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Jordan (Lie) σ-Derivations on Path Algebras

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Abstract. In this paper, we investigate Jordan σ-derivations and Lie σ-derivations on path algebras. This work is motivated by the one of Benkovič done on triangular algebras and the study of Jordan derivations and Lie derivations on path algebras done by Li and Wei. Namely, main results state that every Jordan σ-derivation is a σ-derivation and every Lie σ-derivation is of a standard form on a path algebra when the associated quiver is acyclic and finite.

1. Introduction

Let K be a field of characteristic different than 2. Let A be a unital algebra over K and let σ be an automorphism on A. By $x \circ y = xy + yx$ and $[x, y] = xy - yx$ for every x and y in A, we denote Jordan product and Lie product, respectively. A linear map $d : A \to A$ is called a σ-derivation if it satisfies

$$d(xy) = d(x)y + \sigma(x)d(y) \qquad (\forall x, y \in A). \tag{1}$$

It is clear that when σ equals to the identity map of A, then σ-derivations is nothing but the classical derivations. The set of all σ-derivations on A is denoted by $\mathrm{Der}_\sigma(A)$. A σ-derivation d that satisfies $d(x) = \sigma(x)a - ax$ for every x in A is called an inner σ-derivation, where a is a fixed element in A, the set of all inner σ-derivations on A is denoted by $\mathrm{Inn}_\sigma(A)$. Analogously, a linear map $f : A \to A$ is called a Jordan σ-derivation if it satisfies

$$f(x \circ y) = f(x)y + \sigma(x)f(y) + f(y)x + \sigma(y)f(x) \quad (\forall x, y \in A). \tag{2}$$

Also, a linear map $f : A \to A$ is called a Lie σ-derivation if it satisfies

$$f([x, y]) = f(x)y + \sigma(x)f(y) - f(y)x - \sigma(y)f(x) \quad (\forall x, y \in A). \tag{3}$$

Jordan σ-derivations and Lie σ-derivations are generalizations of Jordan derivations and Lie derivations, respectively. We denote the set of all Jordan σ-derivations on A by $\mathrm{Jor}_\sigma(A)$, and the set of all Lie σ-derivations on A by $\mathrm{Lie}_\sigma(A)$. Clearly, each σ-derivation on A is a Jordan σ-derivation and a Lie σ-derivation, respectively.

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In the sequel, $E = (E^0, E^1, s, t)$ denotes a finite acyclic quiver, where E^0 and E^1 are sets of vertices and edges of E, respectively, and maps $s, t : E^1 \to E^0$ determine the edges of E. We denote by KE the path algebra over K associated with E and by $\mathcal{P}$ the set of all paths in E. Also, we denote by $\mathcal{P}_A$ the set of all non-trivial acyclic paths in E (for more details, see [8]). However, it is important to notice that in our paper the product of two paths in E is defined as follows: A non-trivial path $p = e_1 \cdots e_n$ in E is a sequence of edges such that $t(e_i) = s(e_{i+1})$ for every $1 \leq i < n$, and the product of two paths $p = e_1 \cdots e_n$ and $q = f_1 \cdots f_m$ in E is defined by

$$pq = \begin{cases} e_1 \cdots e_n f_1 \cdots f_m, & \text{if } t(e_n) = s(f_1), \\ 0, & \text{otherwise.} \end{cases}$$

There are some authors (see for instance [6, 7]) who prefer to define the product of paths in the opposite way as follows: A non-trivial path $p = e_n \cdots e_1$ in E is a sequence of edges such that $s(e_{i+1}) = t(e_i)$ for every $1 \leq i < n$, and the product of two paths $p = e_n \cdots e_1$ and $q = f_m \cdots f_1$ in E is define by

$$pq = \begin{cases} e_n \cdots e_1 f_m \cdots f_1, & \text{if } s(e_1) = t(f_m), \\ 0, & \text{otherwise.} \end{cases}$$

The relationship between path algebras and their opposite path algebras was discussed in details by Goodearl in [5, Section 1].

The main aim of this paper is to describe Jordan σ-derivations and Lie σ-derivations on path algebras. The motivations of our research are the papers [2, 3] in which Benkovič studied Jordan σ-derivations and Lie σ-derivations on triangular algebras. Namely, in [2, Theorem 3.1], Benkovič showed that every Jordan σ-derivation on a triangular algebra is a sum of a σ-derivation and an anti-derivation. And, in [3, Theorem 4.3], he characterized when Lie σ-derivations on a triangular algebra have a standard form. In both articles, Benkovič assumed a faithfulness condition. In the case of path algebras, Li and Wei showed in [7] that the condition of faithfulness can be ignored when path algebras can be viewed as one-point extensions (see Section 2 for more details). In [6], Li and Wei studied Jordan derivations of dual extension algebras and generalized one-point extension algebras which are factor algebras of path algebras. Therefore, we are inspired from the studies [2, 3, 6, 7] to investigate Jordan σ-derivations and Lie σ-derivations on path algebras. Namely, we confirm the remark of [7] and we prove it on any path algebra associated with a finite and acyclic quiver.

In Section 2, we inspect the faithfulness property and the loyal property on path algebras, and we state two results related to these properties under some conditions (see Theorem 2.4 and Proposition 2.5). In Section 3, we investigate Jordan σ-derivations on path algebras, and we show that every Jordan σ-derivation on a path algebra is a σ-derivation (see Theorem 3.2). In the last section, we study Lie σ-derivations on path algebras, and we state that every Lie σ-derivation on a path algebra is of a standard form (see Theorem 4.1). Note that, when σ is an inner automorphism on a path algebra, the problem of studying Jordan σ-derivations and Lie σ-derivations on path algebras is reduced to the study of Jordan derivations and Lie derivations, respectively, as stated in [2, Proposition 2.4] and [3, Proposition 2.3].

2. The faithfulness property on path algebras

In this section, we investigate the faithfulness property and the loyal property on path algebras, and we give a construction of a non-trivial idempotent $\mathfrak{e}$ in a path algebra KE such that the bimodule $\mathfrak{e}KE(1-\mathfrak{e})$ is a left faithful $\mathfrak{e}KE\mathfrak{e}$-module as well a right faithful $(1-\mathfrak{e})KE(1-\mathfrak{e})$-module under some constraints.

