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On Stancu Operators Depending on a Non-Negative Integer

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Abstract. In this paper, we deal with Stancu operators which depend on a non-negative integer parameter. Firstly, we define Kantorovich extension of the operators. For functions belonging to the space $L^p[0,1]$, $1 \leq p < \infty$, we obtain convergence in the norm of L^p by the sequence of Stancu-Kantorovich operators, and we give an estimate for the rate of the convergence via first order averaged modulus of smoothness. Moreover, for the Stancu operators; we search variation detracting property and convergence in the space of functions of bounded variation in the variation seminorm.

1. Introduction

Stancu constructed the following Bernstein type positive linear operators

$$L_{n,r}(f;x) := \sum_{k=0}^{n-r} p_{n-r,k}(x)\left[(1-x)f\left(\frac{k}{n}\right) + xf\left(\frac{k+r}{n}\right)\right] \tag{1}$$

for $f \in C[0,1]$, where r is a non-negative integer parameter, n is a natural number such that $n > 2r$ and

$$p_{n,k}(x) = \begin{cases} \binom{n}{k}x^k(1-x)^{n-k}; & 0 \leq k \leq n \\ 0; & k < 0 \text{ or } k > n \end{cases}, \quad x \in [0,1], \tag{2}$$

are the Bernstein basis polynomials satisfying the recurrence

$$p_{n,k}(x) = (1-x)\, p_{n-1,k}(x) + x\, p_{n-1,k-1}(x), \quad 0 \leq k \leq n, \tag{3}$$

[19]. It is clear that for the cases $r = 0$ and $r = 1$, the operators $L_{n,r}$ reduce to the classical Bernstein operators In this paper; Stancu gave uniform convergence $\lim_{n\to\infty} L_{n,r}(f) = f$ on $[0,1]$ for $f \in C[0,1]$ and presented an expression for the remainder $R_{n,r}(f;x)$ of the approximation formula $f(x) = L_{n,r}(f;x) + R_{n,r}(f;x)$ by means of the second order divided differences and also obtained an integral representation for the remainder. Moreover, the author obtained an estimate of Voronovskaja-type and computed the order of approximation

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by the operators $L_{n,r}(f)$ via modulus of continuity. Finally, investigation of the spectral properties of $L_{n,r}$ was given and it was noticed that each $L_{n,r}$ has the variation diminishing property, in the sense of Schoenberg [17].

Yang et al. extended the Stancu operators $L_{n,r}$ given by (1) to the multivariate case on a simplex and using elementary method, the authors proved the preservation of Lipschitz constant and order of a Lipschitz continuous function by the multivariate Stancu operators [21]. Concerning Voronovskaja-type formula; Bustamante and Quesada establihed an asymptotic property for Stancu operators [6]. In [5], we studied an extension of Stancu operators $L_{n,r}$ by using the fundamental functions of Cheney and Sharma operators. In [9], Çetin and Başcanbaz-Tunca studied approximation properties of complex form of the Stancu operators. In [8], Çetin investigated a generalization of the complex Stancu operators depending on non-negative two real parameters.

It is well-known that Bernstein polynomials are not appropriate for approximation of discontinuous functions (see [13, Section 1.9]). In order to get approximation of any Lebesgue integrable function f on $[0,1]$, Kantorovich [11] constructed positive linear operators given by

$$K_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x)(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt,\ x\in[0,1],\ n\in\mathbb{N}. \tag{4}$$

