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On Erdös-Lax Inequality Concerning Polynomials

Irfan Ahmad Wani[a], Ishfaq Nazir[a], Mohammad Ibrahim Mir[a]

[a] *Department of Mathematics, University of Kashmir, South Campus Anantnag-192 101, Jammu and Kashmir, India.*

Abstract. Recently Milovanović et al. [Bulletin T.CLIII de l'Acaémie serbe des sciences et des arts - 2020.] proved that if $P(z) \in \mathbb{P}_n$ with no zeros in $|z| < k$, $k \geq 1$, then,

$$\left|P'(z)\right| \leq \frac{\|P\|}{2}\left[n - \left\{n\left(\frac{k-1}{k+1}\right) + \frac{2}{k+1}\left(\frac{|c_0| - k^n|c_n|}{|c_0| + k^n|c_n|}\right)\right\}\frac{|P(z)|^2}{\|P\|^2}\right], \quad |z| = 1,$$

where $P(z) = c_0 + c_1 z + \cdots + c_n z^n \in \mathbb{P}_n$ is a polynomial of degree n. In this paper, we obtain some results concerning the class of polynomials having $s-$fold zero at origin. These results not only generalizes but also refines many well-known results due to Milovanović.

1. Introduction

For any arbitrary entire function P, let $\|P\| = \max_{|z|=1}|P(z)|$ denoted the uniform norm of P on the unit disk $|z| = 1$. Let $\mathbb{P}_n$ denote the space of all algebraic polynomials of the form $P(z) = \sum_{j=0}^{n} c_j z^j$ of degree n and let $P'(z)$ be its derivative. The study of inequalities for different norms of derivatives of a univariate complex polynomial in terms of the polynomial norm is a classical topic in analysis. A classical inequality that provides an estimate to the size of the derivative of a given polynomial on the unit disk, relative to size of the polynomial itself on the same disk is the famous Bernstein inequality [3]. It states that: if $P(z)$ is a polynomial of degree n, then on $|z| = 1$,

$$\left\|P'(z)\right\| \leq n\left\|P(z)\right\|. \tag{1}$$

Equality holds in (1) if and only if $P(z)$ has all its zeros at the origin.

Concerning the estimation of the upper bound of $\max_{|z|=1}|P'(z)|$, it was conjectured by Erdös and later proved by Lax [4] that if $P(z)$ is a polynomial of degree n and $P(z) \neq 0$ in $|z| < 1$, then,

$$\left\|P'(z)\right\| \leq \frac{n}{2}\left\|P(z)\right\|. \tag{2}$$

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No funding agency
Email addresses: irfanmushtaq62@gmail.com (Irfan Ahmad Wani), ishfaqnazir02@gmail.com (Ishfaq Nazir), ibrahimmath80@gmail.com (Mohammad Ibrahim Mir)

Inequality (2) was further sharpened by Aziz and Dawood [1] in the form

$$\left\|P'(z)\right\| \leq \frac{n}{2}\left\{\left\|P(z)\right\| - \min_{|z|=1} P(z)\right\}. \tag{3}$$

Equality in (2) and (3) holds for any polynomial which has all its zeros on $|z| = 1$. Various extension of (2) and (3) are known in the literature on various regions of the complex plane (see, for example Govil[5], Malik [7], Milovanović *et al.* [9] and Rahman and Schmeisser [10]). Recently Milovanović *et al.*[8] generalize and strengthen the inequalities (2) and (3). In fact, they proved if $P(z) = c_0 + c_1 z + \cdots + c_n z^n$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then for $|z| = 1$,

$$\left|P'(z)\right| \leq \frac{||P||}{2}\left[n - \left\{n\left(\frac{k-1}{k+1}\right) + \frac{2}{k+1}\left(\frac{|c_0| - k^n|c_n|}{|c_0| + k^n|c_n|}\right)\right\}\frac{|P(z)|^2}{||P||^2}\right]. \tag{4}$$

Equality in (4) holds for $P(z) = (z+k)^n$, evaluated at $z = 1$. For $k = 1$ in (4), Milovanović *et al.*[8] proved the following refinement of (2). In fact, they proved if $P(z) = c_0 + c_1 z + \cdots + c_n z^n$ is a polynomial of degree n having no zeros in $|z| < 1$, then for $|z| = 1$,

$$\left|P'(z)\right| \leq \frac{||P||}{2}\left[n - \left(\frac{|c_0| - |c_n|}{|c_0| + |c_n|}\right)\frac{|P(z)|^2}{||P||^2}\right]. \tag{5}$$

In the same paper, they also given the generalization of inequality (4). In fact, they proved if $P(z) = c_0 + c_1 z + \cdots + c_n z^n$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then for $0 \leq t \leq 1$ and $|z| = 1$,

$$|P'(z)| \leq \frac{||P|| - tm}{2}\left[n - \left\{n\left(\frac{k-1}{k+1}\right) + \frac{2C_n(k,t)}{k+1}\right\}\left(\frac{|P(z) - tm|}{||P|| - tm}\right)^2\right], \tag{6}$$

where $m = \min\limits_{|z|=k} P(z)$ and $C_n(k,t) = \dfrac{|c_0| - tm - k^n|c_n|}{|c_0| - tm + k^n|c_n|}$.