Recall that a triangular algebra A is a unital algebra that contain a non-trivial idempotent $\mathfrak{e}$ such that $\mathfrak{e}A(1-\mathfrak{e}) = 0$. Hence, it can be written as $A = \mathfrak{e}A\mathfrak{e} + \mathfrak{e}A(1-\mathfrak{e}) + (1-\mathfrak{e})A(1-\mathfrak{e})$ or in a matrix form

$$\begin{pmatrix} \mathfrak{e}A\mathfrak{e} & \mathfrak{e}A(1-\mathfrak{e}) \\ 0 & (1-\mathfrak{e})A(1-\mathfrak{e}) \end{pmatrix}.$$

When $\mathfrak{e}A\mathfrak{e}$ is isomorphic to a field K, it is called a one-point extension algebra rather than a triangular algebra. Since for a path algebra KE over a field K, the subspace $sKEs$ is isomorphic to K for any source s of E, thus one can view KE as a one-point extension with $sKE(1-s)$ is a vector space over K (see [1, Preliminaries]). Hence, $sKE(1-s)$ is faithful as a left K-module.

Recall that a (A, B)-bimodule M is called loyal if $aMb = \{0\}$ implies that $a = 0$ or $b = 0$ for every a in A and b in B. We have the following immediate result:

Lemma 2.1. *Let KE be a path algebra which admits a source s. Then, $sKE(1-s)$ is a loyal $(K, (1-s)KE(1-s))$-bimodule if and only if E has only one source.*

Since $sKE(1-s)$ is a K-vector space, it is evident that to check $sKE(1-s)$ is a loyal, it suffices to show that it is faithful as a right $(1-s)KE(1-s)$-module. Here, we give some examples and counterexamples.

Example 2.2. *Let E be the following quiver:*

$$\begin{tikzcd} s \arrow[r, "e_1"] & v_1 \arrow[r, "e_2"] & v_2 \arrow[r, "e_3"] \arrow[dr, "e_5"] & v_3 \arrow[r, "e_4"] & v_4 \\ & & & v_5 & \end{tikzcd}$$

Since E has only one source, it follows by Lemma 2.1 that $sKE(1-s)$ is a loyal $(K, (1-s)KE(1-s))$-bimodule.

Example 2.3. *Let E be the following quiver:*

$$\begin{tikzcd} s \arrow[r, "e_1"] & v_1 \arrow[r, "e_2"] & v_2 & t \arrow[l, "e_3"'] \end{tikzcd}$$

Since E has two sources s and t, it follows by Lemma 2.1 that $sKE(1-s)$ is not a loyal $(K, (1-s)KE(1-s))$-bimodule. This can be checked by a straightforward calculations. Without loss of generality, choose the source s as a non-trivial idempotent. Then, we have $sKE(1-s)t = sKEt = \{0\}$, but $t \neq 0$, hence $sKE(1-s)$ is not a right faithful $(1-s)KE(1-s)$-module, so it is not a loyal $(K, (1-s)KE(1-s))$-bimodule.

One may ask what will happen if we choose either the vertex v_1 or the vertex v_2 instead of the source s or the source t in Example 2.3. To investigate this case, we recall first the definition of generalized matrix algebra. Let A and B be two K-algebras, M a (A, B)-bimodule, N a (B, A)-bimodule, and $\Phi_{MN} : M \otimes_B N \to A$ and $\Psi_{NM} : N \otimes_A M \to B$ two bimodule homomorphisms, called the pairings, satisfying the following commutative diagrams:

$$\begin{tikzcd} M \otimes_B N \otimes_A M \arrow[r, "\Phi_{MN}\otimes Id_M"] \arrow[d, "Id_M\otimes\Psi_{NM}"] & A \otimes_A M \arrow[d, "\cong"'] \\ M \otimes_B B \arrow[r, "\cong"] & M \end{tikzcd}, \qquad \begin{tikzcd} N \otimes_A M \otimes_B N \arrow[r, "\Psi_{NM}\otimes Id_N"] \arrow[d, "Id_N\otimes\Phi_{MN}"] & B \otimes_B N \arrow[d, "\cong"'] \\ N \otimes_A A \arrow[r, "\cong"] & N. \end{tikzcd}$$

Then, the set

$$\begin{pmatrix} A & M \\ N & B \end{pmatrix} = \left\{ \begin{pmatrix} a & m \\ n & b \end{pmatrix} \middle| a \in A, m \in M, n \in N, b \in B \right\}$$

forms an K-algebra under matrix-like addition and matrix-like multiplication. There is no constraint condition concerning bimodules M and N. Such a K-algebra is called a generalized matrix algebra. Let $\mathcal{A}$ be a unital K-algebra with a non-trivial idempotent $\mathfrak{e}$, then $\mathcal{A}$ is isomorphic to the generalized matrix algebra

$$\begin{pmatrix} \mathfrak{e}\mathcal{A}\mathfrak{e} & \mathfrak{e}\mathcal{A}(1-\mathfrak{e}) \\ (1-\mathfrak{e})\mathcal{A}\mathfrak{e} & (1-\mathfrak{e})\mathcal{A}(1-\mathfrak{e}) \end{pmatrix},$$

without the assumption of the bimodule $(1-\mathfrak{e})\mathcal{A}\mathfrak{e}$ equals to zero as in triangular algebras.

Now, choose v_1 in Example 2.3 as a non-trivial idempotent. Then, we have

$$KE \cong \begin{pmatrix} v_1 K v_1 & K\{e_2\} \\ K\{e_1\} & K\{s,t,v_2,e_3\} \end{pmatrix} \cong \begin{pmatrix} K & K\{e_2\} \\ K\{e_1\} & K\{s,t,v_2,e_3\} \end{pmatrix}.$$

And, neither $K\{e_2\}$ is a right faithful $K\{s,t,v_2,e_3\}$-module nor $K\{e_1\}$ is a left faithful $K\{s,t,v_2,e_3\}$-module even for Example 2.2. Then, by choosing a vertex which is neither a source nor a sink, we obtain the following result.