Lorentz proved that for $f\in L^p[0,1]$, $1\le p<\infty$, the sequence of Bernstein-Kantorovich operators satisfies $\lim_{n\to\infty} K_n(f) = f$ in $L^p[0,1]$ (see [13], also [2]). In [7], Campiti and Metafune introduced new Bernstein-Kantorovich-type operators $K_{n,\alpha}: L^p[0,1]\to L^p[0,1]$ given by

$$K_{n,\alpha} := \sum_{k=0}^{n} \alpha_{n,k}x^k(1-x)^{n-k}(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt,\ x\in[0,1],\ n\in\mathbb{N},$$

for $f\in L^p[0,1]$, where the coefficients $\alpha_{n,k}$ satisfy the following recurrence

$$\alpha_{n+1,k} = \alpha_{n,k} + \alpha_{n,k-1} \text{ for } k = 1,2,\ldots,n,$$

and for $k = 0, n$;

$$\alpha_{n,0} = \lambda_n,\ \alpha_{n,n} = \rho_n$$

with $\{\lambda_n\}_{n\in\mathbb{N}}$, $\{\rho_n\}_{n\in\mathbb{N}}$ are fixed bounded sequences of real numbers. And the authors studied the convergence $\{K_{n,\alpha}(f)\}_{n\in\mathbb{N}}$ to $f\in L^p[0,1]$ in the norm of L^p as well as the rate of the convergence via τ-modulus. There are several research papers dealing with multivariable design of Kantorovich operators and their generalizations. Here, we only cite [3].

As far as we have searched, for the Stancu operators $L_{n,r}$; the variation detracting property (see, 4th section) and L^p-approximation by their Kantorovich variants have not been studied yet. These problems motivate us in this study.

In this paper, we construct Kantorovich-type extension of the Stancu operators given by (1) and, similar to that for the Bernstein-Kantorovich operators, we obtain approximation of $f\in L^p[0,1]$ by the sequence of Stancu-Kantorovich operators, which will be denoted by $K_{n,r}(f;x)$, in the norm of $L^p[0,1]$, $1\le p<\infty$. For the rate of the convergence; using a result of Popov (Theorem 3.4), we present an estimate in terms of averaged modulus of smoothness of first order. Moreover, we show that for all functions of bounded variation on $[0,1]$, each Stancu operator $L_{n,r}$ has variation detracting property. Finally, using classical technique (see [4]), we present convergence in the variation seminorm of absolutely continuous functions on $[0,1]$ by the sequence of Stancu operators.

For the L^p-convergence, we recall the following definitions which are reproduced here from the survey article of Altomare [2]:

Definition 1.1. *([2]) Let I be a real interval of $\mathbb{R}$. A function $\varphi : I \to \mathbb{R}$ is said to be convex if*

$$\varphi(\lambda x + (1-\lambda) y) \leq \lambda \varphi(x) + (1-\lambda)\varphi(y)$$

for every $x, y \in I$ and $0 \leq \lambda \leq 1$. If I is open and φ is convex, then, for every finite family $(x_k)_{1\leq k\leq n}$ in I and $(\lambda_k)_{1\leq k\leq n}$ in $[0,1]$ such that $\sum\limits_{k=1}^{n} \lambda_k = 1$, the Jensen inequality

$$\varphi\left(\sum_{k=1}^{n} \lambda_k x_k\right) \leq \sum_{k=1}^{n} \lambda_k \varphi(x_k)$$

holds.

Definition 1.2. *([2]) Given a probability space $(\Omega, \mathcal{F}, \mu)$, an open interval I of $\mathbb{R}$ and a μ-integrable function $f : \Omega \to I$, then $\int\limits_{\Omega} f d\mu \in I$. Furthermore, if $\varphi : I \to \mathbb{R}$ is convex and $\varphi \circ f : \Omega \to \mathbb{R}$ is μ-integrable, then the integral Jensen's inequality*

$$\varphi\left(\int_{\Omega} f d\mu\right) \leq \int_{\Omega} (\varphi \circ f)\, d\mu$$

holds.

