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Minimization Problem of Proximal Point Algorithm in Complete CAT(0) Spaces

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Abstract. In this paper a proximal point algorithm for a convex function is considered in complete $CAT(0)$ spaces. We introduce a necessary and sufficient condition for the set of minimizers of the function to be nonempty, and by showing that in this case, this iterative sequence converges strongly to the metric projection of some point onto the set of minimizers of the function.

1. Introduction

Let (X, d) be a $CAT(0)$ space and $f : X \to (-\infty, \infty]$ be a proper, convex and lower semi-continuous function. One of the most important optimization problems is to find $x \in X$ such that

$$f(x) = \min_{y \in X} f(y).$$

We introduce $argmin_{y\in X} f(y)$ by the set of minimizers of f. One of the most important and effective iterative methods for solving this problem is the proximal point algorithm(PPA), which has been initiated and established by Martinet [29] and Rockafellar [31]. G$\ddot{u}$ler [18] showed that the sequence generated by the proximal point algorithm is not necessarily strongly convergent in general.
The PPA with Halpern's algorithm [19] has been combined by Kamimura- Takahashi [22] so that the strong convergence is guaranteed (see also [28, 36]).
The minimizers of the objective convex functional play an important role in the branch of analysis and geometry. Machine learning, electronic structure computation, system balancing and robot manipulation can be considered as an application of problems on manifolds (see for example [1, 32, 34, 35] and the references therein).
Suparatulatorn et al.[33] have recently studied the convergence of a PPA for convex functions. They assumed that the set of minimizers of f is nonempty. In this article, their results are significantly extend and improve by obtaining a necessary and sufficient conditions for the set of minimizers of f to be nonempty, and also by showing that in this case, this PPA converges strongly to an element of the set of minimizers of f, which is nearest to u.

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2. Preliminaries

A geodesic path joining x to y in a metric space (X, d) is an isometry $c : [0, d(x, y)] \to X$ with $c(0) = x, c(d(x, y)) = y$. Is called a geodesic segment between x and y, the image of a geodesic path joining x to y. The metric space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be an uniquely geodesic space if there is exactly one geodesic joining x and y for each $x, y \in X$. A $CAT(0)$ space is a metric space (X, d) such that it is a geodesic space and satisfies the following inequality:
CN − inequality: If $x, y_0, y_1, y_2 \in X$ such that $d(y_0, y_1) = d(y_0, y_2) = \frac{1}{2}d(y_1, y_2)$, then

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2)$$

(For more information on the CAT(0) spaces, the reader is advised to read [3, 6–9, 11, 13, 17, 21, 23] and the references therein). Pre-Hilbert spaces [11], Euclidean buildings [12], $\mathbb{R}$-trees [24], Hadamard manifolds, the complex Hilbert ball with a hyperbolic metric [16], and many other examples, are some kinds of CAT(0) spaces. A complete CAT(0) space is called a Hadamard space. For all x and y belong to a CAT(0) space X, we consider the symbol $(1 - t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that $d(z, x) = td(x, y)$ and $d(z, y) = (1 - t)d(x, y)$.

Example 2.1. *[3] Consider two subgraphs of different quadric surfaces:*

$$\begin{aligned} A &= \{(x, y, z) \in \mathbb{R}^3 : z \leq x^2 + y^2\} \\ B &= \{(x, y, z) \in \mathbb{R}^3 : z \leq -x^2 - y^2\}. \end{aligned}$$

Obviously, A is not a CAT(0) space, but B is a CAT(0) space.

For the proof of our main results, we need the following useful lemma, and for its proof, we refer the reader to [14].

Lemma 2.2. *Let (X, d) be a CAT(0) space. Then, for all $x, y, z \in X$ and all $t \in [0, 1]$:*

(1) $d^2(tx \oplus (1 - t)y, z) \leq td^2(x, z) + (1 - t)d^2(y, z) - t(1 - t)d^2(x, y)$,

(2) $d(tx \oplus (1 - t)y, z) \leq td(x, z) + (1 - t)d(y, z)$,

In addition, by using (1), we have

$$d(tx \oplus (1 - t)y, tx \oplus (1 - t)z) \leq (1 - t)d(y, z)$$

In 1976, Lim [27] introduced a concept of convergence in complete CAT(0) spaces which is called $\triangle$-convergence as follows:
for a bounded sequence (x_n) be in complete CAT(0) space (X, d) and $x \in X$ we set $r(x, (x_n)) := \limsup_{n\to\infty} d(x, x_n)$. The asymptotic radius of (x_n) is given by $r((x_n)) := \inf\{r(x, (x_n)) : x \in X\}$ and the asymptotic center of (x_n) is the set $A((x_n)) := \{x \in X : r(x, (x_n)) = r((x_n))\}$. In the complete CAT(0) spaces (X, d), it is known that , $A((x_n))$ consists of exactly one point (see [24]). Also a sequence (x_n) in the complete CAT(0) space (X, d) is said $\triangle$-convergence to some $x \in X$ if $A((x_{n_k})) = \{x\}$ for every subsequence (x_{n_k}). Many authors has studied the concept of $\triangle$-convergence (see for example [14, 15] and the references therein).
In 2008 Berg and Nikolaev [10] has introduced the concept of quasilinearization for CAT(0) space X. They used the symbol $\overrightarrow{ab}$ instead of $(a, b) \in X \times X$ and they called it a vector. Also they defined the quasilinearization map $\langle .,. \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)), \quad (a, b, c, d \in X).$$