Theorem 2.4. *There is no quiver such that the $(K,(1-v)KE(1-v))$-bimodule $vKE(1-v)$ or the $((1-v)KE(1-v),K)$-bimodule $(1-v)KEv$ is loyal, when v is a vertex which is neither a source nor a sink.*

Proof. Let E be a quiver and v be a vertex which is neither a source nor a sink. Since $vKE(1-v)$ and $(1-v)KEv$ are vector spaces over K, it follows that we only need to show that $vKE(1-v)$ is not a right faithful $(1-v)KE(1-v)$-module and $(1-v)KEv$ is not a left faithful $(1-v)KE(1-v)$-module. By hypotheses all sources and sinks are in $(1-v)KE(1-v)$, therefore by choosing any source s, we obtain that $vKE(1-v)s = 0$, also by choosing any sink t, we obtain $t(1-v)KEv = 0$. Finally, bimodules $vKE(1-v)$ and $(1-v)KEv$ are not loyal. □

Now, we aim to construct a non-trivial idempotent $\mathfrak{e}$ in KE such that the bimodule $\mathfrak{e}KE(1-\mathfrak{e})$ is faithful as a left $\mathfrak{e}KE\mathfrak{e}$-module and also as a right $(1-\mathfrak{e})KE(1-\mathfrak{e})$-module. To this end, we state the following proposition. Recall that a vertex is called isolated if there is no edge that starts or ends at it.

Proposition 2.5. *Let $\mathfrak{e}$ be the sum of all sources in E. Then, the bimodule $\mathfrak{e}KE(1-\mathfrak{e})$ is faithful as a left $\mathfrak{e}KE\mathfrak{e}$-module and also as a right $(1-\mathfrak{e})KE(1-\mathfrak{e})$-module if and only if E does not contain isolated vertices.*

Proof. Let $\mathfrak{e}$ be the sum of all sources in E. Assume that the bimodule $\mathfrak{e}KE(1-\mathfrak{e})$ is faithful as a left $\mathfrak{e}KE\mathfrak{e}$-module and also as a right $(1-\mathfrak{e})KE(1-\mathfrak{e})$-module, hence $\mathfrak{e}KE(1-\mathfrak{e})$ contains all paths that start from all sources in E. Then, for every source s in E, we have $s\mathfrak{e}KE(1-\mathfrak{e}) \neq 0$, and for every vertex v in E not a source, we have $\mathfrak{e}KE(1-\mathfrak{e})v \neq 0$. Hence, E does not contain an isolated vertex.

Now, assume that E contains an isolated vertex w, then $w\mathfrak{e}KE(1-\mathfrak{e}) = \mathfrak{e}KE(1-\mathfrak{e})w = 0$. Hence, $\mathfrak{e}KE(1-\mathfrak{e})$ is not a faithful module as a left module nor as a right module. □

In general, the bimodule $\mathfrak{e}KE(1-\mathfrak{e})$ is not a loyal bimodule even if E is a connected quiver. Indeed, in Example 2.3, we set $\mathfrak{e} = s+t$, then it follows that $\mathfrak{e}KE(1-\mathfrak{e}) = K\{e_1, e_1e_2, e_3\}$, and for the elements t and v_1, we have $t\mathfrak{e}KE(1-\mathfrak{e})v_1 = 0$. So, $\mathfrak{e}KE(1-\mathfrak{e})$ is not a loyal bimodule.

3. Jordan σ-derivations on path algebras

In this section, we study Jordan σ-derivations on path algebras. The main result of this section states that every Jordan σ-derivation is a σ-derivation.

The following lemma is useful throughout the paper, it states that an automorphism on a path algebra cannot translate vertices back and forth on the same non-trivial path.

Lemma 3.1. *Let σ be an automorphism on KE. Then,*

1. *For a non-trivial path $p \in \mathcal{P}_A$, we have $\sigma(t(p))(KE)s(p) = \{0\}$.*
2. *For a vertex $v \in E^0$, we have $\sigma(v)(\mathcal{P}_A)v = \{0\}$.*

Proof. Assume by contradiction that $\sigma(t(p))(KE)s(p) \neq \{0\}$. Then, there exists a path $k \in \mathcal{P}$ such that $\sigma(t(p))kp \neq 0$. Hence, $\sigma(\sigma(t(p))kp)\sigma(t(p))kp = \sigma(\sigma(t(p))kp)kp \neq 0$ with the length of all paths in the linear combination $\sigma(\sigma(t(p))kp)kp$ is greater than the length of p. By repeating the same reasoning recursively, we obtain a contradiction since E is a finite and acyclic quiver.

Now, assume by contradiction that $\sigma(v)kv \neq 0$ for some $k \in \mathcal{P}_A$. Then, we have $\sigma(\sigma(v)kv)kv \neq 0$. By the same reasoning as we did before, we obtain a contradiction. □

The main theorem of this section shows that every Jordan σ-derivation on a path algebra is a σ-derivation without assuming the faithfulness property of the bimodule $sKE(1-s)$, where s is a source in E. A similar result has established in [2, Theorem 3.1] for triangular algebras with the faithfulness condition. But before that, we construct a new Jordan σ-derivation g_f on KE from an arbitrary Jordan σ-derivation f on KE which is constructed by a similar reasoning as in [2, Lemma 3.3] on triangular algebras. Let σ be an automorphism on KE, f be a Jordan σ-derivation on KE, and a_f be an element in KE defined as follows

$$a_f = \sum_{u \in E^0} \sigma(u)f(u)(1-u) - \sigma(1-u)f(u)u.$$

Let d_f be an inner σ-derivation on KE defined by $d_f(x) = \sigma(x)a_f - a_f x$ for every $x \in KE$. Then, we have

$$d_f(v) = \sigma(v)f(v)(1-v) + \sigma(1-v)f(v)v - \sum_{\substack{u \in E^0 \\ u \neq v}} \sigma(v)f(u)u - \sum_{\substack{u \in E^0 \\ u \neq v}} \sigma(u)f(u)v,$$

for every $v \in E^0$. Since $f(v) = f(v)v + \sigma(v)f(v)$ for every vertex $v \in E^0$, it follows that $\sigma(v)f(v)v = 0$ and $f(v)(1-v) = \sigma(v)f(v)(1-v)$ for every vertex $v \in E^0$. Hence, for every vertex $v \in E^0$, we obtain

$$\begin{aligned} f(v) &= f(v)v + f(v)(1-v) \\ &= \sigma(v)f(v)v + \sigma(1-v)f(v)v + f(v)(1-v) \\ &= \sigma(1-v)f(v)v + \sigma(v)f(v)(1-v). \end{aligned}$$

Therefore, we define g_f on KE by $g_f = f - d_f$, then g_f is a Jordan σ-derivation on KE, and it satisfies the following equality:

$$g_f(v) = \sum_{\substack{u \in E^0 \\ u \neq v}} \sigma(v)f(u)u + \sum_{\substack{u \in E^0 \\ u \neq v}} \sigma(u)f(u)v, \tag{4}$$

for every $v \in E^0$. Since for every vertex $w \neq v$ where v is a fixed vertex we have

$$\sigma(v)g_f(v)w = \sigma(v)f(w)w \text{ and } \sigma(w)g_f(v)v = \sigma(w)f(w)v,$$

it follows that the equality (4) can be written as

$$g_f(v) = \sum_{\substack{u \in E^0 \\ u \neq v}} \sigma(v)g_f(v)u + \sum_{\substack{u \in E^0 \\ u \neq v}} \sigma(u)g_f(v)v, \tag{5}$$

for every $v \in E^0$. Note that non-trivial paths are nilpotents in KE.