Equality in (6) holds for $P(z) = (z+k)^n$, evaluated at $z = 1$.

2. Main Results

In this paper, we obtain the results which generalize inequality (4) and (5). First we prove the result on the class of polynomials having s-fold zero at origin and all other zeros in $|z| \geq k, k \geq 1$. In fact, we prove

Theorem 2.1: *Let* $P(z) = z^s(c_0 + c_1 z + \cdots + c_{n-s} z^{n-s}), 0 \leq s \leq n$, *be a polynomial of degree n having no zero in* $|z| < k, k \geq 1$ *except at* $z = 0$, *then for* $|z| = 1$,

$$\left|P'(z)\right| \leq \frac{||P||}{2}\left[n - \left\{\left(\frac{n(k-1) - s(2k)}{k+1}\right) + \frac{2}{k+1}\left(\frac{|c_0| - k^{n-s}|c_{n-s}|}{|c_0| + k^{n-s}|c_{n-s}|}\right)\right\}\frac{|P(z)|^2}{||P||^2}\right]. \tag{7}$$

Moreover, the equality in (7) holds if $P(z) = z^s(z+k)^{n-s}$, evaluated at $z = 1$.

For the case $k = 1$, we get the following refinement of (2).

Corollary 2.2: *Let* $P(z) = z^s(c_0 + c_1 z + \cdots + c_{n-s} z^{n-s}), 0 \leq s \leq n$, *be a polynomial of degree n having no zero in* $|z| < 1$ *except at* $z = 0$, *then for* $|z| = 1$

$$\left|P'(z)\right| \leq \frac{||P||}{2}\left[n + \left\{s - \left(\frac{|c_0| - |c_{n-s}|}{|c_0| + |c_{n-s}|}\right)\right\}\frac{|P(z)|^2}{||P||^2}\right]. \tag{8}$$

Remark 2.3: For $s = 0$ in inequalities (7) and (8), then Theorem 2.1 reduces to inequality (2) and Corollary 2.2 reduces to (5).

Next, we prove the following result which generalize the Theorem 2.1. In fact, we prove

Theorem 2.4: Let $P(z) = z^s(c_0 + c_1z + \cdots + c_{n-s}z^{n-s}), 0 \leq s \leq n$, is a polynomial of degree n having no zero in $|z| < k, k \geq 1$ except at $z = 0$, then for $0 \leq t \leq 1$ and $|z| = 1$

$$|P'(z)| \leq \frac{1}{2}\left[n(||P|| + tm) - \left\{\frac{n(k-1) - s(2k)}{k+1} + \frac{2}{1+k}C_n(k,t)\right\}\left(\frac{(|P(z)| - tm)^2}{||P|| - tm}\right)\right], \tag{9}$$

where $m = \min_{|z|=k} P(z)$ and $C_n(k,t) = \dfrac{|c_0| - k^{n-s}|c_{n-s}| + k^{n-s}tm}{|c_0| + k^{n-s}|c_{n-s}| - k^{n-s}tm}$.

Equality in (9) holds for $P(z) = z^s(z + k)^{n-s}$, evaluated at $z = 1$.

Remark 2.5: For $t = 0$ Theorem 2.4 reduces to Theorem 2.1.

Put $k = 1$ in inequality (9), we obtain the following refinement of (3).

Corollary 2.6: *Let $P(z) = z^s(c_0 + c_1z + \cdots + c_{n-s}z^{n-s}), 0 \leq s \leq n$, be a polynomial of degree n having no zero in $|z| < 1$ except at $z = 0$, then for $0 \leq t \leq 1$ and $|z| = 1$*

$$|P'(z)| \leq \frac{1}{2}\left[n(||P|| + tm) + \left\{s - \frac{|c_0| - |c_{n-s}| + tm}{|c_0| + |c_{n-s}| - tm}\right\}\left(\frac{(|P(z)| - tm)^2}{||P|| - tm}\right)\right], \tag{10}$$

where $m = min_{|z|=1}P(z)$.

3. Auxiliary Results

For the proof of main results we need the following lemmas.

