Filomat 36:18 (2022), 6269–6278
https://doi.org/10.2298/FIL2218269S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Eigenvalue of (p, q)-Laplace System Along the Powers of Mean Curvature Flow

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Abstract. In this paper we derive the evolution equation of the first nonzero eigenvalue of the (p, q)-Laplace system under the unnormalized H^k-flow. By imposing some conditions on the mean curvature we prove the monotonicity of the first nonzero eigenvalue under the unnormalized H^k-flow.

1. Introduction

Let M^n be a smooth, compact manifold without boundary and $F_0 : M^n \to \mathbb{R}^{n+1}$ be a smooth immersion. A smooth family of immersions $F(x, t) : M^n \times [0, T) \to \mathbb{R}^{n+1}$ satisfying

$$\begin{aligned} \frac{\partial F}{\partial t}(x,t) &= -H^k(x,t)v(x,t) \\ F(\cdot, 0) &= F_0(\cdot),\ x \in M^n, \end{aligned} \tag{1}$$

where $k > 0$, H is the mean curvature and v is the normal at the point $F(x, t)$ of the surface $M_t = F(., t)(M)$, is known as unnormalized H^k-flow. For $k = 1$ the flow coincides with the mean curvature flow. In [13], F. Schulze obtained the result about the convergence of the unnormalized H^k-flow.

Let $d\mu$ denotes the volume element of M^n and $u : M^n \to \mathbb{R}$ be a smooth function on M^n or $u \in W^{1,p}(M)$, where $W^{1,p}(M)$ is the completion of the set of smooth functions with respect to the Sobolev norm

$$||u||_{1,p} = \left(\int_M |u|^p d\mu + \int_M |\nabla u|^p d\mu \right)^{\frac{1}{p}}.$$

Then the p-Laplace operator is defined by

$$\begin{aligned} \Delta_p u &= div(|\nabla u|^{p-2} \nabla u) \\ &= |\nabla u|^{p-2} \Delta u + (p-2)|\nabla u|^{p-4} (\text{Hess } u)(\nabla u, \nabla u)\ , \end{aligned}$$

2020 *Mathematics Subject Classification*. 53C15; 53C44
Keywords. Eigenvalue, (p, q)-Laplace system, mean curvature flow.
Received: 21 December 2021; Accepted: 11 March 2022
Communicated by Mića S. Stanković
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where $p \in (1, \infty)$ and Hess denotes the Hessian. If $p = 2$, then the p-Laplace operator is just the Laplace-Beltrami operator.

The spectrum of the Laplace operator and p-Laplace operator on a compact Riemannian manifold has important geometrical meaning. In recent years many mathematicians studied the evolution of the first eigenvalue of Laplace operator and p-Laplace operator under different geometric flows and estimate the spectrum in terms of other geometric quantities see [1], [3], [4], [5], [10] etc.

In this paper, we consider the generalization of the p-Laplace operator, i.e., (p, q)-Laplace system introduced in [9] as

$$\begin{cases} \Delta_p u = -\lambda |u|^\alpha |v|^\beta v, & \text{in } M \\ \Delta_q v = -\lambda |u|^\alpha |v|^\beta u, & \text{in } M \\ (u, v) \in W^{1,p}(M) \times W^{1,q}(M), \end{cases} \tag{2}$$

where $p > 1, q > 1$ and α, β are real numbers satisfying $\alpha > 0, \beta > 0$,

$$\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1.$$

If for some $u \in W_0^{1,p}(M)$ and $v \in W_0^{1,q}(M)$, λ satisfies the following conditions

$$\begin{aligned} \int_M |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle d\mu &= \lambda \int_M |u|^\alpha |v|^\beta v \phi d\mu, \\ \int_M |\nabla v|^{q-2} \langle \nabla v, \nabla \psi \rangle d\mu &= \lambda \int_M |u|^\alpha |v|^\beta u \psi d\mu, \end{aligned}$$

