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On Triangular $\mathfrak{n}-$Matrix Rings Having Multiplicative Lie Type Derivations

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Abstract. Let $1 < \mathfrak{n} \in \mathbb{Z}^+$ and $\mathscr{T}$ be a triangular $\mathfrak{n}-$matrix ring. This manuscript reveals that under a few moderate presumptions, a map $\mathscr{L} : \mathscr{T} \to \mathscr{T}$ could be a multiplicative Lie $\mathtt{N}-$derivation iff $\mathscr{L}(\mathscr{X}) = \mathscr{D}(\mathscr{X}) + \zeta(\mathscr{X})$ holds on every $\mathscr{X} \in \mathscr{T}$, where $\mathscr{D} : \mathscr{T} \to \mathscr{T}$ is an additive derivation and $\zeta : \mathscr{T} \to \mathscr{Z}(\mathscr{T})$ is a central valued map that disappears on all Lie $\mathtt{N}-$products.

1. Introduction

Unless otherwise indicated throughout the manuscript $\mathscr{R}$ could be a commutative ring having identity, $\mathscr{A}$ is an $\mathscr{R}-$algebra and $\mathscr{Z}(\mathscr{A})$ denotes the center of $\mathscr{A}$. A map $\mathscr{L} : \mathscr{A} \to \mathscr{A}$ (not necessary linear) is referred to as a multiplicative derivation on $\mathscr{A}$ if $\mathscr{L}(\mathscr{U}\mathscr{V}) = \mathscr{L}(\mathscr{U})\mathscr{V} + \mathscr{U}\mathscr{L}(\mathscr{V})$ holds for all $\mathscr{U}, \mathscr{V} \in \mathscr{A}$. Further, $\mathscr{L}$ is said to be a derivation on $\mathscr{A}$, if $\mathscr{L}$ is linear on $\mathscr{A}$. A map $\mathscr{L} : \mathscr{A} \to \mathscr{A}$ (not essentially linear) is recognized as a multiplicative Lie derivation (resp. multiplicative Lie triple derivation) on $\mathscr{A}$ if $\mathscr{L}([\mathscr{U}, \mathscr{V}]) = [\mathscr{L}(\mathscr{U}), \mathscr{V}] + [\mathscr{U}, \mathscr{L}(\mathscr{V})]$ (resp. $\mathscr{L}([[\mathscr{U}, \mathscr{V}], \mathscr{W}]) = [[\mathscr{L}(\mathscr{U}), \mathscr{V}], \mathscr{W}] + [[\mathscr{U}, \mathscr{L}(\mathscr{V})], \mathscr{W}] + [[\mathscr{U}, \mathscr{V}], \mathscr{L}(\mathscr{W})]$) holds for all $\mathscr{U}, \mathscr{V}, \mathscr{W} \in \mathscr{A}$.

Here we are characterizing a more specific family of maps through the arrangement of polynomials:

$$\begin{aligned}
\mathscr{P}_1(\mathscr{X}_1) &= \mathscr{X}_1,\\
\mathscr{P}_2(\mathscr{X}_1, \mathscr{X}_2) &= [\mathscr{P}_1(\mathscr{X}_1), \mathscr{X}_2] = [\mathscr{X}_1, \mathscr{X}_2],\\
\mathscr{P}_3(\mathscr{X}_1, \mathscr{X}_2, \mathscr{X}_3) &= [\mathscr{P}_2(\mathscr{X}_1, \mathscr{X}_2), \mathscr{X}_3] = [[\mathscr{X}_1, \mathscr{X}_2], \mathscr{X}_3],\\
&\vdots \qquad \vdots\\
\mathscr{P}_{\mathtt{N}}(\mathscr{X}_1, \mathscr{X}_2, \cdots, \mathscr{X}_{\mathtt{N}}) &= [\mathscr{P}_{\mathtt{N}-1}(\mathscr{X}_1, \mathscr{X}_2, \cdots, \mathscr{X}_{\mathtt{N}-1}), \mathscr{X}_{\mathtt{N}}].
\end{aligned}$$

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For $\mathtt{N} \geqslant 2$, the polynomial $\mathscr{P}_{\mathtt{N}}(\mathscr{X}_1, \mathscr{X}_2, \cdots, \mathscr{X}_{\mathtt{N}})$ is known as $(\mathtt{N}-1)-$th commutator. A map $\mathscr{L}: \mathscr{A} \to \mathscr{A}$ (not essentially linear) is considered a multiplicative Lie $\mathtt{N}-$derivation on $\mathscr{A}$ if

$$\mathscr{L}(\mathscr{P}_{\mathtt{N}}(\mathscr{X}_1, \mathscr{X}_2, \cdots, \mathscr{X}_{\mathtt{N}})) = \sum_{i=1}^{i=\mathtt{N}} \mathscr{P}_{\mathtt{N}}(\mathscr{X}_1, \mathscr{X}_2, \cdots, \mathscr{X}_{i-1}, \mathscr{L}(\mathscr{X}_i), \mathscr{X}_{i+1}, \cdots, \mathscr{X}_{\mathtt{N}})$$

for all $\mathscr{X}_1, \mathscr{X}_2, \cdots, \mathscr{X}_{\mathtt{N}} \in \mathscr{A}$. Along these lines, Abdullaev [1] initiated and conceived the idea of Lie $\mathtt{N}-$derivation on von Neumann algebras. Notice that any multiplicative Lie $2-$derivation is known as multiplicative Lie derivation and multiplicative Lie $3-$derivation is said to be multiplicative Lie triple derivation. Therefore, multiplicative Lie/Lie triple/Lie $\mathtt{N}-$derivation are comprehensively recognized as multiplicative Lie type derivations on $\mathscr{A}$.

Several researchers have investigated the nature of Lie type derivations on various types of rings or algebras [2–4, 11]. In some of these cases, authors have shown that every Lie type derivation has the standard from on that precise ring/algebra contemporary. In 1964, Martindale [11] obtained the first characterization of Lie derivations and he established that "*Every Lie derivation on a primitive ring can be written as a sum of derivation and an additive mapping of a ring to its center that maps commutators into zero, i.e, Lie derivation has the standard form*". In addition, several researchers have addressed the multiplicative mappings on rings and algebras over the last few decades. Martindale [12] has developed a condition on a ring such that multiplicative bijective mappings are all additives on this ring. Notably he demonstrated that "*Every multiplicative bijective mapping from a prime ring containing a nontrivial idempotent onto an arbitrary ring is additive*". Daif [5] examined the additivity of derivable map on a $2-$torsion free prime ring containing a nontrivial idempotent. Besides associative algebras or rings, numerous authors studied multiplicative Lie derivations and multiplicative Lie type derivations on nonassociative rings for example alternative rings see in [7, 9] and references therein.

Amongst these, a ring structure named triangular $n-$matrix ring in [6] was described by Ferreira. In [6], the author studied the additivity of $m-$multiplicative maps and $m-$multiplicative derivations on triangular $n-$matrix rings. Additionally, Ferreira and Guzzo [8] proved the additivity of Lie $\mathtt{N}-$multiplicative mappings on triangular $n-$matrix rings is almost additive. Using the triangular $n-$matrix ring concept for $n = 3$, Chen and Qi [4] gave, within certain premises, a characterization of multiplicative Lie derivations on triangular $n-$matrix rings for any $n \geqslant 2$. Subsequently, Jabeen and Ahmad [10] explained the characterization of multiplicative Lie triple derivations on triangular $3-$matrix rings.