2. Kantorovich-type extension of Stancu operators

Using standard technique in [13, p. 30], we let $f \in L^1[0,1]$ and consider the indefinite integral of f; $F(x) = \int\limits_0^x f(t)\,dt + F(0)$. Differentiating $L_{n+1,r}(F; x)$ with respect to x, we get

$$\begin{aligned}
(L_{n+1,r}(F;x))' = &\sum_{k=0}^{n-r} p_{n-r,k}(x)(n+1-r)\left[(1-x)\left\{F\left(\frac{k+1}{n+1}\right) - F\left(\frac{k}{n+1}\right)\right\} + x\left\{F\left(\frac{k+r+1}{n+1}\right) - F\left(\frac{k+r}{n+1}\right)\right\}\right] \\
&+ \sum_{k=0}^{n+1-r} p_{n+1-r,k}(x)\left[F\left(\frac{k+r}{n+1}\right) - F\left(\frac{k}{n+1}\right)\right] \\
=& K_{n,r}(f;x),
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
K_{n,r}(f;x) := &\sum_{k=0}^{n-r} p_{n-r,k}(x)(n+1-r)\left((1-x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt + x\int_{\frac{k+r}{n+1}}^{\frac{k+r+1}{n+1}} f(t)\,dt\right) \\
&+ \sum_{k=0}^{n+1-r} p_{n+1-r,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+r}{n+1}} f(t)\,dt,
\end{aligned} \tag{6}$$

with r is a non-negative integer parameter and n is a natural number such that $n > 2r$, $f \in L^1[0,1]$ and $x \in [0,1]$. We call the operators $K_{n,r}$ given by (6) as Stancu-Kantorovich operators in the sequel and consider these operators from $L^p[0,1]$ into itself for every $f \in L^p$, $1 \leq p < \infty$. Each $K_{n,r}$ is positive and linear operator and the cases $r = 0$ and $r = 1$ give the Bernstein-Kantorovich operators given by (4), namely,

$$K_{n,1} = K_{n,0} = K_n$$

holds. Indeed, the case $r = 0$ is obvious and the case $r = 1$ readily follows from (2) and (3).

We should note here that a Kantorovich-type extension of the Stancu operators $L_{n,r}$ can be proposed as

$$K^{*}_{n,r}(f;x) = \sum_{k=0}^{n-r} p_{n-r,k}(x)(n+1)\left((1-x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt + x\int_{\frac{k+r}{n+1}}^{\frac{k+r+1}{n+1}} f(t)\,dt\right), \tag{7}$$

where r is a non-negative integer parameter, n is a natural number such that $n > 2r$, $f \in L^1[0,1]$ and $p_{n,k}(x)$ are given by (2). But, for our aims, we prefer to deal with the construction given by (6). For a generalization of the Stancu operator, which depends on two non-negative integer parameters (see [20]); a Kantorovich-type extension for $f \in C[0,1]$, similar to (7), was constructed and studied in [10] by Kajla, and also a generalization of which was studied in [12] by Kumar.

3. L^p-approximation by Stancu-Kantorovich operators

Firstly, we need to show the uniform convergence of real-valued and continuous functions by the sequence $\{K_{n,r}(f)\}_{n\in\mathbb{N}}$ on $[0,1]$. Let us denote $e_v(t) := t^v,\ t \in [0,1],\ v = 0,1,\ldots$.

Theorem 3.1. *If $f \in C[0,1]$ and r is a non-negative fixed integer, then $\lim_{n\to\infty} K_{n,r}(f) = f$ uniformly on $[0,1]$.*

Proof. Making use of first three moments of the Stancu operators $L_{n,r}$ given by

$$L_{n,r}(e_0;x) = 1,\ L_{n,r}(e_1;x) = x,\ L_{n,r}(e_2;x) = x^2 + \left[1 + \frac{r(r-1)}{n}\right]\frac{x(1-x)}{n}$$