where $\phi \in W^{1,p}(M)$ and $\psi \in W^{1,q}(M)$ and $W_0^{1,p}(M)$ is the closure of the $C_0^\infty(M)$ in the Sobolev space $W^{1,p}(M)$. Then we say that λ is the eigenvalue of the system (2) and (u, v) is called an eigenfunction. The first nonzero eigenvalue of the system (2) is given by

$$\inf\{A(u, v) : (u, v) \in W_0^{1,p}(M) \times W_0^{1,q}(M), B(u, v) = 1, C(u, v) = 0, D(u, v) = 0\},$$

where

$$\begin{aligned} A(u, v) &= \frac{\alpha+1}{p} \int_M |\nabla u|^p d\mu + \frac{\beta+1}{q} \int_M |\nabla v|^q d\mu, \\ B(u, v) &= \int_M |u|^\alpha |v|^\beta u v d\mu, \\ C(u, v) &= \int_M |u|^\alpha |v|^\beta v d\mu, \\ D(u, v) &= \int_M |u|^\alpha |v|^\beta u d\mu. \end{aligned}$$

Let $(M^n, g(t))$ be a solution of the H^k-flow (1) on the smooth manifold M^n in the time interval $[0, T)$. Then

$$\lambda(t) = \frac{\alpha+1}{p} \int_M |\nabla u(t)|^p_{g(t)} d\mu + \frac{\beta+1}{q} \int_M |\nabla v(t)|^q_{g(t)} d\mu,$$

defines the evolution of the first nonzero eigenvalue of (2), under the variation of $g(t)$ where the associated eigenfunction is normalized, i.e., $B(u, v) = 1, C(u, v) = 0, D(u, v) = 0$.

In [14], L. Zhao studied the variation of the first nonzero eigenvalue of the p-Laplacian along the powers of mean curvature flow and in [2], S. Azami studied the variation of eigenvalue of (p, q)-Laplace system under the mean curvature flow. In [11, 12], the present authors studied the first eigenvalues of weighted

p-Laplacian and (p, q)-Laplace system along the Cotton flow and forced mean curvature flow respectively. Motivated by the above works in this paper we obtain the variation formula for the first nonzero eigenvalue of the (p, q)-Laplace system on a compact and convex hypersurface M^n of $\mathbb{R}^{n+1}$ under the powers of the mean curvature flow. By imposing some conditions on the mean curvature we also prove that the first eigenvalue of the (p, q)-Laplace system is increasing under the unnormalized H^k-flow given in (1).

2. First eigenvalue of the (p, q)-Laplace system

In this section first we deduce the variation formula for the first nonzero eigenvalue of the system given in (2) under the unnormalized H^k-flow. Before going to deduce the variation formula we recall some evolution formula for the unnormalized H^k-flow as:

Lemma 2.1. *[13] If M_t evolves under H^K-flow, then the geometric quantities associated with M_t satisfy the following quantities:*

$$(i)\frac{\partial}{\partial t}g_{ij} = -2H^k h_{ij} \tag{3}$$

$$(ii)\frac{\partial}{\partial t}g^{ij} = 2H^k h^{ij} \tag{4}$$

$$(iii)\frac{\partial}{\partial t}d\mu = -H^{k+1}d\mu \tag{5}$$

$$(iv)\frac{\partial}{\partial t}H = kH^{k-1}\Delta H + k(k-1)H^{k-2}|\nabla H|^2 + |A|^2 H^k \tag{6}$$

where $A = (h_{ij})$, h_{ij} is the second fundamental form associated with $F : M^n \to \mathbb{R}^{n+1}$.