For all $a, b, c, d, e \in X$, it can be easily shown that $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a, b), \langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$. Also, we can formally add compatible vectors, more precisely $\overrightarrow{ac} + \overrightarrow{cb} = \overrightarrow{ab}$, for all $a, b, c, d \in X$. X satisfies the Cauchy-Schwarz inequality if for all $a, b, c, d \in X$:

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d).$$

It is known ([10, Corollary 3]) that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.
The concept of dual space of a complete CAT(0) space X have been introduced by Ahmadi Kakavandi and Amini [4], based on a work of Berg and Nikolaev [10], as follows.
Suppose that $C(X, \mathbb{R})$ is the space of all continuous real-valued functions on X. Consider the map $\Theta : \mathbb{R} \times X \times X \to C(X, \mathbb{R})$ defined by

$$\Theta(t, a, b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle, \quad (t \in \mathbb{R}, \ \ a, b, x \in X).$$

By using the Cauchy-Schwarz inequality one can shows that $\Theta(t, a, b)$ is a Lipschitz function with Lipschitz semi-norm $L(\Theta(t, a, b)) = |t|d(a, b)$, $(t \in \mathbb{R}, \ \ a, b \in X)$, where

$$L(\phi) = \sup\{\frac{\phi(x) - \phi(y)}{d(x, y)} : \ x, y \in X, \ \ x \neq y\}$$

is the Lipschitz semi-norm for any function $\phi : X \to \mathbb{R}$. A pseudometric D on $\mathbb{R} \times X \times X$ is defined by

$$D((t, a, b), (s, c, d)) = L(\Theta(t, a, b) - \Theta(s, c, d)), \quad (t, s \in \mathbb{R}, \ \ a, b, c, d \in X).$$

For a Hadamard space (X, d), the pseudometric space $(\mathbb{R} \times X \times X, D)$ can be considered as a subspace of the pseudometric space of all real-valued Lipschitz functions $(Lip(X, \mathbb{R}), L)$. It is obtained that $D((t, a, b), (s, c, d)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle$, for all $x, y \in X$ ([4], Lemma 2.1). Then, D can impose an equivalent relation on $\mathbb{R} \times X \times X$, where the equivalence class of (t, a, b) is

$$[\overrightarrow{tab}] = \{\overrightarrow{scd} : \ \ D((t, a, b), (s, c, d)) = 0\}.$$

The set $X^* = \{[\overrightarrow{tab}] : \ \ (t, a, b) \in \mathbb{R} \times X \times X\}$ is a metric space with metric $D([\overrightarrow{tab}], [\overrightarrow{scd}]) := D((t, a, b), (s, c, d))$, which is called the dual space of (X, d).
It is clear that $[\overrightarrow{aa}] = [\overrightarrow{bb}]$ for all $a, b \in X$. Fix $o \in X$, we write $0 = [\overrightarrow{oo}]$ as the zero of the dual space. In [4], it is shown that the dual of a closed and convex subset of Hilbert space H with nonempty interior is H and $t(b - a) \equiv [\overrightarrow{tab}]$ for all $t \in \mathbb{R}, \ \ a, b \in H$. Note that X^* acts on $X \times X$ by

$$\langle x^*, \overrightarrow{xy} \rangle = t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle, \quad (x^* = [\overrightarrow{tab}] \in X, \ \ x, y \in X).$$

Also, we use the following notation:

$$\langle \alpha x^* + \beta y^*, \overrightarrow{xy} \rangle = \alpha\langle x^*, \overrightarrow{xy} \rangle + \beta\langle y^*, \overrightarrow{xy} \rangle, \quad (\alpha, \beta \in \mathbb{R}, \ \ x, y \in X, \ \ x^*, y^* \in X^*).$$

