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Generalized Inverses of Bounded Finite Potent Operators on Hilbert Spaces

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Abstract. The aim of this work is to prove the existence and uniqueness of the Drazin inverse and the DMP inverses of a bounded finite potent endomorphism. In particular, we give the main properties of these generalized inverses, we offer their relationships with the adjoint operator, we study their spectrum, we compute the respective traces and determinants and we relate the Drazin inverse of a bounded finite potent operator with classical definitions of this generalized inverse. Moreover, different properties of the Moore-Penrose inverse of a bounded operator are studied.

Contents

1. Introduction

For an arbitrary $(n \times n)$-matrix A with entries in the complex numbers, the index of A, $i(A) \geq 0$, is the smallest integer such that $\mathrm{rk}(A^{i(A)}) = \mathrm{rk}(A^{i(A)+1})$.

In 1958, M. P. Drazin in [10] showed the existence of a unique $(n \times n)$ complex matrix A^D, called the Drazin inverse, satisfying the equations:

- $A^{r+1}A^D = A^r$ for $r = i(A)$;

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- $A^D A A^D = A^D$;
- $A^D A = A A^D$.

When $i(A) \leq 1$, it is known that the Drazin inverse A^D coincides with the group inverse $A^\#$.

Moreover, given again $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ with $i(A) = r$, in [15] S. B. Malik and N. Thome showed the existence of a unique matrix $A^{d,\dagger} \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ satisfying the equations:

- $A^r A^{d,\dagger} = A^r A^\dagger$;
- $A^{d,\dagger} A A^{d,\dagger} = A^{d,\dagger}$;
- $A^{d,\dagger} A = A^D A$;

$A^\dagger$ being the Moore-Penrose inverse of A.

$A^{d,\dagger}$ is called the Drazin-Moore-Penrose (DMP) inverse of A and its explicit expression is $A^D A A^\dagger$. Moreover, if A has index 1, the matrix $A^{d,\dagger}$ coincides with the core inverse of A offered by O. M. Baksalary and G. Trenkler in [2].

Furthermore, the dual Drazin-Moore-Penrose (dDMP) of A is denoted by $A^{\dagger,d}$ and its explicit expression is $A^\dagger A A^D$. One has that $A^{\dagger,d} \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ is the unique matrix satisfying the conditions:

- $A^{\dagger,d} A^r = A^\dagger A^r$;
- $A^{\dagger,d} A A^{\dagger,d} = A^{\dagger,d}$;
- $A A^{\dagger,d} = A A^D$.

On the other hand, the notion of finite potent endomorphism on an arbitrary vector space was introduced by J. Tate in [26] as a basic tool for his elegant definition of Abstract Residues.

During the last decade, the theory of finite potent endomorphisms have been applied to studying different topics related to Algebra, Arithmetic and Algebraic Geometry. Thus, A. Yekutieli in [27] and O. Braunling in [3] and [4] have addressed problems of arithmetic symbols by using properties of finite potent endomorphism; C. P. Debry in [9] and L. Taelman in [25] have offered results about Drinfeld modules from these linear operators; and V. Cabezas Sánchez and F. Pablos Romo have given explicit solutions of infinite linear systems from reflexive generalized inverses of finite potent endomorphisms in [5]. Moreover, the author of this work has extended to finite potent endomorphisms the notions of Drazin inverse, group inverse and DMP inverses in [18], [20] and [22] and, recently, has studied the properties of bounded finite potent operators on Hilbert spaces in [19]. As far as we know, this last paper is the first approach for studying finite potent endomorphisms from the point of view of the Functional Analysis that has appeared in the literature.

The aim of this work is to prove the existence and uniqueness of the Drazin inverse and the DMP inverses of a bounded finite potent endomorphism. In particular, we give the main properties of these generalized inverses, we study their relationships with the adjoint operator, we show that their spectrum coincides and we compute the respective traces and determinants. Moreover, different properties of the Moore-Penrose inverse of a bounded operator are offered.

The paper is organized as follows. In section 2 we recall basic definitions of Functional Analysis (inner product spaces, Hilbert spaces, bounded operators, orthogonality and the adjoint of a bounded linear map) and we provide a summary of statements of the articles [1], [6], [17], [19], [22] and [26].

For the sake of completeness, Section 3 is devoted to offer different properties of the Moore-Penrose inverse of a bounded linear map between two Hilbert spaces. Most of the given properties are well known to specialists and we provide new proofs of them.

The purpose of Section 4 is to study the Drazin inverse of a bounded finite potent endomorphism on a Hilbert space. Indeed, we show that the Drazin inverse of a bounded finite potent endomorphism is also a bounded finite potent operator, we analyze the Drazin inverse of the adjoint of these operators, we compute their spectrum, trace and determinant and we relate the Drazin inverse of a bounded finite potent operator with the classical definitions of the Drazin inverse for bounded operators on Banach spaces given in [7], [12] and [13]. Moreover, in this section the main properties of the group inverse of a bounded finite potent operator are offered.

Section 5 deals with the main properties of the Drazin-Moore-Penrose inverses of a bounded finite potent operator on Hilbert spaces. Thus, we prove their existence and uniqueness as bounded finite potent linear maps, we characterize their adjoint operator and we calculate the corresponding spectrum, trace and determinant.

Accordingly, Proposition 5.8 shows that

$$\mathrm{Tr}_{\mathcal{H}}(\varphi^{d,\dagger}) = \mathrm{Tr}_{\mathcal{H}}(\varphi^{\dagger,d}) = \mathrm{Tr}_{\mathcal{H}}(\varphi^{D})$$

and

$$\det{}_{\mathcal{H}}(\mathrm{Id} + \varphi^{d,\dagger}) = \det{}_{\mathcal{H}}(\mathrm{Id} + \varphi^{\dagger,d}) = \det{}_{\mathcal{H}}(\mathrm{Id} + \varphi^{D})\,,$$

where φ is a bounded finite potent operator with closed $\mathrm{Im}\,\varphi$ on a Hilbert space $\mathcal{H}$, φ^D is the Drazin inverse of φ and $\varphi^{d,\dagger}$ and $\varphi^{\dagger,d}$ are the DMP inverses of φ.

Finally, we translate to finite square complex matrices the results offered in this section for the DMP inverses.

2. Preliminaries

2.1. Finite Potent Endomorphisms

Let k be an arbitrary field and let V be a k-vector space. Let us now consider an endomorphism φ of V. We say that φ is "finite potent" if $\varphi^n V$ is finite dimensional for some n. This definition was introduced by J. Tate in [26] as a basic tool for his elegant definition of Abstract Residues.

In 2007, M. Argerami, F. Szechtman and R. Tifenbach showed in [1] that an endomorphism φ is finite potent if and only if V admits a φ-invariant decomposition $V = U_\varphi \oplus W_\varphi$ such that $\varphi_{|U_\varphi}$ is nilpotent, W_φ is finite dimensional and $\varphi_{|W_\varphi} \colon W_\varphi \xrightarrow{\sim} W_\varphi$ is an isomorphism.

Indeed, if $k[x]$ is the algebra of polynomials in the variable x with coefficients in k, we may view V as an $k[x]$-module via φ, and the explicit definition of the above φ-invariant subspaces of V is:

- $U_\varphi = \{v \in V \text{ such that } \varphi^m(v) = 0 \text{ for some m }\}$;
- $W_\varphi = \left\{\begin{matrix} v \in V \text{ such that } p(\varphi)(v) = 0 \text{ for some } p(x) \in k[x] \\ \text{relatively prime to } x \end{matrix}\right\}$.

Note that if the annihilator polynomial of φ is $x^m \cdot p(x)$ with $(x, p(x)) = 1$, then $U_\varphi = \mathrm{Ker}\,\varphi^m$ and $W_\varphi = \mathrm{Ker}\, p(\varphi)$.

Hence, this decomposition is unique. We shall call this decomposition the φ-invariant AST-decomposition of V.

Basic examples of finite potent endomorphisms are all endomorphisms of a finite-dimensional vector spaces and finite rank or nilpotent endomorphisms of infinite-dimensional vector spaces.

For a finite potent endomorphism φ, a trace $\mathrm{Tr}_V(\varphi) \in k$ may be defined as

$$\mathrm{Tr}_V(\varphi) = \mathrm{Tr}_{W_\varphi}(\varphi_{|W_\varphi})\,. \tag{1}$$

This trace has the following properties:

1. if V is finite dimensional, then $\mathrm{Tr}_V(\varphi)$ is the ordinary trace;
2. if W is a subspace of V such that $\varphi W \subset W$ then

$$\mathrm{Tr}_V(\varphi) = \mathrm{Tr}_W(\varphi) + \mathrm{Tr}_{V/W}(\varphi)\,;$$

3. if φ is nilpotent, then $\mathrm{Tr}_V(\varphi) = 0$.

Usually, Tr_V is named "Tate's trace".

Moreover, D. Hernández Serrano and the author of this paper have offered in [11] a definition of a determinant for finite potent endomorphisms satisfying the following properties:

- if V is finite dimensional, then $\det_V(\mathrm{Id} + \varphi)$ is the ordinary determinant;
- if W is a subspace of V such that $\varphi W \subset W$, then

$$\det_V(\mathrm{Id} + \varphi) = \det_W(\mathrm{Id} + \varphi) \cdot \det_{V/W}(\mathrm{Id} + \varphi);$$

- if φ is nilpotent, then $\det_V(\mathrm{Id} + \varphi) = 1$.

It is known that

$$\det_V(\mathrm{Id} + \varphi) = \det_{W_\varphi}(\mathrm{Id} + \varphi_{|W_\varphi}). \quad (2)$$

For details readers are referred to [11], [23], [24] and [26].

2.2. *Drazin inverse of Finite Potent Endomorphisms*

Let V be an arbitrary k-vector space and let $\varphi \in \mathrm{End}_k(V)$ be a finite potent endomorphism of V. Let us consider the AST-decomposition $V = U_\varphi \oplus W_\varphi$ induced by φ.