The main result of this section is stated as follows.

Theorem 3.2. *Every Jordan σ-derivation on KE is a σ-derivation.*

Proof. Let f be a Jordan σ-derivation on KE. According to the discussion above, we may assume that f is the sum of an inner σ-derivation on KE and a Jordan σ-derivation g_f on KE satisfies the equality (5). Let v and w be two different vertices. Then, we have

$$\begin{aligned}
0 = g_f(v \circ w) &= g_f(v)w + \sigma(v)g_f(w) + g_f(w)v + \sigma(w)g_f(v) \\
&= \Big(\sum_{\substack{u\in E^0\\ u\neq v}} \sigma(v)g_f(v)u + \sum_{\substack{u\in E^0\\ u\neq v}} \sigma(u)g_f(v)v\Big)w \\
&\quad + \sigma(v)\Big(\sum_{\substack{u\in E^0\\ u\neq w}} \sigma(w)g_f(w)u + \sum_{\substack{u\in E^0\\ u\neq w}} \sigma(u)g_f(w)w\Big) \\
&\quad + \Big(\sum_{\substack{u\in E^0\\ u\neq w}} \sigma(w)g_f(w)u + \sum_{\substack{u\in E^0\\ u\neq w}} \sigma(u)g_f(w)w\Big)v \\
&\quad + \sigma(w)\Big(\sum_{\substack{u\in E^0\\ u\neq v}} \sigma(v)g_f(v)u + \sum_{\substack{u\in E^0\\ u\neq v}} \sigma(u)g_f(v)v\Big) \\
&= \sigma(v)g_f(v)w + \sigma(v)g_f(w)w + \sigma(w)g_f(w)v + \sigma(w)g_f(v)v.
\end{aligned}$$

By multiplying the last line by $\sigma(v)$ (resp. $\sigma(w)$)) from the left and by w (resp. v) from the right, it yields to

$$\sigma(v)g_f(v)w + \sigma(v)g_f(w)w = \sigma(w)g_f(v)v + \sigma(w)g_f(w)v = 0. \tag{6}$$

Hence, we deduce

$$\begin{aligned}
0 &= g_f(v)w + \sigma(v)g_f(w) \\
&= \Big(\sum_{\substack{u\in E^0\\ u\neq v}} \sigma(v)g_f(v)u + \sum_{\substack{u\in E^0\\ u\neq v}} \sigma(u)g_f(v)v\Big)w \\
&\quad + \sigma(v)\Big(\sum_{\substack{u\in E^0\\ u\neq w}} \sigma(w)g_f(w)u + \sum_{\substack{u\in E^0\\ u\neq w}} \sigma(u)g_f(w)w\Big) \\
&= \sigma(v)g_f(v)w + \sigma(v)g_f(w)w.
\end{aligned}$$

Therefore, we obtain $0 = g_f(vw) = g_f(v)w + \sigma(v)g_f(w)$. When vertices v and w are equal, it follows immediately that $g_f(v) = g_f(v)v + \sigma(v)g_f(v)$ for every vertex v. For a non-trivial path $p \in \mathcal{P}_A$, by the equality (5) and Lemma 3.1, we have:

$$\begin{aligned}
g_f(p) &= g_f(s(p) \circ p) \\
&= g_f(s(p))p + \sigma(s(p))g_f(p) + g_f(p)s(p) + \sigma(p)g_f(s(p)) \\
&= \sum_{\substack{u\in E^0\\ u\neq s(p)}} \sigma(u)g_f(s(p))p + \sigma(s(p))g_f(p) + g_f(p)s(p) \qquad (7) \\
&= g_f(t(p) \circ p) \\
&= g_f(t(p))p + \sigma(t(p))g_f(p) + g_f(p)t(p) + \sigma(p)g_f(t(p)) \\
&= \sigma(t(p))g_f(p) + g_f(p)t(p) + \sum_{\substack{u\in E^0\\ u\neq t(p)}} \sigma(p)g_f(t(p))u. \qquad (8)
\end{aligned}$$

By substituting (8) in (7) and using Lemma 3.1, we obtain

$$g_f(p) = \sum_{\substack{u\in E^0\\ u\neq s(p)}} \sigma(u)g_f(s(p))p$$

$$
\begin{aligned}
&+ \sigma(s(p))\Big(\sigma(t(p))g_f(p) + g_f(p)t(p) + \sum_{\substack{u\in E^0\\ u\neq t(p)}} \sigma(p)g_f(t(p))u\Big)\\
&+ \Big(\sigma(t(p))g_f(p) + g_f(p)t(p) + \sum_{\substack{u\in E^0\\ u\neq t(p)}} \sigma(p)g_f(t(p))u\Big)s(p)\\
= &\sum_{\substack{u\in E^0\\ u\neq s(p)}} \sigma(u)g_f(s(p))p + \sigma(s(p))g_f(p)t(p) + \sum_{\substack{u\in E^0\\ u\neq t(p)}} \sigma(p)g_f(t(p))u.
\end{aligned}
$$

Therefore, we obtain $g_f(p) = g_f(s(p))p + \sigma(s(p))g_f(p) = g_f(p)t(p) + \sigma(p)g_f(t(p))$. Let v be a vertex and p a non-trivial path such that $pv = vp = 0$. Then, we have

$$
\begin{aligned}
g_f(p)v + \sigma(p)g_f(v) &= \Big(\sum_{\substack{u\in E^0\\ u\neq s(p)}} \sigma(u)g_f(s(p))p + \sigma(s(p))g_f(p)t(p) + \sum_{\substack{u\in E^0\\ u\neq t(p)}} \sigma(p)g_f(t(p))u\Big)v\\
&\quad + \sigma(p)\Big(\sum_{\substack{u\in E^0\\ u\neq v}} \sigma(v)g_f(v)u + \sum_{\substack{u\in E^0\\ u\neq v}} \sigma(u)g_f(v)v\Big)\\
&= \sigma(p)\Big(\sigma(t(p))g_f(t(p))v + \sigma(t(p))g_f(v)v\Big).
\end{aligned}
$$