(see [19]), it readily follows that

$$\begin{aligned} K_{n,r}(e_0;x) &= 1,\\ K_{n,r}(e_1;x) &= \frac{(n+r)(n-r+1)}{(n+1)^2}x + \frac{n+1+r(r-1)}{2(n+1)^2},\\ K_{n,r}(e_2;x) &= \frac{(n-r)(n-r+1)(n+2r-1)}{(n+1)^3}x^2 + \frac{2(n-r+1)\left(n+r^2\right)}{(n+1)^3}x + \frac{\left(n-r+1+r^3\right)}{3(n+1)^3}. \end{aligned} \tag{8}$$

Then, the result is obtained by using the well-known Korovkin theorem. □

Remark 3.2. *If $f \in C[0,1]$ is a continuously differentiable function on $[0,1]$ and r is a non-negative fixed integer, then, from (5), we get*

$$(L_{n+1,r}(f;x))' = K_{n,r}(f';x)$$

for $n \in \mathbb{N}$ such that $n > 2r$ and $x \in [0,1]$. In view of Theorem 3.1, it readily follows that $\lim_{n\to\infty}(L_{n,r}(f))' = f'$ uniformly on $[0,1]$.

Below, we present the convergence of $\{K_{n,r}(f)\}_{n\in\mathbb{N}}$ in the norm of the space $L^p[0,1]$, $1 \le p < \infty$.

Theorem 3.3. *If $f \in L^p[0,1]$, $1 \le p < \infty$, and r is a non-negative fixed integer, then $\lim_{n\to\infty} K_{n,r}(f) = f$ in $L^p[0,1]$.*

Proof. Denoting the operator norm of $K_{n,r}$, acting from $L^p[0,1]$ into itself, by $\|K_{n,r}\|$, where r is a non-negative fixed integer and $n \in \mathbb{N}$ such that $n > 2r$; it is sufficient to show that there exist an $M > 0$ such that $\|K_{n,r}\| \le M$. Now, by setting

$$S_{n,r}(f;x) := \sum_{k=0}^{n-r} p_{n-r,k}(x)(n+1-r)\left((1-x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt + x\int_{\frac{k+r}{n+1}}^{\frac{k+r+1}{n+1}} f(t)\,dt\right)$$

and

$$S_{n+1,r}(f;x) := \sum_{k=0}^{n+1-r} p_{n+1-r,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+r}{n+1}} f(t)\,dt,$$

we write each of the Stancu-Kantorovich operator $K_{n,r}(f;x)$ given by (6) as

$$K_{n,r}(f;x) = S_{n,r}(f;x) + S_{n+1,r}(f;x)$$

and therefore, we immediately get

$$\left|K_{n,r}(f;x)\right|^p \leq 2^p \left(\left|S_{n,r}(f;x)\right|^p + \left|S_{n+1,r}(f;x)\right|^p\right). \tag{9}$$

Now, taking into accout of the fact that $\varphi(t) = |t|^p$, $1 \leq p < \infty$, $t \in [0,1]$, is convex, we need to estimate each term in (9). For $\left|S_{n,r}(f;x)\right|^p$; firstly, by taking $\lambda_1 = 1 - x$ and $\lambda_2 = x$ for $x \in [0,1]$ such that $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$ and applying the definition of convexity, and secondly, making use of Jensen's inequality for $p_{n-r,k}(x) \geq 0$, where $x \in [0,1]$, $k = 0, \ldots, n-r$, such that $\sum\limits_{k=0}^{n-r} p_{n-r,k}(x) = 1$ and finally, using integral form of Jensen's inequality in the result, with the function $\varphi(t) = |t|^p$, $1 \leq p < \infty$, $t \in [0,1]$ and the measure $(n+1-r)\,dt$, we easily get

$$\left|S_{n,r}(f;x)\right|^p \leq \sum_{k=0}^{n-r} p_{n-r,k}(x)(n+1-r)\left((1-x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \left|f(t)\right|^p dt + x \int_{\frac{k+r}{n+1}}^{\frac{k+r+1}{n+1}} \left|f(t)\right|^p dt\right). \tag{10}$$