We shall call "index of φ", $i(\varphi)$, to the nilpotent order of $\varphi_{|U_\varphi}$, which coincides with the smaller $n \in \mathbb{N}$ such that $\mathrm{Im}\, \varphi^n = W_\varphi$. One has that $i(\varphi) = 0$ if and only if V is a finite-dimensional vector space and φ is an automorphism and, bearing in mind the well-known equivalence between square matrices and endomorphisms of finite-dimensional vector spaces, when V is finite-dimensional and $\varphi \equiv A$, it follows from [22, Lemma 3.2] that $i(\varphi)$ coincides with the index $i(A)$ referred to in Section 1.

For each finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$, it follows from [22, Theorem 3.4] that there exists a unique finite potent endomorphism $\varphi^D \in \mathrm{End}_k(V)$ which satisfies that:

1. $\varphi^{r+1} \circ \varphi^D = \varphi^r$;
2. $\varphi^D \circ \varphi \circ \varphi^D = \varphi^D$;
3. $\varphi^D \circ \varphi = \varphi \circ \varphi^D$,

where r is the index of φ.

2.3. *CN Decomposition of a Finite Potent Endomorphism*

Let V be again an arbitrary k-vector space. Given a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$, there exists a unique decomposition $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1, \varphi_2 \in \mathrm{End}_k(V)$ are finite potent endomorphisms satisfying that:

- $i(\varphi_1) \leq 1$;
- φ_2 is nilpotent;
- $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1 = 0$.

According to [17, Theorem 3.2], if φ^D is the Drazin inverse of φ, one has that $\varphi_1 = \varphi \circ \varphi^D \circ \varphi$ is the core part of φ. Also, φ_2 is named the nilpotent part of φ and one has that

$$\varphi = \varphi_1 \Longleftrightarrow U_\varphi = \mathrm{Ker}\, \varphi \Longleftrightarrow W_\varphi = \mathrm{Im}\, \varphi \Longleftrightarrow (\varphi^D)^D = \varphi \Longleftrightarrow i(\varphi) \leq 1. \quad (3)$$

Moreover, if $V = W_\varphi \oplus U_\varphi$ is the AST-decomposition of V induced by φ, then φ_1 and φ_2 are the unique linear maps such that:

$$\varphi_1(v) = \begin{cases} \varphi(v) & \text{if } v \in W_\varphi \\ 0 & \text{if } v \in U_\varphi \end{cases} \quad \text{and} \quad \varphi_2(v) = \begin{cases} 0 & \text{if } v \in W_\varphi \\ \varphi(v) & \text{if } v \in U_\varphi \end{cases} \,. \tag{4}$$

By definition of Tate's trace, for every finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$, one has that

$$\mathrm{Tr}_V(\varphi) = \mathrm{Tr}_V(\varphi_1)\,.$$

2.4. *Bounded finite potent endomorphisms on Hilbert spaces*

Let k be the field of the real numbers or the field of the complex numbers, and let V be a k-vector space. An inner product on V is a map $g\colon V \times V \to k$ satisfying that:

- g is linear in its first argument:
$$g(\lambda v_1 + \mu v_2, v') = \lambda g(v_1, v') + \mu g(v_2, v') \text{ for every } v_1, v_2, v' \in V\,;$$
- $g(v', v) = \overline{g(v, v')}$ for all $v, v' \in V$, where $\overline{g(v, v')}$ is the complex conjugate of $g(v, v')$;
- g is positive definite:
$$g(v, v) \geq 0 \text{ and } g(v, v) = 0 \Longleftrightarrow v = 0\,.$$

Note that $g(v, v) \in \mathbb{R}$ for each $v \in V$, because $g(v, v) = \overline{g(v, v)}$.
Also, an inner product space is a pair (V, g).

If (V, g) is an inner product vector space, we say that two vectors $v, v' \in V$ are orthogonal when $g(v, v') = 0 = g(v', v)$. Also, given a subspace L of an inner vector space (V, g), we shall call "orthogonal of L", $L^\perp$, to the subset of V consists of all vectors that are orthogonal to every $h \in L$, that is

$$L^\perp = \{v \in V \text{ such that } g(v, h) = 0 \text{ for every } h \in L\}\,.$$

The norm on an inner product space (V, g) is the real-valued function

$$\begin{aligned} \|\cdot\|_g \colon V &\longrightarrow \mathbb{R} \\ v &\longmapsto +\sqrt{g(v, v)}\,, \end{aligned}$$

and the distance is the map

$$\begin{aligned} d_g \colon V \times V &\longrightarrow \mathbb{R} \\ (v, v') &\longmapsto \|v' - v\|_g\,. \end{aligned}$$

Every inner product vector space (V, g) has a natural structure of metric topological space determined by the distance d_g. Complete inner product $\mathbb{C}$-vector spaces are known as "Hilbert spaces". Usually, the inner product of a Hilbert space $\mathcal{H}$ is denoted by $< \cdot, \cdot >_{\mathcal{H}}$. Henceforth, we shall write $\mathcal{H}$ to refer to a Hilbert space and keep the inner product $< \cdot, \cdot >_{\mathcal{H}}$ implicit.

If $L \subseteq \mathcal{H}$ is a subspace of an arbitrary Hilbert space, it is known that $(S^\perp)^\perp = \overline{S}$ where $\overline{L}$ denotes the closure of L. Accordingly, if $L \subseteq \mathcal{H}$ is closed, then $(L^\perp)^\perp = L$ and $\mathcal{H} = L \oplus L^\perp$.

We shall now recall the main properties of bounded operators of Hilbert spaces.

Definition 2.1. *If $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces, a linear map $f\colon \mathcal{H}_1 \to \mathcal{H}_2$ is said "bounded" when there exists $C \in \mathbb{R}^+$ such that*

$$\|f(v)\|_{g_2} \leq C \cdot \|v\|_{g_1}\,,$$

for every $v \in \mathcal{H}_1$.

We shall denote the set of bounded linear maps $f\colon \mathcal{H}_1 \to \mathcal{H}_2$ by $B(\mathcal{H}_1,\mathcal{H}_2)$ and the set of bounded endomorphisms of a Hilbert space $\mathcal{H}$ by $B(\mathcal{H})$. Given a linear map $f\colon \mathcal{H}_1 \to \mathcal{H}_2$, it is known that f is continuous if and only if f is bounded.

The spectrum of a bounded operator $f \in B(\mathcal{H})$ consists of complex numbers λ such that $f - \lambda \mathrm{Id}$ is not invertible. We shall denote the spectrum of f by $\sigma(f)$, and it is clear that every eigenvalue of f is an element of $\sigma(f)$. It is known that it is possible that an element of $\sigma(f)$ is not an eigenvalue.

Recently, the author of this work has studied in [19] the set of bounded finite potent endomorphisms on an arbitrary Hilbert space, which will be denoted by $B_{fp}(\mathcal{H})$.

If $\varphi \in B_{fp}(\mathcal{H})$, $\mathcal{H} = W_\varphi \oplus U_\varphi$ is the AST-decomposition induced by φ and $\varphi = \varphi_1 + \varphi_2$ is the CN-decomposition, then the following properties hold:

1. φ is quasi-compact;
2. $\varphi_1, \varphi_2 \in B_{fp}(\mathcal{H})$ and φ_1 is of trace class;
3. φ is compact if and only if φ_2 is compact;
4. if $\mathrm{Tr}(\varphi_1)$ is the trace of φ_1 as a trace class operator, then $\mathrm{Tr}(\varphi_1) = \mathrm{Tr}_{\mathcal{H}}(\varphi)$;
5. given a non-zero $\lambda \in \mathbb{C}$, one has that λ is an eigenvalue of φ if and only if λ is an eigenvalue of $\varphi_{|_{W_\varphi}}$;
6. if $i(\varphi) \geq 1$, then $\sigma(\varphi) = \{0, \lambda_1, \dots, \lambda_n\}$ where $\{\lambda_1, \dots, \lambda_n\}$ are the eigenvalues of $\varphi_{|_{W_\varphi}}$;
7. $\mathrm{Tr}_{\mathcal{H}}(\varphi) = \mathrm{Tr}_{W_\varphi}(\varphi_{|_{W_\varphi}}) = \mathrm{Tr}(\varphi_1) = \mathrm{Tr}^L_{\mathcal{H}}(\varphi) = \mathrm{Tr}^R_{\mathcal{H}}(\varphi)$;

where $\mathrm{Tr}_{\mathcal{H}}(\varphi)$ is the Tate's trace of φ as a finite potent endomorphism; $\mathrm{Tr}_{W_\varphi}(\varphi_{|_{W_\varphi}})$ is the trace of the endomorphism $\varphi_{|_{W_\varphi}}$ on the finite-dimensional $\mathbb{C}$-vector space W_φ; $\mathrm{Tr}(\varphi_1)$ is the trace of φ_1 of a trace class operator; $\mathrm{Tr}^L_{\mathcal{H}}(\varphi)$ is the Leray trace; and $\mathrm{Tr}^R_{\mathcal{H}}(\varphi)$ is the trace of φ as a Riesz trace class operator.

Moreover, the adjoint operator φ^* satisfies that:

1. $\varphi^* \in B_{fp}(\mathcal{H})$;
2. $i(\varphi^*) = i(\varphi)$;
3. $\varphi^* = (\varphi_1)^* + (\varphi_2)^*$ is the CN-decomposition of φ^*;
4. if $\mathcal{H} = W_{\varphi^*} \oplus U_{\varphi^*}$ is the AST-decomposition induced by φ^*, then one has that $W_{\varphi^*} = [U_\varphi]^\perp$ and $U_{\varphi^*} = [W_\varphi]^\perp$;
5. $\sigma(\varphi^*) = \overline{\sigma(\varphi)}$.

Now we shall recall two statements of [19] that shall be useful for the present work.

