

where ${}_{t}^{C}\mathbf{D}_{0^+}^{\alpha}\xi(t)$ is the Caputo fractional derivative (Caputo FD) of ξ at t with $\alpha \in (1,2]$; $\mathcal{A} : D\big(\mathcal{A}\big) \subset \mathbf{X} \to \mathbf{R}$ is the infinitesimal generator of a strongly continuous fractional cosine family $\big\{C_\alpha(t)\big\}_{t\geq 0}$ on $\mathbf{X}$ (where $\mathbf{X}$ is a Banach space); $f : [0,T]\times\mathbf{X}^3 \to \mathbf{R}$ and $e : [0,T]^2\times\mathbf{X} \to \mathbf{R}$ are continuous functions; $\xi_0, \xi_1 \in \mathbf{X}$. The existence results of a solution for the above problem was investigated through utilizing Banach fixed point theorem and Krasnoselskii's fixed point theorem.

Aissani et al. [4] presented the existence of a mild solution under the conditions concerning Kuratowski's measure of noncompactness of FIDEs with the time delay and non-instantaneous impulsive effects as follows:

$$ \begin{cases} {}_{t}^{C}\mathbf{D}_{0^+}^{\alpha}\xi(t) + \mathcal{A}\xi(t) = \int_0^t a(t,s) f\Big(t, \xi_{\rho(s,\xi_s)}, \xi(s)\Big)ds, \;\; \forall t \in (s_i, t_{i+1}] \subset [0,T], \forall i \in \{0,1,2,...,m\} \\ x(t) = h_i\Big(t, \xi_{\rho(t,\xi_t)}, \xi(t)\Big), \quad \forall t \in (t_i, s_i] \subset [0,T], \forall i \in \{1,2,...,m\} \\ \xi(0) = \phi \in \mathcal{B} \subset \mathbf{X}, \end{cases} $$

where $\alpha \in (1,2]$; $\mathcal{A} : D\big(\mathcal{A}\big) \subset \mathbf{X} \to \mathbf{R}$ is the infinitesimal generator of an analytic semigroup $\big\{S(t)\big\}_{t\geq 0}$ of uniformly bounded linear operators on $\mathbf{X}$; $f : [0,T]\times\mathcal{B}\times\mathbf{X} \to \mathbf{R}$, $a : [0,T]^2 \to \mathbf{R}$ and $\rho : [0,T]\times\mathcal{B} \to (-\infty, T]$ are continuous functions and $h_i \in C\Big([0,T]\times\mathcal{B}\times\mathbf{X}, \mathbf{X}\Big)$ for any $i = \overline{1,m}$. The Darbo's fixed point theorem and the Mönch fixed point theorem are utilized in the authors' paper.

In [27], Shah et al. surveyed the existence results of a solution for FIDEs under Caputo-Fabrizo FD by employing Banach's and Krasnoselskii's fixed point theorems. In addition, the authors also provided necessary conditions for HU and HUR stability of the above problem.

Employing fixed point theorem and the concept of ψ-fractional Bielecki-type norm, D. Pachpatte [24] studied the existence theory of a solution of Fredholm-type FIDEs involving the concept of ψ-Hilfer FD under the form:

$$ \xi(t) = w(t) + \frac{1}{\Gamma(\alpha)}\int_a^b \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1} g\left(t, \tau, \xi(\tau), {}^{H}D_{a+}^{\alpha,\beta;\psi}\xi(\tau)\right) \mathrm{d}\tau $$

where $g \in C([a,b]^2 \times \mathbf{R}^n \times \mathbf{R}^n, \mathbf{R}_+)$and $g \in C([a,b], \mathbf{R}_+)$ satisfy some suitable conditions. The properties of solution (such as continuous dependence, boundedness) are discussed involving ψ-fractional Gronwall type of inequalities.

In recent years, researchers have also developed numerical methods for seeking the solution of FIDEs. For instance, Akbar et al. [5] proceeded to solve the approximate solutions of the FIDEs system by extending the Asymptotic Optimal Homotopy method. Mohammed [19] proposed a numerical approach to seek the approximate solutions of the following linear FIDEs through the least-squares method combined with Chebyshev polynomial:

$$ \begin{cases} {}_{t}^{C}\mathbf{D}_{0^+}^{\alpha}\xi(t) = f(t) + \int_0^1 K(t,s)\xi(s)ds, \quad 0 \leq t, s \leq 1 \\ \varphi^{(i)}(0) = \delta_i, \end{cases} $$

where $n-1 < \alpha \leq n, \quad n \in \mathbf{N}$.

In [12], Ahmed et al. utilized the modified Adomian decomposition method to seek the numerical solutions of the problem below:

$$\begin{cases} {}^{C}_{t}\mathbf{D}^{\alpha}_{0^+}\xi(t) = a(t)\xi(t) + \int_0^t K(t,s)F(\xi(s))ds + g(t) \\ u(0) = u_0. \end{cases}$$

In addition, the authors also investigated the existence, uniqueness results and convergence of a solution to the above problem.

The stability problem of FDEs is one of the important subjects for control theory. An interesting account in the study of the control theory of fractional differential equations through employing the Lyapunov stability theory can be found in [3, 9, 11, 14] and references therein. In recent years, the Ulam stability theory (U-stability) of differential equations has been extensively surveyed and obtained lots of attention since it plays an important role in realistic problems in biology, economics especially numerical analysis where the explicit solution is difficult to find. The benefits of the types of U-stability are that the existence of analytical solutions of the given problems is guaranteed through the appropriate assumptions where the approximate solutions with a determined bound error are assumed first. Consequently, investigating the types of U-stability of differential equations has become a fruitful and reliable approach for approximately seeking the solutions of the given problems because it is shown that the given problem has an almost exact solution if it is U-stable. Recently, there are many noticeable works in the progress of the U-stability theory of functional equations, differential equations, and boundary value problems of FDEs; see the papers by Jung [15], Moslehian et al. [20], Rus [26], Sousa et al. [29], Wang et al. [36], and references cited therein. Meanwhile, the basic results of the U-stability theory for various classes of fractional differential equations concerning another function have been surveyed by the papers of Sousa et al. [29–31].

To our knowledge, the U-stability problem for fractional integro-differential equations is very restricted, and there is no work on surveying the U-stability of (1). So, the motivation for the elaboration of this paper is to investigate the existence of a unique solution, and the U-stability of FIDEs involving ψ-Caputo FD given (1). The outline of the paper is as follows. Some basic definitions and theorems concerning the concept of the $\psi-$Caputo fractional differentiability are recalled in Section 2. In Section 3, by utilizing the fixed point theorem, the existence of a unique solution of problem (1) is discussed. The continuous dependence and boundedness of solution of the problem (1) are also presented in Section 4 through using Pachpatte's inequality. Finally, in Section 5, we prove that the problem (1) are HU-stable and HUR-stable under the appropriate assumptions based on Gronwall inequality.

2. Preliminaries

Denote $C^n\Big([a,b],\mathbf{R}\Big)$ by the family of n-times continuously differentiable functions ξ from $[a,b]$ to $\mathbf{R}$ with the norm

$$\Big\|\xi\Big\|_n = \sum_{i=0}^{n-1} \|\xi^{(n)}\| + \|\xi^{(n)}\|_0,$$

where if $n=0$, one has $C^0\Big([a,b],\mathbf{R}\Big) = C\Big([a,b],\mathbf{R}\Big)$ and $\|\xi\|_0 = \max_{t\in[a,b]} |\xi(t)|$. For convenience, throughout the paper, we will denote $\mathbf{K}_\psi$ the family of the continuously differentiable functions $\psi : [a,b] \to \mathbf{R}_+$ satisfying the conditions as follows: the function ψ is increasing, positive, and $\psi'(t) \neq 0, \forall t \in (a,b)$.

Definition 2.1. [7] Let $\beta > 0$ and $\xi \in \mathbf{K}_\psi$. We set $\Psi^{\beta-1}(t,s) = \psi'(s)\Big(\psi(t)-\psi(s)\Big)^{\beta-1}$. The fractional integral and Caputo FD of ξ with respect to another function $\psi(t)$ of the order β are defined as follows

$${}_{\psi}\mathbf{I}^{\beta}_{a^+}\xi(t) = \frac{1}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\xi(s)ds,$$

where $\xi \in C([a,b],\mathbf{R})$, and

$$ {}^{C}_{\psi}\mathbf{D}^{\beta}_{a^+}\xi(t) = \frac{1}{\Gamma(n-\beta)}\int_a^t \Psi^{n-\beta-1}(t,s)\left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)^n \xi(s)ds, $$

where $\xi \in C^n([a,b],\mathbf{R})$, respectively, $n-1<\beta<n$, and $n=[\beta]+1$.