Now, for $\left|S_{n+1,r}(f;x)\right|^p$; for the function $\varphi(t) = |t|^p$, $t \in [0,1]$, $1 \leq p < \infty$, applying Jensen's inequality with $p_{n+1-r,k}(x) \geq 0$, $x \in [0,1]$, $k = 0, \ldots, n+1-r$, such that $\sum\limits_{k=0}^{n+1-r} p_{n+1-r,k}(x) = 1$, and next, using integral Jensen's inequality, we obtain

$$\left|S_{n+1,r}(f;x)\right|^p \leq \sum_{k=0}^{n+1-r} p_{n+1-r,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+r}{n+1}} \left|f(t)\right|^p dt. \tag{11}$$

Integrating (10) over $[0,1]$, using the well-known beta integral, we get

$$\int_0^1 \left|S_{n,r}(f;x)\right|^p dx \leq \frac{1}{n-r+2} T_{n,r}, \tag{12}$$

where

$$T_{n,r} := \sum_{k=0}^{n-r}\left((n-r-k+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \left|f(t)\right|^p dt + (k+1)\int_{\frac{k+r}{n+1}}^{\frac{k+r+1}{n+1}} \left|f(t)\right|^p dt\right).$$

Since $n > 2r$, $r \in \mathbb{N} \cup \{0\}$, we have $n - r > r$. Hence, we can express $T_{n,r}$ as

$$T_{n,r} = \left(\sum_{k=0}^{r-1} + \sum_{k=r}^{n-r}\right)(n-r-k+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \left|f(t)\right|^p dt + \left(\sum_{k=0}^{n-2r} + \sum_{k=n-2r+1}^{n-r}\right)(k+1)\int_{\frac{k+r}{n+1}}^{\frac{k+r+1}{n+1}} \left|f(t)\right|^p dt.$$

Replacing k with $k-r$ in the sums located in the second term, the above expression of $T_{n,r}$ reduces to

$$T_{n,r} = \sum_{k=0}^{r-1} (n-r-k+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \left|f(t)\right|^p dt + \sum_{k=r}^{n-r} (n-2r+2) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \left|f(t)\right|^p dt$$
$$+ \sum_{k=n-r+1}^{n} (k-r+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \left|f(t)\right|^p dt. \tag{13}$$

On the other hand, integration of (11) over $[0,1]$ gives

$$\int_0^1 \left|S_{n+1,r}(f;x)\right|^p dx \leq \frac{1}{n-r+2} \sum_{k=0}^{n+1-r} \int_{\frac{k}{n+1}}^{\frac{k+r}{n+1}} \left|f(t)\right|^p dt. \tag{14}$$

By collecting like terms, we decompose the sum in (14) into three parts:

$$\begin{aligned} \sum_{k=0}^{n+1-r} \int_{\frac{k}{n+1}}^{\frac{k+r}{n+1}} \left|f(t)\right|^p dt &= \sum_{k=0}^{n+1-r} \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} + \int_{\frac{k+1}{n+1}}^{\frac{k+2}{n+1}} + \cdots + \int_{\frac{k+r-1}{n+1}}^{\frac{k+r}{n+1}} \right) \left|f(t)\right|^p dt \\ &= \sum_{k=0}^{r-1} (k+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \left|f(t)\right|^p dt + r \sum_{k=r}^{n-r} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \left|f(t)\right|^p dt + \sum_{k=n-r+1}^{n} (n-k+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \left|f(t)\right|^p dt. \end{aligned} \tag{15}$$

Making use of (12), (13), (14) and (15); from (9), we arrive at

$$\begin{aligned} \int_0^1 \left|K_{n,r}(f;x)\right|^p dx &\leq 2^p \left(\sum_{k=0}^{r-1} + \sum_{k=r}^{n-r} + \sum_{k=n-r+1}^{n} \right) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \left|f(t)\right|^p dt \\ &= 2^p \sum_{k=0}^{n} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \left|f(t)\right|^p dt = 2^p \int_0^1 \left|f(t)\right|^p dt. \end{aligned}$$