Let M^n be a closed Riemannian manifold and $g(t)$ be the solution of the unnormalized H^k-flow. Let at time $t_0 \in [0, T)$, $(u_0, v_0) = (u(t_0), v(t_0))$ be the eigenfunctions for the eigenvalue $\lambda(t_0)$ of (p, q)-Laplacian system. Assume that

$$h(t) = u_0\left[\frac{\det[g_{ij}(t)]}{\det[g_{ij}(t_0)]}\right]^{\frac{1}{2(\alpha+\beta+1)}}, \qquad l(t) = v_0\left[\frac{\det[g_{ij}(t)]}{\det[g_{ij}(t_0)]}\right]^{\frac{1}{2(\alpha+\beta+1)}},$$

and

$$u(t) = \frac{h(t)}{\left(\int_M |h(t)|^\alpha |l(t)|^\beta h(t)l(t)d\mu\right)^{\frac{1}{p}}}, \qquad v(t) = \frac{l(t)}{\left(\int_M |h(t)|^\alpha |l(t)|^\beta h(t)l(t)d\mu\right)^{\frac{1}{q}}}.$$

Hence $u(t)$, $v(t)$ are smooth functions under the power of mean curvature flow, satisfy

$$\int_M |u|^\alpha |v|^\beta uv d\mu = 1, \quad \int_M |u|^\alpha |v|^\beta v d\mu = 0, \quad \int_M |u|^\alpha |v|^\beta u d\mu = 0,$$

and at time t_0, $(u(t_0), v(t_0))$ is the eigenfunctions for $\lambda(t_0)$ of (p, q)-Laplacian system; i.e., function

$$\lambda(u, v, t) := \frac{\alpha+1}{p}\int_M |\nabla u|^p d\mu + \frac{\beta+1}{q}\int_M |\nabla v|^q d\mu, \tag{7}$$

is a smooth function with respect to t along the power of mean curvature flow where u, v are smooth functions satisfy the normalized condition. If (u, v) are the corresponding eigenfunctions of the first eigenvalue $\lambda(t)$ at t_0 then $\lambda(u, v, t_0) = \lambda(t_0)$.

In the following proposition we compute the evolution formula for the first nonzero eigenvalue along the power of mean curvature flow.

Proposition 2.2. *Let $(M^n, g(t))$ be a solution of (1) on the smooth closed oriented manifold M^n. If $\lambda(t)$ denotes the evolution of the first nonzero eigenvalue under the unnormalized H^k-flow (1), then*

$$\begin{aligned}\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} &= \lambda(t_0)\int_M H^{k+1}|u|^{\alpha}|v|^{\beta}uv\,d\mu + (\alpha+1)\int_M H^k h^{ij}\nabla_i u\nabla_j u|\nabla u|^{p-2}d\mu \\ &\quad -\frac{\alpha+1}{p}\int_M H^{k+1}|\nabla u|^p d\mu + (\beta+1)\int_M H^k h^{ij}\nabla_i v\nabla_j v|\nabla v|^{q-2}d\mu \\ &\quad -\frac{\beta+1}{q}\int_M H^{k+1}|\nabla v|^q d\mu \end{aligned} \tag{8}$$

where (u, v) is the associated eigenfunctions satisfying the normalized conditions.

Proof. Differentiating (7) with respect to time t we get

$$\begin{aligned}\frac{d}{dt}\lambda(u,v,t) &= \frac{\alpha+1}{p}\left(\int_M \frac{d}{dt}(|\nabla u|^p)d\mu + \int_M |\nabla u|^p\frac{d}{dt}(d\mu)\right) \\ &\quad +\frac{\beta+1}{q}\left(\int_M \frac{d}{dt}(|\nabla v|^q)d\mu + \int_M |\nabla v|^q\frac{d}{dt}(d\mu)\right). \end{aligned} \tag{9}$$

Now we have

$$\begin{aligned}\frac{d}{dt}(|\nabla u|^p) &= \frac{d}{dt}((|\nabla u|^2)^{\frac{p}{2}}) \\ &= \frac{p}{2}\{\frac{\partial}{\partial t}(g^{ij})\nabla_i u\nabla_j u + 2g^{ij}\nabla_i u_t\nabla_j u\}|\nabla u|^{p-2} \\ &= p\{H^k h^{ij}\nabla_i u\nabla_j u + g^{ij}\nabla_i u_t\nabla_j u\}|\nabla u|^{p-2}\ , \end{aligned} \tag{10}$$