Lemma 2.2. [7] Let $\beta>0$ and $\alpha>1$, then the following assertions hold:

$$ {}_{\psi}\mathbf{I}^{\beta}_{a^+}\Big(\psi(t)-\psi(a)\Big)^{\alpha-1} = \frac{\Gamma(\alpha)}{\Gamma(\beta+\alpha)}\Big(\psi(t)-\psi(a)\Big)^{\beta+\alpha-1} $$

and

$$ {}^{C}_{\psi}\mathbf{D}^{\beta}_{a^+}\Big(\psi(t)-\psi(a)\Big)^{\alpha-1} = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\Big(\psi(t)-\psi(a)\Big)^{\alpha-\beta-1}. $$

Theorem 2.3. ([7]) Let $\beta>0$, then we have the following properties:

(1) If $\xi \in C\Big([a,b],\mathbf{R}\Big)$, then

$$ {}^{C}_{\psi}\mathbf{D}^{\beta}_{a^+}\,{}_{\psi}\mathbf{I}^{\beta}_{a^+}\xi(t) = \xi(t) $$

(2) If $\xi \in C^{n-1}\Big([a,b],\mathbf{R}\Big)$, then

$$ {}_{\psi}\mathbf{I}^{\beta}_{a^+}\,{}^{C}_{\psi}\mathbf{D}^{\beta}_{a^+}\xi(t) = \xi(t) - \sum_{k=0}^{n-1}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\xi\right)^k\Big(a\Big)\Big(\psi(t)-\psi(a)\Big)^k, \quad k\in\mathbb{N}. $$

Lemma 2.4. (Corollary 2 in[32]) Let $\psi \in \mathbf{K}_{\psi}$, the functions $u(t)$ and $v(t)$ be integrable on $[a,b]$, and $w \in C([a,b],\mathbf{R})$. Assume that the functions v and w are nonnegative and nondecreasing on $[a,b]$. If

$$ u(t) \le v(t) + w(t)\int_a^t \Psi^{\beta-1}(t,s)u(s)ds, $$

then

$$ u(t) \le v(t)\mathbb{E}_{\beta}\Big(w(t)\Gamma(\beta)\Big(\psi(t)-\psi(a)\Big)^{\beta}\Big), $$

where $\mathbb{E}_{\beta}(z)$ is Mittag–Leffler function defined by

$$ \mathbb{E}_{\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+k\beta)}, \quad \beta>0,\ k\in\mathbb{N}. $$

Lemma 2.5. (Pachpatte's inequality) (Theorem 1 in [23]) Let $u, v, c \in C([a,b],\mathbf{R})$ be positive functions, and $h(t)$ be a nonnegative and increasing continuous function. If

$$ u(t) \le h(t) + \int_a^t v(s)\left[u(s) + \int_a^s c(\tau)u(\tau)d\tau\right]ds, $$

then one gets

$$ u(t) \le h(t)\left[1 + \int_a^t v(s)\exp\left(\int_a^s [v(\tau)+c(\tau)]d\tau\right)ds\right] $$

for $t \in \mathbf{R}_+$

The following generalization of Banach's fixed point theorem is used to prove the existence and uniqueness results of problem (1).

Theorem 2.6. Let $\mathbf{X}$ be a Banach space and $A \subset \mathbf{X}$ is a nonempty closed subset. Let $\gamma_n \geq 0, n = 0, 1, 2, \ldots$, be a sequence such that the series $\sum_{n=0}^{\infty} \gamma_n$ converges. If a mapping $\mathbf{U} : A \longrightarrow A$ satisfies the following inequality

$$\|\mathbf{U}^n u - \mathbf{U}^n v\| \leq \gamma_n \|u - v\|$$

for every $n = 0, 1, 2, \ldots$, and every $u, v \in A$, then we deduce that $\mathbf{U}$ has a uniquely defined fixed point u^*. In addition, the sequence $\{\mathbf{U}^n u_0\}_{n=1}^{\infty}$ converges to the fixed point u^* for every $u_0 \in A$.

3. The existence and uniqueness

In this section, we investigate the existence and uniqueness of a solution of FIDEs given by (1). Throughout this paper, we assume that $\mathbf{F} : [a, b] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $\mathbf{K} : [a, b] \times [a, b] \to \mathbf{R}$ satisfy the assumptions below:

(A1) The function $\mathbf{F}$ is continuous on $[a, b]$;

(A2) There exists a non-negative constant M such that

$$\Big|\mathbf{F}(t, \varphi_1, \hat{\varphi}_1) - \mathbf{F}(t, \varphi_2, \hat{\varphi}_2)\Big| \leq M\Big(\Big|\varphi_1 - \varphi_2\Big| + \Big|\hat{\varphi}_1 - \hat{\varphi}_2\Big|\Big),$$

for any $t \in [a, b]$ and $\varphi_1, \varphi_2, \hat{\varphi}_1, \hat{\varphi}_2 \in \mathbf{R}$;

(A3) The function $\mathbf{K}$ is continuous on $[a, b]$. We set $F = \sup_{t\in[a,b]} \mathbf{F}(t, 0, 0)$. Since the function $\mathbf{K}(s, r)$ is continuous on $[a, b]$, we deduce that there is a non-negative constant K satisfying

$$\sup_{(s,r)\in[a,b]\times[a,b]} \mathbf{K}(s, r) \leq K.$$

The equivalence between FIDEs (1) and fractional integral equations is shown by the below lemma.

Lemma 3.1. Let $\mathbf{F}$ and $\mathbf{K}$ be continuous functions, and λ is a non-negative constant. Then, $\xi \in C^1([a, b], \mathbf{R})$ is a solution of FIDEs (1) iff ξ satisfies the fractional integral equation as follows

$$\xi(t) = \xi_a + \frac{\lambda}{\Gamma(\beta)} \int_a^t \Psi^{\beta-1}(t, s)\xi(s)ds + \frac{1}{\Gamma(\beta)} \int_a^t \Psi^{\beta-1}(t, s)\mathbf{F}\Big(s, \xi(s), \int_a^s \mathbf{K}(s, r)\xi(r)dr\Big)ds, \quad \forall t \in [a, b]. \tag{2}$$

Proof. Let $\xi \in C^1([a, b], \mathbf{R})$ be a solution of (1). We set

$$\mathbf{H}(t) = \lambda\xi(t) + \mathbf{F}\Big(t, \xi(t), \int_a^t \mathbf{K}(t, s)\xi(s)ds\Big).$$

Then, from (1) one has

$${}^C_\psi\mathbf{D}^\beta_{a^+}\xi(t) = \mathbf{H}(t).$$

By integrating ${}_\psi\mathbf{I}^\beta_{a^+}$ both sides of the above equation and through Theorem 2.3, one gets

$${}_\psi\mathbf{I}^\beta_{a^+}\,{}^C_\psi\mathbf{D}^\beta_{a^+}\xi(t) = {}_\psi\mathbf{I}^\beta_{a^+}\mathbf{H}(t),$$

which means

$$\xi(t) - \xi(a) = {}_\psi\mathbf{I}^\beta_{a^+}\mathbf{H}(t).$$

Hence, we obtain

$$\xi(t)=\xi_a+\frac{\lambda}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\xi(s)ds+\frac{1}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\mathbf{F}\Big(s,\xi(s),\int_a^s \mathbf{K}(s,r)\xi(r)dr\Big)ds,\quad \forall t\in[a,b].$$

For the converse, we assume that the function $\xi\in C^1([a,b],\mathbf{R})$ satisfies the fractional integral equation (2). Now, we will verify that the function ξ also satisfies FIDEs (1). From Theorem 2.3, taking $\psi-$Caputo FD ${}^{C}_{\psi}\mathbf{D}^{\beta}_{a^+}$ both sides of (2) leads to

$${}^{C}_{\psi}\mathbf{D}^{\beta}_{a^+}\xi(t)={}^{C}_{\psi}\mathbf{D}^{\beta}_{a^+}{}_{\psi}\mathbf{I}^{\beta}_{a^+}\mathbf{H}(t)=\mathbf{H}(t),$$

which means

$${}^{C}_{\psi}\mathbf{D}^{\beta}_{a^+}\xi(t)=\mathbf{H}(t).$$

Hence, the function ξ satisfies (1). □

Theorem 3.2. Let $\beta\in(0,1),\psi\in\mathbf{K}_\psi$. Assume that $\mathbf{F},\mathbf{K}$ satisfy the assumptions (A1)–(A3). If

$$\frac{\big(\psi(b)-\psi(a)\big)^{\beta}\Big[\lambda+M\Big(1+(b-a)K\Big)+F\Big]}{\Gamma(1+\beta)}<1, \tag{3}$$

then FIDE (1) has a unique solution on $[a,b]$.