By equality (6), we obtain $g_f(p)v + \sigma(p)g_f(v) = 0$. Then, $g_f(pv) = g_f(p)v + \sigma(p)g_f(v)$. Similarly, we have

$$
\begin{aligned}
g_f(v)p + \sigma(v)g_f(p) &= \Big(\sum_{\substack{u\in E^0\\ u\neq v}} \sigma(v)g_f(v)u + \sum_{\substack{u\in E^0\\ u\neq v}} \sigma(u)g_f(v)v\Big)p\\
&\quad + \sigma(v)\Big(\sum_{\substack{u\in E^0\\ u\neq s(p)}} \sigma(u)g_f(s(p))p + \sigma(s(p))g_f(p)t(p) + \sum_{\substack{u\in E^0\\ u\neq t(p)}} \sigma(p)g_f(t(p))u\Big)\\
&= \Big(\sigma(v)g_f(v)s(p) + \sigma(v)g_f(s(p))s(p)\Big)p
\end{aligned}
$$

By equality (6), we obtain $g_f(v)p + \sigma(v)g_f(p) = 0$. Then, $g_f(vp) = g_f(v)p + \sigma(v)g_f(p)$. Now, let p and q be two non-trivial paths in $\mathcal{P}_A$. On the first hand, we assume that $p \circ q \neq 0$. Without loss of generality we suppose that $pq \neq 0$, then we have

$$
\begin{aligned}
g_f(pq) &= g_f(p \circ q)\\
&= g_f(p)q + \sigma(p)g_f(q) + g_f(q)p + \sigma(q)g_f(p)\\
&= g_f(p)q + \sigma(p)g_f(q)\\
&\quad + \Big(\sum_{\substack{u\in E^0\\ u\neq s(q)}} \sigma(u)g_f(s(q))q + \sigma(s(q))g_f(p)t(q) + \sum_{\substack{u\in E^0\\ u\neq t(q)}} \sigma(q)g_f(t(p))u\Big)p\\
&\quad + \sigma(q)\Big(\sum_{\substack{u\in E^0\\ u\neq s(p)}} \sigma(u)g_f(s(p))p + \sigma(s(p))g_f(p)t(p) + \sum_{\substack{u\in E^0\\ u\neq t(p)}} \sigma(p)g_f(t(p))u\Big)\\
&= g_f(p)q + \sigma(p)g_f(q).
\end{aligned}
$$

This is due to Lemma 3.1. On the other hand, we assume that $p \circ q = 0$, it follows that $pq = qp = 0$. Then, we have

$$
g_f(p)q + \sigma(p)g_f(q) = \Big(\sum_{\substack{u\in E^0\\ u\neq s(p)}} \sigma(u)g_f(s(p))p + \sigma(s(p))g_f(p)t(p) + \sum_{\substack{u\in E^0\\ u\neq t(p)}} \sigma(p)g_f(t(p))u\Big)q
$$

$$+\,\sigma(p)\Big(\sum_{\substack{u\in E^0\\ u\neq s(q)}}\sigma(u)g_f(s(q))q+\sigma(s(q))g_f(p)t(q)+\sum_{\substack{u\in E^0\\ u\neq t(q)}}\sigma(q)g_f(t(q))u\Big)$$
$$=\sigma(p)g_f(t(p))q+\sigma(p)g_f(s(q))q$$
$$=\sigma(p)\Big(\sigma(t(p))g_f(t(p))s(q)+\sigma(t(p))g_f(s(q))s(q)\Big)q.$$

Hence, by equality (6), we obtain $g_f(pq)=g_f(p)q+\sigma(p)g_f(q)=0$. Which yields that g_f is a σ-derivation on KE, thus f is a σ-derivation on KE. The proof is completed. $\square$

In the next example, we will fix an automorphism σ on KE and we will show that for an arbitrary Jordan σ-derivation on KE, it is a σ-derivation. We express every linear map f on KE by

$$f(p)=\sum_{q\in\mathcal{P}}c_q^p q \quad (\forall p\in\mathcal{P}).$$

This is true due to the fact that the set $\mathcal{P}$ is finite, and it is a basis of KE as a K-vector space.

Example 3.3. *Let E be the following quiver:*

$$\begin{tikzcd} v_1 \arrow[r, bend left, "e_2"] \arrow[r, "e_1"'] & v_3 & v_2. \arrow[l, bend right, "e_4"'] \arrow[l, "e_3"] \end{tikzcd}$$

Let σ be an automorphisms on KE defined as follows: $\sigma(v_1)=v_1+e_1+e_2$, $\sigma(v_2)=v_2+e_3+e_4$, $\sigma(v_3)=v_3-e_1-e_2-e_3-e_4$, $\sigma(e_1)=e_2$, $\sigma(e_2)=e_1$, $\sigma(e_3)=e_3$ and $\sigma(e_4)=e_4$. Let f be a Jordan σ-derivation on KE. We only need to compute the image of the elements of the basis $\mathcal{P}$. The computation is as follows:

$$\begin{aligned} f(v_1) &= 2^{-1}f(v_1\circ v_1)\\ &= 2^{-1}(f(v_1)v_1+\sigma(v_1)f(v_1)+f(v_1)v_1+\sigma(v_1)f(v_1))\\ &= f(v_1)v_1+\sigma(v_1)f(v_1)\\ &= f(v_1)v_1+(v_1+e_1+e_2)f(v_1)\\ &= 2c_{v_1}^{v_1}v_1+c_{e_1}^{v_1}e_1+c_{e_2}^{v_1}e_2+c_{v_3}^{v_1}(e_1+e_2). \end{aligned}$$

Hence, we get $f(v_1)=c_{e_1}^{v_1}e_1+c_{e_2}^{v_1}e_2$. Similarly, we obtain $f(v_2)=c_{e_3}^{v_2}e_3+c_{e_4}^{v_2}e_4$. We have

$$\begin{aligned} f(v_3) &= 2^{-1}f(v_3\circ v_3)\\ &= 2^{-1}(f(v_3)v_3+\sigma(v_3)f(v_3)+f(v_3)v_3+\sigma(v_3)f(v_3))\\ &= f(v_3)v_3+\sigma(v_3)f(v_3)\\ &= f(v_3)v_3+(v_3-e_1-e_2-e_3-e_4)f(v_3)\\ &= 2c_{v_3}^{v_3}v_3+c_{e_1}^{v_3}e_1+c_{e_2}^{v_3}e_2+c_{e_3}^{v_3}e_3+c_{e_4}^{v_3}e_4-c_{v_3}^{v_3}(e_1+e_2+e_3+e_4). \end{aligned}$$