and from Lemma 2.1, we get

$$\frac{\partial}{\partial t}d\mu = -H^{k+1}d\mu\ . \tag{11}$$

Therefore using (10) and (11) in (9) we get

$$\begin{aligned}\frac{d}{dt}\lambda(u,v,t) &= (\alpha+1)\int_M \{H^k h^{ij}\nabla_i u\nabla_j u + g^{ij}\nabla_i u_t\nabla_j u\}|\nabla u|^{p-2}d\mu \\ &\quad +(\beta+1)\int_M \{H^k h^{ij}\nabla_i v\nabla_j v + g^{ij}\nabla_i v_t\nabla_j v\}|\nabla v|^{q-2} \\ &\quad +\frac{\alpha+1}{p}\int_M |\nabla u|^p(-H^{k+1})d\mu \\ &\quad +\frac{\beta+1}{q}\int_M |\nabla v|^q(-H^{k+1})d\mu\ . \end{aligned} \tag{12}$$

From the normalized condition

$$\int_M |u|^{\alpha}|v|^{\beta}uv\,d\mu = 1$$

and taking time derivative, we get

$$(\alpha+1)\int_M |u|^{\alpha}|v|^{\beta}u_t v\,d\mu + (\beta+1)\int_M |u|^{\alpha}|v|^{\beta}uv_t\,d\mu = -\int_M |u|^{\alpha}|v|^{\beta}uv(-H^{k+1})d\mu.$$

Again we have

$$\int_M \langle\nabla u_t, \nabla u\rangle|\nabla u|^{p-2}d\mu = \lambda\int_M |u|^{\alpha}|v|^{\beta}u_t v\,d\mu, \tag{13}$$

$$\int_M \langle\nabla v_t, \nabla v\rangle|\nabla v|^{q-2}d\mu = \lambda\int_M |u|^{\alpha}|v|^{\beta}uv_t\,d\mu\ . \tag{14}$$

Therefore at a time $t = t_0$, we get

$$(\alpha+1)\int_M \langle \nabla u_t, \nabla u\rangle |\nabla u|^{p-2} d\mu + (\beta+1)\int_M \langle \nabla v_t, \nabla v\rangle |\nabla v|^{q-2} d\mu$$
$$= \lambda(t_0)\int_M H^{k+1}|u|^{\alpha}|v|^{\beta} uv d\mu \ .$$

Thus from (12), we get

$$\begin{aligned}\frac{d}{dt}\lambda(u,v,t)|_{t=0} \quad &= \quad \lambda(t_0)\int_M H^{k+1}|u|^{\alpha}|v|^{\beta} uv d\mu + (\alpha+1)\int_M H^k h^{ij}\nabla_i u \nabla_j u |\nabla u|^{p-2} d\mu \\ &-\frac{\alpha+1}{p}\int_M H^{k+1}|\nabla u|^p d\mu + (\beta+1)\int_M H^k h^{ij}\nabla_i v \nabla_j v |\nabla v|^{q-2} d\mu \\ &-\frac{\beta+1}{q}\int_M H^{k+1}|\nabla v|^q d\mu.\end{aligned}$$

□

Theorem 2.3. *Let $(M^n, g(t))$, $t \in [0,T)$ be a solution of the unnormalized H^k-flow on a closed manifold M^n and $\lambda(t)$ be the first nonzero eigenvalue of the (p,q)-Laplace system. If $m = min\{p,q\}$ and there exists a nonnegative constant ϵ such that*

$$H^k h_{ij} - \tfrac{H^{k+1}}{m} g_{ij} \geq -\epsilon g_{ij} \ in \ M \times [0,T)$$

and

$$H^{k+1} > m\epsilon \ in \ M \times [0,T),$$

then $\lambda(t)$ is increasing along the unnormalized H^k-flow.