Proof. Define $\mathcal{B}_\rho:=\Big\{\xi:[a,b]\to\mathbf{R}\;\Big|\;\xi(a)=\xi_a\in\mathbf{R}\text{ and }\sup_{t\in[a,b]}\big|\xi-\xi_a\big|\le\rho\Big\}$, where ρ satisfies

$$\rho\ge\big|\xi_a\big|\left(\frac{\Gamma(1+\beta)}{\big(\psi(b)-\psi(a)\big)^{\beta}\Big[\lambda+M\Big(1+(b-a)K\Big)+F\Big]}-1\right)^{-1}.$$

Consider an operator $\mathbf{U}:\mathcal{B}_\rho\to\mathcal{B}_\rho$ defined by

$$\mathbf{U}\big(\xi\big)(t)=\xi_a+\frac{\lambda}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\xi(s)ds+\frac{1}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\mathbf{F}\Big(s,\xi(s),\int_a^s \mathbf{K}(s,r)\xi(r)dr\Big)ds,\forall t\in[a,b]. \tag{4}$$

We will prove that the operator $\mathbf{U}$ has a uniquely defined fixed point, that is,

- $\mathbf{U}$ maps the set $\mathcal{B}_\rho$ to itself, i.e $\mathbf{U}\xi\in\mathcal{B}_\rho$, for all $\xi\in\mathcal{B}_\rho$.
- $\mathbf{U}:\mathcal{B}_\rho\to\mathcal{B}_\rho$ is a continuous operator.
- $\Big\{\mathbf{U}\xi_m\Big\}_{m=0}^{\infty}$ is a convergent sequence.

- By the definition of $\mathcal{B}_\rho$ and the assumption (A2), one has the following estimates:

$$\big|\xi(t)\big|\le\big|\xi(t)-\xi_a\big|+\big|\xi_a\big|\le\rho+\big|\xi_a\big|,\quad\forall t\in[a,b], \tag{5}$$

which leads to

$$\begin{aligned}\left|\mathbf{F}\Big(s,\xi(s),\int_a^s \mathbf{K}(s,r)\xi(r)dr\Big)\right|&\le\left|\mathbf{F}\Big(s,\xi(s),\int_a^s \mathbf{K}(s,r)\xi(r)dr\Big)-\mathbf{F}(s,0,0)\right|+|\mathbf{F}(s,0,0)|\\&\le M\Big(\big|\xi(s)\big|+\Big|\int_a^s \mathbf{K}(s,r)\xi(r)dr\Big|\Big)+F\\&\le M\Big(\big|\xi(s)\big|+\sup_{(s,r)\in[a,b]\times[a,b]}|\mathbf{K}(s,r)|\int_a^s\big|\xi(r)\big|dr\Big)+F\\&\le M\Big(\rho+\big|\xi_a\big|\Big)\Big(1+(b-a)K\Big)+F,\quad\forall t\in[a,b].\end{aligned} \tag{6}$$

It follows from (4) that

$$\begin{aligned}\Big|\mathbf{U}\Big(\xi\Big)(t)-\xi_a\Big| &\le \frac{\lambda}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\Big|\xi(s)\Big|ds+\frac{1}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\Big|\mathbf{F}\Big(s,\xi(s),\int_a^s \mathbf{K}(s,r)\xi(r)dr\Big)\Big|ds\\ &\le \frac{\lambda}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\Big|\xi(s)\Big|ds+\frac{1}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\Big|\mathbf{F}(s,0,0)\Big|ds\\ &+\frac{1}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\Big|\mathbf{F}\Big(s,\xi(s),\int_a^s \mathbf{K}(s,r)\xi(r)dr\Big)-\mathbf{F}(s,0,0)\Big|ds \end{aligned}\tag{7}$$

Through the estimates (5) and (6) and Lemma 2.2, we have from (7) that

$$\begin{aligned}\Big|\mathbf{U}\Big(\xi\Big)(t)-\xi_a\Big| &\le \frac{\lambda\Big(\rho+\Big|\xi_a\Big|\Big)+F+M\Big(\rho+\Big|\xi_a\Big|\Big)\Big(1+(b-a)K\Big)}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)ds\\ &\le \frac{\Big(\rho+\Big|\xi_a\Big|\Big)\Big[\lambda+M\Big(1+(b-a)K\Big)\Big]+F}{\Gamma(1+\beta)}\Big(\psi(b)-\psi(a)\Big)^{\beta}\le\rho. \end{aligned}\tag{8}$$

Hence, it leads to that $\mathbf{U}\xi\in\mathcal{B}_\rho$ for all $\xi\in\mathcal{B}_\rho$, i.e $\mathbf{U}$ maps the set $\mathcal{B}_\rho$ to itself.

- For any $a\le t_1<t_2\le b$ and $\xi\in\mathcal{B}_\rho$, one has

$$\begin{aligned}|\mathbf{U}\Big(\xi\Big)(t_2)-\mathbf{U}\Big(\xi\Big)(t_1)| &\le \frac{\lambda}{\Gamma(\beta)}\left(\Big|\int_a^{t_1}\Big(\Psi^{\beta-1}(t_2,s)-\Psi^{\beta-1}(t_1,s)\Big)\xi(s)ds-\int_{t_1}^{t_2}\Psi^{\beta-1}(t_2,s)\xi(s)ds\Big|\right)\\ &+\Big|\frac{1}{\Gamma(\beta)}\int_a^{t_1}\Big(\Psi^{\beta-1}(t_2,s)-\Psi^{\beta-1}(t_1,s)\Big)\mathbf{F}\Big(s,\xi(s),\int_a^s \mathbf{K}(s,r)\xi(r)dr\Big)ds\\ &-\frac{1}{\Gamma(\beta)}\int_{t_1}^{t_2}\Psi^{\beta-1}(t_2,s)\mathbf{F}\Big(s,\xi(s),\int_a^s \mathbf{K}(s,r)\xi(r)dr\Big)ds\Big|. \end{aligned}\tag{9}$$

On the other hand, from (5) and (6) one also obtains

$$\int_a^{t_1}\Big(\Psi^{\beta-1}(t_2,s)-\Psi^{\beta-1}(t_1,s)\Big)|\xi(s)|ds\le\frac{\Gamma(\beta)}{\Gamma(\beta+1)}(\rho+|\xi_a|)\Big[(\psi(t_2)-\psi(a))^{\beta}-(\psi(t_1)-\psi(a))^{\beta}\Big];$$

$$\int_{t_1}^{t_2}\Psi^{\beta-1}(t_2,s)|\xi(s)|ds\le\frac{\Gamma(\beta)}{\Gamma(\beta+1)}(\rho+|\xi_a|)(\psi(t_2)-\psi(t_1))^{\beta};$$

$$\begin{aligned}&\int_a^{t_1}(\Psi^{\beta-1}(t_2,s)-\Psi^{\beta-1}(t_1,s))|\mathbf{F}\Big(s,\xi(s),\int_a^s \mathbf{K}(s,r)\xi(r)dr\Big)ds|\\ &\qquad\le\frac{\Gamma(\beta)}{\Gamma(\beta+1)}(M(\rho+|\xi_a|)(1+(b-a)K)+F)\Big[(\psi(t_2)-\psi(a))^{\beta}-(\psi(t_1)-\psi(a))^{\beta}\Big];\end{aligned}$$

$$\int_{t_1}^{t_2}\Psi^{\beta-1}(t_2,s)|\mathbf{F}\Big(s,\xi(s),\int_a^s \mathbf{K}(s,r)\xi(r)dr\Big)|ds\le\frac{\Gamma(\beta)}{\Gamma(\beta+1)}(M(\rho+|\xi_a|)(1+(b-a)K)+F)(\psi(t_2)-\psi(t_1))^{\beta}.$$

Hence, by (9) and by utilizing the inequality $e_1^{\beta}-e_2^{\beta}\le(e_1-e_2)^{\beta}$, where $0\le e_2\le e_1$ and $\beta\in(0,1)$, we infer that

$$\begin{aligned}|\mathbf{U}\Big(\xi\Big)(t_2)-\mathbf{U}\Big(\xi\Big)(t_1)| &\le \frac{\lambda(\rho+|\xi_a|)}{\Gamma(\beta+1)}\Big|(\psi(t_2)-\psi(a))^{\beta}-(\psi(t_1)-\psi(a))^{\beta}+(\psi(t_2)-\psi(t_1))^{\beta}\Big|\\ &+\frac{(M(\rho+|\xi_a|)(1+(b-a)K))+F}{\Gamma(\beta+1)}\Big|(\psi(t_2)-\psi(a))^{\beta}-(\psi(t_1)-\psi(a))^{\beta}+(\psi(t_2)-\psi(t_1))^{\beta}\Big|\\ &\le 2\left[\frac{\lambda(\rho+|\xi_a|)}{\Gamma(\beta+1)}+\frac{M(\rho+|\xi_a|)(1+(b-a)K)+F}{\Gamma(\beta+1)}\right](\psi(t_2)-\psi(t_1))^{\beta}, \end{aligned}\tag{10}$$

which leads to $|\mathbf{U}\Big(\xi\Big)(t_2) - \mathbf{U}\Big(\xi\Big)(t_1)| \to 0$ as $t_1 \to t_1$. Hence, we can infer that $\mathbf{U}$ is a continuous operator.