Motivated by the above literature, our primary aim is characterization of multiplicative Lie type derivation on triangular $n-$matrix rings and to explain that each multiplicative Lie $\mathtt{N}-$derivation on triangular $n-$matrix rings could be the sum of an additive derivation and a central mapping annihilating $(\mathtt{N}-1)-$th commutator with some mild condition.

2. Preliminaries

Some conceptual notions are necessarily demonstrated to develop the proof of the key theorems. Roughly, these ideas are well known and written compactly. Let $\mathscr{R}_1, \mathscr{R}_2, \cdots, \mathscr{R}_n$ be unital rings and $\mathscr{M}_{ij}$ be $(\mathscr{R}_i, \mathscr{R}_j)-$bimodules with $\mathscr{M}_{ii} = \mathscr{R}_i$ for all $1 \leqslant i \leqslant j \leqslant n$. Let $\mho_{ijk}: \mathscr{M}_{ij} \otimes_{\mathscr{R}_j} \mathscr{M}_{jk} \to \mathscr{M}_{ik}$ be $(\mathscr{R}_i, \mathscr{R}_k)-$bimodules homomorphisms with $\mho_{iij}: \mathscr{R}_i \otimes_{\mathscr{R}_i} \mathscr{M}_{ij} \to \mathscr{M}_{ij}$ and $\mho_{ijj}: \mathscr{M}_{ij} \otimes_{\mathscr{R}_j} \mathscr{R}_j \to \mathscr{M}_{ij}$ the canonical multiplication maps for all $1 \leqslant i \leqslant j \leqslant k \leqslant n$. Write $ab = \mho_{ijk}(a \otimes b)$ for all $a \in \mathscr{M}_{ij}$ and $b \in \mathscr{M}_{jk}$. Assume that $\mathscr{M}_{ij}$ is faithful as a left $\mathscr{R}_i-$module and faithful as a right $\mathscr{R}_j-$module for all $1 \leqslant i < j \leqslant n$. Let $\mathscr{T} = \mathscr{T}_n(\mathscr{R}_i; \mathscr{M}_{ij})$ be the set

$$\mathscr{T} = \left\{ \begin{bmatrix} r_{11} & m_{12} & \cdots & m_{1(n-1)} & m_{1n} \\ 0 & r_{22} & \cdots & m_{2(n-1)} & m_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & r_{(n-1)(n-1)} & m_{(n-1)n} \\ 0 & 0 & \cdots & 0 & r_{nn} \end{bmatrix} \middle| \, r_{ii} \in \mathscr{R}_i, m_{ij} \in \mathscr{M}_{ij}, 1 \leqslant i < j \leqslant n \right\}.$$

Alternatively, suppose that $u(vw) = (uv)w$ with all $u \in \mathscr{M}_{ik}$, $v \in \mathscr{M}_{kl}$ including $w \in \mathscr{M}_{lj}$ with all $1 \leqslant i \leqslant k \leqslant l \leqslant j \leqslant n$. Therefore, with the regular matrix operations, $\mathscr{T}$ is termed a $n-$matrix ring. Undoubtedly, upper triangular matrix rings $\mathscr{T}_n(\mathscr{R})$ with $n \geqslant 3$ are triangular $n-$matrix rings over a unital associative ring $\mathscr{R}$. Notice the usual triangular $2-$matrix rings are triangular rings, too. Nonetheless, for $n \geqslant 3$, would not be a triangular $n-$matrix ring. Contrarily, a $n-$matrix triangular ring may not be a triangular ring. We get the following observations for the center of the triangular $n-$matrix rings. For the sake of self contentment of the article, we write the following Proposition from Ferreira's [6] paper as follows:

Proposition 2.1. *[6, Proposition 1.1] "Let $\mathscr{T} = \mathscr{T}_n(\mathscr{R}_i; \mathscr{M}_{ij})$ be a triangular $n-$matrix ring. The centre of $\mathscr{T}$ is*

$$\mathscr{Z}(\mathscr{T}) = \left\{ \bigoplus_{i=1}^{n} r_{ii} \,\middle|\, r_{ii}m_{ij} = m_{ij}r_{jj} \textit{ for all } m_{ij} \in \mathscr{M}_{ij}, i < j \right\}.$$

Furthermore, $\mathscr{Z}(\mathscr{T})_{ii} \cong \psi_{\mathscr{R}_i}(\mathscr{Z}(\mathscr{T})) \subseteq \mathscr{Z}(\mathscr{R}_i)$, and there exists a unique ring isomorphism τ_i^j from $\psi_{\mathscr{R}_i}(\mathscr{Z}(\mathscr{T}))$ to $\psi_{\mathscr{R}_j}(\mathscr{Z}(\mathscr{T}))$ $i = j$ such that $r_{ii}m_{ij} = m_{ij}\tau_i^j(r_{ii})$ for all $m_{ij} \in \mathscr{M}_{ij}$."

Here, $\bigoplus_{i=1}^{n} r_{ii}$ symbolizes the element $\begin{bmatrix} r_{11} & 0 & \cdots & 0 \\ & r_{22} & \cdots & 0 \\ & & \ddots & \vdots \\ & & & r_{nn} \end{bmatrix}$ and $\psi_{\mathscr{R}_i} : \mathscr{T} \to \mathscr{R}_i$ $(1 \leqslant i \leqslant n)$ is the natural projection described by $[m_{ij}] \to r_{ii}$. Now, assume that $\mathscr{T} = \mathscr{T}_n(\mathscr{R}_i; \mathscr{M}_{ij})$ is a triangular $n-$matrix ring. Set $\mathscr{T}_{ij} = \left\{ [m_{kt}] \,\middle|\, m_{kt} = \begin{cases} m_{ij}, & \text{if } (k,t) = (i,j) \\ 0, & \text{if } (k,t) \neq (i,j) \end{cases}, 1 \leqslant i \leqslant j \leqslant n \right\} \subset \mathscr{T}$. Then we can write $\mathscr{T} = \bigoplus_{1 \leqslant i \leqslant j \leqslant n} \mathscr{T}_{ij}$ Henceforth $a_{ij} \in \mathscr{T}_{ij}$ and by the direct computation $a_{ij}a_{kl} = 0$ for $j \neq k$.