Proof. According to the Proposition 2.1 we get

$$\begin{aligned}\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} \quad &= \quad \lambda(t_0)\int_M H^{k+1}|u|^{\alpha}|v|^{\beta} uv d\mu + (\alpha+1)\int_M H^k h^{ij}\nabla_i u \nabla_j u |\nabla u|^{p-2} d\mu \\ &-\frac{\alpha+1}{p}\int_M H^{k+1}|\nabla u|^p d\mu + (\beta+1)\int_M H^k h^{ij}\nabla_i v \nabla_j v |\nabla v|^{q-2} d\mu \\ &-\frac{\beta+1}{q}\int_M H^{k+1}|\nabla v|^q d\mu \ .\end{aligned}$$

Now using the assumption of the theorem we have

$$\begin{aligned}\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} \quad &\geq \quad \lambda(t_0)\int_M H^{k+1}|u|^{\alpha}|v|^{\beta} uv d\mu \\ &+(\alpha+1)\int_M (\frac{H^{k+1}}{m} g^{ij} - \epsilon g^{ij})\nabla_i u \nabla_j u |\nabla u|^{p-2} d\mu \\ &+(\beta+1)\int_M (\frac{H^{k+1}}{m} g^{ij} - \epsilon g^{ij})\nabla_i v \nabla_j v |\nabla v|^{q-2} d\mu \\ &-\frac{\alpha+1}{p}\int_M H^{k+1}|\nabla u|^p d\mu - \frac{\beta+1}{q}\int_M H^{k+1}|\nabla v|^q d\mu \ .\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} \geq\ & \lambda(t_0)\int_M H^{k+1}|u|^{\alpha}|v|^{\beta}uv d\mu \\
&+(\alpha+1)(\frac{1}{m}-\frac{1}{p})\int_M H^{k+1}|\nabla u|^p d\mu \\
&+(\beta+1)(\frac{1}{m}-\frac{1}{q})\int_M H^{k+1}|\nabla v|^q d\mu \\
&-\epsilon(\alpha+1)\int_M |\nabla u|^p d\mu - \epsilon(\beta+1)\int_M |\nabla v|^q d\mu \\
\geq\ & \lambda(t_0)m\epsilon + (\alpha+1)\left\{\left(\frac{1}{m}-\frac{1}{p}\right)m\epsilon - \epsilon\right\}\int_M |\nabla u|^p d\mu \\
&+(\beta+1)\left\{\left(\frac{1}{m}-\frac{1}{q}\right)m\epsilon - \epsilon\right\}\int_M |\nabla v|^q d\mu\ .
\end{aligned}$$

Thus we get

$$\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} \geq 0.$$

Hence we have that $\frac{d}{dt}\lambda(u,v,t) \geq 0$ in any sufficiently small neighborhood of the time $t = t_0$.

Hence $\lambda(t)$ is increasing in any small neighborhood $t = t_0$ in the interval $[0,T)$. Since t_0 is an arbitrary point in the interval $[0,T)$, thus $\lambda(t)$ is increasing along the unnormalized H^k-flow. □

Theorem 2.4. *Let $(M^n, g(t)), t \in [0,T)$ be a solution of the unnormalized H^k-flow (1) on a closed manifold M^n and $\lambda(t)$ be the evolution of the first nonzero eigenvalue of the (p,q)-Laplace system along unnormalized H^k-flow. If $m = \min\{p,q\}$ and there exists a nonnegative constant ϵ such that*

$$h_{ij} \geq \epsilon H g_{ij} (\tfrac{1}{m} \leq \epsilon \leq \tfrac{1}{n}) \text{ in } M\times[0,T)$$

and $H > 0$ at the initial time $t = 0$, then the following quantity

$$\lambda(t)\left(1 - \frac{k+1}{n}H_{min}^{k+1}(0)t\right)^{\frac{n}{k+1}}$$

is nondecreasing along the unnormalized H^k-flow.