Therefore, passing to L^p-norm $\|.\|_p$, we get $\left\|K_{n,r}(f)\right\|_p \leq 2\left\|f\right\|_p$ for every $f \in L^p[0,1]$. Namely, for every $n \in \mathbb{N}$ such that $n > 2r$, $K_{n,r}$ is a bounded operator with $\left\|K_{n,r}\right\| \leq 2$. Now, let $\epsilon > 0$ be arbitrary given. Then, by the density of $C[0,1]$ in $L^p[0,1]$ (with respect to the norm$\|.\|_p$), there is a $g \in C[0,1]$ such that $\left\|f-g\right\|_p < \epsilon$ and, by Theorem 3.1, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ it holds that

$$\left\|K_{n,r}(g) - g\right\| < \epsilon,$$

where $\|.\|$ is the usual sup-norm in $C[0,1]$. Therefore, L^p-convergency is a direct consequence of the above arguments and the inequality

$$\left\|K_{n,r}(f) - f\right\|_p \leq 2\left\|f-g\right\|_p + \left\|K_{n,r}(g) - g\right\|_p + \left\|g-f\right\|_p < 4\epsilon,$$

which completes the proof. □

Concerning rate of the approximation, averaged modulus of smoothness is very useful tool for the error of the convergence in the norm of L^p. Here, we adopt the convention that $M[a,b]$ denotes the space

$$M[a,b] = \{f \mid f \text{ is bounded and measurable on } [a,b]\}.$$

Recall that *for $f \in M[a,b]$ and $\delta > 0$, averaged modulus of smoothness (or τ-modulus) of the first order for step δ in L^p-norm, $1 \le p < \infty$, is denoted by $\tau_1(f;\delta)_p$ and defined as*

$$\tau_1(f;\delta)_p = \left\|\omega_1(f,.;\delta)\right\|_p,$$

where

$$\omega_1(f,x;\delta) = \sup\left\{|f(t+h) - f(t)| : \ t, t+h \in \left[x - \frac{\delta}{2}, x + \frac{\delta}{2}\right] \cap [0,1]\right\}$$

is the local modulus of smoothness of the first order for the function f at the point $x \in [a,b]$ and for step δ (see [16] or, for details, [18]).

For every Borel measurable and bounded function f defined on $[0,1]$, we already have the following results for the Bernstein-Kantorovich operators K_n, $n \in \mathbb{N}$,

$$\left\|K_n(f) - f\right\|_p \le 748\tau_1\left(f; \frac{1}{\sqrt{n+1}}\right)_p$$

(see [1, p.335]) and

$$\left\|K_n(f) - f\right\|_p \le C\tau_1\left(f; \sqrt{\frac{3n+1}{12(n+1)^2}}\right)_p, \tag{16}$$

where C is a positive constant that does not depend on f (see the special case for Proposition 4.2 in [3]). To get an estimate for the approximation error in Theorem 3.3, we shall use the following theorem due to Popov [15], in which averaged modulus of smoothness of the first order is used:

Theorem 3.4. *([15]) Let $L : M[a,b] \to M[a,b]$ be a positive linear operator, having the properties*

$$L(e_0;x) = 1, \ L(e_1;x) = x + \alpha(x), \ L(e_2;x) = x^2 + \beta(x), \ x \in [a,b].$$

Let

$$A := \sup\left\{\left|\beta(x) - 2x\alpha(x)\right|; \ x \in [a,b]\right\} \le 1.$$

Then for $f \in M[a,b]$ and $1 \le p < \infty$, the following estimate holds

$$\left\|L(f) - f\right\|_p \le C\tau_1\left(f; \sqrt{A}\right)_p,$$

where C is a positive constant which does not depend on the operator L, the function f and the L^p-norm.