Lemma 3.1: If $P(z) = c_0 + c_1z + c_2z^2 + \cdots + c_nz^n$ be a polynomial of degree n having no zeros in $|z| < k, k \geq 1$, then for each point z on $|z| = 1$ for which $P(z) \neq 0$, we have

$$Re\left(\frac{zP'(z)}{P(z)}\right) \leq \frac{1}{1+k}\left\{n - \frac{|c_0| - k^n|c_n|}{|c_0| + k^n|c_n|}\right\}. \tag{11}$$

The above lemma is due to Milovanović *et al.* [8].

Lemma 3.2: If $P(z)$ is a polynomial of degree n, then for $|z| = 1$, we have

$$|P'(z)| + |Q'(z)| \leq n||P(z)||, \tag{12}$$

where $Q(z) = \overline{z^nP\left(\frac{1}{\bar{z}}\right)}$.
The above lemma is due to Govil and Rahman [6].

4. Proof of Main Results

Proof of Theorem 2.1: Let $P(z) = z^sF(z) \in \mathbb{P}_n$, where $F(z) = c_0 + c_1z + \cdots + c_{n-s}z^{n-s}$, $0 \leq s \leq n$, then on $|z| = 1$,

$$Re\left(\frac{zP'(z)}{P(z)}\right) = s + Re\left(\frac{zF'(z)}{F(z)}\right). \tag{13}$$

Since $F(z)$ is a polynomial of degree $n-s$ has no zero in $|z|<k, k\geq 1$. Therefore, on applying inequality (11) to $F(z)$, we have for all points z on $|z|=1$, with $P(z)\neq 0$

$$Re\left(\frac{zP'(z)}{P(z)}\right)\leq s+\frac{1}{1+k}\left\{(n-s)-\frac{|c_0|-k^{n-s}|c_{n-s}|}{|c_0|+k^{n-s}|c_{n-s}|}\right\}. \tag{14}$$

If $Q(z)=\overline{z^nP\left(\frac{1}{\bar{z}}\right)}$, then it easily follows that $|Q'(z)|=|nP(z)-zP'(z)|$ for $|z|=1$.
This implies for $|z|=1$,

$$\begin{aligned}\left|\frac{Q'(z)}{P(z)}\right|^2&=\left|n-\frac{zP'(z)}{P(z)}\right|^2\\&=n^2+\left|\frac{zP'(z)}{P(z)}\right|^2-2nRe\left(\frac{zP'(z)}{P(z)}\right).\end{aligned}$$

Which by (14) yields for $|z|=1$,

$$\left|\frac{Q'(z)}{P(z)}\right|^2\geq\left|\frac{zP'(z)}{P(z)}\right|^2+\frac{(k-1)n^2-ns(2k)}{k+1}+\frac{2n}{1+k}\left(\frac{|c_0|-k^{n-s}|c_{n-s}|}{|c_0|+k^{n-s}|c_{n-s}|}\right).$$

This gives for $|z|=1$,

$$|P'(z)|^2+n\left\{\frac{n(k-1)-s(2k)}{k+1}+\frac{2}{1+k}\left(\frac{|c_0|-k^{n-s}|c_{n-s}|}{|c_0|+k^{n-s}|c_{n-s}|}\right)\right\}|P(z)|^2\leq|Q'(z)|^2. \tag{15}$$

Inequality (15) gives, by using lemma 3.2 that for $|z|=1$,

$$|P'(z)|^2+n\left\{\frac{n(k-1)-s(2k)}{k+1}+\frac{2}{1+k}\left(\frac{|c_0|-k^{n-s}|c_{n-s}|}{|c_0|+k^{n-s}|c_{n-s}|}\right)\right\}|P(z)|^2\leq\left(n||P||-|P'(z)|\right)^2. \tag{16}$$

This implies

$$2n||P|||P'(z)|\leq n^2||p||^2-n\left\{\frac{n(k-1)-s(2k)}{k+1}+\frac{2}{1+k}\left(\frac{|c_0|-k^{n-s}|c_{n-s}|}{|c_0|+k^{n-s}|c_{n-s}|}\right)\right\}|P(z)|^2.$$

Which gives

$$|P'(z)|\leq\frac{||P||}{2}\left[n-\left\{\frac{n(k-1)-s(2k)}{k+1}+\frac{2}{1+k}\left(\frac{|c_0|-k^{n-s}|c_{n-s}|}{|c_0|+k^{n-s}|c_{n-s}|}\right)\right\}\left(\frac{P(z)}{||P||}\right)^2\right].$$

That proves the Theorem 2.1 completely. □

Proof of Theorem 2.2: Let $P(z)=z^sF(z)\in\mathbb{P}_n$, where $F(z)=c_0+c_1z+\cdots+c_{n-s}z^{n-s}$, $0\leq s\leq n$, and $F(z)$ has all zeros in $|z|\geq k,\quad k\geq 1$. If $P(z)$ has a zero on $|z|=k$, then $m=min_{|z|=k}P(z)=0$ and the result follows from Theorem 2.1 in this case. Henceforth, we suppose that all the zeros of $F(z)$ lie in $|z|\geq k$ so that $m>0$.