Proof. From Proposition 2.1, we have

$$\begin{aligned}
\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} =\ & \lambda(t_0)\int_M H^{k+1}|u|^{\alpha}|v|^{\beta}uv d\mu + (\alpha+1)\int_M H^k h^{ij}\nabla_i u \nabla_j u |\nabla u|^{p-2} d\mu \\
&-\frac{\alpha+1}{p}\int_M H^{k+1}|\nabla u|^p d\mu + (\beta+1)\int_M H^k h^{ij}\nabla_i v \nabla_j v |\nabla v|^{q-2} d\mu \\
&-\frac{\beta+1}{q}\int_M H^{k+1}|\nabla v|^q d\mu \\
\geq\ & \lambda(t_0)\int_M H^{k+1}|u|^{\alpha}|v|^{\beta}uv d\mu + \epsilon(\alpha+1)\int_M H^{k+1}|\nabla u|^p d\mu \\
&-\frac{\alpha+1}{p}\int_M H^{k+1}|\nabla u|^p d\mu + \epsilon(\beta+1)\int_M H^{k+1}|\nabla v|^q d\mu \\
&-\frac{\beta+1}{q}\int_M H^{k+1}|\nabla v|^q d\mu.
\end{aligned}$$

Hence

$$\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} \geq \lambda(t_0)\int_M H^{k+1}|u|^{\alpha}|v|^{\beta}uvd\mu.$$

F. Schulse [13] got the minimum of $H(x,t)$ according to the evolution equation (6),

$$H_{min}(t) \geq H_{min}(0)\left(1-\frac{k+1}{n}H_{min}^{k+1}(0)t\right)^{-\frac{1}{k+1}}.$$

Using this inequality in the above inequality we get

$$\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} \geq \lambda(t_0)H_{min}^{k+1}(0)\left(1-\frac{k+1}{n}H_{min}^{k+1}(0)t_0\right)^{-1},$$

which implies that in any sufficiently small neighborhood of t_0,

$$\frac{d}{dt}\lambda(u,v,t) \geq \lambda(u,v,t)H_{min}^{k+1}(0)\left(1-\frac{k+1}{n}H_{min}^{k+1}(0)t\right)^{-1}.$$

Now taking integration of the above inequality on $[t_1,t_0] \subset [0,T)$, we have

$$\ln\lambda(u,v,t)|_{t=t_0} - \ln\lambda(u,v,t)|_{t=t_1} \geq -\frac{n}{k+1}\ln\left(1-\frac{k+1}{n}H_{min}^{k+1}(0)t_0\right)+\frac{n}{k+1}\ln\left(1-\frac{k+1}{n}H_{min}^{k+1}(0)t_1\right),$$

which implies that

$$\ln\lambda(u,v,t)|_{t=t_0} + \frac{n}{k+1}\ln\left(1-\frac{k+1}{n}H_{min}^{k+1}(0)t_0\right) \geq \ln\lambda(u,v,t)|_{t=t_1} + \frac{n}{k+1}\ln\left(1-\frac{k+1}{n}H_{min}^{k+1}(0)t_1\right),$$

i.e.,

$$\lambda(t_0)\left(1-\frac{k+1}{n}H_{min}^{k+1}(0)t_0\right)^{\frac{n}{k+1}} \geq \lambda(t_1)\left(1-\frac{k+1}{n}H_{min}^{k+1}(0)t_1\right)^{\frac{n}{k+1}}.$$

This shows that the quantity $\lambda(t)\left(1-\frac{k+1}{n}H_{min}^{k+1}(0)t\right)^{\frac{n}{k+1}}$ is nondecreasing along the unnormalized H^k-flow. □

Theorem 2.5. *Let $(M^n, g(t))$, $t \in [0,T)$ be a solution of the unnormalized H^k-flow on a closed Riemannian manifold M^n. Let $\lambda(t)$ be the evolution of the first nonzero eigenvalue of the (p,q)-Laplace system along unnormalized H^k-flow. If $m = \max\{p,q\}$ and there exists a nonnegative constant ϵ such that*

$$0 \leq h_{ij} \leq \epsilon H g_{ij} \quad \left(\tfrac{1}{n} < \epsilon \leq \tfrac{1}{m}\right) \text{ in } M\times[0,T),$$

and $H > 0$ at the initial time $t = 0$, then the following quantity

$$\lambda(t)(H_{max}^{-(k+1)}(0)-(k+1)t)^{\frac{1}{k+1}}$$

is nonincreasing along the unnormalized H^k-flow.