For the rate of convergence in Theorem 3.3, we present the following estimate:

Theorem 3.5. *If $f \in M[0,1]$, r is a non-negative fixed integer, then, for every $n \in \mathbb{N}$ such that $n > 2r$ and $1 \le p < \infty$,*

$$\left\|K_{n,r}(f) - f\right\|_p \le C\tau_1\left(f; \sqrt{A_{n,r}}\right)_p,$$

where

$$A_{n,r} = \frac{3n^2 + 3nr^2 - 3nr + 4n - 2r^3 + 3r^2 - r + 1}{12(n+1)^3} \le 1 \tag{17}$$

and the positive constant C does not depend on f.

Proof. Assume that $f \in M[0,1]$ and $1 \leq p < \infty$. According to Theorem 3.4, it sufficies to show that

$$A_{n,r} := \sup\left\{K_{n,r}\left((e_1 - xe_0)^2; x\right);\ x \in [0,1]\right\} \leq 1,$$

where r is a non-negative fixed integer and $n \in \mathbb{N}$ such that $n > 2r$. From the linearity of the operators and (8), we obtain, for $x \in [0,1]$,

$$\begin{aligned}
K_{n,r}\left((e_1 - xe_0)^2; x\right) &= \frac{n^2 + \left(r^2 - r\right)n - 2r^3 + r^2 + r - 1}{(n+1)^3} x(1-x) + \frac{n - r + 1 + r^3}{3(n+1)^3} \\
&\leq \frac{n^2 + \left(r^2 - r\right)n - 2r^3 + r^2 + r - 1}{4(n+1)^3} + \frac{n - r + 1 + r^3}{3(n+1)^3} \\
&= \frac{3n^2 + 3nr^2 - 3nr + 4n - 2r^3 + 3r^2 - r + 1}{12(n+1)^3} \\
&= A_{n,r},
\end{aligned}$$

which gives (17). Note that, for the cases $r = 0, 1$ we get

$$A_{n,0} = A_{n,1} = \frac{3n+1}{12(n+1)^2},$$

which is the corresponding result for the Bernstein-Kantorovich operators given by (16). Now, it remains to show that $A_{n,r} \leq 1$. Since $r \geq 0$ and $n > 2r$, we get $n - 1 \geq 2r$. Thus, we can write

$$\begin{aligned}
A_{n,r} &= \frac{1}{12(n+1)^3}\left[3n^2 + 3nr^2 - 3nr + 4n - 2r^3 + 3r^2 - r + 1\right] \\
&\leq \frac{1}{12(n+1)^3}\left[3n^2 + 3nr^2 + 4n + 3r^2 + 1\right] \\
&\leq \frac{1}{12(n+1)^3}\left[3n^2 + 3n\left(\frac{n-1}{2}\right)^2 + 4n + 3\left(\frac{n-1}{2}\right)^2 + 1\right] \\
&= \frac{3n^3 + 9n^2 + 13n + 7}{48(n+1)^3} \leq 1.
\end{aligned}$$

This completes the proof. □

4. Variation detracting property

Recall that the class of all functions of bounded variation on $[0,1]$ is denoted by $TV[0,1]$, with the seminorm $\|f\|_{TV[0,1]} := V_{[0,1]}[f]$, where $V_{[0,1]}[f]$ is total variation of f. It is well-known that for $f \in TV[0,1]$, each Bernstein operator B_n satisfies the inequality $V_{[0,1]}[B_n f] \leq V_{[0,1]}[f]$ (see [13, p.23]) that is called as variation detracting property. Moreover, denoting the class of all absolutely continuous functions on $[0,1]$ by $AC[0,1]$, we have the following result for Bernstein polynomials: For $f \in TV[0,1]$,

$$f \in AC[0,1] \Longleftrightarrow \lim_{n\to\infty} V_{[0,1]}[B_n(f) - f] = 0 \tag{18}$$

(see [14] or [4, p.308]). Below, we show that each Stancu operator $L_{n,r}$ satisfies variation detracting property.