Fix any $i \in \{1, 2, \cdots, n\}$. Let $\mathscr{E}_i$ serve as the non-trivial idempotent of $\mathscr{T}$ whose members were $(i,i)-$th place 1 and indeed the remaining 0. Compose $\mathtt{P}_i = \mathscr{E}_1 + \mathscr{E}_2 + \cdots + \mathscr{E}_i$ and $\mathscr{Q}_i = \mathtt{I} - \mathtt{P}_i$. Denote by $\mathrm{A}_i = \mathtt{P}_i\mathscr{T}\mathtt{P}_i$, $\mathrm{B}_i = \mathscr{Q}_i\mathscr{T}\mathscr{Q}_i$ and $\mathscr{M}_i = \mathtt{P}_i\mathscr{T}\mathscr{Q}_i$. One could also describe $\mathscr{T}$ as $\mathscr{T} = \mathrm{A}_i + \mathscr{M}_i + \mathrm{B}_i$ for each i. In this article, unless there is no uncertainty, for just any $\mathtt{A}_i \in \mathrm{A}_i$, $\mathscr{H}_i \in \mathscr{M}_i$ and $\mathtt{B}_i \in \mathrm{B}_i$, we frequently classify

$$\mathtt{A}_i \cong \begin{bmatrix} r_{11} & m_{12} & \cdots & m_{1i} \\ 0 & r_{22} & \cdots & m_{2i} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{ii} \end{bmatrix}, \ \mathscr{H}_i \cong \begin{bmatrix} m_{1,i+1} & m_{1,i+2} & \cdots & m_{1n} \\ m_{2,i+1} & m_{2,i+2} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{i,i+1} & m_{i,i+2} & \cdots & m_{1n} \end{bmatrix}$$

and

$$\mathtt{B}_i \cong \begin{bmatrix} r_{i+1,i+1} & m_{i+1,i+2} & \cdots & m_{i+1,n} \\ 0 & r_{i+2,i+2} & \cdots & m_{i+2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{nn} \end{bmatrix}.$$

For this depiction, it's really easy to confirm that A_i is a triangular $i-$matrix ring and B_i is a triangular $(n-i)-$matrix ring. Therefore, by Proposition 2.1, we get

$$\mathscr{Z}(\mathrm{A}_i) = \left\{ \bigoplus_{k=1}^{i} r_{kk} \,\middle|\, r_{kk} \in \mathscr{R}_k \text{ and } r_{kk}m_{kl} = m_{kl}r_{ll} \text{ for all } m_{kl} \in \mathscr{M}_{kl}, 1 \leqslant k < l \leqslant i \right\}$$

and

$$\mathscr{Z}(\mathrm{B}_i) = \left\{ \bigoplus_{k=i+1}^{n} r_{kk} \,\middle|\, r_{kk} \in \mathscr{R}_k \text{ and } r_{kk}m_{kl} = m_{kl}r_{ll} \text{ for all } m_{kl} \in \mathscr{M}_{kl}, i+1 \leqslant k < l \leqslant n \right\}.$$

Thence, by some premise that $\mathscr{M}_{ij}$ is faithful as a left $\mathscr{R}_i-$module and faithful as a right $\mathscr{R}_j-$module for all $1 \leqslant i < j \leqslant n$, a direct computation gives, for $\mathtt{A}_i = \bigoplus_{k=1}^{i} r_{kk} \in \mathrm{A}_i$ and $\mathtt{B}_i = \bigoplus_{k=i+1}^{n} r_{kk} \in \mathrm{B}_i$, we see that

$$\mathtt{A}_i\mathscr{M}_i = \{0\} \implies \mathtt{A}_i = 0 \text{ and } \mathscr{M}_i\mathtt{B}_i = \{0\} \implies \mathtt{B}_i = 0. \quad (1)$$

Let's develop natural projections $\psi_{\mathrm{A}_i} : \mathscr{T} \to \mathrm{A}_i$ and $\psi_{\mathrm{B}_i} : \mathscr{T} \to \mathrm{B}_i$ by $\mathscr{X} = \mathtt{A}_i + \mathscr{H}_i + \mathtt{B}_i \to \mathtt{A}_i$ and $\mathscr{X} = \mathtt{A}_i + \mathscr{H}_i + \mathtt{B}_i \to \mathtt{B}_i$ respectively. Undoubtedly, $\psi_{\mathrm{A}_i}(\mathscr{Z}(\mathscr{T})) \subseteq \mathscr{Z}(\mathrm{A}_i)$ and $\psi_{\mathrm{B}_i}(\mathscr{Z}(\mathscr{T})) \subseteq \mathscr{Z}(\mathrm{B}_i)$.

In addition, we pick out the following lemma from Chen and Qi's [4] study as follows:

Lemma 2.2. *[4, Lemma 2.2] "Let $\mathscr{T} = \mathscr{T}_n(\mathscr{R}_i; \mathscr{M}_{ij})$ be a triangular $n-$matrix ring. Then there exists a unique ring isomorphism $\psi : \psi_{\mathrm{A}_i}(\mathscr{Z}(\mathscr{T})) \to \psi_{\mathrm{B}_i}(\mathscr{Z}(\mathscr{T}))$ such that $\mathtt{A}_i\mathscr{H}_i = \mathscr{H}_i\psi(\mathtt{A}_i)$ for all $\mathscr{H}_i \in \mathscr{M}_i$ and $\mathtt{A}_i \in \psi_{\mathrm{A}_i}(\mathscr{Z}(\mathscr{T}))$; and moreover, $\mathtt{A}_i \oplus \psi(\mathtt{A}_i) \in \mathscr{Z}(\mathscr{T})$."*

3. Main Results

In this subdivision, we exhibit the primary result of the manuscript.

Theorem 3.1. *Let $1 < n \in \mathbb{Z}^+$ and $\mathscr{T}$ be a $(\mathtt{N}-1)-$torsion free triangular $n-$matrix ring. Expect that*

- *(♯) $P_{[n/2]}\mathscr{Z}(\mathscr{T})P_{[n/2]} = \mathscr{Z}(P_{[n/2]}\mathscr{T}P_{[n/2]})$ and $\mathscr{Q}_{[n/2]}\mathscr{Z}(\mathscr{T})\mathscr{Q}_{[n/2]} = \mathscr{Z}(\mathscr{Q}_{[n/2]}\mathscr{T}\mathscr{Q}_{[n/2]})$;*
- *(♮) $\mathrm{A}_{[n/2]}$ and $\mathrm{B}_{[n/2]}$ contains no nonzero central ideal.*

Therefore, $\mathscr{L}$ has a standard form iff $\mathscr{L} : \mathscr{T} \to \mathscr{T}$ would be a multiplicative Lie $\mathtt{N}-$derivation ($\mathtt{N} \geqslant 3$) i.e., $\mathscr{L}(\mathscr{X}) = \mathscr{D}(\mathscr{X}) + \zeta(\mathscr{X})$ holds for all $\mathscr{X} \in \mathscr{T}$, where $\mathscr{D} : \mathscr{T} \to \mathscr{T}$ is an additive derivation and $\zeta : \mathscr{T} \to \mathscr{Z}(\mathscr{T})$ is a map that vanishes on all Lie $\mathtt{N}-$products. Here $[n]$ is the integer part of n.

By carrying out a series of lemmas, we are offering the proof of our key theorem.