Since $m\leq|P(z)|$ for $|z|=1$, therefore if β is any complex number with $|\beta|\leq 1$, then for $|z|=1$ we have

$$|m\beta z^n|<|P(z)| \tag{17}$$

Since $n-s$ zeros of $P(z)$ are $|z|\geq k,\quad k\geq 1$, it follows by Rouche's Theorem $n-s$ zeros of $P(z)-m\beta z^n$ are $|z|\geq k,\quad k\geq 1$. On applying inequality (4.3) to the polynomial $P(z)-m\beta z^n$, we get for every $|\beta|<1$ and $|z|=1$,

$$|P'(z)-nm\beta z^{n-1}|^2+n\left\{\frac{n(k-1)-s(2k)}{k+1}+\frac{2}{1+k}\left(\frac{|c_0|-k^{n-s}|c_{n-s}-\beta m|}{|c_0|+k^{n-s}|c_{n-s}-\beta m|}\right)\right\}|P(z)-\beta mz^n|^2\leq|Q'(z)|^2 \tag{18}$$

By Triangles inequality, we have for $|z| = 1$,

$$|P(z) - \beta m z^n| \geq |P(z)| - |\beta| m \tag{19}$$

and note that

$$||P(z)| - |\beta| m|^2 = (|P(z)| - |\beta| m)^2 . \tag{20}$$

So that, from (19) and (20), we obtain for $|z| = 1$

$$|P(z) - \beta m z^n|^2 \geq (|P(z)| - |\beta| m)^2 . \tag{21}$$

Similarly, we get for $|z| = 1$

$$|P'(z) - \beta n m z^{n-1}|^2 \geq (|P'(z)| - |\beta| m n)^2 . \tag{22}$$

Also, for every β, we have

$$|c_{n-s} - \beta m| \geq |c_{n-s}| - |\beta| m,$$

and since the function

$$x \to \frac{x - k^{n-s}|c_{n-s}|}{x + k^{n-s}|c_{n-s}|} \quad (x \geq 0)$$

is non-decreasing, it follows that

$$\frac{|c_0| - k^{n-s}|c_{n-s} - \beta m|}{|c_0| + k^{n-s}|c_{n-s} - \beta m|} \geq \frac{|c_0| - k^{n-s}|c_{n-s}| + k^{n-s}|\beta| m}{|c_0| + k^{n-s}|c_{n-s}| - k^{n-s}|\beta| m} = C_n(k, |\beta|) \tag{23}$$

where

$$C_n(k, |\beta|) = \frac{|c_0| - k^{n-s}|c_{n-s}| + k^{n-s}|\beta| m}{|c_0| + k^{n-s}|c_{n-s}| - k^{n-s}|\beta| m}.$$

Using (21), (22) and (23) in (18), we get for every $|\beta < 1$ and $|z| = 1$,

$$(|P'(z)| - |\beta| m n)^2 + n\left\{\frac{n(k-1) - s(2k)}{k+1} + \frac{2}{1+k} C_n(k, |\beta|)\right\}(|P(z)| - |\beta| m)^2 \leq |Q'(z)| \tag{24}$$

which gives by using Lemma 3.2 for $|z| = 1$, that

$$(|P'(z)| - |\beta| m n)^2 + n\left\{\frac{n(k-1) - s(2k)}{k+1} + \frac{2}{1+k} C_n(k, |\beta|)\right\}(|P(z)| - |\beta| m)^2 \leq (n||P|| - |P'(z)|)^2 \tag{25}$$

equivalently for $|z| = 1$,

$$2n\,(||P|| - |\beta| m)\,|P'(z)| \leq n^2(||P||^2 - (|\beta| m)^2) - n\left\{\frac{n(k-1) - s(2k)}{k+1} + \frac{2}{1+k} C_n(k, |\beta|)\right\}(|P(z)| - |\beta| m)^2$$

which gives for $|z| = 1$ and $|\beta| < 1$,

$$|P'(z)| \leq \frac{1}{2}\left[n(||P|| + |\beta| m) - \left\{\frac{n(k-1) - s(2k)}{k+1} + \frac{2}{1+k} C_n(k, |\beta|)\right\}\left(\frac{(|P(z)| - |\beta| m)^2}{||P|| - |\beta| m}\right)\right]$$

taking $|\beta| = t$, so that $0 \leq t \leq 1$, we get

$$|P'(z)| \leq \frac{1}{2}\left[n(||P|| + t m) - \left\{\frac{n(k-1) - s(2k)}{k+1} + \frac{2}{1+k} C_n(k, t)\right\}\left(\frac{(|P(z)| - t m)^2}{||P|| - t m}\right)\right].$$

That proves the Theorem 2.2 completely. □

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