- The next step is to show that for every $m \in \{0, 1, 2, ...\}$ and every $t \in [a, b]$, one has

$$\|\mathbf{U}^m\xi - \mathbf{U}^m\widetilde{\xi}\|_0 \leq \frac{\Big(\lambda + M\Big(1 + (b-a)K\Big)\Big)^m}{\Gamma(1+m\beta)}\Big(\psi(t) - \psi(a)\Big)^{m\beta}\|\xi - \widetilde{\xi}\|_0. \tag{11}$$

This assertion can be proved by induction. It is clear that in the case $m = 0$, the assertion (11) is trivially true. For the induction steps, one defines

$$\begin{aligned}
\|\mathbf{U}^m\xi - \mathbf{U}^m\widetilde{\xi}\|_0 &= \|\mathbf{U}(\mathbf{U}^{m-1}\xi) - \mathbf{U}(\mathbf{U}^{m-1}\widetilde{\xi})\|_0 \\
&= \xi_a + \max_{t\in[a,b]} \frac{\lambda}{\Gamma(\beta)} \int_a^t \Psi^{\beta-1}(t,s)|\mathbf{U}^{m-1}\xi(s) - \mathbf{U}^{m-1}\widetilde{\xi}(s)|ds \\
&+ \frac{1}{\Gamma(\beta)} \max_{t\in[a,b]} \int_a^t \Psi^{\beta-1}(t,s)\left|\mathbf{F}\Big(s, \mathbf{U}^{m-1}\xi(s), \int_a^s \mathbf{K}(s,r)\mathbf{U}^{m-1}\xi(r)dr\Big)\right. \\
&\left. \qquad -\mathbf{F}\Big(s, \mathbf{U}^{m-1}\widetilde{\xi}(s), \int_a^s \mathbf{K}(s,r)\mathbf{U}^{m-1}\widetilde{\xi}(r)dr\Big)\right| ds.
\end{aligned} \tag{12}$$

Now, by virtue of the assumption (A2) and for any $t \in [a, b]$, and $\xi, \widetilde{\xi} \in \mathcal{B}_\rho$, we have

$$\begin{aligned}
\Big\|\mathbf{U}^1\xi - \mathbf{U}^1\widetilde{\xi}\Big\|_0 &\leq \frac{\lambda + M}{\Gamma(\beta)} \max_{t\in[a,b]} \int_a^t \Psi^{\beta-1}(t,s)\Big|\xi(s) - \widetilde{\xi}(s)\Big|ds + \frac{MK}{\Gamma(\beta)} \max_{t\in[a,b]} \int_a^t \Psi^{\beta-1}(t,s)\Big(\int_a^s \Big|\xi(r) - \widetilde{\xi}(r)\Big|dr\Big)ds \\
&\leq \Big(\frac{\lambda + M}{\Gamma(\beta)} + \frac{MK(b-a)}{\Gamma(\beta)}\Big)\Big(\int_a^t \Psi^{\beta-1}(t,s)ds\Big)\Big\|\xi - \widetilde{\xi}\Big\|_0 \\
&\leq \frac{\lambda + M\Big(1 + (b-a)K\Big)}{\Gamma(1+\beta)}\Big(\psi(t) - \psi(a)\Big)^\beta \Big\|\xi - \widetilde{\xi}\Big\|_0.
\end{aligned} \tag{13}$$

For each $m \geq 2$, from (A2) and (12), we obtain that

$$\begin{aligned}
\Big|\mathbf{U}^m\xi(t) - \mathbf{U}^m\widetilde{\xi}(t)\Big| &\leq \frac{\lambda + M}{\Gamma(\beta)} \int_a^t \Psi^{\beta-1}(t,s)\Big|\mathbf{U}^{m-1}\xi(s) - \mathbf{U}^{m-1}\widetilde{\xi}(s)\Big|ds \\
&+ \frac{MK}{\Gamma(\beta)} \int_a^t \Psi^{\beta-1}(t,s)\Big(\int_a^s \Big|\mathbf{U}^{m-1}\xi(r) - \mathbf{U}^{m-1}\widetilde{\xi}(r)\Big|dr\Big)ds.
\end{aligned} \tag{14}$$

In particular, for $m = 2$, it follows from Lemma 2.2 and (13) that

$$\begin{aligned}
\Big\|\mathbf{U}^2\xi - \mathbf{U}^2\widetilde{\xi}\Big\|_0 &\leq \frac{\lambda + M}{\Gamma(\beta)} \int_a^t \Psi^{\beta-1}(t,s)\Big\|\mathbf{U}^1\xi - \mathbf{U}^1\widetilde{\xi}\Big\|_0 ds + \frac{MK}{\Gamma(\beta)} \int_a^t \Psi^{\beta-1}(t,s)\Big(\int_a^s \Big\|\mathbf{U}^1\xi - \mathbf{U}^1\widetilde{\xi}\Big\|_0 dr\Big)ds \\
&\leq \frac{\Big(\lambda + M\Big(1 + (b-a)K\Big)\Big)^2}{\Gamma(1+2\beta)}\Big(\psi(t) - \psi(a)\Big)^{2\beta}\Big\|\xi - \widetilde{\xi}\Big\|_0, \quad \forall t \in [a,b].
\end{aligned}$$

By induction, we assume that (11) is true for the case $m = k$, i.e.,

$$\Big\|\mathbf{U}^k\xi - \mathbf{U}^k\widetilde{\xi}\Big\|_0 \leq \frac{\Big(\lambda + M\Big(1 + (b-a)K\Big)\Big)^k}{\Gamma(1+k\beta)}\Big(\psi(t) - \psi(a)\Big)^{k\beta}\Big\|\xi - \widetilde{\xi}\Big\|_0. \tag{15}$$

Then, from the assumption (A2), Lemma 2.2, the formulas (12), (14) and (15), we obtain that

$$\begin{aligned}\left\|\mathbf{U}^{k+1}\xi - \mathbf{U}^{k+1}\widetilde{\xi}\right\|_0 &\le \frac{\lambda + M}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\left\|\mathbf{U}^k\xi - \mathbf{U}^k\widetilde{\xi}\right\|_0 ds + \frac{MK}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\Big(\int_a^s \left\|\mathbf{U}^k\xi - \mathbf{U}^k\widetilde{\xi}\right\|_0 dr\Big)ds\\ &\le \frac{\Big(\lambda + M\big(1 + (b-a)K\big)\Big)^{k+1}}{\Gamma(1 + (1+k)\beta)}\Big(\psi(t) - \psi(a)\Big)^{(k+1)\beta}\left\|\xi - \widetilde{\xi}\right\|_0, \end{aligned} \tag{16}$$

for any $t \in [a, b]$. Thus, by using mathematical induction, one infers that the formula (11) holds for any $m \ge 0$. In addition, if we set

$$\sum_{m=0}^{\infty}\gamma_m = \sum_{m=0}^{\infty}\frac{\Big(\lambda + M\big(1 + (b-a)K\big)\Big)^m}{\Gamma(1 + m\beta)}\Big(\psi(b) - \psi(a)\Big)^{m\beta},$$

then by the definition of Mittag-Leffler function and (11), one gets

$$\left\|\mathbf{U}^m\xi - \mathbf{U}^m\widetilde{\xi}\right\|_0 \le E_\beta\Big(\Big(\lambda + M\big(1 + (b-a)K\big)\Big)\Big(\psi(b) - \psi(a)\Big)^\beta\Big)\left\|\xi - \widetilde{\xi}\right\|_0.$$

We observe that all conditions of Theorem 2.6 are satisfied. Hence, we can conclude that the operator $\mathbf{U}$ has a uniquely defined fixed point ξ^*, that is, the problem (1) has a unique solution ξ^*. □