3. Convex functions

Let X be a CAT(0) space and C be a convex subset of X. A function $f : C \subset X \to (-\infty, \infty]$ with domain $D(f) = \{x \in C : f(x) < +\infty\}$ is convex if, for any geodesic $\gamma : [a, b] \to C$, the function $fo\gamma$ is convex, i.e. $f(\alpha x \oplus (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$ for all $x, y \in C$ and $\alpha \in (0, 1)$. Recall that f is proper if and only if $D(f) \neq \emptyset$.
A function $f : C \subset X \to (-\infty, \infty]$ is said to be lower semi-continuous at $x \in D(f)$ if

$$f(x) \leq \liminf_{n\to\infty} f(x_n)$$

for each sequence x_n such that $x_n \to x$ as $n \to \infty$. A function f is called lower semi-continuous if it is lower semi-continuous at any point in $D(f)$. For any $\lambda > 0$,the Moreau-Yosida resolvent of f in $CAT(0)$ spaces is defined by:

$$J_\lambda(x) = argmin_{y\in C}\Big(f(y) + \frac{1}{\lambda}d^2(y,x)\Big),$$

for all $x \in C$. The mapping J_λ is well defined for all $\lambda > 0$ (see for example [20, 30]). Let $f : X \to (-\infty, +\infty]$ be a proper convex and lower semi-continuous function. It was shown in [5] that the set $F(J_\lambda)$ of fixed points of the resolvent associated with f coincides with the set $argmin_{y\in C}f(y)$ of minimizers of f. We say that a function $T : X \to X$ is a nonexpansive mapping, if $d(Tx, Ty) \le d(x, y)$ for all $x, y \in X$. Also, for any $\lambda > 0$, the resolvent J_λ of f is non-expansive; see [20].
To reach the main results of this paper, the following definitions and useful lemmas are required.

Definition 3.1. *[26] Let X be a CAT(0) space. A function $f : C \to (-\infty, \infty]$ is $\triangle$-lower semi-continuous at a point $x \in D(f)$ if*

$$f(x) \le \liminf_{n\to\infty} f(x_n)$$

for each sequence x_n such that x_n is $\triangle$-convergence to x as $n \to \infty$.

A function f is said to be $\triangle$-lower semi-continuous if it is $\triangle$-lower semi-continuous at any point in $D(f)$. It can be easily seen that, every lower semi-continuous and convex function is $\triangle$-lower semicontinuous.

Lemma 3.2. *[2] A bounded sequence (x_n) in Hadamard space (X, d), $\triangle$-convergence to $x \in X$ if and only if $\limsup_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle \le 0, \quad$ for all $y \in X$.*

Lemma 3.3. *[20] Every bounded sequence in a $CAT(0)$ space has a $\triangle$-convergent subsequence.*

4. Main results

Let (X, d) be a $CAT(0)$ space and $f : X \to (-\infty, \infty]$ be a proper, convex and lower semi-continuous function. One of the important problems in optimization is to find $x \in X$ such that

$$f(x) = \min_{y\in X} f(y).$$

We introduce $argmin_{y\in X}f(y)$ by the set of minimizers of f.
Suppose that (λ_n) is a sequence of positive real numbers, (α_n) is a sequence in $]0, 1[$ and $u \in X$ to be fixed. Proximal point algorithm for proper, convex and lower semi-continuous $f : X \to (-\infty, \infty]$, is the sequence generated by

$$\begin{cases} x_{n+1} = J_{\lambda_n}(\alpha_n u \oplus (1-\alpha_n)x_n), \\ x_0 \in X. \end{cases} \tag{1}$$

The inexact version of (1) can be formulated as follow

$$\begin{cases} z_{n+1} = J_{\lambda_n}(\alpha_n u \oplus (1-\alpha_n)y_n), \\ d(z_n, y_n) \le e_n, \\ y_0 \in X, \end{cases} \tag{2}$$

where (e_n) is a sequence in $]0, \infty[$.

Lemma 4.1. *Let X be a Hadamard space and $f : X \to (-\infty, +\infty]$ be a proper, convex and lower semi-continuous function. Assume that the sequences $(x_n), (y_n)$ are generated by the algorithms (1) and (2), respectively, and that $\alpha_n \to 1$ and $\Sigma_{n=1}^{\infty} e_n < +\infty$. Then:*

(1) *(x_n) is bounded, if and only if (y_n) is bounded.*

(2) *If $F = argmin_{y\in X} f(y) \neq \emptyset$, then the sequence (x_n) converges strongly to $P_F u$ if and only if the sequence (y_n) converges strongly to $P_F u$.*

Proof. From the nonexpansiveness of the resolvent operator, for all $n \geq 0$:

$$\begin{aligned} d(x_n, y_n) \leq & d(x_n, z_n) + d(z_n, y_n), \\ = & d\Big(J_{\lambda_{n-1}}(\alpha_{n-1}u \oplus (1-\alpha_{n-1})x_{n-1}), J_{\lambda_{n-1}}(\alpha_{n-1}u \oplus (1-\alpha_{n-1})y_{n-1})\Big) + d(z_n, y_n) \\ \leq & (1-\alpha_{n-1})d(x_{n-1}, y_{n-1}) + e_n \\ \leq & d(x_{n-1}, y_{n-1}) + e_n \\ & \vdots \\ \leq & d(x_0, y_0) + \Sigma_{i=1}^{n} e_i < \infty. \end{aligned}$$