Proof. According to the Proposition 2.1, we get

$$\begin{aligned}\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} &= \lambda(t_0)\int_M H^{k+1}|u|^{\alpha}|v|^{\beta}uvd\mu + (\alpha+1)\int_M H^k h^{ij}\nabla_i u\nabla_j u|\nabla u|^{p-2}d\mu \\ &\quad -\frac{\alpha+1}{p}\int_M H^{k+1}|\nabla u|^p d\mu + (\beta+1)\int_M H^k h^{ij}\nabla_i v\nabla_j v|\nabla v|^{q-2}d\mu \\ &\quad -\frac{\beta+1}{q}\int_M H^{k+1}|\nabla v|^q d\mu.\end{aligned}$$

Using the assumption of the theorem in the above equation we get

$$\begin{aligned}\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} &\leq \lambda(t_0)\int_M H^{k+1}|u|^{\alpha}|v|^{\beta}uv d\mu + (\alpha+1)(\epsilon - \frac{1}{p})\int_M H^{k+1}|\nabla u|^p d\mu \\ &\quad (\beta+1)(\epsilon - \frac{1}{q})\int_M H^{k+1}|\nabla v|^q d\mu \\ &\leq \lambda(t_0)\int_M H^{k+1}|u|^{\alpha}|v|^{\beta}uv d\mu.\end{aligned}$$

F. Schulze in [13] got the maximum of $H(x,t)$ according to the evolution equation (6)

$$H(x,t) \leq \left(H_{max}^{-(k+1)}(0) - (k+1)t\right)^{-\frac{1}{k+1}}.$$

Thus we have

$$\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} \leq \lambda(t_0)\left(H_{max}^{-(k+1)}(0) - (k+1)t_0\right)^{-1},$$

which implies that in any small neighborhood of t_0,

$$\frac{d}{dt}\lambda(u,v,t) \leq \lambda(u,v,t)\left(H_{max}^{-(k+1)}(0) - (k+1)t\right)^{-1}.$$

Now taking integration in the last inequality in the interval $[t_1,t_0] \subset [0,T)$, we have

$$\begin{aligned}\ln\lambda(u,v,t)|_{t=t_0} - \ln\lambda(u,v,t)|_{t=t_1} &\leq -\frac{1}{k+1}\ln\left(H_{max}^{-(k+1)}(0) - (k+1)t_0\right) \\ &\quad +\frac{1}{k+1}\ln\left(H_{max}^{-(k+1)}(0) - (k+1)t_1\right),\end{aligned}$$

i.e.,

$$\ln\lambda(u,v,t)|_{t=t_0} + \frac{1}{k+1}\ln\left(H_{max}^{-(k+1)}(0) - (k+1)t_0\right) \leq \ln\lambda(u,v,t)|_{t=t_1} + \frac{1}{k+1}\ln\left(H_{max}^{-(k+1)}(0) - (k+1)t_1\right),$$

which implies that

$$\lambda(t_0)(H_{max}^{-(k+1)}(0) - (k+1)t_0)^{\frac{1}{k+1}} \leq \lambda(t_1)(H_{max}^{-(k+1)}(0) - (k+1)t_1)^{\frac{1}{k+1}}.$$

This shows that the quantity $\lambda(t)(H_{max}^{-(k+1)}(0)-(k+1)t)^{\frac{1}{k+1}}$ in nonincreasing along the unnormalized H^k-flow. □

Theorem 2.6. *Let $(M^n, g(t))$ be a solution of the power of mean curvature flow on the smooth closed oriented manifold (M^n, g_0) and $\lambda(t)$ denotes the evolution of the first eigenvalue under the power of mean curvature flow. Let $q \leq p$ and there exist positive constants $a_1, a_2, \cdots, a_n$ such that the initial hypersurface M_0 satisfies*

$$h_{ij} = a_i H g_{ij}, \quad \text{where } \sum_{i=1}^{n} a_i = 1, \text{ and } |a_i - \frac{1}{n}| \leq \epsilon \tag{15}$$

for small enough ϵ only depending on n, q and $H > 0$. Then under the power of mean curvature flow $\lambda(t)$ is nondecreasing for $\epsilon \leq \frac{1}{n} - \frac{1}{q}$.