Lemma 3.2. *On assumption that $\mathscr{L} : \mathscr{T} \to \mathscr{T}$ is a multiplicative Lie $\mathtt{N}-$derivation and $\mathscr{T}$ is a triangular $n-$matrix ring. This provides an additive derivation $\mathscr{D}_i : \mathscr{T} \to \mathscr{T}$ and a multiplicative Lie $\mathtt{N}-$derivation $\mathscr{L}_i : \mathscr{T} \to \mathscr{T}$ such that $\mathtt{P}_i\mathscr{L}_i(\mathscr{Q}_i)\mathscr{Q}_i = 0$ and $\mathscr{L}(\mathscr{X}) = \mathscr{L}_i(\mathscr{X}) + \mathscr{D}_i(\mathscr{X})$ for all $\mathscr{X} \in \mathscr{T}$ $(i \in \{1, \cdots, n-1\})$.*

Proof. Firstly, we recognize the mappings $\mathscr{D}_i, \mathscr{L}_i : \mathscr{T} \to \mathscr{T}$, where $i \in \{1, \cdots, n-1\}$ such that

$$\mathscr{D}_i(\mathscr{X}) = [\mathscr{L}(\mathscr{Q}_i), \mathscr{X}] \text{ and } \mathscr{L}_i(\mathscr{X}) = \mathscr{L}(\mathscr{X}) - \mathscr{D}_i(\mathscr{X}) \text{ for all } \mathscr{X} \in \mathscr{T}.$$

It becomes easier to analyse $\mathscr{D}_i$ to be an additive derivation and $\mathscr{L}_i$, a multiplicative Lie $\mathtt{N}-$derivation. Further, since

$$\mathscr{L}_i(\mathscr{Q}_i) = \mathscr{L}(\mathscr{Q}_i) - \mathscr{D}_i(\mathscr{Q}_i) = \mathscr{L}(\mathscr{Q}_i) - [\mathscr{L}(\mathscr{Q}_i), \mathscr{Q}_i] = \mathscr{L}(\mathscr{Q}_i) - \mathtt{P}_i\mathscr{L}(\mathscr{Q}_i)\mathscr{Q}_i.$$

Multiplying by $\mathtt{P}_i$ and $\mathscr{Q}_i$ from the left and the right, respectively, we get $\mathtt{P}_i\mathscr{L}_i(\mathscr{Q}_i)\mathscr{Q}_i = 0$. □

Lemma 3.3. *For each one $\mathscr{H}_i \in \mathscr{M}_i$, $\mathtt{B}_i \in \mathrm{B}_i$ and $\mathtt{A}_i \in \mathrm{A}_i$ the following statements have always been:*

1. $P_i\mathscr{L}_i(A_i)\mathscr{Q}_i = 0$,
2. $P_i\mathscr{L}_i(B_i)\mathscr{Q}_i = 0$,
3. $P_i\mathscr{L}_i(\mathscr{H}_i)P_i = \mathscr{Q}_i\mathscr{L}_i(\mathscr{H}_i)\mathscr{Q}_i = 0$.

Proof. It is noticeable that $\mathscr{L}_i(0) = 0$. For each one $\mathscr{X} \in \mathscr{T}$, note that $\mathscr{P}_{\mathtt{N}}(\mathscr{X}, \mathscr{Q}_i, \cdots, \mathscr{Q}_i) = \mathtt{P}_i\mathscr{X}\mathscr{Q}_i$. We have

$$\begin{aligned} \mathscr{L}_i(\mathtt{P}_i\mathscr{X}\mathscr{Q}_i) &= \mathscr{L}_i(\mathscr{P}_{\mathtt{N}}(\mathscr{X}, \mathscr{Q}_i, \cdots, \mathscr{Q}_i)) \\ &= \mathscr{P}_{\mathtt{N}}(\mathscr{L}_i(\mathscr{X}), \mathscr{Q}_i, \cdots, \mathscr{Q}_i) + \cdots + \mathscr{P}_{\mathtt{N}}(\mathscr{X}, \mathscr{Q}_i, \cdots, \mathscr{L}_i(\mathscr{Q}_i)) \\ &= \mathtt{P}_i\mathscr{L}_i(\mathscr{X})\mathscr{Q}_i + (\mathtt{N}-1)[\mathtt{P}_i\mathscr{X}\mathscr{Q}_i, \mathscr{L}_i(\mathscr{Q}_i)]. \end{aligned} \quad (2)$$

Especially, if $\mathscr{X} = \mathtt{A_i} \in \mathrm{A_i}$ multiplying by $\mathtt{P_i}$ and $\mathscr{Q}_i$ from the left and the right in (2) respectively, we get $\mathtt{P_i}\mathscr{L}_i(\mathtt{A_i})\mathscr{Q}_i = 0$. If $\mathscr{X} = \mathtt{B_i} \in \mathrm{B_i}$, we reveal that $\mathtt{P_i}\mathscr{L}_i(\mathtt{B_i})\mathscr{Q}_i = 0$. Hence, truly justifying the statements (1) and (2).

Forthwith, if $\mathscr{X} = \mathscr{H}_i \in \mathscr{M}_i$, we have

$$\begin{aligned}\mathscr{L}_i(\mathscr{H}_i) &= \mathscr{L}_i(\mathscr{P}_{\mathtt{N}}(\mathscr{H}_i, \mathscr{Q}_i, \cdots, \mathscr{Q}_i))\\ &= \mathscr{P}_{\mathtt{N}}(\mathscr{L}_i(\mathscr{H}_i), \mathscr{Q}_i, \cdots, \mathscr{Q}_i) + \cdots + \mathscr{P}_{\mathtt{N}}(\mathscr{H}_i, \mathscr{Q}_i, \cdots, \mathscr{L}_i(\mathscr{Q}_i))\\ &= \mathtt{P_i}\mathscr{L}_i(\mathscr{H}_i)\mathscr{Q}_i + (\mathtt{N}-1)[\mathscr{H}_i, \mathscr{L}_i(\mathscr{Q}_i)]. \end{aligned} \tag{3}$$

Multiplying by $\mathtt{P_i}$ the left and the $\mathscr{Q}_i$ from right in (3), we have $(\mathtt{N}-1)[\mathscr{H}_i, \mathscr{L}_i(\mathscr{Q}_i)] = 0$ and hence $[\mathscr{H}_i, \mathscr{L}_i(\mathscr{Q}_i)] = 0$. Then (3) imply $\mathscr{L}_i(\mathscr{H}_i) = \mathtt{P_i}\mathscr{L}_i(\mathscr{H}_i)\mathscr{Q}_i$. Hence holding, the statement (3). □

Lemma 3.4. *$P_i\mathscr{L}_i(\mathrm{B_i})P_i \subseteq \mathscr{Z}(\mathrm{A_i})$ and $\mathscr{Q}_i\mathscr{L}_i(\mathrm{A_i})\mathscr{Q}_i \subseteq \mathscr{Z}(\mathrm{B_i})$.*