Theorem 4.1. *If $f \in TV[0,1]$, r is a non-negative fixed integer, then, for every $n \in \mathbb{N}$ such that $n > 2r$,*

$$V_{[0,1]}[L_{n,r}(f)] \leq V_{[0,1]}[f].$$

Proof. Since the cases for $r = 0$ and $r = 1$ give the Bernstein operators, we consider for $0 < r \neq 1$. For $f \in TV[0,1]$, we have $L_{n,r}(f;x)$ is continuous on $[0,1]$ and $(L_{n,r}(f;x))'$ is bounded on $(0,1)$. Therefore, it follows that $L_{n,r}(f) \in AC[0,1]$. Writing the formula $(L_{n,r}(f;x))'$ simply by replacing n with $n-1$ and F with f in (5) and using beta integral, we obtain the following inequality for the total variation of $L_{n,r}(f)$:

$$\begin{aligned} V_{[0,1]}\left[L_{n,r}(f)\right] =& \int_0^1 \left|(L_{n,r}(f;x))'\right| dx \\ \leq& \frac{1}{n-r+1} \sum_{k=0}^{n-1-r} \left[(n-r-k)\left|f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right| + (k+1)\left|f\left(\frac{k+r+1}{n}\right) - f\left(\frac{k+r}{n}\right)\right|\right] \\ &+ \frac{1}{n-r+1} \sum_{k=0}^{n-r} \left|f\left(\frac{k+r}{n}\right) - f\left(\frac{k}{n}\right)\right|. \end{aligned} \tag{19}$$

As in the proof of Theorem 3.3, by using the similar decomposition technique to the first and second sums in (19), we reach to

$$\begin{aligned} V_{[0,1]}\left[L_{n,r}(f)\right] \leq& \frac{1}{n-r+1} \left\{ \sum_{k=0}^{r-1} (n-r-k) + \sum_{k=r}^{n-1-r} (n-2r+1) + \sum_{k=n-r}^{n-1} (k-r+1) \right. \\ &\left. + \sum_{k=0}^{r-1} (k+1) + \sum_{k=r}^{n-1-r} r + \sum_{k=n-r}^{n-1} (n-k) \right\} \left|f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right| \\ =& \sum_{k=0}^{n-1} \left|f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right| \\ \leq& V_{[0,1]}[f], \end{aligned}$$

which completes the proof. □

Now, we present the similar result in (18) for the Stancu polynomials:

Proposition 4.2. *If $f \in TV[0,1]$ and r is a non-negative fixed integer, then*

$$f \in AC[0,1] \Longleftrightarrow \lim_{n\to\infty} V_{[0,1]}\left[L_{n,r}(f) - f\right] = 0.$$

Proof. Since $AC[0,1]$ is a closed subspace of $TV[0,1]$ according to the seminorm $\|.\|_{TV[0,1]}$ (see [4, Lemma 2.1]), for $f \in TV[0,1]$,

$$\lim_{n\to\infty} V_{[0,1]}\left[L_{n,r}(f) - f\right] = \lim_{n\to\infty} \left\|L_{n,r}(f) - f\right\|_{TV[0,1]} = 0$$

implies that $f \in AC[0,1]$. Conversely, we let $f \in AC[0,1]$ (which gives $f' \in L^1[0,1]$ and $f(x) = \int_0^x f'(t)\,dt + f(0)$). Since, $L_{n,r}(f) \in AC[0,1]$, we have $[L_{n,r}(f) - f] \in AC[0,1]$. Thus, we arrive at

$$\lim_{n\to\infty} V_{[0,1]}\left[L_{n,r}(f) - f\right] = \lim_{n\to\infty} \left\|K_{n-1,r}(f') - f'\right\|_1 = 0$$

by (5), (6) and Theorem 3.3. This completes the proof. □

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