Thus (x_n) is bounded if and only if (y_n) is bounded.
(2): Using the above inequality, the following result can be obtained, for all $n \geq 0$:

$$d(x_n, y_n) \leq (1-\alpha_{n-1})d(x_{n-1}, y_{n-1}) + e_n.$$

By letting $n \to \infty$ and by using (1), we conclude that (x_n) converges strongly to $P_F u$ if and only if (y_n) converges strongly to $P_F u$. This completes the proof. □

The proof of our main result is based on the following useful lemma.

Lemma 4.2. *Let X be a Hadamard space and $f : X \to (-\infty, +\infty]$ be a proper, convex and lower semi-continuous function and $F = argmin_{y\in X} f(y) \neq \emptyset$. Suppose $t \in (0, \infty)$ and $u \in X$. Then $\lim_{t\to\infty} J_t u = p$, where $p = P_F u$.*

Proof. By the definition of resolvent function, for all $q \in F$

$$f(J_t u) + \frac{1}{2t} d^2(J_t u, u) \leq f(q) + \frac{1}{2t} d^2(q, u).$$

On the other hand

$$f(q) \leq f(J_t u).$$

Hence

$$d(J_t u, u) \leq d(q, u), \quad \forall q \in F. \tag{3}$$

In particular, $(J_t u)$ is bounded. By using the Lemma 3.3, there exists a subsequence of $(J_t u)$ that is Δ-convergent. Suppose (t_{n_k}) is a sequence in $(0, \infty)$ such that $t_{n_k} \to \infty$ and $(J_{t_{n_k}} u)$ is Δ-convergent to some $z \in X$. By the definition of resolvent operator, for all $x \in X$

$$f(J_{t_{n_k}} u) + \frac{1}{2t_{n_k}} d^2(J_{t_{n_k}} u, u) \leq f(x) + \frac{1}{2t_{n_k}} d^2(x, u), \ \forall k \in \mathbb{N}.$$

Now by letting $k \to \infty$ in above inequality, we get

$$\liminf_{k\to\infty} f(J_{t_{n_k}} u) \leq f(x). \tag{4}$$

Since f is convex and lower semi-continuous then, f is Δ- lower semi-continuous. Thus

$$f(z) \leq \liminf_{k\to\infty} f(J_{t_{n_k}} u). \tag{5}$$

From 4 and 5

$$f(z) = \liminf_{k\to\infty} f(J_{t_{n_k}} u),$$

and hence $z \in F$. Since $(J_{t_{n_k}} u)$ is Δ-convergent to z, for every $x \in X$

$$\limsup_{k\to\infty} \langle \overrightarrow{z(J_{t_{n_k}} u)}, \overrightarrow{zx} \rangle \leq 0,$$

witch implies that

$$\limsup_{k\to\infty} \langle \overrightarrow{z(J_{t_{n_k}} u)}, \overrightarrow{zu} \rangle \leq 0.$$

Therefore

$$\limsup_{k\to\infty} [d^2(z,u) + d^2(J_{t_{n_k}} u, z) - d^2(J_{t_{n_k}} u, u)] \leq 0. \tag{6}$$

Also by (3),

$$d^2(z,u) - d^2(J_{t_{n_k}} u, u) \geq 0. \tag{7}$$

Therefore by using 6 and 7,

$$\limsup_{k\to\infty} d^2(J_{t_{n_k}} u, z) = 0,$$

and this shows that, $(J_{t_{n_k}} u)$ converges strongly to z. On the other hand by using (6), for all $q \in F$

$$d(u,z) \leq d(u,q),$$

witch implies that, $z = p = P_F u$. This completes the proof. □

We are now prepared to give a necessary and sufficient condition for the completion of set of minimizers of f to be nonempty.

Theorem 4.3. *Let X be a Hadamard space and $f : X \to (-\infty, +\infty]$ be a proper convex and lower semi-continuous function. If (x_n) and (y_n) are generated by the algorithms (1) and (2), respectively, such that, $\alpha_n \to 1$ and $\lambda_n \to \infty$ as $n \to \infty$ and $\Sigma_{n=1}^{\infty} e_n < \infty$, then $F = argmin_{y\in X} f(y) \neq \emptyset$ if and only if (x_n) is bounded if and only if (y_n) is bounded.*

