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Generalized Form of Fixed-Point Theorems in Generalized Banach Algebra Relative to the Weak Topology with an Application

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Abstract. In this paper, a general hybrid fixed point theorem for the contractive mappings in generalized Banach spaces is proved via measure of weak non-compactness and it is further applied to fractional integral equations for proving the existence results for the solutions under mixed Lipschitz and weakly sequentially continuous conditions. Finally, an example is given to illustrate the result.

1. Introduction

Many nonlinear problems involve the study of nonlinear equations of the form

$$x = Ax \cdot Bx + Cx, \quad x \in S \tag{1}$$

where S is a nonempty, closed, and convex subset of a Banach algebra X, see for example [6, 7, 9–11, 22] and the references therein. These studies were mainly based on the Schauder fixed point theorem, and properties of the operators A, B and C (cf. completely continuous, k-set contractive, condensing, and the potential tool of the axiomatic measure of noncompactness,...). Various attempts have been made in the literature to extend this study to a weak topology, see for example [2, 4, 15]. Since the weak topology is the practice setting, significant advances have been made in the development of fixed point theory in Banach algebras using this topology [4, 16–22]. In general, these arguments are based on the space's condition ($\mathcal{GP}$) (see Definition 2.14), to ensure the sequential weak continuity of the product of the entree operators. In recent years, the authors Ben Amar et al. have established in [2] some fixed point theorems for the operator equation (1) in Banach algebra under the weak topology, their results were based on the class of weak sequential continuity, weakly compact and weakly condensing conditions. In other direction, the classical Banach contraction principle was extended for contractive maps on spaces endowed with vector-valued metrics by Perov [26, 27]. Schauder's fixed point theorem has been extended from Banach spaces to generalized Banach spaces by Viorel [32] and Krasnosel'skii's fixed point theorem has been extended by Petru and ouahab in [25, 28]. More recently, the authors Nieton et al. [24] have also established some new variants of fixed-point theorems for operator equation (1) in vector-valued metrics endowed with an internal composition law $(\cdot)$. These studies were mainly based on the convexity and the closure of the

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bounded domain and properties of the operators A, B and C. Their analysis was carried out via arguments of strong topology.

This paper is centered around the following question: under which conditions on its entries, the operator equation (1) acting on a generalized Banach algebras with respect to the weak topology, has a solution? Our main results are applied to investigate the existence of solutions for the following coupled system of quadratic integral equations of fractional order

$$x_i(t) = f_i(t, x_1(t), x_2(t)) \cdot \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} g_i(s, x_1(s), x_2(s))ds + \sum_{k=1}^{m} I^{\beta_i^k} h_i^k(t, x_1(t), x_2(t)), \quad i = 1, 2 \tag{2}$$

where $k \in \{1, ..., m\}$, $t, \alpha_i \in (0, 1)$, $I^{\beta_i^k}$ is the fractional Pettis integral of order $\beta_i^k > 0$, $\Gamma(\cdot)$ is the Gamma function and the functions f_i, g_i, h_i^k are given functions, whereas $x_1 = x_1(t)$ and $x_2 = x_2(t)$ are unknown functions.

The present paper is arranged as follows. The next section, we give some preliminary results for future use. Moreover, we shall extend the result of B. Amar, Jeribi and Mnif [[5], Theorem 2.5]. In addition, we establish the fixed point theorem by using the concept of the measure of weak non compactness in generalized Banach space (see Theorem 2.12). Note our result (Theorem 2.12) improves and generalizes Theorem 3.2 in [3]. Apart from that we introduce a class of generalized Banach algebras satisfying certain sequential conditions called here the condition $(\mathcal{GP})$ (see Definition 2.14). In section 3, we present a collection of new fixed point theorems in generalized Banach algebras satisfying condition $(\mathcal{GP})$. Our results extend and improve well-known results in [2, 4, 24]. In the last section of this manuscript, we apply Theorem 3.4 to discuss the existence of solutions for a system of fractional integral equations (2) and an example is given to explain the applicability of the results.

2. Preliminaries and mains results

Let $(X, \|\cdot\|)$ be a generalized Banach space in the sense of Perov such that the vector-valued norm $\|\cdot\| : X \longrightarrow \mathbb{R}_+^n$ is given by

$$\|x\| = \begin{pmatrix} \|x\|_1 \\ \vdots \\ \|x\|_n \end{pmatrix}, \quad x \in X$$

with $\|\cdot\|_i, i = 1, ..., n$ define n norm on X. We denote by θ the zero element of X and $B_r = B(\theta, r)$ the closed ball centered at θ with radius $r \in \mathbb{R}_+^n$. For more details, the reader may consult the monograph of John R. Graef, Johnny Henderson and Abdelghani Ouahab [13].
Let $(Y_i, \tau_i)_{i\in I}$ be a family of topological spaces and let $f_i : X \longrightarrow Y_i, \quad i \in I$ be a linear and continuous mappings. We define the smallest topology on X such that all the mappings f_i remain continuous with respect to this topology. Its basic open sets are of the form $\cap_{i\in J} f_i^{-1} u_i$, with J a finite subset of I and $u_i \in \tau_i$ for each i. This topology is called the weak topology on X generated by the $(f_i)_{i\in I}$ and we denote it by $\sigma(X, (f_i)_{i\in I})$. So that a sequence $(x_n)_n$ in X converges to x in $\sigma(X, (f_i)_{i\in I})$ if and only if $(f_i(x_n))_n$ converges to $f_i(x)$, for all $i \in I$. In generalized Banach spaces, weakly open subset, weakly closed subset and weak compactness, are similar to those for usual Banach spaces. We denote by $\mathcal{B}(X)$ the collection of all nonempty bounded subsets of X and $\mathcal{W}(X)$ is a subset of $\mathcal{B}(X)$ consisting of all weakly compact subset of X. If $x, y \in \mathbb{R}^n$, $x = (x_1, ..., x_n), \quad y = (y_1, ..., y_n)$ by $x \le y$ we means $x_i \le y_i$ for all $i = 1, ..., n$.

Definition 2.1. *A square matrix of real numbers M is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$, with $det(M - \lambda I) = 0$, where I denotes the unit matrix of $\mathcal{M}_{n\times n}(\mathbb{R}_+)$.*

Lemma 2.2. *[31] Let M The following assertions are equivalent:*
(i) M is a matrix convergent to zero,

(ii) $M^k \to 0$ *as* $k \to \infty$,
(iii) The matrix $(I - M)$ *is nonsingular and*

$$(I - M)^{-1} = I + M + M^2 + \ldots + M^k + \ldots,$$

(iv) The matrix $(I - M)$ *is nonsingular and* $(I - M)^{-1}$ *has nonnegative elements.*

Definition 2.3. *[30] Let X be a generalized Banach space. An operator* $T : X \longrightarrow X$ *is said to be contractive if there exists a matrix* $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ *convergent to zero such that*

$$\|Tx - Ty\| \leq M\|x - y\|, \quad \text{for all } x, y \text{ in } X.$$

For $n = 1$, *we recover the classical Banach's contraction fixed point result.*

Definition 2.4. *An operator* $T : X \longrightarrow X$ *is called weakly sequentially continuous on X, if for every sequence* $(x_n)_{n\in\mathbb{N}}$ *with* $x_n \rightharpoonup x$ *we have* $Tx_n \rightharpoonup Tx$, *here* $\rightharpoonup$ *denotes weak convergence.*

Definition 2.5. *An operator* $T : X \longrightarrow X$ *is said to be weakly compact if* $T(V)$ *is relatively weakly compact for every bounded subset* $V \subseteq X$.