Thus, it follows from [19, Lemma 3.3] that

Lemma 2.2. *If $\mathcal{H}$ is a Hilbert space, $f \in \mathrm{End}_{\mathbb{C}}(\mathcal{H})$ and $U \subseteq \mathcal{H}$ is a closed subspace of finite codimension such that $f_{|_U} = 0$, then $f \in B(\mathcal{H})$.*

Moreover, from [19, Proposition 4.1] we know that

Proposition 2.3. *If $\mathcal{H}$ is a Hilbert space and we consider $\varphi \in B_{fp}(\mathcal{H})$, then the adjoint φ^* is also a bounded finite potent endomorphism.*

2.5. Moore-Penrose Inverse

2.5.1. Moore-Penrose Inverse of an $(n \times m)$-Matrix

Let $\mathbb{C}$ be the field. Given a matrix $A \in \mathrm{Mat}_{n\times m}(\mathbb{C})$, the Moore-Penrose inverse of A is a matrix $A^\dagger \in \mathrm{Mat}_{m\times n}(\mathbb{C})$ such that:

- $A\,A^\dagger\,A = A$;
- $A^\dagger\,A\,A^\dagger = A^\dagger$;
- $(A\,A^\dagger)^* = A\,A^\dagger$;

- $(A^\dagger A)^* = A^\dagger A$;

B^* being the conjugate transpose of the matrix B.

The Moore-Penrose inverse of A always exists, it is unique, $[A^\dagger]^\dagger = A$, and, if $A \in \mathbb{C}^{n\times n}$ is non-singular, then the Moore-Penrose inverse of A coincides with the inverse matrix A^{-1}.

For details, readers are referred to [8].

2.5.2. Moore-Penrose Inverse of a Linear Map over Arbitrary Inner Product Spaces

Let (V, g) and $(W, \bar{g})$ be inner product vector spaces over k, with $k = \mathbb{C}$ or $k = \mathbb{R}$.

Given a linear map $f: V \to W$, a linear map $f^+: W \to V$ is a reflexive generalized inverse of f when

- $f \circ f^+ \circ f = f$;
- $f^+ \circ f \circ f^+ = f^+$.

Definition 2.4. *Given a linear map $f: V \to W$, we say that f is admissible for the Moore-Penrose inverse when $V = \operatorname{Ker} f \oplus [\operatorname{Ker} f]^\perp$ and $W = Im\, f \oplus [Im\, f]^\perp$.*

According to [6, Theorem 3.12], if (V, g) and $(W, \bar{g})$ are inner product spaces over k, then $f: V \to W$ is a linear map admissible for the Moore-Penrose inverse if and only if there exists a unique linear map $f^\dagger: W \to V$ such that:

1. $f^\dagger$ is a reflexive generalized inverse of f;
2. $f^\dagger \circ f$ and $f \circ f^\dagger$ are self-adjoint, that is:
   - $g([f^\dagger \circ f](v), v') = g(v, [f^\dagger \circ f](v'))$;
   - $\bar{g}([f \circ f^\dagger](w), w') = \bar{g}(w, [f \circ f^\dagger](w'))$;

for all $v, v' \in V$ and $w, w' \in W$. The operator $f^\dagger$ is named the Moore-Penrose inverse of f and it is the unique linear map satisfying that

$$f^\dagger(w) = \begin{cases} (f_{|_{[\operatorname{Ker} f]^\perp}})^{-1}(w) & \text{if} \quad w \in \operatorname{Im} f \\ 0 & \text{if} \quad w \in [\operatorname{Im} f]^\perp \end{cases}.$$

The Moore-Penrose inverse $f^\dagger: W \to V$ also satisfies the following properties:

- $f^\dagger$ is admissible for the Moore-Penrose inverse and $(f^\dagger)^\dagger = f$;
- If $f \in \operatorname{End}_k(V)$ and f is an isomorphism, then $f^\dagger = f^{-1}$;
- $f^\dagger \circ f = P_{[\operatorname{Ker} f]^\perp}$;
- $f \circ f^\dagger = P_{\operatorname{Im} f}$;

where $P_{[\operatorname{Ker} f]^\perp}$ and $P_{\operatorname{Im} f}$ are the projections induced by the decompositions $V = \operatorname{Ker} f \oplus [\operatorname{Ker} f]^\perp$ and $W = \operatorname{Im} f \oplus [\operatorname{Im} f]^\perp$, respectively.

Lemma 2.5. *If V is k-vector space, $f \in \operatorname{End}_k(V)$ is an endomorphism admissible for the Moore-Penrose inverse and $g \in \operatorname{End}_k(V)$ is such that $Im\, g \subseteq Im\, f$, then*

$$f \circ f^\dagger \circ g = g\,.$$

Proof. The statement is immediately deduced from the equality $f \circ f^\dagger = P_{\operatorname{Im} f}$. □

Analogously, we can easily check that

Lemma 2.6. *If V is k-vector space, $f \in \operatorname{End}_k(V)$ is an endomorphism admissible for the Moore-Penrose inverse and $\tilde{g} \in \operatorname{End}_k(V)$ is such that $Im\, \tilde{g} \subseteq [\operatorname{Ker} f]^\perp$, then*

$$f^\dagger \circ f \circ \tilde{g} = \tilde{g}\,.$$

For more details on the Moore-Penrose inverse over arbitrary inner product spaces readers are referred to [6].

3. Moore-Penrose inverse of a bounded linear map

For the sake of completeness, in this section we shall now offer different properties of the Moore-Penrose inverse of a bounded linear map between two Hilbert spaces. Most of these properties are well known and we provide in this work new proofs of them.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ two Hilbert spaces. Given a linear map $f\colon \mathcal{H}_1 \to \mathcal{H}_2$, recall from Definition 2.4 that f is admissible for the Moore-Penrose inverse when $\mathcal{H}_1 = \operatorname{Ker} f \oplus [\operatorname{Ker} f]^\perp$ and $\mathcal{H}_2 = \operatorname{Im} f \oplus [\operatorname{Im} f]^\perp$.

Lemma 3.1. *If $f \in B(\mathcal{H}_1, \mathcal{H}_2)$, then f is admissible for the Moore-Penrose inverse if and only if $\operatorname{Im} f$ is a closed subspace of $\mathcal{H}_1$.*

Proof. Since f is bounded, we have that $\operatorname{Ker} f$ is closed and, therefore,

$$\mathcal{H}_1 = \operatorname{Ker} f \oplus [\operatorname{Ker} f]^\perp .$$

Thus, f is admissible for the Moore-Penrose inverse if and only if

$$\mathcal{H}_2 = \operatorname{Im} f \oplus [\operatorname{Im} f]^\perp .$$

However, bearing in mind that $\operatorname{Ker} f^* = [\operatorname{Im} f]^\perp$, one has that $[\operatorname{Im} f]^\perp$ is closed and

$$\mathcal{H}_2 = ([\operatorname{Im} f]^\perp)^\perp \oplus [\operatorname{Im} f]^\perp = \overline{\operatorname{Im} f} \oplus [\operatorname{Im} f]^\perp .$$

Accordingly, we conclude that f is admissible for the Moore-Penrose inverse if and only if $\overline{\operatorname{Im} f} = \operatorname{Im} f$, from where the claim is proved. □

Proposition 3.2. *If $f \in B(\mathcal{H}_1, \mathcal{H}_2)$ is admissible for the Moore-Penrose inverse, then $f^\dagger \in B(\mathcal{H}_2, \mathcal{H}_1)$.*

Proof. Since $[\operatorname{Ker} f]^\perp$ and $\operatorname{Im} f$ are closed with the hypothesis of the proposition, we have that $f_{|_{[\operatorname{Ker} f]^\perp}} \in B([\operatorname{Ker} f]^\perp, \operatorname{Im} f)$. Since $f_{|_{[\operatorname{Ker} f]^\perp}}$ is bijective, from the Bounded Inverse Theorem one has that $(f_{|_{[\operatorname{Ker} f]^\perp}})^{-1} = (f^\dagger)_{|_{\operatorname{Im} f}} \in B(\operatorname{Im} f, [\operatorname{Ker} f]^\perp)$ and there exists $\tilde{C} \in \mathbb{R}^+$ such that

$$||f^\dagger(v_2)||_{\mathcal{H}_1} \leq \tilde{C} \cdot ||v_2||_{\mathcal{H}_2}$$

for every $v_2 \in \operatorname{Im} f$.

Now, for every $h_2 \in \mathcal{H}_2$, if we write $h_2 = v_2 + u_2$ with $v_2 \in \operatorname{Im} f$ and $u_2 \in (\operatorname{Im} f)^\perp$, bearing in mind that $||v_2||_{\mathcal{H}_2} \leq ||h_2||_{\mathcal{H}_2}$, one has that

$$||f^\dagger(h_2)||_{\mathcal{H}_1} = ||f^\dagger(v_2)||_{\mathcal{H}_1} \leq \tilde{C} \cdot ||v_2||_{\mathcal{H}_2} \leq \tilde{C} \cdot ||h_2||_{\mathcal{H}_2} ,$$

and the assertion is proved. □

Proposition 3.3. *If $f \in B(\mathcal{H}_1, \mathcal{H}_2)$ is admissible for the Moore-Penrose inverse, then f^* is also admissible for the Moore-Penrose inverse and $(f^*)^\dagger = (f^\dagger)^*$.*

Proof. From Proposition 3.2, one deduces the existence of $(f^\dagger)^*$ because $f^\dagger$ is a bounded linear map.

Moreover, bearing in mind the conditions that uniquely determine the Moore-Penrose inverse of f, from the properties of the adjoint operator of a bounded linear map, one has that:

- $f^* \circ (f^\dagger)^* \circ f^* = f^*$;
- $(f^\dagger)^* \circ f^* \circ (f^\dagger)^* = (f^\dagger)^*$;
- $f^* \circ (f^\dagger)^* = (f^\dagger \circ f)^* = f^\dagger \circ f = [f^* \circ (f^\dagger)^*]^*$;

- $(f^{\dagger})^* \circ f^* = (f \circ f^{\dagger})^* = f \circ f^{\dagger} = [(f^{\dagger})^* \circ f^*]^*$,

from where we conclude that $(f^{\dagger})^*$ satisfies the conditions of the Moore-Penrose inverse of f^* and [6, Theorem 3.12] shows that f^* is also admissible for the Moore-Penrose inverse and $(f^*)^{\dagger} = (f^{\dagger})^*$. □

Corollary 3.4. *If $f \in B(\mathcal{H}_1, \mathcal{H}_2)$ such that Im f is a closed subspace of $\mathcal{H}_2$, then Im f^* is a closed subspace of $\mathcal{H}_1$.*

Proof. The statement is a direct consequence of Lemma 3.1 and Proposition 3.3. □

From the properties of the Moore-Penrose inverse of a linear map, if $f \in B(\mathcal{H}_1, \mathcal{H}_2)$ with Im f being a closed subspace of $\mathcal{H}_2$, one has that:

- $f^* \circ (f^*)^{\dagger} = P_{[\mathrm{Ker}\, f]^{\perp}}$;
- $(f^*)^{\dagger} \circ f^* = P_{\mathrm{Im}\, f}$,

where $P_{[\mathrm{Ker}\, f]^{\perp}}$ and $P_{\mathrm{Im}\, f}$ are the projections induced by the decompositions $\mathcal{H}_1 = \mathrm{Ker}\, f \oplus [\mathrm{Ker}\, f]^{\perp}$ and $\mathcal{H}_2 = \mathrm{Im}\, f \oplus [\mathrm{Im}\, f]^{\perp}$ respectively.