Proof. Assume that $F \neq \emptyset$ and $q \in F$. From the algorithm (1), and the fact that the resolvent operator is nonexpansive, for all $m \geq 0$ we have:

$$\begin{aligned} d(x_{m+1}, q) &= d(x_{m+1}, J_{\lambda_m} q) \\ &\leq d(\alpha_m u \oplus (1-\alpha_m) x_m, q) \\ &\leq \alpha_m d(u,q) + (1-\alpha_m) d(x_m, q). \end{aligned}$$

Using the above inequality, by induction, we get for all $m \geq 0$:

$$d(x_{m+1}, q) \leq \max\{d(u,q), d(x_1, q)\}.$$

This inequality shows that (x_n) is bounded.
Conversely, assume that (x_n) is bounded. Hence $J_{\lambda_n}(\alpha_n u \oplus (1-\alpha_n)x_n)$ is bounded. Too by using the Lemma 3.3, there exists a subsequence (n_k) of $\mathbb{N}$ and some $z \in X$ such that the subsequence $J_{\lambda_{n_k}}(\alpha_{n_k} u \oplus (1-\alpha_{n_k})x_{n_k})$ is Δ-convergent to z.
By the definition of resolvent function, if $w_k = J_{\lambda_{n_k}}(\alpha_{n_k} u \oplus (1-\alpha_{n_k})x_{n_k})$, then for all $x \in X$ and $k \geq 1$

$$f(w_k) + \frac{1}{2\lambda_{n_k}} d^2(w_k, \alpha_{n_k} u \oplus (1-\alpha_{n_k})x_{n_k}) \leq f(x) + \frac{1}{2\lambda_{n_k}} d^2(x, \alpha_{n_k} u \oplus (1-\alpha_{n_k})x_{n_k})).$$

Therefore,

$$\begin{aligned} & f(J_{\lambda_{n_k}}(\alpha_{n_k} u \oplus (1-\alpha_{n_k})x_{n_k})) \\ & + \frac{1}{2\lambda_{n_k}} d^2(J_{\lambda_{n_k}}(\alpha_{n_k} u \oplus (1-\alpha_{n_k})x_{n_k}), \alpha_{n_k} u \oplus (1-\alpha_{n_k})x_{n_k})) \\ & \leq f(x) + \frac{1}{2\lambda_{n_k}} d^2(x, \alpha_{n_k} u \oplus (1-\alpha_{n_k})x_{n_k})). \end{aligned}$$

By taking $k \to \infty$ in above inequality, for all $x \in X$, we get:

$$\liminf_{k\to\infty} f(J_{\lambda_{n_k}}(\alpha_{n_k} u \oplus (1-\alpha_{n_k})x_{n_k})) \leq f(x). \tag{8}$$

On the other hand f is convex and lower semi-continuous, we conclude that f is Δ-lower semi-continuous. And hence since $J_{\lambda_{n_k}}(\alpha_{n_k} u \oplus (1-\alpha_{n_k})x_{n_k})$ is Δ-convergent to z, we get

$$f(z) \leq \liminf_{k\to\infty} f(J_{\lambda_{n_k}}(\alpha_{n_k} u \oplus (1-\alpha_{n_k})x_{n_k})). \tag{9}$$

Therefore by using the inequalities (8) and (9),

$$f(z) = \liminf_{k\to\infty} f(J_{\lambda_{n_k}}(\alpha_{n_k} u \oplus (1-\alpha_{n_k})x_{n_k}))$$

and thus $z \in F$. This completes the proof of the first conclusion. The second conclusion follows from Lemma 4.1 and the first conclusion. □

The following theorem, is another main result of this paper.

Theorem 4.4. *Let X be a Hadamard space and $f : X \to (-\infty, +\infty]$ be a proper, convex and lower semi-continuous function. If (x_n) and (y_n) are generated by the algorithms (4.1) and (4.2), respectively, such that, $\alpha_n \to 1$, $\lambda_n \to \infty$ and $\Sigma_{n=1}^{\infty} e_n < \infty$, then*

(1) *if $F = argmin_{y\in X} f(y) \neq \emptyset$ then (x_n) converges strongly to $P_F u$,*

(2) *if $F = argmin_{y\in X} f(y) \neq \emptyset$ then (y_n) converges strongly to $P_F u$.*

Proof. (1): Assume that $F \neq \emptyset$ and $P_F u = p$. For all $n \geq 1$,

$$\begin{aligned} d(x_{n+1}, p) & \leq d(x_{n+1}, J_{\lambda_n} u) + d(J_{\lambda_n} u, p) \\ & \leq d(\alpha_n u \oplus (1-\alpha_n)x_n, u) + d(J_{\lambda_n} u, p) \\ & \leq (1-\alpha_n) d(x_n, u) + d(J_{\lambda_n} u, p). \end{aligned}$$

where the second inequality follows from the algorithm (1) and the fact that the resolvent operator is nonexpansive. Now the result follows immediately by letting $n \to \infty$ in the above inequality and by using Lemma 4.2.