Theorem 2.6. *[14, 32] Let X be a generalized Banach space, S be a nonempty, compact and convex subset of X and* $A : S \longrightarrow S$ *be a continuous operator. Then A has at least a fixed point in S.*

Now, we will prove the following theorem.

Theorem 2.7. *Let S be a nonempty, closed and convex subset of a generalized Banach space X. Let* $T : S \longrightarrow S$ *be a weakly sequentially continuous mapping. If* $T(S)$ *is relatively weakly compact, then T has a fixed point.*

Proof. Let $M = \overline{co}(T(S))$ the closed convex hull of $T(S)$. Because $T(S)$ is relatively weakly compact, then M is weakly compact convex subset of X. On the other hand, $T(M) \subset T(S) \subset \overline{co}(T(S) = M$ ie : T maps M into itself. Since T is weakly sequentially continuous on M, we infer that T is weakly continue on M. (consider $X = (X, \sigma(X, (f_i)_{i\in I}))$ the space endowed with the weak topology and note that $T : M \longrightarrow M$ is continuous with M is compact). The use of Theorem 2.6 achieves the proof Q.E.D.

Definition 2.8. *the measure of weak non-compactness of the generalized Banach space X is the map* $\mu : \mathcal{B}(X) \longrightarrow \mathbb{R}^n_+$ *defined in the following way*

$$\mu(S) = \inf\Big\{r \in \mathbb{R}^n_+, \quad \text{there exists } K \in \mathcal{W}(X) \text{ such that } S \subseteq K + B_r\Big\},$$

for all $S \in \mathcal{B}(X)$.

Lemma 2.9. *Let* S_1, S_2 *be two elements of* $\mathcal{B}(X)$. *Then the functional* μ *has the properties:*
(i) $S_1 \subseteq S_2$ *implies* $\mu(S_1) \leq \mu(S_2)$,
(ii) $\mu(\overline{S_1^w}) = \mu(S_1)$,
(iii) $\mu(S_1) = 0$, *if and only if,* $\overline{S_1^w} \in \mathcal{W}(X)$, *where* $\overline{S_1^w}$ *is the weak closure of* S_1,
(iv) $\mu(co(S_1)) = \mu(S_1)$, *where* $co(S_1)$ *is the convex hull of* S_1,
(v) $\mu(S_1 + S_2) \leq \mu(S_1) + \mu(S_2)$, $\mu(\{a\} + S_1) \leq \mu(S_1)$,
(vi) $\mu(S_1 \cup S_2) = \max\{\mu(S_1), \mu(S_2)\}$,
(vii) $\mu(\lambda S_1) = |\lambda|\mu(S_1)$, *for all* $\lambda \in \mathbb{R}$.

Proof. The statements *(i)* and *(iv)* - *(viii)* are immediate consequences of the definition of μ . Let us prove *(ii)*. From the definition of μ there exists a subset $K_1 \in \mathcal{W}(X)$ and $r_1 \in \mathbb{R}^n_+ \setminus \{0_{\mathbb{R}^n}\}$ such that $S_1 \subseteq K_1 + B(\theta, r_1 + \mu(S_1))$, then $S_1 \subseteq \overline{co}K_1 + B(\theta, r_1 + \mu(S))$. By the Kerein-Šmulian theorem $\overline{co}K_1$ is weakly compact. Since $B(\theta, r_1 + \mu(S_1))$ is weakly closed, we infer that $\overline{co}K_1 + B(\theta, r_1 + \mu(S_1))$ is weakly closed. So $\overline{S_1}^w \subseteq \overline{co}K_1 + B(\theta, r_1 + \mu(S_1))$ implies that $\mu(\overline{S_1}^w) \leq r_1 + \mu(S_1)$. Letting $r_1 \to 0_{\mathbb{R}^n}$ in the above inequality, we get $\mu(\overline{S_1}^w) \leq \mu(S_1)$. The reverse inequality is obvious . The proof of the "only if" part of (iii). By the definition of μ, there exists a subset $K_1 \in \mathcal{W}(X)$ and $r_1 \in \mathbb{R}^n_+ \setminus \{0_{\mathbb{R}^n}\}$ such that

$$\overline{S_1}^w \subseteq \overline{K_1}^w + B(\theta, r_1).$$

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence on $\overline{S_1}^w$. So there existe sebsequences $\{y_n\}_{n\in\mathbb{N}}$ and $\{z_n\}_{n\in\mathbb{N}}$ of $\overline{K_1}^w$ and $B(\theta, r_1)$ respectively, such that $x_n = y_n + z_n$. If $r_1 \longrightarrow 0_{\mathbb{R}^n}$, we deduce that $\{x_n\}$ has a weakly convergence subsequence, So $\overline{S_1}^w$ is weakly compact, while the "if" part is trivial.

Q.E.D.

For $n = 1$, we recover the classical De Blasi measure of weak non-compactness.

Definition 2.10. *Let S be a nonempty subset of a generalized Banach space X and $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ is a matrix convergent to zero. If T maps S into X, we say that*
(i) T is M-set contraction with respect to μ if T is bounded and for any bounded subset V in S and $\mu(T(V)) \leq M\mu(V)$,
(ii) T is condensing with μ if T is bounded and $\mu(T(V)) < \mu(V)$ for all bounded subsets V of S with $\mu(V) > 0_{\mathbb{R}^n}$.

Remark 2.11. *If we assume that T is M-set contraction, then T is condensing. Indeed, let V be a bounded subset on S with $\mu(V) > 0_{\mathbb{R}^n}$. We claim that $M\mu(V) < \mu(V)$. If not we obtain $(I - M)\mu(V) \leq 0_{\mathbb{R}^n}$. Since $(I - M)^{-1}$ has nonnegative elements , it follows that $\mu(V) = 0_{\mathbb{R}^n}$. which is a contradiction with $\mu(V) > 0_{\mathbb{R}^n}$.*

By using the concept of a measure of weak noncompactness in vector-valued Banach spaces, we obtain the following fixed point theorem.

Theorem 2.12. *Let S be a nonempty closed convex subset of a generalized Banach space X. Let $T : S \longrightarrow S$ be a weakly seqeuntially continuous operator and condensing with respect to μ. If $T(S)$ is bounded, then T has a fixed point in S.*

Proof. let x_0 be fixed in S and define the set

$$\Sigma = \{K : \;\; x_0 \in K \subseteq S, \;\; K \text{ is closed, convex, bonded and } T(K) \subseteq K\}.$$

Clearly, $\Sigma \neq \emptyset$ since $\overline{co}(T(S) \cup \{x_0\}) \subseteq S$ and we have

$$T(\overline{co}(T(S) \cup \{x_0\}) \subseteq T(S) \subseteq \overline{co}(T(S) \cup \{x_0\}).$$

Which shows that $\overline{co}(T(S) \cup \{x_0\}) \in \Sigma$. If we consider $M = \cap_{K\in\Sigma} K$, then $x_0 \in M \subseteq S, M$ is also a closed convex subset and $T(M) \subseteq M$. Therefore, we have that $M \in \Sigma$. We will prove that M is weakly compact. Denoting by $M_0 = \overline{co}(T(M) \cup x_0)$, we have $M_0 \subseteq M$, which implies that $T(M_0) \subseteq T(M) \subseteq M_0$. Therefore $M_0 \in \Sigma$, $M \subseteq M_0$. Hence $M = M_0$. Since M is weakly closed, it suffices to show that M is relatively weakly compact. If $\mu(M) > 0_{\mathbb{R}^n}$, we obtain

$$\mu(M) = \mu(\overline{co}(T(M) \cup \{x_0\})) < \mu(M)$$

which is a contradiction. Hence, $\mu(M) = 0_{\mathbb{R}^n}$ and so M is relatively weakly compact. Now, T is weakly continuous on M. Consider $X = (X, \sigma(X, (f_i)_{i\in I}))$ the space endowed with the weak topology. Hence, an application of Theorem 2.6 shows that T has at least one fixed point in M. Q.E.D.