4. The continuous dependence and boundedness of solutions

Theorem 4.1. (The boundedness of solutions) Let $\psi \in \mathbf{K}_\psi$ and $\beta \in (0, 1)$. Assume that $\mathbf{F}, \mathbf{K}$ satisfy the hypotheses (A1)-(A3). If the function $\xi(t)$ is a solution of (1), one gets the following estimate, for all $t \in [a, b]$,

$$\left|\xi(t)\right| \le \Big(\left|\xi_a\right| + \frac{F}{\Gamma(1+\beta)}\Big(\psi(b) - \psi(a)\Big)^\beta\Big)\Big(1 + \frac{\lambda + M}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\exp\Big(\frac{\lambda + M}{\Gamma(1+\beta)}\Big(\psi(t) - \psi(a)\Big)^\beta + \frac{MK}{\lambda + M}(t-a)\Big)ds\Big).$$

Proof. Since $\xi(t)$ is a solution of (1), from Lemma 3.1 one has

$$\xi(t) = \xi_a + \frac{\lambda}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\xi(s)ds + \frac{1}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\mathbf{F}\Big(s, \xi(s), \int_a^s \mathbf{K}(s,r)\xi(r)dr\Big)ds, \quad \forall t \in [a, b]. \tag{17}$$

By virtue of the assumptions (A1)–(A3), one gets the estimate below

$$\begin{aligned}\left|\xi(t)\right| &\le \left|\xi_a\right| + \frac{\lambda}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\left|\xi(s)\right|ds + \frac{1}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\left|\mathbf{F}\Big(s, \xi(s), \int_a^s \mathbf{K}(s,r)\xi(r)dr\Big)\right|ds\\ &\le \left|\xi_a\right| + \frac{\lambda}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\left|\xi(s)\right|ds + \frac{1}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\left|\mathbf{F}(t,0,0)\right|ds\\ &+ \frac{1}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\left|\mathbf{F}\Big(s, \xi(s), \int_a^s \mathbf{K}(s,r)\xi(r)dr\Big) - \mathbf{F}(t,0,0)\right|ds\\ &\le \left|\xi_a\right| + \frac{F}{\Gamma(1+\beta)}\Big(\psi(b) - \psi(a)\Big)^\beta + \int_a^t \frac{\lambda + M}{\Gamma(\beta)}\Psi^{\beta-1}(t,s)\left|\xi(s)\right|ds\\ &+ \int_a^t \frac{\lambda + M}{\Gamma(\beta)}\Psi^{\beta-1}(t,s)\Big(\int_a^s \frac{MK}{\lambda + M}\left|\xi(r)\right|dr\Big)ds, \quad \forall t \in [a, b]. \end{aligned} \tag{18}$$

If we put

$$u(t) = \left|\xi(t)\right|,\ h(t) = \left|\xi_a\right| + \frac{F}{\Gamma(1+\beta)}\Big(\psi(b) - \psi(a)\Big)^\beta,\ v(s) = \frac{\lambda + M}{\Gamma(\beta)}\Psi^{\beta-1}(t,s),\ c(r) = \frac{MK}{\lambda + M},$$

then the inequality (18) is rewritten as

$$u(t) \le h(t) + \int_a^t v(s)\left[u(s) + \int_a^s c(\tau)u(\tau)d\tau\right]ds.$$

Utilizing Lemma 2.5 leads to

$$u(t) \le h(t)\left[1 + \int_a^t v(s)\exp\left(\int_a^s [v(\tau) + c(\tau)]d\tau\right)ds\right], \quad \forall t \in [a,b].$$

This infers that

$$\begin{aligned}\left|\xi(t)\right| \le \left(\left|\xi_a\right| + \frac{F}{\Gamma(1+\beta)}\left(\psi(b) - \psi(a)\right)^\beta\right)&\left(1 + \frac{\lambda + M}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\right.\\ &\left.\times \exp\left(\frac{\lambda+M}{\Gamma(1+\beta)}\left(\psi(t) - \psi(a)\right)^\beta + \frac{MK}{\lambda+M}(t-a)\right)ds\right),\end{aligned}$$

for any $t \in [a,b]$. □

Theorem 4.2. (The continuous dependence of solutions on the initial condition) Assume that $\mathbf{F}, \mathbf{K}$ satisfy the hypotheses (A1)–(A3). Let $\xi(t)$ and $\xi^*(t)$ be two any solutions of problem (1) with $\xi(a) = \xi_a$ and $\xi^*(a) = \xi_a^*$, respectively. Then, the estimate below holds

$$\begin{aligned}\left|\xi(t) - \xi^*(t)\right| \le \left|\xi_a - \xi_a^*\right| \times &\left(1 + \frac{\lambda+M}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\right.\\ &\left.\times \exp\left(\frac{\lambda+M}{\Gamma(1+\beta)}\left[\left(\psi(t) - \psi(a)\right)^\beta\right] + \frac{MK}{\lambda+M}(t-a)\right)ds\right), \ \forall t \in [a,b].\end{aligned}$$

Proof. Since $\xi(t)$ and $\xi^*(t)$ are two solutions of FIDEs (1) with the initial conditions $\xi(a) = \xi_a$ and $\xi^*(a) = \xi_a^*$, respectively, we have

$$\xi(t) = \xi_a + \frac{\lambda}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\xi(s)ds + \frac{1}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\mathbf{F}\left(s, \xi(s), \int_a^s \mathbf{K}(s,r)\xi(r)dr\right)ds, \forall t \in [a,b], \tag{19}$$

and

$$\xi^*(t) = \xi_a^* + \frac{\lambda}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\xi^*(s)ds + \frac{1}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\mathbf{F}\left(s, \xi^*(s), \int_a^s \mathbf{K}(s,r)\xi^*(r)dr\right)ds, \forall t \in [a,b]. \tag{20}$$

Using the assumptions (A1)–(A3), one gets

$$\begin{aligned}\left|\xi(t) - \xi^*(t)\right| &\le \left|\xi_a - \xi_a^*\right| + \frac{\lambda}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\left|\xi(s) - \xi^*(s)\right|ds\\ &+ \frac{M}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\left(\left|\xi(s) - \xi^*(s)\right| + K\int_a^s \left|\xi(r) - \xi^*(r)\right|dr\right)ds\\ &\le \left|\xi_a - \xi_a^*\right| + \int_a^t \frac{\lambda+M}{\Gamma(\beta)}\Psi^{\beta-1}(t,s)\left|\xi(s) - \xi^*(s)\right|ds\\ &+ \int_a^t \frac{\lambda+M}{\Gamma(\beta)}\Psi^{\beta-1}(t,s)\left(\int_a^s \frac{MK}{\lambda+M}\left|\xi(r) - \xi^*(r)\right|dr\right)ds, \quad \forall t \in [a,b].\end{aligned}$$

By virtue of Lemma 2.5, we obtain

$$\left|\xi(t) - \xi^*(t)\right| \le \left|\xi_a - \xi_a^*\right| \times \left(1 + \frac{\lambda+M}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\exp\left(\frac{\lambda+M}{\Gamma(1+\beta)}\left(\psi(t) - \psi(a)\right)^\beta + \frac{MK}{\lambda+M}(s-a)\right)ds\right),$$

for any $t \in [a,b]$. □

5. The HU-stability and HUR-stability

Definition 5.1. The problem (1) is called HU-stable if there exists a non-negative constant $\widetilde{M}$ such that for any $\epsilon > 0$ and each $\xi \in C^1\Big([a,b], \mathbf{R}\Big)$ satisfying the following inequality

$$\left|{}^{C}_{\psi}\mathbf{D}^{\beta}_{a^+}\xi(t) - \lambda\xi(t) - \mathbf{F}\Big(t, \xi(t), \int_a^t \mathbf{K}(t,s)\xi(s)ds\Big)\right| \le \epsilon, \quad \forall t \in [a,b], \tag{21}$$

then there exists a solution $\xi^* \in C^1([a,b], \mathbf{R})$ of problem (1) satisfying

$$\Big|\xi(t) - \xi^*(t)\Big| \le \epsilon\widetilde{M}, \quad \forall t \in [a,b].$$

Definition 5.2. The problem (1) is called HUR-stable with respect to $\phi \in C\Big([a,b], \mathbf{R}\Big)$ if there exists a non-negative constant $\widehat{M}$ such that for any $\epsilon > 0$ and each $\xi \in C^1\Big([a,b], \mathbf{R}\Big)$ satisfying the following inequality

$$\left|{}^{C}_{\psi}\mathbf{D}^{\beta}_{a^+}\xi(t) - \lambda\xi(t) - \mathbf{F}\Big(t, \xi(t), \int_a^t \mathbf{K}(t,s)\xi(s)ds\Big)\right| \le \epsilon\phi(t), \quad \forall t \in [a,b], \tag{22}$$

then there exists a solution $\xi^* \in C^1\Big([a,b], \mathbf{R}\Big)$ of problem (1) satisfying

$$\Big|\xi(t) - \xi^*(t)\Big| \le \epsilon\widehat{M}\phi(t), \quad \forall t \in [a,b].$$

Theorem 5.3. If the assumptions of Theorem 3.2 hold, then problem (1) is HU-stable.