Lemma 3.5. *If $f \in B(\mathcal{H}_1, \mathcal{H}_2)$ such that Im f is a closed subspace of $\mathcal{H}_2$, then one has that:*

- $f^* \circ f \circ f^{\dagger} = f^*$;
- $f^{\dagger} \circ f \circ f^* = f^*$;
- $(f^*)^{\dagger} \circ f^* \circ f = f$;
- $f \circ f^* \circ (f^*)^{\dagger} = f$.

Proof. The assertion follows immediately from Proposition 3.3 and the above properties of the Moore-Penrose inverse for bounded finite potent linear maps. □

Given a Hilbert space $\mathcal{H}$, let us recall now that a bounded endomorphism $f \in B(\mathcal{H})$ with closed Im f is said to be "EP" when $f^{\dagger} \circ f = f \circ f^{\dagger}$.

Lemma 3.6. *If $\mathcal{H}$ is a Hilbert space and $f \in B(\mathcal{H})$ is a bounded endomorphism with closed Im f, one has that f is EP if and only if f^* is EP.*

Proof. Bearing in mind the properties of the adjoint operator, the assertion is immediately deduced from Proposition 3.3. □

If V is an arbitrary inner product vector space, it is known that if a finite potent endomorphism φ is EP, then $i(\varphi) \leq 1$.

4. Drazin inverse of a bounded finite potent operator

In this section, we shall now study the structure and basic properties of the Drazin inverse of a bounded finite potent endomorphism of an arbitrary Hilbert space. In particular, we shall show that the Drazin inverse of a finite potent endomorphism offered in [22] coincides with the definitions of the Drazin inverse for bounded operators given in [7] and [13].

4.1. Classical definitions of the Drazin inverse of a bounded operator

Let f be a bounded operator on a Banach space X. According to [14, Section 1], the "ascent of f", $\alpha(f)$, and the "descent of f", $\delta(f)$, are given by

$$\begin{aligned} \alpha(f) &= \inf\{n \in \mathbb{N} \text{ with } \operatorname{Ker} f^n = \operatorname{Ker} f^{n+1}\} \text{ and} \\ \delta(f) &= \inf\{n \in \mathbb{N} \text{ with } \operatorname{Im} f^n = \operatorname{Im} f^{n+1}\}, \end{aligned} \tag{5}$$

where the infimum over the empty set is taken as $+\infty$.

If $\alpha(f) = p < \infty$ and $\delta(f) = q < \infty$, it follows from [14, Theorem 1.2] that $p = q$ and

$$X = \operatorname{Ker} f^p \oplus \operatorname{Im} f^p . \tag{6}$$

Accordingly, writing

$$f_1 = f_{|_{\operatorname{Im} f^p}} \colon \operatorname{Im} f^p \to \operatorname{Im} f^p$$

and

$$f_2 = f_{|_{\operatorname{Ker} f^p}} \colon \operatorname{Ker} f^p \to \operatorname{Ker} f^p ,$$

D. C. Lay in [13] defined the Drazin inverse of f as the unique bounded linear operator f^d that equals to f_1^{-1} on $\operatorname{Im} f^p$ and is zero on $\operatorname{Ker} f^p$, and showed that f^d satisfies the following system of equations:

$$\begin{aligned} g \circ f \circ g &= g \\ g \circ f &= f \circ g \\ f^{s+1} \circ g &= f^s \end{aligned} \tag{7}$$

for every $s \geq p$.

Moreover, S. L. Campbell in [7] gives an alternative generalization of the Drazin inverse of bounded linear operators on Banach spaces. Thus, if f is a bounded operator on a Banach space X such that the hyperrange

$$R(f^\infty) = \bigcap_{n=1}^{\infty} \operatorname{Im} f^n$$

is closed and complemented by a closed f-invariant subspace $M \subset X$, and bearing in mind the f-invariant decomposition

$$X = R(f^\infty) \oplus M , \tag{8}$$

when $f_{|_{R(f^\infty)}} \colon R(f^\infty) \to R(f^\infty)$ is invertible, the generalized Drazin inverse provided by S. L. Campbell is the unique bounded operator f^δ that equals to $(f_{|_{R(f^\infty)}})^{-1}$ on $R(f^\infty)$ and zero on M.

Furthermore, J. J. Koliha in [12, Section 7] showed that if f^d and f^δ both exist, then $f^d = f^\delta$ if and only if $f_{|_M} \colon M \to M$ is quasinilpotent.

4.2. Structure of the Drazin inverse of a bounded finite potent operator

Let V be an arbitrary k-vector space and let $\varphi \in \operatorname{End}_k(V)$ be a finite potent endomorphism. Recall from Section 2.2 that the Drazin inverse of φ is the unique finite potent endomorphism $\varphi^D \in \operatorname{End}_k(V)$ satisfying that:

1. $\varphi^{r+1} \circ \varphi^D = \varphi^r$;
2. $\varphi^D \circ \varphi \circ \varphi^D = \varphi^D$;
3. $\varphi^D \circ \varphi = \varphi \circ \varphi^D$,

where r is the index of φ.

The map φ^D is the Drazin inverse of φ and is the unique linear map such that:

$$\varphi^D(v) = \begin{cases} (\varphi_{|W_\varphi})^{-1}(v) & \text{if} \quad v \in W_\varphi \\ 0 & \text{if} \quad v \in U_\varphi \end{cases} . \tag{9}$$

Note that for finite potent endomorphisms, the system of equations (7) has the Drazin inverse φ^D as its unique solution.

Moreover, φ^D satisfies the following properties:

- $(\varphi^D)^D = \varphi$ if and only if $i(\varphi) \leq 1$;
- $\varphi = \varphi^D$ if and only if $\varphi_{|U_\varphi} = 0$ and $(\varphi_{|W_\varphi})^2 = \mathrm{Id}_{|W_\varphi}$;
- if ψ is a projection finite potent endomorphism, then $\psi^D = \psi$.

We shall now study the Drazin inverse of a bounded finite potent operator on a Hilbert space.

Lemma 4.1. *If $\mathcal{H}$ is a Hilbert space and $\varphi \in B_{fp}(\mathcal{H})$, then the Drazin inverse φ^D is also a bounded finite potent endomorphism.*

Proof. Since φ^D is finite potent, we only need to prove that φ^D is bounded. However, bearing in mind the explicit expression of φ^D offered in (9), the boundedness of φ is immediately deduced from Lemma 2.2. □

Proposition 4.2. *If $\mathcal{H}$ is a Hilbert space and $\varphi \in B_{fp}(\mathcal{H})$, then $(\varphi^D)^* = (\varphi^*)^D$.*

Proof. Since from Lemma 4.1 one has that φ^D is bounded finite potent, Proposition 2.3 shows that $(\varphi^D)^*$ is also a bounded finite potent endomorphism.

Accordingly, bearing in mind that $i(\varphi) = i(\varphi^*)$, from the definition of the Drazin inverse of a finite potent endomorphism and the properties of the adjoint operator of bounded linear maps, we have that:

- $(\varphi^D)^* \circ \varphi^* = \varphi^* \circ (\varphi^D)^*$;
- $(\varphi^*)^{r+1} \circ (\varphi^D)^* = (\varphi^*)^r$;
- $(\varphi^D)^* \circ \varphi^* \circ (\varphi^D)^* = (\varphi^D)^*$,

where $r = i(\varphi^*)$.

Hence, the uniqueness of the Drazin inverse of a finite potent endomorphism implies that $(\varphi^D)^* = (\varphi^*)^D$. □

Moreover, it is immediately deduced from the basic properties of the Drazin inverse of a finite potent endomorphism that

Corollary 4.3. *If $\mathcal{H}$ is a Hilbert space and $\varphi \in B_{fp}(\mathcal{H})$, then one has that:*

- $([\varphi^*]^D)^D = \varphi^*$ *if and only if* $i(\varphi) \leq 1$;
- $\varphi^* = (\varphi^*)^D$ *if and only if* $\varphi_{|U_\varphi} = 0$ *and* $(\varphi_{|W_\varphi})^2 = Id_{|W_\varphi}$;
- *if ψ is a projection finite potent endomorphism, then $(\psi^*)^D = \psi^*$.*

Furthermore, if $\varphi \in B_{fp}(\mathcal{H})$ and $\mathcal{H} = W_{(\varphi^*)^D} \oplus W_{(\varphi^D)^*}$ is the AST-decomposition determined by $(\varphi^*)^D$, we have that

- $W_{(\varphi^*)^D} = [U_{(\varphi^D)}]^\perp = [U_\varphi]^\perp$;

- $U_{(\varphi^*)^D} = [W_{(\varphi^D)}]^\perp = [W_\varphi]^\perp$.

We shall now study the spectrum of φ^D.