Definition 2.13. *[24] A generalized normed algebra X is an algebra endowed with a norm satisfying the following property*

$$\text{for all } x, y \in X \;\; \|x.y\| \leq \|x\|\|y\|,$$

where

$$\|x.y\| = \begin{pmatrix} \|x.y\|_1 \\ \vdots \\ \|x.y\|_n \end{pmatrix}$$

and

$$\|x\|\|y\| = \begin{pmatrix} \|x\|_1\|y\|_1 \\ \vdots \\ \|x\|_n\|y\|_n \end{pmatrix}.$$

A complete generalized normed algebra is called a generalized Banach algebra.

Because the product of two sequentially weakly continuous functions in generalized Banach algaebra is not necessarily sequentially weakly continuous, we will introduce:

Definition 2.14. *we will say that the generalized Banach algebra X satisfies conditions* $(\mathcal{GP})$ *if*

$$(\mathcal{GP})\left\{\begin{array}{l} \text{For any sequences } \{x_n\} \text{ and } \{y_n\} \text{ of } X \text{ such that } x_n \rightharpoonup x \text{ and } y_n \rightharpoonup y, \\ \text{then } x_n \cdot y_n \rightharpoonup x \cdot y;\ \text{here } \rightharpoonup \text{ denotes weak convergence .} \end{array}\right.$$

If X is Banach algebra, we recover the classical sequential condition $(\mathcal{P})$ [[4],Definition 3.1]

Lemma 2.15. *If K and K' are weakly compact subsets of generalized Banach algebra X satisfying the condition $(\mathcal{GP})$, then $K \cdot K'$ is weakly compact.*

Proof. We will show that $K \cdot K'$ is weakly sequentially compact. For that, let $\{z_n\}_n$ be any sequence of $K \cdot K'$. So, there exist sequences $\{x_n\}_n$ and $\{x'_n\}_n$ of K and K' respectively. By hypothesis, there is a renamed subsequence $\{x_n\}_n$ such that $x_n \rightharpoonup x \in K$. Again, there is a renamed subsequence $\{x'_n\}_n$ of K' such that $x'_n \rightharpoonup x' \in K$. This, together with condition $(\mathcal{GP})$ yields that

$$z_n \rightharpoonup z = x \cdot x'.$$

This implies that $K \cdot K'$ is weakly sequentially compact. Hence, an application of the Eberlein-Šmulian's theorem yields that $K \cdot K'$ is weakly compact. Q.E.D.

3. Existence solutions of Equation (1)

In this section, we are prepared to state our first fixed point theorems in generalized Banach algebra in order to found the existence of solutions for the operator equation (1) under weak topology in the case where A, B and C are weakly sequentially continuous operators.

Theorem 3.1. *Let S be a nonempty, closed, and convex subset of a generalized Banach algebra X satisfying the sequential condition $(\mathcal{GP})$. Assume that $A, C : X \longrightarrow X$ and $B : S \longrightarrow X$ are three weakly sequentially continuous operators satisfying the following conditions:*
(i) A and C are contractive with Lipschitz matrix M_A and M_C respectively,
(ii) A is regular on X, (i.e. A maps X into the set of all invertible elements of X),
(iii) $B(S)$ is bounded,
(iv) $x = Ax \cdot By + Cx,\ \ y \in S \Rightarrow x \in S$, and
(v) $\left(\dfrac{I - C}{A}\right)^{-1}$ is weakly compact.
Then, the operator equation (1) has, at least, a solution in S as soon as $\|B(S)\|M_A + M_C \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ is a matrix converging to zero.

Remark 3.2. *If $M_A = (a_{ij})_{1\le i,j\le n},\ \ M_C = (\overline{a}_{ij})_{1\le i,j\le n}$ and $\|B(S)\| = (b_j)_{1\le j\le n}$ then*

$$\|B(S)\|M_A + M_C = \begin{pmatrix} b_1a_{11} + \overline{a}_{11} & \dots & b_na_{1n} + \overline{a}_{1n} \\ \vdots & \ddots & \vdots \\ b_1a_{n1} + \overline{a}_{n1} & \dots & b_na_{nn} + \overline{a}_{nn}, \end{pmatrix}.$$

Proof of Theorem 3.1. It is easy to check that the vector $x \in S$ is a solution for the operator equation $x = Ax \cdot Bx + Cx$, if and only if x is a fixed point for the inverse operator $T := \left(\frac{I-C}{A}\right)^{-1} B$. The use of assumption (i) as well as perov's theorem [29] leads to for each $y \in S$, there is a unique $x_y \in X$ such that $x_y = Ty$. Indeed, let $y \in S$ be arbitrary and let us define the mapping $\varphi_y : X \longrightarrow X$ by the formula

$$\varphi_y(x) = Ax \cdot By + Cx.$$

Notice that φ_y is contractive with a Lipschitz matrix $\|B(S)\|M_A + M_C$. Applying Perov's theorem [29], we obtain that φ_y has a unique fixed point in S, say x_y. From assumption (ii), it follows that the operator $\left(\frac{I-C}{A}\right)^{-1}$ is well defined on $B(S)$.
Let $\{x_n,\ n \in \mathbb{N}\}$ be a weakly convergent sequence of S to a point $x \in S$ and let $y_n = Tx_n$. Then the relation $y_n = Ay_n \cdot Bx_n + Cy_n$ holds and, therefore $\{y_n,\ n \in \mathbb{N}\} \subset S$ in view of assumption (iv). Since $T(S)$ is relatively weakly compact, there is a subsequence (x_{n_k}) of $\{x_n,\ n \in \mathbb{N}\}$ such that $y_{n_k} = Tx_{n_k} \rightharpoonup y$, for some $y \in X$. The weak sequential closedness of S gives $y \in S$. Making use of the condition $(\mathcal{GP})$, together with the assumptions on A, B and C, enables us to have

$$y_{n_k} = Ay_{n_k} \cdot Bx_{n_k} + Cy_{n_k} \rightharpoonup Ay \cdot Bx + Cy.$$

This implies that $y = Ay \cdot Bx + Cy$. Consequently $Tx_{n_k} \rightharpoonup Tx$ in light of assumption (ii).
Now a standard argument shows that $Tx_n \rightharpoonup Tx$. Suppose the contrary, then there exists a weakly neighborhood V^w of Tx and a subsequence $(x_{n_j})_j$ of $\{x_n,\ n \in \mathbb{N}\}$ such that $Tx_{n_j} \notin V^w$, for all $j \geq 1$. Arguing as before, we may extract a subsequence $(x_{n_{j_k}})_{k\in\mathbb{N}}$ of $\{x_{n_j},\ j \in \mathbb{N}\}$ verifying $Tx_{n_{j_k}} \rightharpoonup y$, which is a contradiction and consequently T is weakly sequentially continuous. Hence, T has, at least, one fixed point x in S in view of Theorem 2.7. Q.E.D.
An interesting fixed point result of Theorem 3.1 is

Corollary 3.3. *Let S be a nonempty, closed, and convex subset of a generalized Banach algebra X satisfying the sequential condition $(\mathcal{GP})$. Assume that $A, C : X \longrightarrow X$ and $B : S \longrightarrow X$ are three weakly sequentially continuous operators satisfying the following conditions:*
(i) *A and C are contractive with Lipschitz matrix M_A and M_C respectively,*
(ii) *A is regular on X,*
(iii) *$A(S), B(S)$ and $C(S)$ are relatively weakly compact,*
(iv) $x = Ax{\cdot}By + Cx, \quad y \in S \Rightarrow x \in S.$
Then, the operator equation (1) has, at least, a solution in S as soon as $\|B(S)\|M_A + M_C$ is a matrix converging to zero.