Proof. By (15) we have

$$h_{ij} = a_i H g_{ij}, \quad \text{on } M_0. \tag{16}$$

On the other hand, under the power of mean curvature flow, we get

$$\begin{aligned}\frac{\partial}{\partial t}(h_{ij}-a_iHg_{ij}) &= kH^{k-1}\Delta(h_{ij}-a_iHg_{ij})+k(k-1)H^{k-2}\nabla_iH\nabla_jH \\ &\quad -a_ik(k-1)H^{k-1}|\nabla H|^2g_{ij}-(k+1)H^kh_{ir}h_j^r \\ &\quad -a_i|A|^2H^kg_{ij}+2a_iH^{k+1}h_{ij}+kH^{k-1}|A|^2h_{ij},\end{aligned} \tag{17}$$

where $\nabla H=\frac{g^{ij}\nabla(h_{ij}-a_iHg_{ij})}{1-a_ig_{ij}g^{ij}}$.
Using Hamilton's maximum principle for tensors on manifolds we can conclude that

$$h_{ij}=a_iHg_{ij},\quad \text{on } M_t. \tag{18}$$

So, from (8) we obtain

$$\begin{aligned}\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} &= \lambda(t_0)\int_M H^{k+1}|u|^\alpha|v|^\beta uv d\mu+(\alpha+1)\int_M H^{k+1}a_ig^{ij}\nabla_iu\nabla_ju|\nabla u|^{p-2}d\mu \\ &\quad -\frac{\alpha+1}{p}\int_M|\nabla u|^pH^{k+1}d\mu+(\beta+1)\int_M H^{k+1}a_ig^{ij}\nabla_iv\nabla_jv|\nabla v|^{q-2}d\mu \\ &\quad -\frac{\beta+1}{q}\int_M|\nabla v|^qH^{k+1}d\mu \\ &\geq \lambda(t_0)\int_M H^{k+1}|u|^\alpha|v|^\beta uv d\mu+(\alpha+1)(\frac{1}{n}-\epsilon-\frac{1}{p})\int_M H^{k+1}|\nabla u|^pd\mu \\ &\quad +(\beta+1)(\frac{1}{n}-\epsilon-\frac{1}{q})\int_M H^{k+1}|\nabla v|^qd\mu.\end{aligned}$$

Thus for $\epsilon\leq\frac{1}{n}-\frac{1}{q}$ we get $\frac{d}{dt}\lambda(u,v,t)|_{t=t_0}\geq 0$ and since t_0 is arbitrary, it implies that $\lambda(t)$ is nondecreasing along this flow. □

3. Conclusion

The (p,q)-Laplace system arises in several fields of application. For instance, in the case where $p>2$, (p,q)-Laplace system appears in the study of non-Newtonian fluids, pseudoplastics for $1<p<2$, and in reaction-diffusion problems, flows through porous media, nonlinear elasticity, and glaciology for $p=\frac{4}{3}$ [6, 8]. Moreover Khalil et al. [9] has been studied and introduced the first eigenvalue of (p,q)-Laplace system. In this paper we generalized the results of [2, 10, 14].

Acknowledgement: The first author (A. Saha) gratefully acknowledges to the CSIR (File No.: 09/025(0273)/2019-EMR-I), Government of India for the award of Senior Research Fellowship. The third author (S. K. Hui) sincerely acknowledges the partial financial support of DST FIST programme (No.: SR/FST/MSII/2017/10(C)).

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