(2): The result (2) follows by using the Lemma 4.1 and (1). □

Remark 4.5. *In Theorem 4.3 and Theorem 4.4, we provided necessary and sufficient conditions for $F = argmin_{y \in X} f(y)$ to be nonempty, and proved the strong convergence of the sequence (x_n) to an element of F. In particular, Theorem 4.3 and Theorem 4.4 extends the previous results given by Suparatulatorn et al.[33] who assumed that the set of minimizers of f is nonempty.*

Example 4.6. *Let $X = \mathbb{R}$ and d be the Euclidean metric and let $f : X \longrightarrow (-\infty, +\infty]$ be defined by*

$$f(x) = \begin{cases} |x| & x \in [-1, 1] \\ +\infty & o.w. \end{cases} \tag{10}$$

Obviously, f is a proper, convex and lower semi-continuous function and for every $\lambda \geq 1$

$$J_\lambda(x) = argmin_{y \in \mathbb{R}}\{f(y) + \frac{1}{2\lambda}|x - y|^2\} = \frac{x^2}{2\lambda}.$$

By taking $\alpha_n = \dfrac{n}{n+1}$, $u = 1$, $\lambda_n = \dfrac{n+1}{2}$ and $x_0 = 1$ we get

$$x_{n+1} = J_{\lambda_n}(\alpha_n u + (1 - \alpha_n)x_n) = \frac{(n + x_n)^2}{(n+1)^3}.$$

Obviously, $x_n \geq 0$ and by using the Induction, we can conclude that $x_n \leq 1$ for all $n \in \mathbb{N}$. Hence the sequence (x_n) is bounded. Also, from $x_n \leq 1$

$$0 \leq x_{n+1} = \frac{(n + x_n)^2}{(n+1)^3} \leq \frac{(n+1)^2}{(n+1)^3}$$

for all $n \in \mathbb{N}$. Therefore $\lim_{n \to \infty} x_n = 0$

In the following, we introduce a new iteration (algorithm (11)) and show that (x_n) converges strongly to $P_F u$.

Let (X, d) be a $CAT(0)$ space and $f : X \to (-\infty, \infty]$ be a proper, convex and lower semi-continuous function and $T : X \to X$ be a nonexpansive mapping. Also let (λ_n) be a sequence of positive real numbers, (α_n) be a sequence in $]0, 1[$ and $u \in X$ to be fixed. Suppose x_n is the sequence generated by

$$\begin{cases} x_{n+1} = T\Big(J_{\lambda_n}(\alpha_n u \oplus (1 - \alpha_n)x_n)\Big), \\ x_0 \in X. \end{cases} \tag{11}$$

The inexact version of (11) can be formulated as follow:

$$\begin{cases} z_{n+1} = T\Big(J_{\lambda_n}(\alpha_n u \oplus (1 - \alpha_n)y_n)\Big), \\ d(z_n, y_n) \leq e_n, \\ y_0 \in X, \end{cases} \tag{12}$$

where (e_n) is a sequence in $]0, \infty[$.

Theorem 4.7. *Let X be a Hadamard space, $f : X \to (-\infty, +\infty]$ be a proper, convex and lower semi-continuous function, T be a nonexpansive mapping on X and $\Omega = F(T) \cap F$, where $F = argmin_{y \in X} f(y)$. Assume that the sequences (x_n) and (y_n) are generated by the algorithms (11) and (12), respectively, and that $\alpha_n \to 1$ and $\Sigma_{n=1}^{\infty} e_n < +\infty$. Then:*

(1) *(x_n) is bounded, if and only if (y_n) is bounded.*

(2) *If $\Omega \neq \emptyset$, then the sequence (x_n) converges strongly to $T(P_F u)$ if and only if the sequence (y_n) converges strongly to $T(P_F u)$.*

Proof. (1): From the nonexpansiveness of the resolvent operator and T, for all $n \geq 0$:

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, z_n) + d(z_n, y_n) \\ &= d\Big(T\Big(J_{\lambda_{n-1}}(\alpha_{n-1}u \oplus (1-\alpha_{n-1})x_{n-1})\Big), T\Big(J_{\lambda_{n-1}}(\alpha_{n-1}u \oplus (1-\alpha_{n-1})y_{n-1})\Big)\Big) + d(z_n, y_n) \\ &\leq (1-\alpha_{n-1})d(x_{n-1}, y_{n-1}) + e_n \\ &\leq d(x_{n-1}, y_{n-1}) + e_n \\ &\vdots \\ &\leq d(x_0, y_0) + \Sigma_{i=1}^{n} e_i < \infty. \end{aligned}$$

Thus (x_n) is bounded if and only if (y_n) is bounded.