Theorem 3.4. *Let S be a nonempty, bounded, closed, and convex subset of a generalized Banach algebra X satisfying the sequential condition $(\mathcal{GP})$. Assume that $A, C : X \longrightarrow X$ and $B : S \longrightarrow X$ are three weakly sequentially continuous operators satisfying the following conditions:*
(i) *A and C are contractive with Lipschitz matrix M_A and M_C respectively,*
(ii) *A is regular on X,*
(iii) *B is weakly compact,*
(iv) $x = Ax \cdot By + Cx, \quad y \in S \Rightarrow x \in S.$
Then, the operator equation (1) has, at least, a solution in S as soon as $\|B(S)\|M_A + M_C$ is a matrix converging to zero.

Before proving the theorem, we need to establish two lemmas.

Lemma 3.5. *Let X be a generalized Banach algebra. If $S \in \mathcal{B}(X)$ and $K \in \mathcal{W}(X)$, then $\mu(S \cdot K) \leq \|K\|\mu(S)$.*

Proof. Assume that $\|K\| > 0_{\mathbb{R}^n}$. By the definition of μ, there exists a subset K' of $\mathcal{W}(X)$ and $r \in \mathbb{R}^n_+ \setminus \{0_{\mathbb{R}^n}\}$ such that

$$S \cdot K \subseteq K' \cdot K + B\left(\theta, \mu(S) + r\|K^{-1}\|\right) \cdot K,$$

where

$$r\|K^{-1}\| := \begin{pmatrix} \frac{r_1}{\|K\|_1} \\ \vdots \\ \frac{r_n}{\|K\|_n} \end{pmatrix}.$$

So

$$S \cdot K \subseteq K' \cdot K + B\left(\theta, \|K\|\mu(S) + r\right).$$

Keeping in mind the subadditivity of the measure of weak noncompactness and using lemma 2.15, we get

$$\begin{array}{rcl} \mu(S \cdot K) & \leq & \mu\left(K \cdot K'\right) + \mu\left(B\left(0, \|K\|\mu(S) + r\right)\right) \\ & \leq & \|K\|\mu(S) + r. \end{array}$$

Since ε is arbitrary, we deduce that

$$\mu(S \cdot K) \leq \|K\|\mu(S).$$

Q.E.D.

Lemma 3.6. *Let X be a generalized Banach algebra. Assume that $T : X \longrightarrow X$ is weakly sequentially continuous. If T is contractive with Lipschitz matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$, then for any bounded subset S of X, one has*

$$\mu\left(T(S)\right) \leq M\mu\left(S\right).$$

Proof. Let S be a bounded subset of X and $\varepsilon \in \mathbb{R}^n$ such that $\varepsilon > \mu(S)$ and let $M = (a_{ij})_{1\leq i,j\leq n}$. By the definition of μ, we have there exists $r \in \mathbb{R}^n$ and a weakly compact subset K of X such that $0 < r < \varepsilon$ and $S \subseteq K + B(\theta, r)$. Let $y \in T\left(K + B(\theta, r)\right)$, then there exists $x \in K + B(\theta, r)$ such that $y = Tx$. Since $x \in K + B(\theta, r)$, there are $k \in K$ and $b \in B\left(\theta, r\right))$ such that $x = k + b$, and so

$$\|y - Tk\|_i = \|Tx - Tk\|_i \leq \sum_{j=1}^{n} a_{ij}\|x - k\|_j = \sum_{j=1}^{n} a_{ij}\|b\|_j \leq \sum_{j=1}^{n} a_{ij}r_j.$$

This means that

$$\|y - Tk\| \leq \begin{pmatrix} \sum_{j=1}^{n} a_{1j}r_j \\ \sum_{j=1}^{n} a_{2j}r_j \\ \vdots \\ \sum_{j=1}^{n} a_{nj}r_j \end{pmatrix} = Mr.$$

That is, $y \in TK + B(\theta, Mr)$ and consequently $TS \subset TK + B(\theta, Mr)$. Moreover, since T is sequentially weakly continuous we have $\overline{TK}^{w} \in \mathcal{W}(X)$. Accordingly,

$$\mu(TS) \leq Mr.$$

Letting $\varepsilon \to \mu(S)$ in the above inequality, we get $\mu(TS) \leq M\mu(S)$. Q.E.D.

Proof of Theorem 3.4. Following the same procedures as in the proof of Theorem 3.1, it can be proved that the inverse operator $T := \left(\frac{I - C}{A}\right)^{-1} B$ exists on S. Now, we claim that $T(S)$ is a relatively compact subset of

X. If this is not the case, then $\mu(TS) > 0_{\mathbb{R}^n}$. Keeping in mind the subadditivity of the De Blasi's measure of weak noncompactness and using the equality:

$$T = AT \cdot B + CT \tag{3}$$

we obtain

$$\mu(T(S)) \leq \mu\left(AT(S) \cdot B(S)\right) + \mu\left(CT(S)\right).$$

The use of Lemma 3.5 as well as Lemma 3.6 leads to

$$\mu(TS) \leq \left(\|B(S)\| M_A + M_C\right) \mu(TS).$$

Since $\|B(S)\| M_A + M_C$ is a matrix converging to zero, we get a contradiction and consequently the claim is approved. Q.E.D.

As easy consequences of Theorem 3.4 we obtain the following result.

Corollary 3.7. *Let S be a nonempty, bounded, closed and convex subset of a generalized Banach algebra X satisfying the sequential condition $(\mathcal{GP})$. Assume that $A, C : X \longrightarrow X$ and $B : S \longrightarrow X$ are three weakly sequentially continuous operators satisfying the following conditions:*
(i) A is regular and is contractive with a Lipschitz matrix M,
(ii) B and C are weakly compact,
(iii) $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(S)$,
(iv) $x = Ax \cdot By + Cx, \quad y \in S \Rightarrow x \in S$.
Then, the operator equation (1) has, at least, one solution in S as soon as $\|B(S)\|M$ is a matrix converging to zero.

In the following result, we will consider that A and B are weakly compact operators.

Theorem 3.8. *Let S be a nonempty, bounded, closed and convex subset of a generalized Banach algebra X satisfying the condition $(\mathcal{GP})$. Assume that $A, B, C : S \longrightarrow X$ are three weakly sequentially continuous operators satisfying the following conditions:*
(i) A is regular,
(ii) A and B are weakly compact,
(iii) If $(I - C)x_n \rightharpoonup y$, then there exists a weakly convergent subsequence of $(x_n)_n$,
(iv) $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(S)$, and
(v) $x = Ax \cdot By + Cx \in S, \quad y \in S \Rightarrow x \in S$.
Then, the operator equation (1) has, at least, one solution in S.