Hence, we get $f(v_3)=c_{e_1}^{v_3}e_1+c_{e_2}^{v_3}e_2+c_{e_3}^{v_3}e_3+c_{e_4}^{v_3}e_4$. We compute the images of all edges as follows

$$\begin{aligned} f(e_1) &= f(v_1\circ e_1)\\ &= f(v_1)e_1+\sigma(v_1)f(e_1)+f(e_1)v_1+\sigma(e_1)f(v_1)\\ &= (v_1+e_1+e_2)f(e_1)+f(e_1)v_1\\ &= 2c_{v_1}^{e_1}v_1+c_{e_1}^{e_1}e_1+c_{e_2}^{e_1}e_2+c_{v_3}^{e_1}(e_1+e_2) \end{aligned}$$

Hence, we get $f(e_1)=c_{e_1}^{e_1}e_1+c_{e_2}^{e_1}e_2$. Similarly, we obtain $f(e_2)=c_{e_1}^{e_2}e_1+c_{e_2}^{e_2}e_2$, $f(e_3)=c_{e_3}^{e_3}e_3+c_{e_4}^{e_3}e_4$ and $f(e_4)=c_{e_3}^{e_4}e_3+c_{e_4}^{e_4}e_4$. By straightforward verification, we deduce that f is a σ-derivation on KE.

Denote by Id$([A, A])$ the ideal of A generated by all commutators in A. As in [2], an algebra A is not of a triangular form if for each idempotent $\mathfrak{e}$ in A the condition $(1-\mathfrak{e})A\mathfrak{e} = \{0\}$ implies that $\mathfrak{e}A(1-\mathfrak{e}) = \{0\}$. In the next theorem, we assume that $\mathfrak{e}A(1-\mathfrak{e})$ is faithful as a left $\mathfrak{e}A\mathfrak{e}$-module and also as a right $(1-\mathfrak{e})A(1-\mathfrak{e})$-module.

Theorem 3.4 ([2, Theorem 4.1]). *Let A be a 2-torsion free triangular matrix algebra. Let us assume that one of the following statements holds:*

1. $\mathfrak{e}A\mathfrak{e}$ *is not of a triangular form,*
2. $(1-\mathfrak{e})A(1-\mathfrak{e})$ *is not of a triangular form,*
3. $\mathfrak{e}A\mathfrak{e} = Id([\mathfrak{e}A\mathfrak{e}, \mathfrak{e}A\mathfrak{e}])$,
4. $(1-\mathfrak{e})A(1-\mathfrak{e}) = Id([(1-\mathfrak{e})A(1-\mathfrak{e}), (1-\mathfrak{e})A(1-\mathfrak{e})])$,
5. $\mathfrak{e}A(1-\mathfrak{e})$ *is a loyal* $(\mathfrak{e}A\mathfrak{e}, (1-\mathfrak{e})A(1-\mathfrak{e}))$*-bimodule.*

Then, any Jordan σ-derivation on A is a σ-derivation.

Here, an immediate consequence of Lemma 2.1, Proposition 2.5, and Theorem 3.2.

Corollary 3.5. *In the case when $A = KE$ with E is a quiver without isolated vertices, the faithfulness constraint for Theorem 3.4 is unnecessary and one of the conditions* (1), (2) *or* (5) *is always satisfied.*

4. Lie σ-derivations on path algebras

In this section, we study Lie σ-derivations on path algebras. The main result of this section states that every Lie σ-derivation is of a standard form. Also, we show that $\mathrm{Lie}_\sigma(KE) = \mathrm{Der}(KE)_\sigma \oplus \mathrm{L}_\sigma(KE)$, where $\mathrm{L}_\sigma(KE)$ is the set of all maps that vanish on all commutators of KE and their values are in the σ-center of KE.

Recall a σ-centre $Z_\sigma(A)$ of A is the set defined by $Z_\sigma(A) = \{\lambda \in A : \sigma(x)\lambda = \lambda x, \forall x \in A\}$, where σ is an automorphism on A. A more detailed discussion about σ-centres was provided in [3, Section 2].

The main result of this section shows that every Lie σ-derivation a standard form without assuming the faithfulness property of the bimodule $sKE(1-s)$ with s is a source in E. A similar result established in [3, Theorem 3.5] for triangular algebras with the condition of faithfulness. As in the previous section, we construct a new Lie σ-derivation g_f on KE from an arbitrary Lie σ-derivation f on KE, and it is done by a similar reasoning as in [3, Lemma 3.2] on triangular algebras. Let σ be an automorphism on KE, f be a Lie σ-derivation on KE, and a_f be an element in KE defined as follows

$$a_f = \sum_{u \in E^0} \sigma(u)f(u)(1-u) - \sigma(1-u)f(u)u.$$

Let d_f be an inner σ-derivation on KE defined by $d_f(x) = \sigma(x)a_f - a_f x$ for every $x \in KE$. Then, we have

$$d_f(v) = \sigma(v)f(v)(1-v) + \sigma(1-v)f(v)v - \sum_{\substack{u \in E^0 \\ u \neq v}} \sigma(v)f(u)u - \sum_{\substack{u \in E^0 \\ u \neq v}} \sigma(u)f(u)v,$$

for every $v \in E^0$. We define g_f on KE by

$$g_f = f - d_f, \tag{9}$$

hence g_f is a Lie σ-derivation on KE satisfies the following equality:

$$g_f(v) = \sigma(v)f(v)v + \sigma(1-v)f(v)(1-v) + \sum_{\substack{u \in E^0 \\ u \neq v}} \sigma(v)f(u)u + \sum_{\substack{u \in E^0 \\ u \neq v}} \sigma(u)f(u)v, \tag{10}$$

for every $v \in E^0$. Now, let δ_f be a linear map on KE defined as follows:

$$\delta_f(v) = \sum_{\substack{u\in E^0\\ u\neq v}} \sigma(v)f(u)u + \sum_{\substack{u\in E^0\\ u\neq v}} \sigma(u)f(u)v, \tag{11a}$$

$$\delta_f(p) = g_f(p), \tag{11b}$$

for every $v \in E^0$ and $p \in \mathcal{P}_A$. We claim that δ_f is a σ-derivation on KE. Indeed, let $v \in E^0$. Then, we have

$$\begin{aligned}\delta_f(v)v + \sigma(v)\delta_f(v) &= \sum_{\substack{u\in E^0\\ u\neq v}} \sigma(u)f(u)v + \sum_{\substack{u\in E^0\\ u\neq v}} \sigma(v)f(u)u\\ &= \delta_f(v).\end{aligned}$$