(2): For all $n \geq 0$, by the above inequality, we get

$$d(x_n, y_n) \leq (1-\alpha_{n-1})d(x_{n-1}, y_{n-1}) + e_n.$$

By letting $n \to \infty$ and by using (1), we conclude that (x_n) converges strongly to $T(P_F u)$ if and only if (y_n) converges strongly to $T(P_F u)$. □

Theorem 4.8. *Let X be a Hadamard space, $f : X \to (-\infty, +\infty]$ be a proper convex and lower semi-continuous function and T be a nonexpansive mapping on X such that $\Omega = F(T) \cap F \neq \emptyset$, where $F = argmin_{y \in X} f(y)$. Assume that the sequences $(x_n), (y_n)$ are generated by the algorithms (11) and (12), respectively, such that $\alpha_n \to 1$, $\lambda_n \to \infty$, and $\Sigma_{n=1}^{\infty} e_n < +\infty$, then (x_n) and (y_n) converges strongly to $T(P_F u)$.(i.e. (x_n) and (y_n) not converges to $P_\Omega u$)*

Proof. At first we show that (x_n) is a bounded sequence. Assume that $q \in \Omega$. From the algorithm (11), and the fact that the resolvent operator and T are nonexpansive, for all $m \geq 0$ we have:

$$\begin{aligned} d(x_{m+1}, q) &= d(x_{m+1}, TJ_{\lambda_m} q) \\ &\leq d(\alpha_m u \oplus (1-\alpha_m)x_m, q) \\ &\leq \alpha_m d(u, q) + (1-\alpha_m)d(x_m, q). \end{aligned}$$

Using the above inequality, by induction, we get for all $m \geq 0$,

$$d(x_{m+1}, q) \leq \max\{d(u, q), d(x_1, q)\}.$$

This inequality shows that (x_n) is bounded.

Now assume that $P_F u = p$. For all $n \geq 1$,

$$\begin{aligned} d(x_{n+1}, Tp) &\leq d(x_{n+1}, TJ_{\lambda_n} u) + d(TJ_{\lambda_n} u, Tp) \\ &\leq d(\alpha_n u \oplus (1-\alpha_n)x_n, u) + d(J_{\lambda_n} u, p) \\ &\leq (1-\alpha_n)d(x_n, u) + d(J_{\lambda_n} u, p). \end{aligned}$$

where the second inequality follows from the algorithm (11) and the fact that the resolvent operator and T are nonexpansive. Now by letting $n \to \infty$ in the above inequality and by using Theorem 4.7 and Lemma 4.2 we get (x_n) and (y_n) converges strongly to $T(P_F u)$. □

Conclusions

In this paper, we studied a PPA with error sequences for a convex function in complete $CAT(0)$ spaces. Also in this paper, we provided necessary and sufficient conditions for the set of minimizers of a proper, convex and lower semi-continuous function f to be nonempty, and proved the strong convergence of the PPA to the metric projection of some point onto the set of minimizers of the function f. Also, a new algorithm (algorithm (11)) was introduced and studied, and the necessary and sufficient conditions for this new algorithm were also provided. As a future direction for research, it might be interesting to investigate the possibility of implementing the ideas and methods developed in this paper to these other PPA and other algorithms.