Proposition 4.4. *If $\mathcal{H}$ is an arbitrary Hilbert space, $\varphi \in B_{fp}(\varphi)$ and $\mathcal{H} = W_\varphi \oplus U_\varphi$ is the AST-decomposition determined by φ, one has that the spectrum of φ^D is:*

- $\sigma(\varphi^D) = \{\lambda_1^{-1}, \dots, \lambda_n^{-1}\}$ *when* $i(\varphi) = 0$;
- $\sigma(\varphi^D) = \{0, \lambda_1^{-1}, \dots, \lambda_n^{-1}\}$ *when* $i(\varphi) \geq 1$;

where $\{\lambda_1, \dots, \lambda_n\}$ are the eigenvalues of $\varphi_{|W_\varphi}$.

Proof. Given a Hilbert space, it follows from [19, Proposition 3.14] that the spectrum of a bounded finite potent endomorphism $\varphi \in B_{fp}(\varphi)$ is:

- $\sigma(\varphi) = \{\lambda_1, \dots, \lambda_n\}$ when $i(\varphi) = 0$;
- $\sigma(\varphi) = \{0, \lambda_1, \dots, \lambda_n\}$ when $i(\varphi) \geq 1$;

where $\{\lambda_1, \dots, \lambda_n\}$ are the eigenvalues of $\varphi_{|W_\varphi}$.

Accordingly, the statement is immediately deduced from the explicit expression of φ^D and Lemma 4.1 bearing in mind that, if E is a finite dimensional k-vector space and $f \in \mathrm{Aut}_k(E)$, a non-zero $\lambda \in k$ is an eigenvalue of f if and only if λ^{-1} is an eigenvalue of f^{-1}. □

Lemma 4.5. *Keeping the previous notation, if $\{\lambda_i(\varphi)\}_{i\in\{1,\dots,s\}}$ is the listing of all non-zero eigenvalues of $\varphi \in B_{fp}(\mathcal{H})$, counted up to algebraic multiplicity, we have that:*

- $\mathrm{Tr}_{\mathcal{H}}(\varphi^D) = \sum_{i=1}^s [\lambda_i(\varphi)]^{-1}$;
- $\det_{\mathcal{H}}(\varphi^D) = \prod_{i=1}^s (1 + [\lambda_i(\varphi)]^{-1})$.

Proof. If E is a finite-dimensional k-vector space and $f \in \mathrm{Aut}_k(E)$, it is clear that

$$\dim_k \mathrm{Ker}(f - \lambda \mathrm{Id})^n = \dim_k \mathrm{Ker}(f^{-1} - \lambda^{-1}\mathrm{Id})^n$$

for every non-zero $\lambda \in k$ and for each $n \in \mathbb{N}$.

Hence, the algebraic multiplicity of a non-zero eigenvalue λ of $\varphi_{|W_\varphi}$ coincides with the algebraic multiplicity of λ^{-1} as an eigenvalue of $(\varphi_{|W_\varphi})^{-1}$ and, therefore, the statement follows from (1), (2) and (9). □

Example 4.6. *Let $\{u_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of a separable Hilbert space $\mathcal{H}$. If we consider $\varphi \in B_{fp}(\mathcal{H})$ determined by the conditions*

$$\varphi(u_i) = \begin{cases} u_1 + u_2 + u_4 & \text{if} \quad i = 1 \\ 2u_1 + u_3 & \text{if} \quad i = 2 \\ u_1 - 2u_2 + 3u_3 - 2u_4 & \text{if} \quad i = 3 \\ 0 & \text{if} \quad i = 4 \\ \frac{1}{i^2} u_4 & \text{if} \quad i \geq 5 \end{cases},$$

an easy computation shows that

$$\varphi^*(u_i) = \begin{cases} u_1 + 2u_2 + u_3 & \text{if} \quad i = 1 \\ u_1 - 2u_3 & \text{if} \quad i = 2 \\ u_2 + 3u_3 & \text{if} \quad i = 3 \\ u_1 - 2u_3 + \sum_{j\geq 5} \frac{1}{j^2} u_j & \text{if} \quad i = 4 \\ 0 & \text{if} \quad i \geq 5 \end{cases}.$$

Thus, since $W_\varphi = \langle u_1, u_2 + u_4, u_3 \rangle$ and $U_\varphi = \overline{\langle u_i \rangle_{i\geq 4}}$, one has that:

- $W_{\varphi^*} = U_{\varphi}^{\perp} = \langle u_1, u_2, u_3 \rangle$;
- $U_{\varphi^*} = W_{\varphi}^{\perp} = \langle u_2 - u_4 \rangle \oplus \overline{\langle u_j \rangle_{j \geq 5}}$.

Also, it is clear that $i(\varphi) = i(\varphi^*) = 2$.

In this case we have that

$$\varphi^D(u_i) = \begin{cases} -\frac{2}{3}u_1 + u_2 - \frac{1}{3}u_3 + u_4 & \text{if} \quad i = 1 \\ \frac{5}{3}u_1 - u_2 + \frac{1}{3}u_3 - u_4 & \text{if} \quad i = 2 \\ \frac{4}{3}u_1 - u_2 + \frac{2}{3}u_3 - u_4 & \text{if} \quad i = 3 \\ 0 & \text{if} \quad i \geq 4 \end{cases}$$

and

$$(\varphi^D)^*(u_i) = \begin{cases} -\frac{2}{3}u_1 + \frac{5}{3}u_2 + \frac{4}{3}u_3 & \text{if} \quad i = 1 \\ u_1 - u_2 - u_3 & \text{if} \quad i = 2 \\ -\frac{1}{3}u_1 + \frac{1}{3}u_2 + \frac{2}{3}u_3 & \text{if} \quad i = 3 \\ u_1 - u_2 - u_3 & \text{if} \quad i = 4 \\ 0 & \text{if} \quad i \geq 5 \end{cases},$$

and one can easily check that $(\varphi^D)^* = (\varphi^*)^D$.

Moreover, since $W_{\varphi^D} = W_{\varphi}$ *and*

$$(\varphi^D)_{|_{W_{\varphi^D}}} \equiv \begin{pmatrix} -\frac{2}{3} & \frac{5}{3} & \frac{4}{3} \\ 1 & -1 & -1 \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

in the basis $\{u_1, u_2 + u_4, u_3\}$ *of* W_{φ^D}, *one has that*

$$\mathrm{Tr}_{\mathcal{H}}(\varphi^D) = -1 \quad \text{and} \quad \det_{\mathcal{H}}(Id + \varphi^D) = -\frac{15}{9}.$$

4.2.1. Relationship of the Drazin inverse of a bounded finite potent endomorphism with the classical definitions of the generalized Drazin inverse for bounded operators

Our task is now to relate the Drazin inverse of a bounded finite potent operator on a Hilbert space studied above to the classical definitions of the Drazin inverse for bounded operators on Banach spaces referred to in Section 4.1.

Lemma 4.7. *If* $\mathcal{H}$ *is a Hilbert space and* $\varphi \in B_{fp}(\mathcal{H})$, *then*

$$i(\varphi) = \alpha(\varphi) = \delta(\varphi),$$

where $\alpha(\varphi)$ *and* $\delta(\varphi)$ *are the ascent and the descent of* φ *respectively.*

Proof. If $\mathcal{H} = W_{\varphi} \oplus U_{\varphi}$ is the AST-decomposition of $\mathcal{H}$ induced by φ, bearing in mind that $\mathrm{Im}\, \varphi^m = W_{\varphi}$ and $\mathrm{Ker}\, \varphi^m = U_{\varphi}$ for every $m \geq i(\varphi)$, the claim is deduced from the definitions of $\alpha(\varphi)$ and $\delta(\varphi)$ given in (5). □

Proposition 4.8. *If* $\mathcal{H}$ *is a Hilbert space and* $\varphi \in B_{fp}(\mathcal{H})$, *then* φ *satisfies the Lay condition for the existence of the generalized Drazin inverse* φ^d *and* $\varphi^d = \varphi^D$.

Proof. It follows from Lemma 4.7 that the AST decomposition $\mathcal{H} = W_{\varphi} \oplus U_{\varphi}$ coincides with the Lay decomposition (6). Hence, the explicit expression of φ^D given by (9) shows that $\varphi^d = \varphi^D$. □

Proposition 4.9. *If* $\mathcal{H}$ *is a Hilbert space and* $\varphi \in B_{fp}(\mathcal{H})$, *then* φ *satisfies the Campbell conditions for the existence of the generalized Drazin inverse* φ^δ *and* $\varphi^\delta = \varphi^D$.

Proof. If $\mathcal{H} = W_\varphi \oplus U_\varphi$ is again the AST-decomposition of $\mathcal{H}$ induced by φ, since $\operatorname{Im} \varphi^m = W_\varphi$ for all $m \geq i(\varphi)$, one has that $R(\varphi^\infty) = W_\varphi$. Hence, bearing in mind that the AST-decomposition satisfies the conditions of the Campbell decomposition (8) with $M = U_\varphi$ and $\varphi_{|W_\varphi}$ is invertible, from (9) we deduce that $\varphi^\delta = \varphi^D$. □

We can summarize the statements of Propositions 4.8 and 4.9 in

Theorem 4.10. *If $\mathcal{H}$ is a Hilbert space and $\varphi \in B_{fp}(\mathcal{H})$, then φ^d and φ^δ exist and we have that*

$$\varphi^d = \varphi^\delta = \varphi^D .$$

4.3. *Group Inverse of a bounded finite potent endomorphism*

Given an arbitrary k-vector space V and a finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$, we say that $\varphi^\# \in \operatorname{End}_k(V)$ is the group inverse of φ when

- $\varphi \circ \varphi^\# \circ \varphi = \varphi$;
- $\varphi^\# \circ \varphi \circ \varphi^\# = \varphi^\#$;
- $\varphi^\# \circ \varphi = \varphi \circ \varphi^\#$.

It is known from [18, Lemma 3.4 and Theorem 3.5] that the group inverse of a finite potent endomorphism φ exists if and only if $i(\varphi) \leq 1$ and, in this case, $\varphi^\# = \varphi^D$.