Let v and w be two different vertices. Then, we have

$$\begin{aligned}0 &= \delta_f(v)w + \sigma(v)\delta_f(w)\\ &= \sigma(v)f(w)w + \sigma(v)f(v)w\\ &= \delta_f(vw).\end{aligned}$$

This is due to the fact that

$$\begin{aligned}0 &= g_f([v,w])\\ &= g_f(v)w + \sigma(v)g_f(w) - g_f(w)v - \sigma(w)g_f(v)\\ &= \sigma(1-v)f(v)w + \sigma(v)f(w)w + \sigma(v)f(w)(1-w) + \sigma(v)f(v)w\\ &\quad - \sigma(1-w)f(w)v - \sigma(w)f(v)v - \sigma(w)f(v)(1-v) - \sigma(w)f(w)v.\end{aligned} \tag{12}$$

Hence, by multiplying (12) from the left by $\sigma(v)$ and from right by w, we obtain

$$\sigma(v)f(w)w + \sigma(v)f(v)w = 0, \tag{13}$$

for every two different vertices v and w. Let p be a non-trivial path in $\mathcal{P}_A$. Then, we have

$$\delta_f(s(p))p + \sigma(s(p))\delta_f(p) = \sum_{\substack{u\in E^0\\ u\neq s(p)}} \sigma(u)f(u)p + \sigma(s(p))g_f(p),$$

$$\delta_f(p)t(p) + \sigma(p)\delta_f(t(p)) = g_f(p)t(p) + \sum_{\substack{u\in E^0\\ u\neq t(p)}} \sigma(p)f(u)u.$$

Also, we have

$$\begin{aligned}g_f(p) &= g_f([s(p),p])\\ &= g_f(s(p))p + \sigma(s(p))g_f(p) - g_f(p)s(p) - \sigma(p)g_f(s(p)) \qquad (14)\\ &= g_f([p,t(p)])\\ &= g_f(p)t(p) + \sigma(p)g_f(t(p)) - g_f(t(p))p - \sigma(t(p))g_f(p). \qquad (15)\end{aligned}$$

Then, by multiplying (14) from the left by $\sigma(t(p))$ and multiplying (15) from the right by $s(p)$, and by applying Lemma 3.1, we obtain

$$g_f(p)s(p) = \sigma(t(p))g_f(p) = 0.$$

Also, we have by equality (10):

$$g_f(s(p))p = \sigma(s(p))f(s(p))p + \sum_{\substack{u\in E^0\\ u\neq s(p)}} \sigma(u)f(u)p,$$

$$\sigma(p)g_f(t(p)) = \sigma(p)f(t(p))t(p) + \sum_{\substack{u\in E^0\\ u\neq t(p)}} \sigma(p)f(u)u,$$
$$\sigma(p)g_f(s(p)) = \sigma(s(p))g_f(s(p))p = \sigma(s(p))f(s(p))p,$$
$$g_f(t(p))p = \sigma(p)g_f(t(p))t(p) = \sigma(p)f(t(p))t(p).$$

Hence, we deduce that

$$g_f(p) = \sum_{\substack{u\in E^0\\ u\neq s(p)}} \sigma(u)f(u)p + \sigma(s(p))g_f(p) = g_f(p)t(p) + \sum_{\substack{u\in E^0\\ u\neq t(p)}} \sigma(p)f(u)u. \tag{16}$$

Which yields by equality (10) to

$$\begin{aligned}\delta_f(p) &= \delta_f(s(p))p + \sigma(s(p))\delta_f(p)\\ &= \delta_f(p)t(p) + \sigma(p)\delta_f(t(p)).\end{aligned}$$

Let v be a vertex in E^0 and p be a non-trivial path in $\mathcal{P}_A$ such that $[v,p] = 0$. Then, we have

$$\begin{aligned}0 &= \delta_f(v)p + \sigma(v)\delta_f(p)\\ &= \sigma(v)f(s(p))p + \sigma(v)f(v)p\\ &= \delta_f(p)v + \sigma(p)\delta_f(v)\\ &= \sigma(p)f(v)v + \sigma(p)f(t(p))v.\end{aligned}$$

Hence, by equality (13), we obtain $\delta_f(vp) = \sigma(v)f(v)p+\sigma(v)f(s(p))p = 0$ and $\delta_f(pv) = \sigma(p)f(v)v+\sigma(p)f(t(p))v = 0$. Let p and q be non-trivial paths in $\mathcal{P}_A$ such that $pq \neq 0$. By Lemma 3.1 and the equality (16), we have $\delta_f(q)p = g_f(q)p = \sigma(s(q))g_f(q)p = 0$ and $\sigma(q)\delta_f(p) = \sigma(q)g_f(p) = \sigma(q)g_f(p)t(p) = 0$. Thus, we obtain

$$\delta_f(pq) = \delta_f(p)q + \sigma(p)\delta_f(q).$$

Let p and q be non-trivial paths in $\mathcal{P}_A$ such that $pq = qp = 0$. Then, we have

$$\begin{aligned}0 = \delta_f(pq) &= \delta_f(p)q + \sigma(p)\delta_f(q)\\ &= \sigma(s(p))\delta_f(p)q + \sigma(p)\delta_f(s(q))q.\end{aligned}$$

This due to the fact that

$$\begin{aligned}0 &= g_f([p,q])\\ &= g_f(p)q + \sigma(p)g_f(q) - g_f(q)p - \sigma(q)g_f(p)\\ &= \sigma(s(p))\delta_f(p)q + \sigma(p)\delta_f(s(q))q - \sigma(s(q))\delta_f(q)p - \sigma(q)\delta_f(s(p))p.\end{aligned}$$

Which yields to $\sigma(s(p))\delta_f(p)q + \sigma(p)\delta_f(s(q))q = 0$. Finally, we obtain that δ_f is a σ-derivation on KE and the claim is proved. Therefore, g_f is the sum of δ_f and the linear map l_f on KE defined by

$$l_f(v) = \sigma(v)f(v)v + \sigma(1-v)f(v)(1-v), \text{ and } l_f(p) = 0, \tag{17}$$

for every vertex v and a non-trivial path p. Now, we are in a position to state the main result of this section.