Proof. Consider, for every $\mathtt{A_i} \in \mathrm{A_i}$ and $\mathtt{B_i} \in \mathrm{B_i}$. Although $[\mathtt{A_i}, \mathtt{B_i}] = 0$, by Lemma 3.3 (1)-(2), we've got

$$\begin{aligned} 0 &= \mathscr{L}_i(\mathscr{P}_{\mathtt{N}}(\mathtt{A_i}, \mathtt{B_i}, \mathscr{X}, \mathscr{Q}_i, \cdots, \mathscr{Q}_i))\\ &= \mathscr{P}_{\mathtt{N}-1}([\mathscr{L}_i(\mathtt{A_i}), \mathtt{B_i}], \mathscr{X}, \mathscr{Q}_i, \cdots, \mathscr{Q}_i) + \mathscr{P}_{\mathtt{N}-1}([\mathtt{A_i}, \mathscr{L}_i(\mathtt{B_i})], \mathscr{X}, \mathscr{Q}_i, \cdots, \mathscr{Q}_i)\\ &= \mathtt{P_i}[[\mathscr{L}_i(\mathtt{A_i}), \mathtt{B_i}] + [\mathtt{A_i}, \mathscr{L}_i(\mathtt{B_i})], \mathscr{X}]\mathscr{Q}_i. \end{aligned}$$

Thus, $[[\mathscr{L}_i(\mathtt{A_i}), \mathtt{B_i}] + [\mathtt{A_i}, \mathscr{L}_i(\mathtt{B_i})], \mathscr{M}_i] = 0$. From Lemma 3, we see that $[\mathtt{A_i}, \mathscr{L}_i(\mathtt{B_i})] = [\mathtt{A_i}, \mathtt{P_i}\mathscr{L}_i(\mathtt{B_i})\mathtt{P_i}] \in \mathrm{A_i}$ and $[\mathscr{L}_i(\mathtt{A_i}), \mathtt{B_i}] = [\mathscr{Q}_i\mathscr{L}_i(\mathtt{A_i})\mathscr{Q}_i, \mathtt{B_i}] \in \mathrm{B_i}$. Now, assumption (♮) implies that $[[\mathscr{L}_i(\mathtt{A_i}), \mathtt{B_i}] + [\mathtt{A_i}, \mathscr{L}_i(\mathtt{B_i})], \mathrm{A_i}] = 0$ and $[[\mathscr{L}_i(\mathtt{A_i}), \mathtt{B_i}] + [\mathtt{A_i}, \mathscr{L}_i(\mathtt{B_i})], \mathtt{B_i}] = 0$. It follows that

$$[\mathtt{A_i}, \mathtt{P_i}\mathscr{L}_i(\mathtt{B_i})\mathtt{P_i}] + [\mathscr{Q}_i\mathscr{L}_i(\mathtt{A_i})\mathscr{Q}_i, \mathtt{B_i}] \in \mathscr{Z}(\mathscr{T}) \text{ for all } \mathtt{A_i} \in \mathrm{A_i},\ \mathtt{B_i} \in \mathrm{B_i}$$

and hence $\mathtt{P_i}\mathscr{L}_i(\mathtt{B_i})\mathtt{P_i} \in \mathscr{Z}(\mathrm{A_i})$ and $\mathscr{Q}_i\mathscr{L}_i(\mathtt{A_i})\mathscr{Q}_i \in \mathscr{Z}(\mathrm{B_i})$ for all $\mathtt{A_i} \in \mathrm{A_i}, \mathtt{B_i} \in \mathrm{B_i}$. □

Lemma 3.5. *$\mathscr{L}_i(P_i), \mathscr{L}_i(\mathscr{Q}_i) \in \mathscr{Z}(\mathscr{T})$.*

Proof. Mark that, for each one $\mathscr{X} \in \mathscr{T}$, we get

$$\begin{aligned}\mathscr{L}_i(\mathtt{P_i}\mathscr{X}\mathscr{Q}_i) &= \mathscr{L}_i(\mathscr{P}_{\mathtt{N}}(\mathtt{P_i}, \mathscr{X}, \mathscr{Q}_i, \cdots, \mathscr{Q}_i))\\ &= \mathscr{P}_{\mathtt{N}-1}([\mathscr{L}_i(\mathtt{P_i}), \mathscr{X}], \mathscr{Q}_i, \cdots, \mathscr{Q}_i)\\ &\quad + \mathscr{P}_{\mathtt{N}-1}([\mathtt{P_i}, \mathscr{L}_i(\mathscr{X})], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) + \cdots + \mathscr{P}_{\mathtt{N}}(\mathtt{P_i}, \mathscr{X}, \mathscr{Q}_i, \cdots, \mathscr{L}_i(\mathscr{Q}_i))\\ &= \mathtt{P_i}[\mathscr{L}_i(\mathtt{P_i}), \mathscr{X}]\mathscr{Q}_i + \mathtt{P_i}\mathscr{L}_i(\mathscr{X})\mathscr{Q}_i + (\mathtt{N}-2)[\mathtt{P_i}\mathscr{X}\mathscr{Q}_i, \mathscr{L}_i(\mathscr{Q}_i)]. \end{aligned} \tag{4}$$

From Lemma 3, we can see $\mathtt{P_i}\mathscr{L}_i(\mathtt{P_i})\mathscr{Q}_i = 0$. Replacing $\mathscr{X}$ by $\mathtt{P_i}\mathscr{X}\mathscr{Q}_i$ in (4), and by Lemma 3.3(3), we get $\mathscr{L}_i(\mathtt{P_i}\mathscr{X}\mathscr{Q}_i) = [\mathscr{L}_i(\mathtt{P_i}), \mathtt{P_i}\mathscr{X}\mathscr{Q}_i] + \mathscr{L}_i(\mathtt{P_i}\mathscr{X}\mathscr{Q}_i)$, that is, $[\mathscr{L}_i(\mathtt{P_i}), \mathtt{P_i}\mathscr{X}\mathscr{Q}_i] = 0$. By Lemma 4, we obtain $[\mathscr{L}_i(\mathtt{P_i}), \mathtt{P_i}\mathscr{X}\mathtt{P_i}] = 0$ and $[\mathscr{L}_i(\mathtt{P_i}), \mathscr{Q}_i\mathscr{X}\mathscr{Q}_i] = 0$. So $[\mathscr{L}_i(\mathtt{P_i}), \mathscr{X}] = 0$ for all $\mathscr{X}$, and hence $\mathscr{L}_i(\mathtt{P_i}) \in \mathscr{Z}(\mathscr{T})$. Therefore, $\mathscr{L}_i(\mathscr{Q}_i) \in \mathscr{Z}(\mathscr{T})$ can be shown by a congruent altercation to that of the above. □

Lemma 3.6. *For any $\mathscr{X} \in \mathscr{T}$, we have $\mathscr{L}_i(P_i\mathscr{X}\mathscr{Q}_i) = P_i\mathscr{L}_i(\mathscr{X})\mathscr{Q}_i$.*