Lemma 4.11. *If $\mathcal{H}$ is a Hilbert space and $\varphi \in B_{fp}(\mathcal{H})$, then the group inverse $\varphi^\# \in B(\mathcal{H})$ exists, is unique and coincides with φ^D if and only if $i(\varphi) \leq 1$.*

Proof. Bearing in mind the properties of the group inverse of a finite potent endomorphism of a vector space, we only need to check that $\varphi^\# = \varphi^D \in B(\mathcal{H})$, which is immediately deduced from Lemma 4.1. □

Corollary 4.12. *If $\mathcal{H}$ is a Hilbert space and $\varphi \in B_{fp}(\mathcal{H})$ with $i(\varphi) \leq 1$, then $(\varphi^*)^\# \in B_{fp}(\mathcal{H})$ and $(\varphi^*)^\# = (\varphi^\#)^* = (\varphi^D)^*$.*

Proof. Bearing in mind that from the properties of the adjoint operator we have that

- $\varphi^* \circ (\varphi^\#)^* \circ \varphi^* = \varphi^*$;
- $(\varphi^\#)^* \circ \varphi^* \circ (\varphi^\#)^* = (\varphi^\#)^*$;
- $(\varphi^\#)^* \circ \varphi^* = \varphi^* \circ (\varphi^\#)^*$,

the claim is immediately deduced from the uniqueness of the group inverse of finite potent endomorphisms with an index less than or equal to 1, from Proposition 2.3 and from Lemma 4.11. □

Example 4.13. *Using the data from Example 4.6 and an easy computation we have that*

$$\varphi_1(u_i) = \begin{cases} u_1 + u_2 + u_4 & \text{if} \quad i = 1 \\ 2u_1 + u_3 & \text{if} \quad i = 2 \\ u_1 - 2u_2 + 3u_3 - 2u_4 & \text{if} \quad i = 3 \\ 0 & \text{if} \quad i \geq 4 \end{cases}$$

and, since $(\varphi_1)^\# = (\varphi_1)^D = \varphi^D$, from Corollary 4.12 and the explicit expression of $(\varphi^D)^*$ offered in Example 4.6 one has that

$$(\varphi_1^*)^\#(u_i) = \begin{cases} -\frac{2}{3}u_1 + \frac{5}{3}u_2 + \frac{4}{3}u_3 & \text{if} \quad i = 1 \\ u_1 - u_2 - u_3 & \text{if} \quad i = 2 \\ -\frac{1}{3}u_1 + \frac{1}{3}u_2 + \frac{2}{3}u_3 & \text{if} \quad i = 3 \\ u_1 - u_2 - u_3 & \text{if} \quad i = 4 \\ 0 & \text{if} \quad i \geq 5 \end{cases} .$$

5. DMP inverses of a bounded finite potent endomorphism

This final section deals with the study of the DMP inverses of finite potent endomorphisms on arbitrary Hilbert spaces.

Given an arbitrary k-vector space V and a finite potent endomorphism $\varphi \in \text{End}_k(V)$ admissible for the Moore-Penrose inverse, according to [20, Theorem 3.2], there exists a unique finite potent endomorphism $\varphi^{d,\dagger} \in \text{End}_k(V)$ verifying that:

1. $\varphi^{d,\dagger} \circ \varphi \circ \varphi^{d,\dagger} = \varphi^{d,\dagger}$;
2. $\varphi^r \circ \varphi^{d,\dagger} = \varphi^r \circ \varphi^\dagger$ with $r = i(\varphi)$;
3. $\varphi^{d,\dagger} \circ \varphi = \varphi^D \circ \varphi$,

where φ^D is the Drazin inverse and $\varphi^\dagger$ is the Moore-Penrose inverse of φ.

The map $\varphi^{d,\dagger}$ is called the left-Drazin Moore-Penrose (lDMP) inverse of φ.

Moreover, from [20, Theorem 3.17] one has the existence and uniqueness of a finite potent endomorphism $\varphi^{\dagger,d} \in \text{End}_k(V)$ satisfying that:

1. $\varphi^{\dagger,d} \circ \varphi \circ \varphi^{\dagger,d} = \varphi^{\dagger,d}$;
2. $\varphi^{\dagger,d} \circ \varphi^r = \varphi^\dagger \circ \varphi^r$ with $r = i(\varphi)$;
3. $\varphi \circ \varphi^{\dagger,d} = \varphi \circ \varphi^D$.

The map $\varphi^{\dagger,d}$ is the right-Drazin Moore-Penrose (rDMP) inverse of φ.

The lDMP-inverse and the rDMP-inverse of a finite potent endomorphism respectively generalize the notions of DMP-inverse and dual DMP-inverse of a finite complex matrix introduced by S. B. Malik and N. Thome in [15].

According to the statements of [20, Section 3.A], if $\varphi \in \text{End}_k(V)$ is finite potent admissible for the Moore-Penrose inverse, then one has that:

1. $\varphi \circ \varphi^{d,\dagger} \circ \varphi = \varphi_1$;
2. $\varphi^{d,\dagger} = \varphi^D \circ P_{\text{Im}\,\varphi}$;
3. $(\varphi^{d,\dagger})^n = \begin{cases} (\varphi^D \circ \varphi^\dagger)^{\frac{n}{2}} & \text{if } n \text{ is even} \\ \varphi \circ (\varphi^D \circ \varphi^\dagger)^{\frac{n+1}{2}} & \text{if } n \text{ is odd} \end{cases}$;
4. $i(\varphi^{d,\dagger}) \leq 1$;
5. $((\varphi^{d,\dagger})^D)^D = \varphi^{d,\dagger}$;
6. If $\varphi^{d,\dagger} = \varphi$, then $\varphi^\dagger = \varphi^D$;
7. $\varphi^{d,\dagger} = \varphi$ if and only if φ is EP and tripotent;
8. $\varphi^{d,\dagger} = 0$ if and only if φ is nilpotent or $\varphi = 0$.

Moreover, given a finite potent endomorphism $\varphi \in \text{End}_k(V)$ admissible for the Moore-Penrose inverse, from [20, Section 3.B] we have that:

1. $\varphi \circ \varphi^{\dagger,d} \circ \varphi = \varphi_1$;
2. $\varphi^{\dagger,d} = P_{[\text{Ker}\,\varphi]^\perp} \circ \varphi^D$;
3. $(\varphi^{\dagger,d})^n = \begin{cases} (\varphi^\dagger \circ \varphi^D)^{\frac{n}{2}} & \text{if n is even} \\ (\varphi^\dagger \circ \varphi^D)^{\frac{n+1}{2}} \circ \varphi & \text{if n is odd} \end{cases}$;
4. $i(\varphi^{\dagger,d}) \leq 1$;
5. $((\varphi^{\dagger,d})^D)^D = \varphi^{\dagger,d}$;
6. $\varphi^{\dagger,d} = 0$ if and only if φ is nilpotent or $\varphi = 0$;
7. if $\varphi^{\dagger,d} = \varphi$, then $\varphi^\dagger = \varphi^D$;
8. $\varphi^{\dagger,d} = \varphi$ if and only φ is EP and tripotent.

Lemma 5.1. *If $\mathcal{H}$ is a Hilbert space and $\varphi \in B_{fp}(\mathcal{H})$ with closed Im φ, then one has that $\varphi^{d,\dagger}, \varphi^{\dagger,d} \in B_{fp}(\mathcal{H})$.*

Proof. Since $\varphi^{d,\dagger}$ and $\varphi^{\dagger,d}$ are finite potent, we only have to check that both linear maps are bounded to prove the claim. Thus, bearing in mind that from [20, Theorem 3.2] we know that $\varphi^{d,\dagger} = \varphi^D \circ \varphi \circ \varphi^\dagger$ and [20, Theorem 3.17] shows that $\varphi^{\dagger,d} = \varphi^\dagger \circ \varphi \circ \varphi^D$, the boundedness of these endomorphisms are immediately deduced from Lemma 4.1 and Proposition 3.2. □

Proposition 5.2. *If $\mathcal{H}$ is a Hilbert space and $\varphi \in B_{fp}(\mathcal{H})$ with closed Im φ, then the lDMP-inverse $(\varphi^*)^{d,\dagger}$ and the rDMP-inverse $(\varphi^*)^{\dagger,d}$ of the adjoint linear map φ^* exist. Moreover, $(\varphi^*)^{d,\dagger}, (\varphi^*)^{\dagger,d} \in B_{fp}(\mathcal{H})$, $(\varphi^*)^{d,\dagger} = (\varphi^{\dagger,d})^*$ and $(\varphi^*)^{\dagger,d} = (\varphi^{d,\dagger})^*$.*

Proof. The existence and uniqueness of the bounded finite potent endomorphisms $(\varphi^*)^{d,\dagger}$ and $(\varphi^*)^{\dagger,d}$ follow from Proposition 2.3 and Lemma 5.1. Now, if $i(\varphi) = r$, from Proposition 4.2 and Proposition 3.3 we have that

- $(\varphi^{d,\dagger})^* \circ \varphi^* \circ (\varphi^{d,\dagger})^* = (\varphi^{d,\dagger})^*$;
- $(\varphi^{d,\dagger})^* \circ (\varphi^*)^r = (\varphi^*)^\dagger \circ (\varphi^*)^r$;
- $\varphi^* \circ (\varphi^{d,\dagger})^* = \varphi^* \circ (\varphi^*)^D$;

and

- $(\varphi^{\dagger,d})^* \circ \varphi^* \circ (\varphi^{\dagger,d})^* = (\varphi^{\dagger,d})^*$;
- $(\varphi^*)^r \circ (\varphi^{\dagger,d})^* = (\varphi^*)^r \circ (\varphi^*)^\dagger$;
- $(\varphi^{\dagger,d})^* \circ \varphi^* = (\varphi^*)^D \circ \varphi^*$.