Proof. Assume that $\xi(t)$ is a solution of (21). Through Theorem 3.2, there exists a unique solution $\xi^*(t)$ of (1). So, one has

$$\xi^*(t) = \xi_a + \frac{\lambda}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\xi^*(s)ds + \frac{1}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\mathbf{F}\Big(s, \xi^*(s), \int_a^s \mathbf{K}(s,r)\xi^*(r)dr\Big)ds. \tag{23}$$

By the assumption (A1)–(A3) and for any $t \in [a,b]$, one has

$$\begin{aligned}
\Big|\xi(t) - \xi^*(t)\Big| &\le \left|\xi(t) - \frac{\lambda}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\xi(s)ds - \frac{1}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\mathbf{F}\Big(s, \xi(s), \int_a^s \mathbf{K}(s,r)\xi(r)dr\Big)ds\right| \\
&+ \left|\frac{\lambda}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\xi(s)ds + \frac{1}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\mathbf{F}\Big(s, \xi(s), \int_a^s \mathbf{K}(s,r)\xi(r)dr\Big)ds\right. \\
&\left. - \frac{\lambda}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\xi^*(s)ds - \frac{1}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\mathbf{F}\Big(s, \xi^*(s), \int_a^s \mathbf{K}(s,r)\xi^*(r)dr\Big)ds\right| \\
&\le \left|\xi(t) - \frac{\lambda}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\xi(s)ds - \frac{1}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\mathbf{F}\Big(s, \xi(s), \int_a^s \mathbf{K}(s,r)\xi(r)dr\Big)ds\right| \\
&+ \frac{\lambda + M\Big(1 + (b-a)K\Big)}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\Big|\xi(s) - \xi^*(s)\Big|ds.
\end{aligned} \tag{24}$$

Because $\xi(t)$ satisfies the inequality (21), we infer that there exists a function $\mathbf{P} \in C([a,b], \mathbf{R})$ such that $\Big|\mathbf{P}(t)\Big| \le \epsilon$ satisfying

$${}^{C}_{\psi}\mathbf{D}^{\beta}_{a^+}\xi(t) - \lambda\xi(t) - \mathbf{F}\Big(t, \xi(t), \int_a^t \mathbf{K}(t,s)\xi(s)ds\Big) = \mathbf{P}(t), \quad \forall t \in [a,b]. \tag{25}$$

Then, by integrating ${}_{\psi}\mathbf{I}^{\beta}_{a^+}$ both sides of Eq. (25) and by doing some calculations, one gets

$$\begin{aligned}&\left|\xi(t)-\xi_a-\frac{\lambda}{\Gamma(\beta)}\int_a^t\Psi^{\beta-1}(t,s)\xi(s)ds-\frac{1}{\Gamma(\beta)}\int_a^t\Psi^{\beta-1}(t,s)\mathbf{F}\Big(s,\xi(s),\int_a^s\mathbf{K}(s,r)\xi(r)dr\Big)ds\right|\\&\qquad={}_{\psi}\mathbf{I}^{\beta}_{a^+}\Big|\mathbf{P}(t)\Big|\le{}_{\psi}\mathbf{I}^{\beta}_{a^+}\epsilon=\frac{\Big(\psi(t)-\psi(a)\Big)^{\beta}}{\Gamma(\beta+1)}\epsilon,\quad\forall t\in[a,b].\end{aligned}\tag{26}$$

Combining the estimates (24) and (26) leads to

$$\Big|\xi(t)-\xi^*(t)\Big|\le\frac{\Big(\psi(t)-\psi(a)\Big)^{\beta}}{\Gamma(\beta+1)}\epsilon+\frac{\lambda+M\Big(1+(b-a)K\Big)}{\Gamma(\beta)}\int_a^t\Psi^{\beta-1}(t,s)\Big|\xi(s)-\xi^*(s)\Big|ds\tag{27}$$

for any $t\in[a,b]$. Hence, utilizing Lemma 2.4 to (27), one receives

$$\Big|\xi(t)-\xi^*(t)\Big|\le\widetilde{M}\epsilon,\quad\forall t\in[a,b],$$

where

$$\widetilde{M}:=\frac{\Big(\psi(b)-\psi(a)\Big)^{\beta}}{\Gamma(\beta+1)}\mathbb{E}_{\beta}\Big(\Big(\lambda+M\Big(1+(b-a)K\Big)\Big)\Big(\psi(b)-\psi(a)\Big)^{\beta}\Big).$$

□

Theorem 5.4. We assume the assumptions of Theorem 3.2 hold. If there exists a non-negative constant $\widehat{M}$ and $\phi\in C\Big([a,b],\mathbf{R}\Big)$ satisfying

$$\frac{1}{\Gamma(\beta)}\int_a^t\Psi^{\beta-1}(t,s)\phi(s)ds\le\widehat{M}_{\phi}\phi(t),\quad\forall t\in[a,b],\tag{28}$$

then problem (1) is HUR-stable.

Proof. Assume that $\xi(t)$ is a solution of the inequality (22) and the function $\xi^*(t)$ is a solution of problem (1). With the same manner as in the proof of Theorem 4.2, one obtains

$$\begin{aligned}\Big|\xi(t)-\xi^*(t)\Big|\le&\left|\xi(t)-\frac{\lambda}{\Gamma(\beta)}\int_a^t\Psi^{\beta-1}(t,s)\xi(s)ds-\frac{1}{\Gamma(\beta)}\int_a^t\Psi^{\beta-1}(t,s)\mathbf{F}\Big(s,\xi(s),\int_a^s\mathbf{K}(s,r)\xi(r)dr\Big)ds\right|\\&+\frac{\lambda+M\Big(1+(b-a)K\Big)}{\Gamma(\beta)}\int_a^t\Psi^{\beta-1}(t,s)\Big|\xi(s)-\xi^*(s)\Big|ds.\end{aligned}\tag{29}$$

Since ξ satisfies the inequality (22), one infers that there exists a function $\phi\in C([a,b],\mathbf{R})$ such that

$${}^{C}_{\psi}\mathbf{D}^{\beta}_{a^+}\xi(t)-\lambda\xi(t)-\mathbf{F}\Big(t,\xi(t),\int_a^t\mathbf{K}(t,s)\xi(s)ds\Big)=\epsilon\phi(t).\tag{30}$$

Then, by integrating ${}_{\psi}\mathbf{I}^{\beta}_{a^+}$ both sides of (30) and by the condition (28), one gets

$$\begin{aligned}&\left|\xi(t)-\xi_a-\frac{\lambda}{\Gamma(\beta)}\int_a^t\Psi^{\beta-1}(t,s)\xi(s)ds-\frac{1}{\Gamma(\beta)}\int_a^t\Psi^{\beta-1}(t,s)\mathbf{F}\Big(s,\xi(s),\int_a^s\mathbf{K}(s,r)\xi(r)dr\Big)ds\right|\\&\qquad=\epsilon{}_{\psi}\mathbf{I}^{\beta}_{a^+}\phi(t)\le\epsilon\widehat{M}_{\phi}\phi(t),\quad\forall t\in[a,b].\end{aligned}\tag{31}$$

Combining the inequalities (29) and (31) leads to

$$\left|\xi(t)-\xi^*(t)\right| \le \epsilon\widehat{M}_\phi\phi(t)+\frac{\lambda+M\Big(1+(b-a)K\Big)}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\left|\xi(s)-\xi^*(s)\right|ds. \tag{32}$$

By utilizing Lemma 2.4, we receive

$$\left|\xi(t)-\xi^*(t)\right| \le \widetilde{M}\epsilon, \quad \forall t\in[a,b],$$

where

$$\widetilde{M} := \widehat{M}_\phi\mathbb{E}_\beta\Big(\Big(\lambda+M\Big(1+(b-a)K\Big)\Big)\Big(\psi(b)-\psi(a)\Big)^\beta\Big).$$