Proof. Let $\{y_n,\ n \in \mathbb{N}\} \subset T(S)$, there is a sequence $\{x_n,\ n \in \mathbb{N}\} \subset S$ such that

$$y_n = Tx_n = \left(\frac{I-C}{A}\right)^{-1} Bx_n.$$

Or equivalently, $y_n = Ay_n \cdot Bx_n + Cy_n$. Taking into account the weak compactness of the weak closure of $A(S)$ and $B(S)$, we infer that

$$ATx_{n_k} \rightharpoonup x \qquad \text{and} \qquad Bx_{n_{k_j}} \rightharpoonup x', \text{ for some } x, x' \in S,$$

where (x_{n_k}) is a subsequence of $\{x_n,\ n \in \mathbb{N}\}$ and $(x_{n_{k_j}})$ is a subsequence of $\{x_{n_k},\ k > n\}$. Using the condition $(\mathcal{GP})$, we get

$$Ay_{n_{k_j}} \cdot Bx_{n_{k_j}} = (I - C)y_{n_{k_j}} \rightharpoonup x \cdot x' \quad \text{in } X.$$

Based on assumption (iii), it follows that there exists a weakly convergent subsequence of $(y_{n_{k_j}})$ and consequently $T(S)$ is a relatively weakly compact subset of X. The use of Theorem 3.1 achieves the proof. Q.E.D.

Remark 3.9. *If we assume that C is sequentially weakly continuous and contractive with a Lipschitz Matrix M, then C satisfies the condition (iii) of Theorem 3.8. In fact, let $\{x_n,\ n \in \mathbb{N}\}$ be a sequence in S such that $(I-C)x_n \rightharpoonup y$, for some $y \in X$. Based on the subadditivity of the De Blasi's measure of weak non-compactness it is shown that*

$$\beta(\{x_n,\ n \in \mathbb{N}\}) \leq \beta(\{(I-C)x_n,\ n \in \mathbb{N}\}) + \beta(\{Cx_n,\ n \in \mathbb{N}\}).$$

If we consider the weak compactness of the weak closure of $\{(I-C)x_n,\ n \in \mathbb{N}\}$ and deploy Lemma 3.6, we get

$$\beta(\{x_n,\ n \in \mathbb{N}\}) \leq M\beta(\{x_n,\ n \in \mathbb{N}\}).$$

Since M is a matrix converging to zero, then there is a weakly convergent subsequence of $\{x_n,\ n \in \mathbb{N}\}$.

A consequence of Theorem 3.8 is

Corollary 3.10. *Let S be a nonempty, bounded, closed and convex subset of a generalized Banach algebra X satisfying the condition $(\mathcal{GP})$. Assume that $A, B, C : S \longrightarrow X$ are three weakly sequentially continuous operators satisfying the following conditions:*
(i) A is regular and B is weakly compact,
(ii) If $\left(\frac{I-C}{A}\right)x_n \rightharpoonup y$, then there exists a weakly convergent subsequence of $(x_n)_n$,
(iii) $\left(\frac{I-C}{A}\right)^{-1}$ exists and $\left(\frac{I-C}{A}\right)^{-1} B(S) \subseteq S$.
Then, the operator equation (1) has, at least, one solution in S.

Let us study the case where the operators B and $\left(I - \frac{I-C}{A}\right)$ are M_1-μ-contraction and M_2-μ-contraction respectively.

Theorem 3.11. *Let S be a nonempty, bounded, closed and convex subset of a generalized Banach algebra X. Assume that $A, C : X \longrightarrow X$ and $B : S \longrightarrow X$ are three operators satisfying the following conditions:*
(i) A is regular,
(ii) B and $\left(I - \frac{I-C}{A}\right)$ are M_1-μ-contraction and M_2-μ-contraction respectively,
(iii) B and $\left(\frac{I-C}{A}\right)$ are weakly sequentially continuous,
(iv) $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(S)$, and
(v) $x = Ax \cdot By + Cx,\ \ y \in S \Rightarrow x \in S$.
Then, the operator equation (1) has, at least, one solution in S as soon as $(I-M_2)^{-1}M_1$ is a matrix converging to zero.

Proof. It is easy to see that the operator $T := \left(\frac{I-C}{A}\right)^{-1} B : S \longrightarrow S$ is well defined. Let $\{x_n,\ \ n \in \mathbb{N}\}$ be a weakly convergent sequence of S to a point $x \in S$. Keeping in mind the weak sequential continuity of the operator B and using the equality

$$T = B + \left(I - \frac{I-C}{A}\right)T, \tag{4}$$

we obtain

$$\begin{aligned}
\mu\left(\{Tx_n,\ \ n \in \mathbb{N}\}\right) &\leq \mu\left(Bx_n,\ \ n \in \mathbb{N}\right) + \mu\left(\left(I - \frac{I-C}{A}\right)(\{Tx_n,\ \ n \in \mathbb{N}\})\right) \\
&\leq M_2\mu\left(\{Tx_n,\ \ n \in \mathbb{N}\}\right).
\end{aligned}$$

This inequality means, in particular, that $\{Tx_n,\ \ n \in \mathbb{N}\}$ is relatively weakly compact. Consequently, there is a subsequence $(x_{n_k})_k$ of $\{Tx_n,\ \ n \in \mathbb{N}\}$ such that $Tx_{n_k} \rightharpoonup y$, fore some $y \in S$. Making use of equality (4), together with the assumptions on B and $\left(\frac{I-C}{A}\right)$; enables us to have $y = Tx$. Now a standard argument shows that $Tx_n \rightharpoonup Tx$. Suppose the contrary, then there exists a weakly neighborhood V^w of Tx and a subsequence $(x_{n_j})_j$ of $\{x_n,\ n \in \mathbb{N}\}$ such that $Tx_{n_j} \notin V^w$, for all $j \geq 1$. Arguing as before, we may extract a subsequence $(x_{n_{j_k}})_{k\in\mathbb{N}}$ of $\{x_{n_j},\ j \in \mathbb{N}\}$ verifying $Tx_{n_{j_k}} \rightharpoonup y$, which is a contradiction and consequently T is sequentially weakly continuous. Next, T is μ-condensing. In fact, let V be a bounded subset of S with $\mu(V) > 0$. Using the subadditivity of the De Blasi's measure of weak noncompactness, we get

$$\begin{aligned} \mu\left(T(V)\right) &\leq \mu\left(B(V)\right) + \mu\left(\left(I - \frac{I-C}{A}\right)T(V)\right) \\ &\leq M_1\mu(V) + M_2\mu(T(V)). \end{aligned}$$

This implies that

$$\mu\left(T(V)\right) \leq (I - M_2)^{-1}M_1\mu\left(V\right).$$

Hence, T has, at least, one fixed point x in S in view of Theorem 2.12 . Q.E.D.
If we take $A = 1_X$ in the above result, where 1_X is the unit element of the generalized Banach algebra X, we obtain the following Corollary.

Corollary 3.12. *Let S be a nonempty, bounded, closed and convex subset of a generalized Banach algebra X. Assume that $C : X \longrightarrow X$ and $B : S \longrightarrow X$ are two sequentially weakly continuous operators satisfying the following conditions:*
(i) B and C are M_1-μ-contraction and M_2-μ-contraction respectively,
(ii) $(I - C)^{-1}$ exists on $B(S)$,
(iii) $x = By + Cx,\ \ y \in S \Rightarrow x \in S$.
Then, the operator equation (1) has, at least, one solution in S as soon as $(I - M_2)^{-1}M_1$ is a matrix converging to zero.

Remark 3.13. *Note that condition (iii) in Theorem 3.11 may be replaced by "A, B and C are weakly sequentially continuous", but the generalized Banach algebra must satisfy condition $(\mathcal{GP})$. Now, we can study the folllwing result.*

Theorem 3.14. *Let S be a nonempty, bounded, closed, and convex subset of a generalized Banach algebra X satisfying the sequential condition $(\mathcal{GP})$. Assume that $A, C : X \longrightarrow X$ and $B : S \longrightarrow X$ are three weakly sequentially continuous operators satisfying the following conditions:*
(i) A is regular and weakly compact,
(ii) B and C are M_1-μ-contraction and M_2-μ-contraction respectively,
(iii) $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(S)$, and
(iv) $x = Ax \cdot By + Cx,\ \ y \in S \Rightarrow x \in S$.
Then, the operator equation (1) has, at least, one solution in S as soon as $(I - M_2)^{-1}\|A(S)\|M_1$ is a matrix converging to zero.