Proof. By Lemma 3.5, we have

$$\mathscr{L}_i(\mathtt{P_i}\mathscr{X}\mathscr{Q}_i) = \mathtt{P_i}[\mathscr{L}_i(\mathtt{P_i}), \mathscr{X}]\mathscr{Q}_i + \mathtt{P_i}\mathscr{L}_i(\mathscr{X})\mathscr{Q}_i = \mathtt{P_i}\mathscr{L}_i(\mathscr{X})\mathscr{Q}_i.$$

for all $\mathscr{X} \in \mathscr{T}$ □

Lemma 3.7. *$\mathscr{L}_i$ is additive on $\mathscr{M}_i$.*

Proof. Start taking every $\mathscr{H}_i, \mathscr{H}_i' \in \mathscr{M}_i$, $\mathtt{A}_i, \mathtt{A}_i' \in \mathrm{A}_i$ and $\mathtt{B}_i, \mathtt{B}_i' \in \mathrm{B}_i$. Next, we demonstrate that the following equalities hold:

$$[\mathscr{L}_i(\mathtt{A}_i + \mathscr{H}_i) - \mathscr{L}_i(\mathtt{A}_i) - \mathscr{L}_i(\mathscr{H}_i), \mathscr{H}_i'] = 0, \tag{5}$$

$$[\mathscr{L}_i(\mathtt{B}_i + \mathscr{H}_i) - \mathscr{L}_i(\mathtt{B}_i) - \mathscr{L}_i(\mathscr{H}_i), \mathscr{H}_i'] = 0. \tag{6}$$

In point of fact, since

$$\begin{aligned}
[\mathscr{L}_i(\mathtt{A}_i), \mathscr{H}_i'] &= \mathscr{P}_{\mathtt{N}}(\mathscr{L}_i(\mathtt{A}_i), \mathscr{H}_i', \mathscr{Q}_i, \cdots, \mathscr{Q}_i)\\
&= \mathscr{L}_i(\mathscr{P}_{\mathtt{N}}(\mathtt{A}_i, \mathscr{H}_i', \mathscr{Q}_i, \cdots, \mathscr{Q}_i))\\
&\quad -\mathscr{P}_{\mathtt{N}}(\mathtt{A}_i, \mathscr{L}_i(\mathscr{H}_i'), \mathscr{Q}_i, \cdots, \mathscr{Q}_i) - \cdots - \mathscr{P}_{\mathtt{N}}(\mathtt{A}_i, \mathscr{H}_i', \mathscr{Q}_i, \cdots, \mathscr{L}_i(\mathscr{Q}_i))\\
&= \mathscr{L}_i(\mathscr{P}_{\mathtt{N}}(\mathtt{A}_i + \mathscr{H}_i, \mathscr{H}_i', \mathscr{Q}_i, \cdots, \mathscr{Q}_i)) - \mathscr{P}_{\mathtt{N}}(\mathtt{A}_i + \mathscr{H}_i, \mathscr{L}_i(\mathscr{H}_i'), \mathscr{Q}_i, \cdots, \mathscr{Q}_i)\\
&\quad -\mathscr{P}_{\mathtt{N}}(\mathtt{A}_i + \mathscr{H}_i, \mathscr{H}_i', \mathscr{L}_i(\mathscr{Q}_i), \cdots, \mathscr{Q}_i) - \cdots - \mathscr{P}_{\mathtt{N}}(\mathtt{A}_i + \mathscr{H}_i, \mathscr{H}_i', \mathscr{Q}_i, \cdots, \mathscr{L}_i(\mathscr{Q}_i))\\
&= \mathscr{P}_{\mathtt{N}}(\mathscr{L}_i(\mathtt{A}_i + \mathscr{H}_i), \mathscr{H}_i', \mathscr{Q}_i, \cdots, \mathscr{Q}_i)\\
&= [\mathscr{L}_i(\mathtt{A}_i + \mathscr{H}_i), \mathscr{H}_i']
\end{aligned}$$

It follows that (5) holds. Symmetrically, one can show that (6) holds.

$$\begin{aligned}
\mathscr{L}_i(\mathscr{H}_i) &= \mathscr{L}_i(\mathscr{P}_{\mathtt{N}}(\mathtt{P}_i + \mathscr{H}_i, \mathscr{Q}_i, \cdots, \mathscr{Q}_i))\\
&= \mathscr{P}_{\mathtt{N}}(\mathscr{L}_i(\mathtt{P}_i + \mathscr{H}_i), \mathscr{Q}_i, \cdots, \mathscr{Q}_i) + \mathscr{P}_{\mathtt{N}}(\mathtt{P}_i + \mathscr{H}_i, \mathscr{L}_i(\mathscr{Q}_i), \cdots, \mathscr{Q}_i)\\
&\quad + \cdots + \mathscr{P}_{\mathtt{N}}(\mathtt{P}_i + \mathscr{H}_i, \mathscr{Q}_i, \cdots, \mathscr{L}_i(\mathscr{Q}_i))\\
&= \mathscr{P}_{\mathtt{N}}(\mathscr{L}_i(\mathtt{P}_i + \mathscr{H}_i), \mathscr{Q}_i, \cdots, \mathscr{Q}_i).
\end{aligned}$$

Also, we have

$$\begin{aligned}
\mathscr{L}_i(\mathscr{H}_i') &= \mathscr{L}_i(\mathscr{P}_{\mathtt{N}}(\mathtt{P}_i + \mathscr{H}_i, \mathscr{H}_i', \mathscr{Q}_i, \cdots, \mathscr{Q}_i))\\
&= \mathscr{P}_{\mathtt{N}}(\mathscr{L}_i(\mathtt{P}_i + \mathscr{H}_i), \mathscr{H}_i', \mathscr{Q}_i, \cdots, \mathscr{Q}_i) + \mathscr{P}_{\mathtt{N}}(\mathtt{P}_i + \mathscr{H}_i, \mathscr{L}_i(\mathscr{H}_i'), \mathscr{Q}_i, \cdots, \mathscr{Q}_i)\\
&\quad + \cdots + \mathscr{P}_{\mathtt{N}}(\mathtt{P}_i + \mathscr{H}_i, \mathscr{H}_i', \mathscr{Q}_i, \cdots, \mathscr{L}_i(\mathscr{Q}_i))\\
&= \mathscr{P}_{\mathtt{N}}(\mathscr{L}_i(\mathtt{P}_i + \mathscr{H}_i), \mathscr{H}_i', \mathscr{Q}_i, \cdots, \mathscr{Q}_i) + \mathscr{L}_i(\mathscr{H}_i')\\
0 &= \mathscr{P}_{\mathtt{N}}(\mathscr{L}_i(\mathtt{P}_i + \mathscr{H}_i), \mathscr{H}_i', \mathscr{Q}_i, \cdots, \mathscr{Q}_i).
\end{aligned}$$