□

Example 5.5. Let $[a,b]=[1,2], \psi=\ln t, \beta=0.5$. Consider the following FIDE:

$$^{C}_{\ln t}\mathbf{D}^{0.5}_{1^+}\xi(t)=\frac{1}{8}\xi(t)+\int_1^t e^{-s}s\xi(s)ds, \quad t\in[1,2], \tag{33}$$

with the initial condition $\xi(1)=0$. We observe that $\lambda=\frac{1}{8}$, $\mathbf{K}(t,s)=e^{-s}s$; $\mathbf{F}\Big(t,\xi(t),\int_a^t\mathbf{K}(t,s)\xi(s)ds\Big)=\int_1^t e^{-s}s\xi(s)ds$ and $\mathbf{F}(t,0,0)=0$. We observe that the functions $\mathbf{K},\mathbf{F}$ are continuous on $[1,2]$ and $\mathbf{F}$ satisfies the Lipschitz condition with $M=\frac{1}{e}$. Indeed, for $\xi_1,\xi_2\in C\Big([1,2],\mathbf{R}\Big)$, we have

$$\begin{aligned}\left|\mathbf{F}\Big(s,\xi_1(s),\int_1^s\mathbf{K}(s,r)\xi_1(r)dr\Big)ds-\mathbf{F}\Big(s,\xi_2(s),\int_1^s\mathbf{K}(s,r)\xi_2(r)dr\Big)ds\right| &= \left|\int_1^t e^{-s}s\xi_1(s)ds-\int_1^t e^{-s}s\xi_2(s)ds\right|\\ &\le \int_1^t \left|e^{-s}s\right|\left|\xi_1(s)-\xi_2(s)\right|ds\\ &\le \left|\frac{1}{e}-(1+t)e^{-t}\right|\left\|\xi_1-\xi_2\right\|_0\\ &\le \frac{1}{e}\left\|\xi_1-\xi_2\right\|_0.\end{aligned}$$

Furthermore, we also see that the condition (3) holds, i.e.,

$$\frac{\Big(\psi(b)-\psi(a)\Big)^\beta\Big[\lambda+M\Big(1+(b-a)K\Big)+F\Big]}{\Gamma(1+\beta)}=\frac{\Big(\ln 2-\ln(1)\Big)^{0.5}\Big[\frac{1}{8}+\frac{1}{e}\Big(1+(2-1)\frac{1}{e}\Big)+0\Big]}{\Gamma(1+0.5)}\approx 0.59017<1.$$

Hence, the hypotheses of Theorem 3.2 hold, which leads to that problem (33) has a unique solution on $[1,2]$.

Next, in order to check the HU-stability and HUR-stability of problem (33), we consider the following inequality, for any $\varepsilon>0$,

$$\left|^{C}_{\ln t}\mathbf{D}^{0.5}_{1^+}\xi(t)-\frac{1}{8}\xi(t)-\int_1^t e^{-s}s\xi(s)ds\right|\le\epsilon\phi(t), \quad \forall t\in[1,2]. \tag{34}$$

Firstly, if we take $\phi(t)=1$, then through Theorem 5.3, it is easy to see that problem (33) is HU-stable with

$$\left|\xi(t)-\xi^*(t)\right|\le\widetilde{M}\epsilon, \quad \forall t\in[1,2].$$

where ξ and ξ^* are two solutions of inequality (34) and problem (33), respectively, and

$$\widetilde{M} := \frac{\big(\ln(2)\big)^{0.5}}{\Gamma(0.5)}\mathbb{E}_{0.5}\Big(\Big(\frac{1}{8}+\frac{1}{e}+\frac{1}{e^2}\Big)\big(\ln(2)\big)^{0.5}\Big).$$

Secondly, if we take $\phi(t) = e^2\sqrt{\ln t}$ for any $t \in [1,2]$ and $\widehat{M}_\phi = \dfrac{2e^2}{\Gamma(0.5)}$, then

$$\frac{1}{\Gamma(\beta)}\int_a^t \Psi^{\beta-1}(t,s)\phi(s)ds = \frac{e^2}{\Gamma(0.5)}\int_1^t \Big(\ln\frac{t}{s}\Big)^{-0.5}\sqrt{\ln t}\frac{ds}{s} \le \frac{e^2}{\Gamma(0.5)}\int_1^t \Big(\ln\frac{t}{s}\Big)^{-0.5}\frac{ds}{s} = \frac{2e^2}{\Gamma(0.5)}\sqrt{\ln t} = \widehat{M}_\phi\phi(t),$$

for any $t \in [1,2]$. Thus, through Theorem 5.4, we deduce problem (33) is HUR-stable with

$$\big|\xi(t) - \xi^*(t)\big| \le \widetilde{M}\epsilon,$$

where

$$\widetilde{M} := \frac{2e^2}{\Gamma(0.5)}\mathbb{E}_{0.5}\Big(\Big(\frac{1}{8}+\frac{1}{e}+\frac{1}{e^2}\Big)\big(\ln(2)\big)^{0.5}\Big).$$

In the sequel, we will consider the problem concerning the well-known model of RLC circuit in the form of FIDEs involving the ψ−Caputo derivative as follows:

$$L\,{}^C_\psi\mathbf{D}^\beta_{0^+}I(t) + RI(t) + \frac{1}{C}\int_0^t \mathbf{K}(t,s)I(s)ds = E(t), \quad t \in [0,b], \tag{35}$$

where $\beta \in (0,1), \psi \in \mathbf{K}_\psi$; the non-negative elements R, L, C denote resistor, inductor, and capacitor, respectively; $E(t)$ is called the impressed voltage; $I(t)$ denote the current of circuit. In the model (35), it is well-known that $E(t) > 0$ if the anode potential is greater than the cathode voltage; $E(t) = 0$ if the potential is the same at the two terminals; $E(t) < 0$ if the anode potential is less than the cathode potential. In addition to this, we will say that $I(t) = 0$ if no current passes through at t; $I(t) > 0$ if the direction of flow is around the circuit from the positive terminal of the battery; $I(t) < 0$ if the flow is in the opposite direction. To our knowledge, there is no work on surveying the problem (35). Normally, it is not easy to seek the exact solutions of (35), so in the example below it is really necessary to investigate the HU-stability and HUR-stability of (35).

Example 5.6. Let $\beta \in (0,1)$, $\alpha > \beta$ and $\mathbf{K}(t,s) = \psi'(s)/\Gamma(\alpha+1)$. The following input data are considered: $R = 120\Omega, C = 15\mu\text{F}, L = 230\text{mH}, I_{\text{am}} = 0.0164, \omega = 1/\sqrt{LC}$. Assume that the function $I^* \in C^1([0,b],\mathbf{R})$ satisfies the following inequality:

$$\Big|L\,{}^C_\psi\mathbf{D}^\beta_{0^+}I(t) + RI(t) + \frac{1}{C}\int_0^t \mathbf{K}(t,s)I(s)ds - E(t)\Big| \le \varepsilon\phi(t), \quad \forall t \in [0,b], \tag{36}$$

where $I(0) = 0$. Based on the comparison between problem (1) and problem (36), it is easy to determine that $\lambda = R/L = 0.5217$, and

$$\mathbf{F}\Big(t, I(t), \int_a^t \mathbf{K}(t,s)I(s)ds\Big) = \frac{1}{LC}\int_0^t \mathbf{K}(t,s)I(s)ds - \frac{1}{L}E(t).$$

We observe that the functions $\mathbf{F}$ and $\mathbf{K}$ are continuous on $[0,b]$, $F = \sup_{t\in[0,b]}|\mathbf{F}(t,0,0)| = (1/L)\sup_{t\in[0,b]}|E(t)|$, and

$$K = \sup_{(s,t)\in[0,b]\times[0,b]}\mathbf{K}(t,s) = \sup_{s\in[0,b]}(\psi'(s)/\Gamma(\alpha+1)), \tag{37}$$

and

$$\left|\mathbf{F}\Big(t, I_1(t), \int_a^t \mathbf{K}(t,s)I_1(s)ds\Big) - \mathbf{F}\Big(t, I_2(t), \int_a^t \mathbf{K}(t,s)I_2(s)ds\Big)\right| \leq \frac{(\psi(b)-\psi(0))}{LC\,\Gamma(\alpha+1)}\|I_1 - I_2\|_0. \tag{38}$$

Now, in order to survey the HUR-stability of problem (35) with the given data as in this example, we will consider $\varepsilon = I_{\text{am}}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)}$; $\phi(t) = (\psi(t)-\psi(0))^{\alpha-\beta}$, and the voltage $E(t) = \Big[RI_{\text{am}}(\psi(t)-\psi(0))^{\alpha} + \frac{I_{\text{am}}}{C\Gamma(\alpha+2)}(\psi(t)-\psi(0))^{\alpha+1}\Big]$, where I_{am} is current amplitude. The values of the function $\psi \in \mathbf{K}_{\psi}$ are taken as follows: $\psi(t) = t, \psi(t) = \log(t+1), \psi(t) = t^2, \psi(t) = \sin(\omega t)$. First, based on Theorem 3.2, the below condition will be checked.

$$\mathbb{C} := \frac{\big(\psi(b)-\psi(a)\big)^{\beta}\Big[\lambda + M\Big(1+(b-a)K\Big)\Big]}{\Gamma(1+\beta)} < 1, \tag{39}$$

In this example, we consider $\alpha = 0.75, \beta = 0.5$, and $b = \pi/2$, and we will calculate the values of $\mathbb{C}$ in (39) with the function $\psi(t)$ as the below.