4. Integral Equations of Fractional Order

Let $(X, \|\cdot\|)$ be a reflexive Banach and let $C(J, X)$ be the Banach algebra of all X-valued continuous functions defined on $J = [0, 1]$, endowed with the norm $\|f\|_\infty = \sup_{t\in J}\|f(t)\|$. We will use Theorem 3.4 to examine the existence of solutions to the coupled system of quadratic integral equations of fractional order (2). We need the following definition and proposition in the sequel.

Definition 4.1. *[1] Let $f : [0, T] \to X$ be a function. The fractional Pettis integral of the function f of order $\alpha \in \mathbb{R}_+$ is defined by*

$$I^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds,$$

where the sign "$\int$" denotes the Pettis integral.

Proposition 4.2. *[1] If $f : [0,T] \to X$ is Riemann integrable on $[0,T]$, then $I^\alpha f$ exists on $[0,T]$ and fractional Pettis integral.*

Let us now introduce the following assumptions:
$(\mathcal{H}_0)$ The function $f_i : J \times X \times X \longrightarrow X, i = 1,2$ is such that:
(a) The partial function $x \longrightarrow f_1(t,x,y)$ is regular on X
(b) The partial function $y \longrightarrow f_2(t,x,y)$ is regular on X
(c) The partial function $t \mapsto f_i(t,x,y)$ is continuous,
(d) The partial function $(x,y) \mapsto f_i(t,x,y)$ is weakly sequentially continuous,
(e) There is nonnegative real numbers a_{i1} and a_{i2}, $i = 1,2$ such that

$$\|f_i(t,x,y) - f_i(t,\tilde{x},\tilde{y})\| \leq a_{i1}\|x-\tilde{x}\| + a_{i2}\|y-\tilde{y}\|.$$

$(\mathcal{H}_1)$ The function $g_i : J \times X \times X \longrightarrow X, i = 1,2$ is such that:
(a)The partial function $t \mapsto g_i(t,x,y)$ is Riemann integrable,
(b) The partial function $(x,y) \mapsto g_i(t,x,y)$ is weakly sequentially continuous.

$(\mathcal{H}_2)$ The function $h_i^k : J \times X \times X \longrightarrow X, k = 1,\dots,m$ is such that:
(a)The partial function $t \mapsto h_i^k(t,x,y)$ is Riemann integrable,
(b) The partial function $(x,y) \mapsto h_i^k(t,x,y)$ is weakly sequentially continuous,
(c) There is nonnegative real numbers b_{i1}^k and b_{i2}^k such that

$$\|h_i^k(t,x,y) - h_i^k(t,\tilde{x},\tilde{y})\| \leq b_{i1}^k\|x-\tilde{x}\| + b_{i2}^k\|y-\tilde{y}\|.$$

Theorem 4.3. *Suppose that the assumptions $(\mathcal{H}_0) - (\mathcal{H}_3)$ are satisfied. Moreover, assume that there exists a real number $r_0 > 0$ and $P \in \mathbb{R}_+^*$ such that*

$$\begin{cases} \|g_i(t,s,x,y)\| \leq P \quad \textit{with } \|x\| \leq r_0 \textit{ and } \|y\| \leq r_0 \\ \rho\Big[P\Big(\frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{T^{\alpha_2}}{\Gamma(\alpha_2+1)}\Big) + \sum_{k=1}^m \Big(\frac{T^{\beta_1^k}}{\Gamma(\beta_1^k+1)} + \frac{T^{\beta_2^k}}{\Gamma(\beta_2^k+1)}\Big)\Big] < 1, \textit{ and} \\ M_A, M_C \textit{ and } \|B(S)\|M_A + M_C \textit{ are three matrices converging to zero, where} \end{cases} \tag{5}$$

$$\rho = \max\{a_{11}, a_{12}, a_{21}, a_{22}, b_{11}^k, b_{12}^k, b_{21}^K, b_{21}^k, b_{22}^k;\ \ k = 1,\dots,m\},$$

$$M_A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},\ M_C = \begin{pmatrix} \sum_{k=1}^m \frac{T^{\beta_1^k}}{\Gamma(\beta_1^k+1)} b_{11}^k & \sum_{k=1}^m \frac{T^{\beta_1^k}}{\Gamma(\beta_1^k+1)} b_{12}^k \\ \sum_{k=1}^m \frac{T^{\beta_2^k}}{\Gamma(\beta_2^k+1)} b_{21}^k & \sum_{k=1}^m \frac{T^{\beta_2^k}}{\Gamma(\beta_2^k+1)} b_{22}^k \end{pmatrix} \text{ and}$$

$$\|B(S)\|M_A + M_C = \begin{pmatrix} \dfrac{PT^{\alpha_1}}{\Gamma(\alpha_1+1)} a_{11} + \sum_{k=1}^m \frac{T^{\beta_1^k}}{\Gamma(\beta_1^k+1)} b_{11}^k & \dfrac{PT^{\alpha_2}}{\Gamma(\alpha_2+1)} a_{12} + \sum_{k=1}^m \frac{T^{\beta_2^k}}{\Gamma(\beta_2^k+1)} b_{12}^k \\ \dfrac{PT^{\alpha_1}}{\Gamma(\alpha_1+1)} a_{21} + \sum_{k=1}^m \frac{T^{\beta_1^k}}{\Gamma(\beta_1^k+1)} b_{21}^k & \dfrac{PT^{\alpha_2}}{\Gamma(\alpha_2+1)} a_{22} + \sum_{k=1}^m \frac{T^{\beta_2^k}}{\Gamma(\beta_2^k+1)} b_{22}^k \end{pmatrix}.$$

Then the problem (2) *has a solution in* $C(J,X) \times C(J,X)$.

Proof. We recall that the problem (2) is equivalent to the operator equation

$$(x,y) = (A_1(x,y), A_2(x,y))(B_1(x,y), B_2(x,y)) + (C_1(x,y), C_2(x,y)),$$

where the operators A_i, B_i and C_i, $i = 1,2$ are defined by

$$\begin{cases} A_i(x,y)(t) = f_i(t,x(t),y(t)) \\ B_i(x,y)(t) = \int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_i(s,x(s),y(s))ds \\ C_i(x,y)(t) = \sum_{k=1}^m I^{\beta_1^i} h_i^k(t,x(t),y(t)). \end{cases}$$

Let us define the subset S of $C(J,X)\times C(J,X)$ by

$$S=\left\{(x,y)\in C(J,X)\times C(J,X),\ \ \|(x,y)\|=\begin{pmatrix}\|x\|_\infty\\ \|y\|_\infty\end{pmatrix}\leq\begin{pmatrix}r_0\\ r_0\end{pmatrix}\right\},$$

where

$$r_0\geq\frac{F_0P\Big(\frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)}+\frac{T^{\alpha_2}}{\Gamma(\alpha_2+1)}\Big)+H_0\sum_{k=1}^m\Big(\frac{T^{\beta_1^k}}{\Gamma(\beta_1^k+1)}+\frac{T^{\beta_2^k}}{\Gamma(\beta_2^k+1)}\Big)}{1-\rho\Big[P\Big(\frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)}+\frac{T^{\alpha_2}}{\Gamma(\alpha_2+1)}\Big)+\sum_{k=1}^m\Big(\frac{T^{\beta_1^k}}{\Gamma(\beta_1^k+1)}+\frac{T^{\beta_2^k}}{\Gamma(\beta_2^k+1)}\Big)\Big]},$$

with

$$\begin{cases}F_0=\max\left\{\|f_1(\cdot,0,0)\|,\|f_2(\cdot,0,0)\|\right\}\\ H_0=\max\left\{\|h_1^k(\cdot,0,0)\|,\|h_2^k(\cdot,0,0)\|;\ \ k=1,\ldots,m\right\}\end{cases}$$