Likewise, $\mathscr{L}_i(\mathscr{H}_i') = \mathscr{P}_{\mathtt{N}}(\mathtt{P}_i, \mathscr{L}_i(\mathscr{H}_i' + \mathscr{Q}_i), \mathscr{Q}_i, \cdots, \mathscr{Q}_i)$ and $\mathscr{P}_{\mathtt{N}}(\mathscr{H}_i, \mathscr{L}_i(\mathscr{H}_i' + \mathscr{Q}_i), \mathscr{Q}_i, \cdots, \mathscr{Q}_i) = 0$. Using $\mathscr{H}_i + \mathscr{H}_i' = [\mathtt{P}_i + \mathscr{H}_i, \mathscr{H}_i' + \mathscr{Q}_i]$, the fact $\mathscr{L}_i(\mathscr{Q}_i) \in \mathscr{Z}(\mathscr{T})$ we see that

$$\begin{aligned}
\mathscr{L}_i(\mathscr{H}_i + \mathscr{H}_i') &= \mathscr{L}_i([\mathtt{P}_i + \mathscr{H}_i, \mathscr{H}_i' + \mathscr{Q}_i])\\
&= \mathscr{L}_i(\mathscr{P}_{\mathtt{N}}(\mathtt{P}_i + \mathscr{H}_i, \mathscr{H}_i' + \mathscr{Q}_i, \mathscr{Q}_i, \cdots, \mathscr{Q}_i))\\
&= \mathscr{P}_{\mathtt{N}}(\mathscr{L}_i(\mathtt{P}_i + \mathscr{H}_i), \mathscr{H}_i' + \mathscr{Q}_i, \mathscr{Q}_i, \cdots, \mathscr{Q}_i)\\
&\quad +\mathscr{P}_{\mathtt{N}}(\mathtt{P}_i + \mathscr{H}_i, \mathscr{L}_i(\mathscr{H}_i' + \mathscr{Q}_i), \mathscr{Q}_i, \cdots, \mathscr{Q}_i)\\
&\quad +\mathscr{P}_{\mathtt{N}}(\mathtt{P}_i + \mathscr{H}_i, \mathscr{H}_i' + \mathscr{Q}_i, \mathscr{L}_i(\mathscr{Q}_i), \cdots, \mathscr{Q}_i)\\
&\quad + \cdots + \mathscr{P}_{\mathtt{N}}(\mathtt{P}_i + \mathscr{H}_i, \mathscr{H}_i' + \mathscr{Q}_i, \mathscr{Q}_i, \cdots, \mathscr{L}_i(\mathscr{Q}_i))\\
&= \mathscr{P}_{\mathtt{N}}(\mathscr{L}_i(\mathtt{P}_i + \mathscr{H}_i), \mathscr{H}_i', \mathscr{Q}_i, \cdots, \mathscr{Q}_i) + \mathscr{P}_{\mathtt{N}}(\mathscr{L}_i(\mathtt{P}_i + \mathscr{H}_i), \mathscr{Q}_i, \mathscr{Q}_i, \cdots, \mathscr{Q}_i)\\
&\quad +\mathscr{P}_{\mathtt{N}}(\mathtt{P}_i, \mathscr{L}_i(\mathscr{H}_i' + \mathscr{Q}_i), \mathscr{Q}_i, \cdots, \mathscr{Q}_i) + \mathscr{P}_{\mathtt{N}}(\mathscr{H}_i, \mathscr{L}_i(\mathscr{H}_i' + \mathscr{Q}_i), \mathscr{Q}_i, \cdots, \mathscr{Q}_i)\\
&= \mathscr{L}_i(\mathscr{H}_i) + \mathscr{L}_i(\mathscr{H}_i').
\end{aligned}$$

That is, $\mathscr{L}_i$ is additive on $\mathscr{M}_i$. □

Lemma 3.8. *For every* $A_i \in \mathrm{A}_i$ *and* $B_i \in \mathrm{B}_i$, *we have* $\mathscr{L}_i(A_i + B_i) - \mathscr{L}_i(A_i) - \mathscr{L}_i(B_i) \in \mathscr{Z}(\mathscr{T})$.

Proof. With every $\mathscr{H}_i' \in \mathscr{M}_i$, we provide

$$\begin{aligned}
&\mathscr{L}_i(\mathscr{P}_N(A_i + B_i, \mathscr{H}_i', \mathscr{Q}_i, \cdots, \mathscr{Q}_i)) \\
&= \mathscr{P}_{N-1}([\mathscr{L}_i(A_i + B_i), \mathscr{H}_i'], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) + \mathscr{P}_{N-1}([A_i + B_i, \mathscr{L}_i(\mathscr{H}_i')], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) \\
&\quad + \mathscr{P}_N(A_i + B_i, \mathscr{H}_i', \mathscr{L}_i(\mathscr{Q}_i), \cdots, \mathscr{Q}_i) + \cdots + \mathscr{P}_N(A_i + B_i, \mathscr{H}_i', \mathscr{Q}_i, \cdots, \mathscr{L}_i(\mathscr{Q}_i)) \\
&= [\mathscr{L}_i(A_i + B_i), \mathscr{H}_i'] + [A_i + B_i, \mathscr{L}_i(\mathscr{H}_i')].
\end{aligned}$$

On the other way, it follows that

$$\begin{aligned}
&\mathscr{L}_i(\mathscr{P}_N(A_i + B_i, \mathscr{H}_i', \mathscr{Q}_i, \cdots, \mathscr{Q}_i)) \\
&= \mathscr{L}_i([A_i, \mathscr{H}_i']) + \mathscr{L}_i([B_i, \mathscr{H}_i']) \\
&= \mathscr{L}_i(\mathscr{P}_N(A_i, \mathscr{H}_i', \mathscr{Q}_i, \cdots, \mathscr{Q}_i)) + \mathscr{L}_i(\mathscr{P}_N(B_i, \mathscr{H}_i', \mathscr{Q}_i, \cdots, \mathscr{Q}_i)) \\
&= \mathscr{P}_{N-1}([\mathscr{L}_i(A_i), \mathscr{H}_i'], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) + \mathscr{P}_{N-1}([A_i, \mathscr{L}_i(\mathscr{H}_i')], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) \\
&\quad + \cdots + \mathscr{P}_N(A_i, \mathscr{H}_i', \mathscr{Q}_i, \cdots, \mathscr{L}_i(\mathscr{Q}_i)) + \mathscr{P}_{N-1}([\mathscr{L}_i(B_i), \mathscr{H}_i'], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) \\
&\quad + \mathscr{P}_{N-1}([B_i, \mathscr{L}_i(\mathscr{H}_i')], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) + \cdots + \mathscr{P}_N(B_i, \mathscr{H}_i', \mathscr{Q}_i, \cdots, \mathscr{L}_i(\mathscr{Q}_i)) \\
&= [\mathscr{L}_i(A_i), \mathscr{H}_i'] + [A_i, \mathscr{L}_i(\mathscr{H}_i')] + [\mathscr{L}_i(B_i), \mathscr{H}_i'] + [B_i, \mathscr{L}_i(\mathscr{H}_i')].
\end{aligned}$$

Now from above two expressions, we arrive at $[\mathscr{L}_i(A_i+B_i)-\mathscr{L}_i(A_i)-\mathscr{L}_i(B_i), \mathscr{H}_i'] = 0$ for all $\mathscr{H}_i' \in \mathscr{M}_i$. With Lemma 3.4, we have $[\mathscr{L}_i(A_i+B_i)-\mathscr{L}_i(A_i)-\mathscr{L}_i(B_i), B_i'] = 0$ for all and $[\mathscr{L}_i(A_i+B_i)-\mathscr{L}_i(A_i)-\mathscr{L}_i(B_i), A_i'] = 0$ for all $A_i' \in \mathrm{A}_i$ and $B_i' \in \mathrm{B}_i$. These together implies that

$$[\mathscr{L}_i(A_i + B_i) - \mathscr{L}_i(A_i) - \mathscr{L}_i(B_i), \mathscr{T}] = 0 \text{ and hence } \mathscr{L}_i(A_i + B_i) - \mathscr{L}_i(A_i) - \mathscr{L}_i(B_i) \in \mathscr{Z}(\mathscr{T})$$

for all $A_i \in \mathrm{A}_i$ and $B_i \in \mathrm{B}_i$. □

Lemma 3.9. *For every $A_i \in \mathrm{A}_i$, $\mathscr{H}_i \in \mathscr{M}_i$ and $B_i \in \mathrm{B}_i$, we get*

$$[\mathscr{L}_i(A_i + \mathscr{H}_i + B_i) - \mathscr{L}_i(A_i) - \mathscr{L}_i(\mathscr{H}_i) - \mathscr{L}_i(B_i), \mathscr{M}_i] \equiv 0.$$