References

[1] R. Adler, J.P. Dedieu, J.Y. Margulies, M. Martens, M. Shub, Newton׳s method on Reimannian manifolds and a geometric model for human spine, IMA Journal of Numerical Analysis, 22(2002) 359–390.
[2] B. Ahmadi Kakavandi, Weak topologies in complete CAT(0) metric spaces, Proceedings of the American Mathematical Society, 141(2013) 1029–1039.
[3] S. Alexander, V. Kapovitch, A. Petrunin, An Invitation to Alexandrov Geometry CAT(0) Spaces, SpringerBriefs in Mathematics, 2019.
[4] B. Ahmadi Kakavandi, M. Amini, Duality and subdifierential for convex functions on complete CAT(0) metric spaces, Nonlinear Analysis(TMA), 73(2010) 3450–3455.
[5] D. Ariza-Ruiz, L. Leu¸stean, G. Lόpez, Firmly nonexpansive mappings in classes of geodesic spaces, Transactions of the American Mathematical Society, 366(2014) 4299–4322.
[6] M. Asadi, Fixed points and common fixed points of mappings on CAT(0) spaces, Fixed Point Theory, 14(1)(2013) 29–38.
[7] M. Asadi, Some notes on fixed point sets in CAT(0) spaces, Advances in Fixed Point Theory, 4(3)(2014) 395–401.
[8] M. Asadi, S. Ghasemzadehdibagi, S. Haghayeghe and N. Ahmad, Fixed point theorem for (α,p)-nonexpansive mappings in CAT(0) spaces, Nonlinear Functional Analysis and Applications, 26(5)(2021), 1045–1057.
[9] M. Asadi, S. M. Vaezpour and H. Soleimani α-nonexpansive mappings on CAT(0) spaces, World Applied Sciences Journal, 11(10)(2010) 1303–1306.
[10] I. D. Berg, I. G. Nikolaev, Quasilinearization and curvature of Alexandrov spaces, Geometriae Dedicata, 133(2008) 195–218.
[11] M. Bridson and A. Haefliger, Metric spaces of non-positive curvature, Vol. 319, Springer, Berlin, 1999.
[12] K. S. Brown, Buildings, Springer, New York, 1989.
[13] D. Burago, Y. Burago and S. Ivanov, A course in metric geometry, Graduate Studies in Mathematics, 33(American Mathematical Society, Providence, RI, 2001).
[14] S. Dhompongsa and B. Panyanak, On $\triangle$-convergence theorems in CAT(0) spaces, Computers and Mathematics with Applications, 56(2008) 2572–2579.
[15] R. Espίnola, A. Fernάndez-Leόn, CAT(k)-spaces, weak convergence and fixed points, Journal of Mathematical Analysis and Applications, 353(2009) 410–427.
[16] K. Goebel, S. Reich, Uniform convexity, hyperbolic geometry, and nonexpansive mappings, Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, Inc, New York, 83 1984.
[17] M. Gromov, S. M. Bates, Metric structures for riemannian and non-riemannian spaces, Progress in Mathematics 152(eds. J. Lafontaine and P. Pansu) (Birkhӓuser, Boston, 1999), with appendices by M. Katz, P. Pansu and S. Semmes.
[18] O. Gϋler, On the convergence of the proximal point algorithm for convex minimization, SIAM Journal on Control and Optimization 29(1991) 403–419.
[19] B. Halpern, Fixed points of nonexpanding maps, Bulletin of the American Mathematical Society, 73(1967) 957–961.
[20] J. Jost, Convex functionals and generalized harmonic maps into spaces of nonpositive curvature, Commentarii Mathematici Helvetici 70(1995) 659–673.
[21] J. Jӧst, Nonpositive curvature: geometric and analytic aspects, IMA J. Numer. Anal, Lectures in Mathematics (Birkhӓuser, Basel, 1997).
[22] S. Kamimura and W. Takahashi, Approximating solutions of maximal monotone operators in Hilbert spaces, Journal of Approximation Theory 106(2000) 226–240.
[23] E. KarapJnar, H. Salahifard, and S. M. Vaezpour, Demiclosedness Principle for Total Asymptotically Nonexpansive Mappings in CAT(0) Spaces, Journal of Applied Mathematics, (2014), Article ID 738150.
[24] W. A. Kirk, Fixed point theorems in CAT(0) spaces and $\mathbb{R}$-tree, Fixed Point Theory and Applications 4(2004) 309–316.
[25] W. A. Kirk, B. Panyanak, A concept of convergence in geodesic spaces, Nonlinear Analysis (TMA) 68(2008) 3689–3696.
[26] H. Khatibzadeh, V. Mohebbi, Proximal Point Algorithm for Quasi-convex Minimization Problems in metric spaces, Numerical Functional Analysis and Optimization 39(4)(2018) 438–448.
[27] T. C. Lim, Remarks on some fixed point theorems, Proceedings of the American Mathematical Society 60(1976) 179–182.
[28] G. Marino, H. K. Xu, Convergence of generalized proximal point algorithm, Communications on Pure and Applied Analysis 3(2004) 791–808.
[29] B. Martinet, Rέgularisation d׳inέquations variationnelles par approximations successive, Revue Francέaise d׳Informatique et de Recherche Opέrationnelle 4(1970) 154–158.
[30] U. F. Mayer, Gradient flows on nonpositively curved metric spaces and harmonic maps, Communications in Analysis and Geometry 6(1998) 199–253.
[31] R. T. Rockafellar, Monotone operators and the proximal point algorithm, SIAM Journal on Control and Optimization 14(1976) 877–898.
[32] S. T. Smith, Optimization techniques on Riemannian manifolds in Fields Institute Communications, American Mathematical Society 3(1994) 113–146.
[33] R. Suparatulatorn, P. Cholamjiak, S. Suantai, On solving the minimization problem and the fixed-point problem for nonexpansive mappings in CAT(0) spaces, Optimization Methods and Software 32(1)(2016) 182–192.
[34] C. Udriste, Convex Functions and Optimization Methods on Riemannian Manifolds, Mathematics and its Applications, 297 Kluwer Academic, Dordrecht, 1994.
[35] J. H. Wang, C. Li, Convergence of the family of Euler-Halley type methods on Riemannian manifolds under the γ-condition, Taiwanese Journal of Mathematics 13(2009) 585–606.
[36] H. K. Xu, A regularization method for the proximal point algorithm, Journal of Global Optimization 36 (2006) 115–125.