Theorem 4.1. *Every Lie σ-derivation on KE is of a standard form.*

Proof. Let f be a Lie σ-derivation on KE. According to equality (9), we may assume that f is a sum of a Lie σ-derivation g_f on KE and an inner σ-derivation d_f on KE. And, by the discussion above, we assume that g_f is a sum of σ-derivation δ_f on KE and linear map l_f on KE defined as in (17). Since l_f vanishes on commutators of KE by construction, we only need to show that $l_f(v) \in Z_\sigma(KE)$ for every vertex v in E to prove that f is of a standard form.

Let $v \in E^0$ be a fixed vertex, from the hypotheses, we obtain that $l_f = g_f - \delta_f$ is a Lie σ-derivation on KE. Hence, for every non-trivial path $p \in \mathcal{P}_A$, we have $l_f([v,p]) = l_f(v)p - \sigma(p)l_f(v) = 0$. Which yields to $\sigma(p)l_f(v) = l_f(v)p$ for every $p \in \mathcal{P}_A$. Let $v \neq u \in E^0$, then we have

$$0 = l_f([v,u]) = l_f(v)u + \sigma(v)l_f(u) - l_f(u)v - \sigma(u)l_f(v). \tag{18}$$

Multiplying (18) from the left by $\sigma(u)$, we obtain

$$0 = \sigma(u)f(v)u - \sigma(u)f(v)(1-v). \tag{19}$$

And, multiplying (18) from the right by u, we obtain

$$0 = \sigma(1-v)f(v)u - \sigma(u)f(v)u. \tag{20}$$

Hence, we have $\sigma(u)f(v)(1-v) = \sigma(1-v)f(v)u$, which yields to $\sigma(u)l_f(v) = l_f(v)u$. For the case when $v = u$, we have $\sigma(v)l_f(v) = \sigma(v)f(v)v = l_f(v)v$. Therefore, $\sigma(u)l_f(v) = l_f(v)u$ for every vertex $u \in E^0$. Finally, we deduce that $l_f(v) \in Z_\sigma(KE)$ for every vertex $v \in E^0$. $\square$

Denote by $W(A)$ the algebra generated by idempotents and commutators of A. In the next proposition, we assume that $\mathfrak{e}A(1-\mathfrak{e})$ is faithful as a left $\mathfrak{e}A\mathfrak{e}$-module and also as a right $(1-\mathfrak{e})A(1-\mathfrak{e})$-module, where $\mathfrak{e}$ is a non-trivial idempotent in A.

Proposition 4.2 ([3, Corollary 4.4]). *Let A be a 2-torsion free triangular algebra such that $A = W(A)$. Then, any Lie σ-derivation d of A is of the form $d = \Delta + \gamma$, where $\Delta : A \to A$ is a σ-derivation and $\gamma : A \to Z_\sigma(A)$ is a linear map the vanishes on $[A,A]$.*

One can state a similar result to Corollary 3.5. Since every non-trivial path in E can be viewed as a commutator and all vertices are idempotents, we have the following consequence of Theorem 4.1 and Proposition 2.5.

Corollary 4.3. *In the case when $A = KE$ with E is a quiver without isolated vertices, then the faithfulness constraint for Proposition 4.2 is unnecessary.*

In the next example, we use a quiver from [4], which we can define on it an automorphism does not map a vertex to vertex.

Example 4.4. *Let $K = F_5$ and let E be the following quiver $v_1 \xrightarrow{\ e\ } v_2$ and σ an automorphism on KE defined as in [4, Page 1398], i.e.:*

$$\sigma(v_1) = v_1 + e, \sigma(v_2) = v_2 - e \text{ and } \sigma(e) = e.$$

Then, the σ-centre of KE is $Z_\sigma(KE) = \{0, v_1 + v_2 + 4e, 2v_1 + 2v_2 + 3e, 3v_1 + 3v_2 + 2e, 4v_1 + 4v_2 + e\}$. Let f be the Lie σ-derivation on KE defined by

$$f(v_1) = v_1 + v_2 + e, f(v_2) = 4v_1 + 4v_2 + 4e \text{ and } f(e) = 3e.$$

Then, f can be written as a sum of a σ-derivation d on KE defined by

$$d(v_1) = 2e, d(v_2) = 3e \text{ and } d(e) = 4e,$$

and a linear map l on KE that vanishes on all commutators of KE and its values are in σ-center of KE, which is defined by

$$l(v_1) = v_1 + v_2 + 4e, l(v_2) = 4v_1 + 4v_2 + e \text{ and } l(e) = 0.$$

Denote the set of all maps that vanishes on all commutators and their values are in the σ-center of A by $L_\sigma(A)$. Notice that every element of $L_\sigma(A)$ is a Lie σ-derivation on A. Indeed, let l be an element in $L_\sigma(A)$, thus by definition, we have $l([x, y]) = 0$ for every x and y in A. And, we have

$$l(x)y + \sigma(x)l(y) - l(y)x - \sigma(y)l(x) = 0,$$

due to the fact that Im(l) is a subset of σ-center of A. Hence, we obtain the next result.

Corollary 4.5. *The following sequence is exact and split as K-vector spaces.*

$$0 \longrightarrow L_\sigma(KE) \xrightarrow{\varphi} Lie_\sigma(KE) \xrightarrow{\psi} Der_\sigma(KE) \longrightarrow 0, \tag{21}$$

where φ is a canonical inclusion and $\psi : f \mapsto f - l_f$ with l_f is the associated map with f that vanishes on all commutators and its values are in the σ-center of KE as defined in (17).

Proof. On the first hand, the map φ is K-linear by definition. And, on the other hand, by Theorem 4.1, ψ is an epimorphism and Im(φ) = Ker(ψ). Thus, the sequence (21) is exact. To show it splits, define $\overline{\psi} : \mathrm{Der}_\sigma(KE) \to \mathrm{Lie}_\sigma(KE)$ to be the canonical injection. Then, for every derivation d in $\mathrm{Der}_\sigma(KE)$, we have $\psi\overline{\psi}(d) = \psi(d) = d - l_d = d$, due to the fact that $l_d = 0$. Therefore, $\psi\overline{\psi}$ is the identity on $\mathrm{Der}_\sigma(KE)$ and the sequence (21) is split. □

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