Hence, bearing in mind that [19, Corollary 4.4] shows that $i(\varphi^*) = r$, we conclude that $(\varphi^*)^{d,\dagger} = (\varphi^{\dagger,d})^*$ and $(\varphi^*)^{\dagger,d} = (\varphi^{d,\dagger})^*$. □

From the above-mentioned properties of the lDMP-inverse and the rDMP-inverse of finite potent endomorphisms, given $\varphi \in B_{fp}(\mathcal{H})$ with closed Im φ, one has that:

1. $\varphi^* \circ (\varphi^*)^{d,\dagger} \circ \varphi^* = \varphi^* \circ (\varphi^*)^{\dagger,d} \circ \varphi^* = (\varphi_1)^*$;
2. $(\varphi^*)^{\dagger,d} = P_{\mathrm{Im}\,\varphi} \circ (\varphi^*)^D$;
3. $(\varphi^*)^{d,\dagger} = (\varphi^*)^D \circ P_{[\mathrm{Ker}\,\varphi]^\perp}$;
4. $(((\varphi^*)^{d,\dagger})^D)^D = (\varphi^{\dagger,d})^*$;
5. $(((\varphi^*)^{\dagger,d})^D)^D = (\varphi^{d,\dagger})^*$;
6. If $(\varphi^*)^{\dagger,d} = \varphi^*$ or $(\varphi^*)^{d,\dagger} = \varphi^*$, then $\varphi^\dagger = \varphi^D$;
7. $(\varphi^*)^{\dagger,d} = \varphi^* = (\varphi^*)^{d,\dagger}$ if and only if φ is EP and tripotent;
8. $(\varphi^*)^{\dagger,d} = 0 = (\varphi^*)^{d,\dagger}$ if and only if φ is nilpotent or $\varphi = 0$.

Lemma 5.3. *If $\mathcal{H}$ is a Hilbert space and $\varphi \in B_{fp}(\mathcal{H})$ with closed Im φ, then $(\varphi^*)^{\dagger,d} = (\varphi^D)^*$ if and only if* $\mathrm{Ker}\,\varphi^\dagger \subseteq \mathrm{Ker}\,\varphi^D$.

Proof. The assertion follows from Proposition 5.2, because [21, Proposition 3.1 and Corollary 3.2] show that $\varphi^{d,\dagger} = \varphi^D$ if and only if $\mathrm{Ker}\,\varphi^\dagger \subseteq \mathrm{Ker}\,\varphi^D$. □

Lemma 5.4. *If $\mathcal{H}$ is a Hilbert space and $\varphi \in B_{fp}(\mathcal{H})$ with closed Im φ, then $(\varphi^*)^{d,\dagger} = (\varphi^D)^*$ if and only if* $W_\varphi \subseteq [\mathrm{Ker}\,\varphi]^\perp$.

Proof. The claim is again immediately deduced from Proposition 5.2, because from [21, Proposition 3.1 and Corollary 3.2] we know that $\varphi^{\dagger,d} = \varphi^D$ if and only if $W_\varphi \subseteq [\mathrm{Ker}\,\varphi]^\perp$. □

We shall now study the spectrum of the DMP inverses of bounded finite potent endomorphisms.

Lemma 5.5. *If $\mathcal{H}$ is a Hilbert space and $\varphi \in B_{fp}(\mathcal{H})$ with closed Im φ, given a non-zero $\lambda \in \mathbb{C}$, one has that λ is an eigenvalue of $\varphi^{d,\dagger}$ if and only if λ^{-1} is an eigenvalue of φ. Moreover, the multiplicity of λ as an eigenvalue of $\varphi^{d,\dagger}$ coincides with the multiplicity of λ^{-1} as an eigenvalue of φ.*

Proof. Recall from [19, Proposition 3.12] that λ is an eigenvalue of $\varphi^{d,\dagger}$ if and only if λ is an eigenvalue of $(\varphi^{d,\dagger})_{|_{W_{\varphi^{d,\dagger}}}}$. Thus, since from [20, Lemma 3.5] we know that $W_{\varphi^{d,\dagger}} = W_{\varphi}$, bearing in mind that

$$(\varphi^{d,\dagger})_{|_{W_{\varphi^{d,\dagger}}}} = (\varphi_{|_{W_\varphi}})^{-1}\,,$$

we deduce that λ is an eigenvalue of $\varphi^{d,\dagger}$ if and only if λ^{-1} is an eigenvalue of φ.

Moreover, given a non-zero $\lambda \in \mathbb{C}$, bearing in mind that

$$\dim_{\mathbb{C}} \operatorname{Ker}(\varphi_{|_{W_\varphi}} - \lambda \mathrm{Id})^n = \dim_{\mathbb{C}} \operatorname{Ker}((\varphi_{|_{W_\varphi}})^{-1} - \lambda^{-1}\mathrm{Id})^n$$

for every $n \in \mathbb{N}$, the statement is proved. □

Corollary 5.6. *If $\mathcal{H}$ is an arbitrary Hilbert space, $\varphi \in B_{fp}(\varphi)$ and $\mathcal{H} = W_\varphi \oplus U_\varphi$ is the AST-decomposition determined by φ, one has that the spectrum of $\varphi^{d,\dagger}$ is:*

- $\sigma(\varphi^{d,\dagger}) = \{\lambda_1^{-1}, \dots, \lambda_n^{-1}\}$ *when* $i(\varphi) = 0$;
- $\sigma(\varphi^{d,\dagger}) = \{0, \lambda_1^{-1}, \dots, \lambda_n^{-1}\}$ *when* $i(\varphi) \geq 1$;

where $\{\lambda_1, \dots, \lambda_n\}$ are the eigenvalues of $\varphi_{|_{W_\varphi}}$.

Proof. The claim is a direct consequence of [19, Proposition 3.14] and Lemma 5.5. □

Lemma 5.7. *If $\mathcal{H}$ is a Hilbert space and $\varphi \in B_{fp}(\mathcal{H})$ with closed Im φ, then $\sigma(\varphi^{\dagger,d}) = \sigma(\varphi^{d,\dagger})$.*

Proof. Bearing in mind that from [19, Proposition 4.8] we know that $\sigma(\varphi^*) = \overline{\sigma(\varphi)}$, one has that

$$\sigma(\varphi^{\dagger,d}) = \sigma([(\varphi^*)^*]^{\dagger,d}) = \sigma([(\varphi^*)^{d,\dagger}]^*) = \overline{\sigma((\varphi^*)^{d,\dagger})}$$

and the assertion is deduced from [19, Proposition 3.14] and Corollary 5.6. □

Proposition 5.8. *Given a Hilbert space $\mathcal{H}$ and a bounded finite potent endomorphism $\varphi \in B_{fp}(\mathcal{H})$ with closed Im φ, one has that*

$$\mathrm{Tr}_{\mathcal{H}}(\varphi^{d,\dagger}) = \mathrm{Tr}_{\mathcal{H}}(\varphi^{\dagger,d}) = \mathrm{Tr}_{\mathcal{H}}(\varphi^{D})$$

and

$$\det{}_{\mathcal{H}}(Id + \varphi^{d,\dagger}) = \det{}_{\mathcal{H}}(Id + \varphi^{\dagger,d}) = \det{}_{\mathcal{H}}(Id + \varphi^{D})\,.$$

Proof. If $\mathcal{H} = W_\varphi \oplus U_\varphi$ is the AST-decomposition of $\mathcal{H}$ induced by φ, it is known from [20, Section 3.B] that

$$W_{\varphi^{\dagger,d}} = P_{[\operatorname{Ker}\varphi]^\perp}(W_\varphi) = \varphi^\dagger(W_\varphi)\,.$$

Now, bearing in mind that

$$(\varphi^{\dagger,d} \circ \varphi^\dagger)(W_\varphi) = (\varphi^\dagger \circ \varphi^D)[(\varphi \circ \varphi^\dagger \circ \varphi)(W_\varphi)] = (\varphi^\dagger \circ \varphi^D)(W_\varphi)\,,$$

we have the commutative diagram of isomorphisms of linear maps

$$\begin{tikzcd} W_{\varphi} \arrow[r, "\varphi^D"] \arrow[d, "\varphi^{\dagger}"'] & W_{\varphi} \arrow[d, "\varphi^{\dagger}"] \\ W_{\varphi^{\dagger,d}} \arrow[r, "\varphi^{\dagger,d}"] & W_{\varphi^{\dagger,d}} \end{tikzcd}$$

and we deduce that

$$\mathrm{Tr}_{\mathcal{H}}(\varphi^{\dagger,d}) = \mathrm{Tr}_{\mathcal{H}}(\varphi^D) \quad \text{and} \quad \det_{\mathcal{H}}(\mathrm{Id} + \varphi^{\dagger,d}) = \det_{\mathcal{H}}(\mathrm{Id} + \varphi^D)\,.$$

Thus, the statement is immediately deduced from (1), (2), Proposition 4.4 and Corollary 5.6.
□

Example 5.9. *If $\mathcal{H}$ and $\varphi \in B_{fp}(\mathcal{H})$ are as in Example 4.6, since a non-difficult computation shows that $\varphi^{\dagger}$ is determined by the assignations*

$$\varphi^{\dagger}(u_i) = \begin{cases} -\frac{2}{3}u_1 + u_2 - \frac{1}{3}u_3 & \text{if} \quad i = 1 \\ \frac{5}{3}u_1 - u_2 + \frac{1}{3}u_3 - \frac{1}{\lambda}\sum_{j\geq 5}(\frac{1}{j^2}u_j) & \text{if} \quad i = 2 \\ \frac{4}{3}u_1 - u_2 + \frac{2}{3}u_3 & \text{if} \quad i = 3 \\ \frac{1}{\lambda}\sum_{j\geq 5}(\frac{1}{j^2}u_j) & \text{if} \quad i = 4 \\ 0 & \text{if} \quad i \geq 5 \end{cases}$$

with $\lambda = \sum_{j\geq 5}\frac{1}{j^4} = \frac{\pi^4}{90} - \frac{827}{768}$, from the explicit expression of φ^D obtained in Example 4.6, we have that

$$\varphi^{d,\dagger}(u_i) = \begin{cases} -\frac{2}{3}u_1 + u_2 - \frac{1}{3}u_3 + u_4 & \text{if} \quad i = 1 \\ \frac{5}{3}u_1 - u_2 + \frac{1}{3}u_3 - u_4 & \text{if} \quad i = 2 \\ \frac{4}{3}u_1 - u_2 + \frac{2}{3}u_3 - u_4 & \text{if} \quad i = 3 \\ 0 & \text{if} \quad i \geq 4 \end{cases}$$

and

$$\varphi^{\dagger,d}(u_i) = \begin{cases} -\frac{2}{3}u_1 + u_2 - \frac{1}{3}u_3 & \text{if} \quad i = 1 \\ \frac{5}{3}u_1 - u_2 + \frac{1}{3}u_3 & \text{if} \quad i = 2 \\ \frac{4}{3}u_1 - u_2 + \frac{2}{3}u_3 & \text{if} \quad i = 3 \\ 0 & \text{if} \quad i \geq 4 \end{cases}.$$