+ If we take $\psi(t) = t$, then from (37) and (38) one obtains

$$K = \sup_{t\in[0,b]}(\psi'(t)/\Gamma(\alpha+1)) = 1.0881; \quad M = \frac{(\psi(b)-\psi(0))}{LC\,\Gamma(\alpha+1)} = 4.9540\times 10^{-4}.$$

Then, it implies from (39) that $\mathbb{C} = 0.7397 < 1$.

+ If we take $\psi(t) = \log(t+1)$, then from (37) and (38) one obtains

$$K = \sup_{t\in[0,b]}(\psi'(t)/\Gamma(\alpha+1)) = 1.0881; \quad M = \frac{(\psi(b)-\psi(0))}{LC\,\Gamma(\alpha+1)} = 2.9779\times 10^{-4}.$$

Hence, from (39) we deduce that $\mathbb{C} = 0.5729 < 1$. In the same way, one can obtain the values of $\mathbb{C}$ in (39) with the different functions $\psi(t)$ as shown in Tab. 1.

The function ψ	t	$\log(t+1)$	t^2	$\sin(\omega t)$
The values of $\mathbb{C}$ in (39)	0.7397	0.5729	0.9335	0.0963

Table 1: The condition (39) in the case $\alpha = 0.75, \beta = 0.5$, and $b = \pi/2$.

Then, we deduce from Tab. 1 that the condition (39) is satisfied. Therefore, the hypotheses of Theorem 3.2 hold which lead to problem (35) with the given data as in this example has a unique solution on $[0, b]$. Next, the condition (28) in Theorem 5.4 is checked as follows:

$$\frac{1}{\Gamma(\beta)}\int_0^t \Psi^{\beta-1}(t,s)\phi(s)ds = \frac{1}{\Gamma(\beta)}\int_0^t \Psi^{\beta-1}(t,s)(\psi(s)-\psi(0))^{\alpha-\beta}ds \leq \hat{M}_{\phi}\phi(t), \tag{40}$$

where $\hat{M}_{\phi} := (\psi(b)-\psi(0))^{\beta}\frac{\Gamma(\alpha-\beta+1)}{\Gamma(\alpha+1)}$, and we use the property in Lemma 2.2. Therefore, by Theorem 5.4, we deduce that problem (35) is HUR-stable with

$$\left|I(t) - I^*(t)\right| \leq \widetilde{M}\epsilon, \tag{41}$$

where $\varepsilon = I_{\text{am}}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)}$ and

$$\widetilde{M} := (\psi(b)-\psi(0))^{\beta}\frac{\Gamma(\alpha-\beta+1)}{\Gamma(\alpha+1)}\mathbb{E}_{\beta}\Big((\lambda + M\Big(1+(b-a)K\Big))(\psi(b)-\psi(0))^{\beta}\Big).$$

Table 2 shows the upper bound of the estimate (41), and the numerical solutions of problem (35) with the bound error (41) have also plotted in Figs. 1-4. In numerical illustration, the function $I^*(t) = I_{\text{am}}(\psi(t)-\psi(0))^{\alpha}$ holds the inequality (36) and it satisfies the estimate (41) with the exact solution $I(t)$ given by Tab. 2.

The function ψ	t	$\log(t+1)$	t^2	$\sin(\omega t)$
The bound error (41)	0.1121	0.0336	0.7619	0.0027

Table 2: The bound errors between the exact solution and the numerical solution of problem (35) with $\alpha = 0.5, \beta = 0.75$, and $b = \pi/2$.

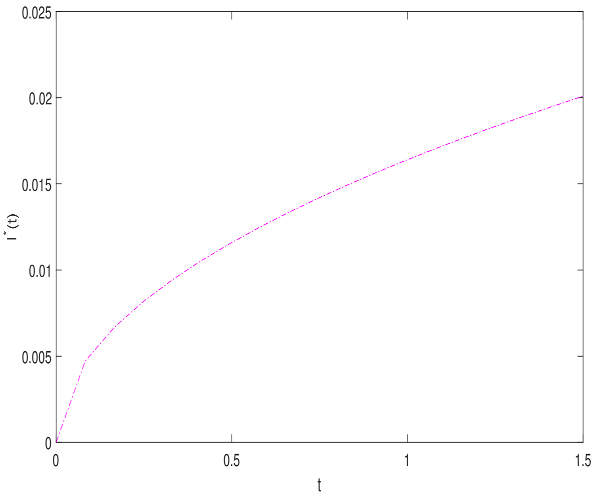

Figure 1: The numerical representation of the solution of problem (35) with $\psi(t) = t$.

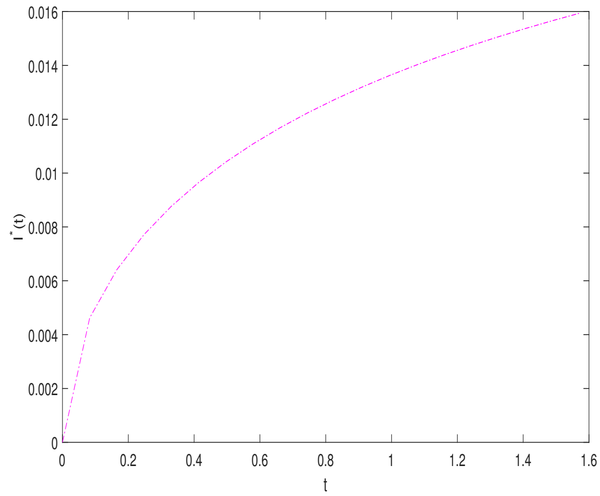

Figure 2: The numerical representation of the solution of problem (35) with $\psi(t) = \log(t+1)$.

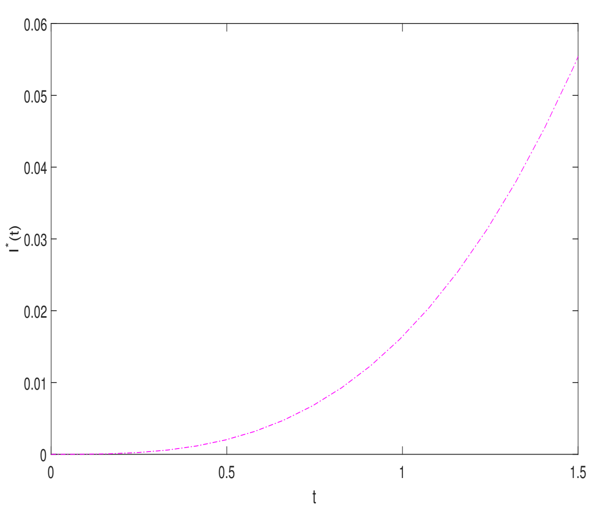

Figure 3: The numerical representation of the solution of problem (35) with $\psi(t) = t^2$.

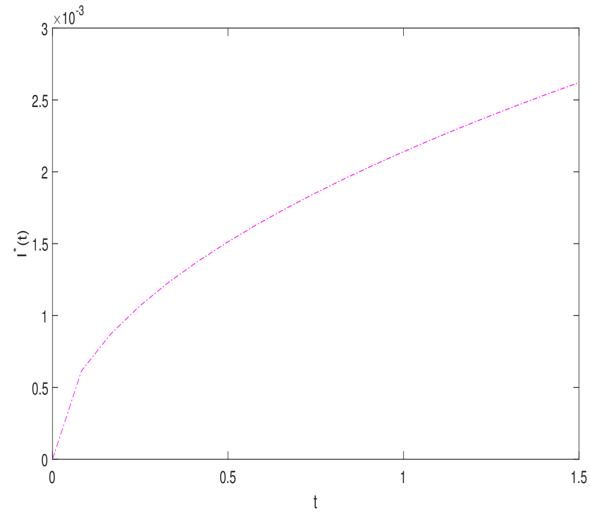

Figure 4: The numerical representation of the solution of problem (35) with $\psi(t) = \sin(\omega t)$.

6. Conclusion

The results of this paper have introduced a standard framework to investigate the existence, uniqueness, and stability of the solution for a class of FIDEs via the concept of the generalized Caputo FD. The HU-stability and HUR-stability is a reliable approach for approximately seeking FIDEs when we cannot seek the explicit solutions of the problem.

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Conflict of interests

The authors declare that they have no competing interests regarding this research work.

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