Our strategy is to apply Theorem 3.4 to prove the existence of a fixed point for the nonlinear equation (2) in S. Then, we need to verify the following steps:

Claim 1: We start by showing that the operators $A,C:C(J,X)\times C(J,X)\longrightarrow C(J,X)\times C(J,X)$ and $B:S\longrightarrow C(J,X)\times C(J,X)$ are weakly sequentially continuous. Firstly, we verify that the operator $A_i(x,y),$ for $i=1,2$ is continuous on J for all $(x,y)\in C(J,X)\times C(J,X)$. To see this, let $\{t_n,\ \ n\in\mathbb{N}\}$ be any sequence in J converging to a point t in J. Then,

$$\begin{aligned}\|A_i(x,y)(t_n)-A_i(x,y)(t)\| &= \|f_i(t_n,x(t_n),y(t_n))-f_i(t,x(t),y(t))\|\\ &\leq a_{i1}\|x(t_n)-x(t)\|+a_{i2}\|y(t_n)-y(t)\|\\ &+ \|f_i(t_n,x(t),y(t)-f_i(t,x(t),y(t)\|.\end{aligned}$$

The continuity of x,y and $t\mapsto f_i(t,x,y)$ on $[0,1]$ implies that the function $A_i(x,y)$ is continuous. Let $\{(x_n,y_n),n\in\mathbb{N}\}$ be a weakly convergent sequence of $C(J,X)\times C(J,X)$ to a point (x,y). In this case, the set $\{(x_n,y_n),n\in\mathbb{N}\}$ is bounded and so, we can apply the Dobrakov's theorem [12] in order to get

$$(x_n(t),y_n(t))\rightharpoonup(x(t),y(t))\text{ in }X\times X.$$

Based on assumption $(\mathcal{H}_0)(d)$, it is shown that $A_i(x_n,y_n)(t)\rightharpoonup A_i(x,y)(t)$ and then, we can again apply the Dobrakov's theorem to obtain the weak sequential continuity of the operator A. Besides, the use of assumption $(\mathcal{H}_1)(b)$ and assumption $(\mathcal{H}_2)(b)$ as well as the Dobrakov's theorem [12, page 36] leads to the two maps B and C are weakly sequentially continuous.

Claim 2: The operators A and C are contractive. The claim regarding the operator A is immediate, from assumption $(\mathcal{H}_0)(e)$. Let us fix arbitrary $(x,y),(\tilde{x},\tilde{y})\in C(J,X)\times C(J,X)$. If we take an arbitrary $t\in[0,1]$, then we get

$$\begin{aligned}\|C_i(x,y)(t)-C_i(\tilde{x},\tilde{y})(t)\| &= \left\|\sum_{k=1}^m I^{\beta_i^k}h_i^k(t,x(t),y(t)-\sum_{k=1}^m I^{\beta_i^k}h_i^k(t,\tilde{x}(t),\tilde{y}(t))\right\|\\ &\leq \sum_{k=1}^m I^{\beta_i^k}\|h_i^k(t,x(t),y(t)-h_i^k(t,\tilde{x}(t),\tilde{y}(t))\|\\ &\leq \sum_{k=1}^m I^{\beta_i^k}\left(b_{i1}^k\|x(t)-\tilde{x}(t)\|+b_{i1}^k\|y(t)-\tilde{y}(t)\|\right)\\ &\leq \sum_{k=1}^m\frac{T^{\beta_i^k}}{\Gamma(\beta_i^k+1)}\left(b_{i1}^k\|x(t)-\tilde{x}(t)\|+b_{i1}^k\|y(t)-\tilde{y}(t)\|\right).\end{aligned}$$

This implies that $\left\|C(x,y)-C(\tilde{x},\tilde{y})\right\| \leq M_C \left\|(x,y)-(\tilde{x},\tilde{y})\right\|$, where

$$M_C = \begin{pmatrix} \sum_{k=1}^{m} \frac{T^{\beta_1^k}}{\Gamma(\beta_1^k+1)} b_{11}^k & \sum_{k=1}^{m} \frac{T^{\beta_1^k}}{\Gamma(\beta_1^k+1)} b_{12}^k \\ \sum_{k=1}^{m} \frac{T^{\beta_2^k}}{\Gamma(\beta_2^k+1)} b_{21}^k & \sum_{k=1}^{m} \frac{T^{\beta_2^k}}{\Gamma(\beta_2^k+1)} b_{22}^k \end{pmatrix}.$$

Claim 3: Let $\varepsilon > 0$, $(x,y) \in S$ and $t, t' \in J$ such that $|t'-t| < \varepsilon$. From the Hahn-Banach theorem there exists a linear function $\phi \in X^*$ with $\|\phi\| = 1$ such that

$$\left\|B_i(x,y)(t') - B_i(x,y)(t)\right\| = \phi\left(B_i(x,y)(t) - B_i(x,y)(t)\right), \quad i = 1, 2.$$

Based on the first inequality in (5) it is shown that

$$\begin{aligned} \left\|B_i(x,y)(t') - B_i(x,y)(t)\right\| &= \phi\left(\int_0^{t'} \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_j)} g_i(s,x(s),y(s))ds - \int_0^{t} \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} g_i(s,x(s),y(s))\right) \\ &\leq \int_0^{t} \frac{|(t'-s)^{\alpha_i-1} - (t-s)^{\alpha_i-1}|}{\Gamma(\alpha_i)} \phi(g_i(s,x(s),y(s)))ds \\ &+ \int_t^{t'} \frac{(t'-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} \phi(g_i(s,x(s),y(s)))ds \\ &\leq P\int_0^{t} \frac{|(t'-s)^{\alpha_i-1} - (t-s)^{\alpha_i-1}|}{\Gamma(\alpha_i)} ds + P\int_t^{t'} \frac{(t'-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} ds. \end{aligned}$$

This implies that $\left\|B_i(x,y)(t') - B_i(x,y)(t)\right\| \to 0$ as $\varepsilon \to 0$, and consequently $B(S)$ is a weakly equi-continuous subset. Let now $\{(x_n,y_n), n \in \mathbb{N}\}$ be any sequence in S. From the first inequality in (5), it follows that

$$\begin{aligned} \left\|B_i(t,x_n(t),y_n(t))\right\| &\leq \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} \left\|g_i(s,x_n(s),y_n(s))\right\| ds \\ &\leq \frac{PT^{\alpha_i}}{\Gamma(\alpha_i+1)}, \end{aligned}$$

for all $t \in [0,1]$. This demonstrate that $\{B_i(x_n,y_n),\ n \in \mathbb{N}\}$ is a uniformly bounded sequence in $B(S)$ and so, $B(S)(t)$ is sequentially relatively weakly compact. Hence, $B(S)$ is sequentially relatively weakly compact in light of the Arzelà-Ascoli's theorem [33]. An application of Eberlein-Šmulian's theorem [8] yields that $B(S)$ is relatively weakly compact.