Furthermore, according to Proposition 5.2, one has that

$$(\varphi^*)^{d,\dagger} = (\varphi^{\dagger,d})^* = \begin{cases} -\frac{2}{3}u_1 + \frac{5}{3}u_2 + \frac{4}{3}u_3 & \text{if} \quad i = 1 \\ u_1 - u_2 - u_3 & \text{if} \quad i = 2 \\ -\frac{1}{3}u_1 + \frac{1}{3}u_2 + \frac{2}{3}u_3 & \text{if} \quad i = 3 \\ 0 & \text{if} \quad i \geq 4 \end{cases}$$

and

$$(\varphi^*)^{\dagger,d} = (\varphi^{d,\dagger})^* = \begin{cases} -\frac{2}{3}u_1 + \frac{5}{3}u_2 + \frac{4}{3}u_3 & \text{if} \quad i = 1 \\ u_1 - u_2 - u_3 & \text{if} \quad i = 2 \\ -\frac{1}{3}u_1 + \frac{1}{3}u_2 + \frac{2}{3}u_3 & \text{if} \quad i = 3 \\ u_1 - u_2 - u_3 & \text{if} \quad i = 4 \\ 0 & \text{if} \quad i \geq 5 \end{cases}.$$

Now, since $W_{\varphi^{d,\dagger}} = W_{\varphi} = \langle u_1, u_2 + u_4, u_3 \rangle$, $W_{\varphi^{\dagger,d}} = \varphi^{\dagger}(W_{\varphi}) = \langle u_1, u_2, u_3 \rangle$ *and*

$$(\varphi^{d,\dagger})_{|_{W_{\varphi^{d,\dagger}}}} \equiv \begin{pmatrix} -\frac{2}{3} & \frac{5}{3} & \frac{4}{3} \\ 1 & -1 & -1 \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \equiv (\varphi^{\dagger,d})_{|_{W_{\varphi^{\dagger,d}}}}$$

in the bases $\{u_1, u_2 + u_4, u_3\}$ *of* $W_{\varphi^{d,\dagger}}$ *and* $\{u_1, u_2, u_3\}$ *of* $W_{\varphi^{\dagger,d}}$ *respectively, we obtain that*

$$\mathrm{Tr}_{\mathcal{H}}(\varphi^{d,\dagger}) = \mathrm{Tr}_{\mathcal{H}}(\varphi^{\dagger,d}) = -1 \quad \textit{and} \quad \det_{\mathcal{H}}(Id + \varphi^{d,\dagger}) = \det_{\mathcal{H}}(Id + \varphi^{\dagger,d}) = -\frac{15}{9}\,.$$

Finally, from the computations made in Example 4.6, readers can check that this example allows us to illustrate the statements of Lemma 5.3 and Proposition 5.8.

Remark 5.10. *Given an arbitrary k-vector space* V*, recall that a finite potent endomorphism* $\varphi \in \mathrm{End}_k(V)$ *with CN-decomposition* $\varphi = \varphi_1 + \varphi_2$ *and admissible for the Moore-Penrose inverse is Core-EP when* $\varphi_1 \circ \varphi^{\dagger} = \varphi^{\dagger} \circ \varphi_1$ *([17, Definition 4.12]). Moreover, from [20, Section 3.C] it is known that a finite potent endomorphism* $\varphi \in \mathrm{End}_k(V)$ *admissible for the Moore-Penrose inverse is Core-EP if and only if*

$$\varphi^{\dagger,d} = \varphi^{d,\dagger}\,.$$

Accordingly, the endomorphism $\varphi \in B_{fp}(\mathcal{H})$ *studied in Example 5.9 is not Core-EP because* $\varphi^{\dagger,d} \neq \varphi^{d,\dagger}$.

Remark 5.11 (Final Remark). *Bearing in mind the equivalence between finite square matrices and endomorphisms of finite-dimensional vector spaces, given an* $(n \times n)$ *complex matrix* A *with core-nilpotent decomposition* $A = A_1 + A_2$*, from the statements of this work we have obtained proofs of the following assertions:*

1. $A^* \cdot (A^*)^{d,\dagger} \cdot A^* = A^* \cdot (A^*)^{\dagger,d} \cdot A^* = (A_1)^*$;
2. $(A^*)^{\dagger,d} = P_{R(A)} \cdot (A^*)^D$;
3. $(A^*)^{d,\dagger} = (A^*)^D \cdot P_{[N(A)]^\perp}$;
4. $(((A^*)^{d,\dagger})^D)^D = (A^{\dagger,d})^*$;
5. $(((A^*)^{\dagger,d})^D)^D = (A^{d,\dagger})^*$;
6. *If* $(A^*)^{\dagger,d} = A^*$ *or* $(A^*)^{d,\dagger} = A^*$*, then* $A^\dagger = A^D$;
7. $(A^*)^{\dagger,d} = A^* = (A^*)^{d,\dagger}$ *if and only if* A *is EP and tripotent;*
8. $(A^*)^{\dagger,d} = 0 = (A^*)^{d,\dagger}$ *if and only if* A *is nilpotent or* $A = 0$;

where $R(A)$ *is the range of* A*,* $N(A)$ *is the nullspace of* A*,* $P_{R(A)}$ *is the orthogonal projection onto* $R(A)$ *and* $P_{[N(A)]^\perp}$ *is the orthogonal projection onto* $[N(A)]^\perp$.

References

[1] M. Argerami, F. Szechtman, R. Tifenbach, On Tate's trace, Linear Multilinear Algebra 55(6) (2007) 515-520.
[2] O. M. Baksalary, G. Trenkler, Core inverse of matrices, Linear and Multilinear Algebra 58 (6) (2010) 681–697.
[3] O. Braunling, Adele residue symbol and Tate's central extension for multiloop Lie algebras, Algebra Number Theory 8(1) (2014) 19–52.
[4] O. Braunling, On the local residue symbol in the style of Tate and Beilinson, New York J. Math. 24 (2018) 458-513.
[5] V. Cabezas Sánchez, F. Pablos Romo, Explicit Solutions of Infinite Systems of Linear Equations from Reflexive Generalized Inverses of finite potent endomorphisms, Linear Algebra Appl. 559 (2018) 125-144.
[6] V. Cabezas Sánchez, F. Pablos Romo, Moore-Penrose Inverse of Some Linear Maps on Infinite-Dimensional Vector Spaces, Electron. J. Linear Algebra 36 (2020) 570 - 586.
[7] S. L. Campbell, The Drazin inverse of an operator, Recent Applications of Generalized Inverses, S. L. Campbell, ed., Research Notes in Mathematics 66, Pitman, London, 1982, pp.250–259.
[8] S. L. Campbell, Jr. Meyer, Generalized Inverses of Linear Transformations, Dover, London, 1991. ISBN 978-0-486-66693-8.
[9] C. P. Debry, Towards a class number formula for Drinfeld modules, Thesis (2016) University of Amsterdam.

[10] M. P. Drazin, Pseudo-inverses in associative rings and semigroups, The American Math. Monthly. 65(7) (1958) 506-514.
[11] D. Hernández Serrano, F. Pablos Romo, Determinants of finite potent endomorphisms, symbols and reciprocity laws, Linear Alg. Appl. 469 (2013) 239-261.
[12] J. J. Koliha, A generalized Drazin inverse, Glasgow Math. J. 38 (3) (1996) 367 - 381.
[13] D. C. Lay, Spectral properties of generalized inverses of linear operators, SIAM J. Appl. Math. 29 (1975) 103–109.
[14] D. C. Lay, Spectral analysis using ascent, descent, nullity and defect, Math. Ann. 184 (1970) 197-214.
[15] S. B. Malik, N. Thome, On a new generalized inverse for matrices of an arbitrary index, Appl. Math Comput. 226 (1) (2014) 575–580.
[16] M. Mehdipour, A. Salemi, On a new generalized inverse of matrices, Linear and Multilinear Algebra 66(5) (2018) 1046-1053.
[17] F. Pablos Romo, Core-Nilpotent Decomposition and new generalized inverses of Finite Potent Endomorphisms, Linear and Multilinear Algebra 68(11) (2020) 2254-2275.
[18] F. Pablos Romo, Group Inverse of Finite Potent Endomorphisms on Arbitrary Vector Spaces, Oper. Matrices 14(4) (2020) 1029-1042.
[19] F. Pablos Romo, On Bounded Finite Potent Operators on arbitrary Hilbert Spaces, Bull. Malays. Math. Sci. Soc. 44 (6) (2021) 4085–4107.
[20] F. Pablos Romo, On Drazin-Moore-Penrose Inverses of Finite Potent Endomorphisms, Linear and Multilinear Algebra 69(4) (2021) 627-647.
[21] F. Pablos Romo, On the coincidence of the Drazin inverse and the Drazin-Moore-Penrose inverses, Linear and Multilinear Algebra (2020) DOI: 10.1080/03081087.2020.1849004.
[22] F. Pablos Romo, On the Drazin Inverse of Finite Potent Endomorphisms, Linear and Multilinear Algebra 67(10) (2019) 2135-2146.
[23] F. Pablos Romo, On the linearity property of Tate's trace, Linear Multilinear Algebra 55(4) (2007) 523-526.
[24] J. Ramos González, F. Pablos Romo, A negative answer to the question of the linearity of Tate's Trace for the sum of two endomorphisms, Linear Multilinear Algebra 62(4) (2014) 548-552.
[25] L. Taelman, Special L-values of Drinfeld modules, Annals of Math. 175(1) (2012) 369-391.
[26] J. Tate, Residues of Differentials on Curves, Ann. Scient. Éc. Norm. Sup. 1, 4a série, (1968) 149-159.
[27] A. Yekutieli, Local Beilinson–Tate operators, Algebra and Number Theory 9(1) (2015) 173 - 224.