Claim 4: The operators A, B and C satisfy assumption (iv) of Theorem 3.4. To see this, let $(x,y) \in C(J,X) \times C(J,X)$ and $(u,v) \in S$ with $(x,y) = A(x,y) \cdot B(u,v) + C(x,y)$. We shall show that $(x,y) \in S$. For all $t \in [0,1]$, we have

$$\begin{aligned} \|x(t)\| &\leq \left\|f_1(t,x(t),y(t))\right\| \int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} \left\|g_1(s,u(s),v(s))\right\| ds + \sum_{i=1}^{m} I^{\beta_1^i} \|h_1^i(t,x(t),y(t))\| \\ &\leq \left[\rho(\|x\|_\infty + \|y\|_\infty) + F_0\right] \frac{PT^{\alpha_1}}{\Gamma(\alpha_1+1)} + \sum_{k=1}^{m} \frac{T^{\beta_1^k}}{\Gamma(\beta_1^k+1)} \left[\rho(\|x\|_\infty + \|y\|_\infty) + H_0\right] \end{aligned}$$

This implies that

$$\begin{aligned} \|x(t)\| + \|y(t)\| &\leq \Big[\rho(\|x\|_\infty + \|y\|_\infty) + F_0\Big] P\left(\tfrac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} + \tfrac{T^{\alpha_2}}{\Gamma(\alpha_2+1)}\right) \\ &+ \textstyle\sum_{k=1}^{m} \left(\tfrac{T^{\beta_1^k}}{\Gamma(\beta_1^k+1)} + \tfrac{T^{\beta_2^k}}{\Gamma(\beta_2^k+1)}\right) \Big[\rho(\|x\|_\infty + \|y\|_\infty) + H_0\Big] \end{aligned}$$

Consequently,

$$\|x\|_\infty \le r_0 \text{ and } \|y\|_\infty \le r_0.$$

To end the proof, we apply Theorem (4.3), we deduce that the problem (2) has, at least, one solution in $C(J,X)\times C(J,X)$.

Example : Let $C(J,\mathbb{R})$ be the Banach algebra of all continuous functions from J to $\mathbb{R}$ endowed with the sup-norm $\|\cdot\|$ defined by $\|x\|_\infty = \sup_{0\le t\le T} |x(t)|$, for each $x \in C(J,\mathbb{R})$.

$$\begin{cases} D^{\frac{1}{2}}\Big[\frac{x(t)-\sum_{k=1}^{2} I^{\beta_1^k} h_1^k(t,x(t),y(t))}{f_1(t,x(t),y(t))}\Big] = \frac{3}{35(13-t^2)}(7|x(t)|+15|y(t)|), \\ D^{\frac{1}{2}}\Big[\frac{y(t)-\sum_{k=1}^{2} I^{\beta_2^k} h_2^k(t,x(t),y(t))}{f_2(t,x(t),y(t))}\Big] = \frac{3}{35(13-t^2)}(7|x(t)|+15|y(t)|), \\ x(0)=y(0)=0, \qquad\qquad t\in[0,1]. \end{cases} \tag{6}$$

Note that this problem may be transformed into the fixed point problem (2) in view of lemma 2.5 in [23], where

$$\sum_{k=1}^{2} I^{\beta_1^k} h_1^k(t,x(t),y(t)) = I^{1/3}\frac{2te^{-3t}}{15(3+t)}\left(\frac{x^2(t)+9|x(t)|}{|x(t)|+5}+\frac{12e^{3t}}{5}\right)+I^{10/3}\frac{2t\sin\pi t}{14+t^2}\left(\frac{x^2(t)+5|x(t)|}{|x(t)|+8}+\frac{1}{3}\right),$$

$$\sum_{k=1}^{2} I^{\beta_2^k} h_2^k(t,x(t),y(t)) = I^{7/4}\frac{t\sin t}{7(4+e^t)}\left(\frac{y^2(t)+4|y(t)|}{|y(t)|+3}+\cos t\right)+I^{29/6}\frac{3t\cos t}{10(4-t^2)}\left(\frac{y^2(t)+5|y(t)|}{|y(t)|+4}+\frac{t}{t+2}\right),$$

and

$$f_1(t,x(t),y(t)) = \frac{3cos\pi t+2t}{5(2+10t^2)(|x(t)|+3)},\ f_2(t,x(t),y(t)) = \frac{4cos\pi t+3t}{7(3+8t^2)(|y(t)|+6)},$$

and

$$g_1(t,x(t),y(t)) = g_2(t,x(t),y(t)) = \frac{3}{35(13-t^2)}(7|x(t)|+15|y(t)|,$$

here $\alpha_1=\alpha_2=\dfrac{1}{2}, T=1, m=2, \beta_1^1=\dfrac{1}{3}, \beta_1^2=\dfrac{10}{3}, \beta_2^1=\dfrac{7}{4}$ and $\beta_2^2=\dfrac{29}{6}$.

We can show that

$$|f_1(t,x,y)-f_1(t,\tilde{x},\tilde{y})| \le \frac{3+2t}{5(2+10t^2)}|x-\tilde{x}|,$$

$$|f_2(t,x,y)-f_2(t,\tilde{x},\tilde{y})| \le \frac{4+3t}{7(1+5t^2)}|y-\tilde{y}|,$$

and

$$|h_1^1(t,x,y)-h_1^1(t,\tilde{x},\tilde{y})| \le \frac{18t}{75(3+t)}|x-\tilde{x}|,$$

$$|h_1^2(t,x,y)-h_1^2(t,\tilde{x},\tilde{y})| \le \frac{10t}{8(14+t^2)}|x-\tilde{x}|,$$

$$|h_2^1(t,x,y)-h_2^1(t,\tilde{x},\tilde{y})| \le \frac{4t}{21(4+e^t)}|y-\tilde{y}|,$$

$$|h_2^2(t,x,y)-h_2^2(t,x,\tilde{y})| \le \frac{3t}{8(4-t^2)}|y-\tilde{y}|.$$

It follows that $a_{11} = \dfrac{1}{12}$, $a_{21} = 0$, $a_{12} = 0$, $a_{22} = \dfrac{1}{6}$, $b^1_{11} = \dfrac{3}{50}$, $b^1_{12} = 0$, $b^1_{21} = 0$, $b^1_{22} = \dfrac{4}{21(4+e)}$, $b^2_{11} = \dfrac{1}{12}$, $b^2_{12} = 0$, $b^2_{21} = 0$, $b^2_{22} = \dfrac{1}{8}$. Now

$$|g_1(t,x,y)| = |g_2(t,x,y)| \leq \frac{1}{4}$$

It is easy to verify that $\rho = \dfrac{1}{6}$, $F_0 = \dfrac{1}{36}$, $H_0 = \dfrac{2}{25}$. We see that condition $(\mathcal{H}_3)$ is followed with a number $r_0 = 2$. Moreover, $\dfrac{PT^{\alpha_1}}{\Gamma(\alpha_1+1)}a_{11} + \sum_{k=1}^m \frac{T^{\beta_1^k}}{\Gamma(\beta_1^k+1)} b^k_{11} \simeq 0.0990$, $\dfrac{PT^{\alpha_2}}{\Gamma(\alpha_2+1)}a_{12} + \sum_{k=1}^m \frac{T^{\beta_2^k}}{\Gamma(\beta_2^k+1)} b^k_{12} = \dfrac{PT^{\alpha_1}}{\Gamma(\alpha_1+1)}a_{21} + \sum_{k=1}^m \frac{T^{\beta_1^k}}{\Gamma(\beta_1^k+1)} b^k_{21} = 0$, $\dfrac{PT^{\alpha_2}}{\Gamma(\alpha_2+1)}a_{22} + \sum_{k=1}^m \frac{T^{\beta_2^k}}{\Gamma(\beta_2^k+1)} b^k_{22} \simeq 0.0653$. Theorem (4.3) proves the existence of a solution to system (6) . Q.E.D.

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