Proof. For every $\mathscr{H}_i' \in \mathscr{M}_i$, we can write

$$\begin{aligned}
&\mathscr{L}_i(\mathscr{P}_N(A_i + \mathscr{H}_i + B_i, \mathscr{H}_i', \mathscr{Q}_i, \cdots, \mathscr{Q}_i)) \\
&= \mathscr{P}_{N-1}([\mathscr{L}_i(A_i + \mathscr{H}_i + B_i), \mathscr{H}_i'], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) + \mathscr{P}_{N-1}([A_i + \mathscr{H}_i + B_i, \mathscr{L}_i(\mathscr{H}_i')], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) \\
&\quad + \mathscr{P}_N(A_i + \mathscr{H}_i + B_i, \mathscr{H}_i', \mathscr{L}_i(\mathscr{Q}_i), \cdots, \mathscr{Q}_i) + \cdots + \mathscr{P}_N(A_i + \mathscr{H}_i + B_i, \mathscr{H}_i', \mathscr{Q}_i, \cdots, \mathscr{L}_i(\mathscr{Q}_i)) \\
&= [\mathscr{L}_i(A_i + \mathscr{H}_i + B_i), \mathscr{H}_i'] + [A_i + \mathscr{H}_i + B_i, \mathscr{L}_i(\mathscr{H}_i')].
\end{aligned}$$

On the other way, it follows that

$$\begin{aligned}
&\mathscr{L}_i(\mathscr{P}_N(A_i + \mathscr{H}_i + B_i, \mathscr{H}_i', \mathscr{Q}_i, \cdots, \mathscr{Q}_i)) \\
&= \mathscr{L}_i(\mathscr{P}_N(A_i + B_i, \mathscr{H}_i', \mathscr{Q}_i, \cdots, \mathscr{Q}_i)) \\
&= \mathscr{P}_{N-1}([\mathscr{L}_i(A_i + B_i), \mathscr{H}_i'], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) + \mathscr{P}_{N-1}([A_i + B_i, \mathscr{L}_i(\mathscr{H}_i')], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) \\
&\quad + \mathscr{P}_N(A_i + B_i, \mathscr{H}_i', \mathscr{L}_i(\mathscr{Q}_i), \cdots, \mathscr{Q}_i) + \cdots + \mathscr{P}_N(A_i + B_i, \mathscr{H}_i', \mathscr{Q}_i, \cdots, \mathscr{L}_i(\mathscr{Q}_i)) \\
&= [\mathscr{L}_i(A_i + B_i), \mathscr{H}_i'] + [A_i + B_i, \mathscr{L}_i(\mathscr{H}_i')].
\end{aligned}$$

The combination of the aforementioned both equations provides $[\mathscr{L}_i(A_i + \mathscr{H}_i + B_i) - \mathscr{L}_i(A_i + B_i), \mathscr{H}_i'] = 0$. It concludes through Lemmas 3.3 and 3.8. □

Especially, $\mathscr{T} = \mathrm{A}_{[n/2]} + \mathscr{M}_{[n/2]} + \mathrm{B}_{[n/2]}$ when $i = [n/2]$. Within that scenario, $\mathscr{L}_{[n/2]}$ is indeed a multiplicative Lie derivation with $P_{[n/2]}\mathscr{L}_{[n/2]}(\mathscr{Q}_{[n/2]})\mathscr{Q}_{[n/2]} = 0$; in comparison, for just about any

$\mathscr{X} = \mathtt{A}_{[n/2]} + \mathscr{H}_{[n/2]} + \mathtt{B}_{[n/2]} \in \mathscr{T}$, through Lemmas 3.3 and 3.8 can be written as,

$$\begin{aligned} \mathtt{H}_0 &= \mathscr{L}_{[n/2]}(\mathscr{X}) - \mathscr{L}_{[n/2]}(\mathtt{A}_{[n/2]}) - \mathscr{L}_{[n/2]}(\mathscr{H}_{[n/2]}) - \mathscr{L}_{[n/2]}(\mathtt{B}_{[n/2]}) \\ &= \begin{bmatrix} r_{11} & \cdots & r_{1,[n/2]} & 0 & \cdots & 0 \\ & \ddots & \vdots & \vdots & \ddots & \vdots \\ & & r_{[n/2],[n/2]} & 0 & \cdots & 0 \\ & & & r_{[n/2]+1,[n/2]+1} & \cdots & r_{[n/2]+1,n} \\ & & & & \ddots & \vdots \\ & & & & & r_{nn} \end{bmatrix}. \end{aligned} \tag{7}$$

Over the next portion, we will illustrate that $\mathtt{H}_0 \in \mathscr{Z}(\mathscr{T})$ mostly with the subsequent Lemmas 3.10-3.13. For this, we consider $\tau_i : \mathscr{T} \to \mathscr{T}$ and $h_i : \mathscr{T} \to \mathscr{T}$, where $i \in \{1, \cdots, [n/2]-1, [n/2]+1, \cdots, n-1\}$, such that

$$h_i(\mathscr{X}) = [\mathscr{L}_{[n/2]}(\mathscr{Q}_i), \mathscr{X}] \text{ and } \tau_i(\mathscr{X}) = \mathscr{L}_{[n/2]}(\mathscr{X}) - h_i(\mathscr{X}) \text{ for all } \mathscr{X} \in \mathscr{T}. \tag{8}$$

Hence, by offering the same premises like those of Lemmas 3.2-3.9 for $\mathscr{L}_i$, it can be shown that τ_i would also be a multiplicative Lie $\mathtt{N}$−derivation enjoying $\mathtt{P}_i\tau_i(\mathscr{Q}_i)\mathscr{Q}_i = 0$ as well as Lemmas 3.3-3.9 even now holds the map τ_i.

Lemma 3.10. *For any $\mathscr{H}_i \in \mathscr{M}_i = \mathtt{P}_i\mathscr{T}\mathscr{Q}_i$, we have $\mathscr{L}_{[n/2]}(\mathscr{H}_i) = \mathtt{P}_i\mathscr{L}_{[n/2]}(\mathscr{H}_i)\mathscr{Q}_i = \tau_i(\mathscr{H}_i)$; and moreover, $\mathscr{L}_{[n/2]}$ is additive on $\mathscr{M}_i$. Here $i \in \{1, \cdots, [n/2]-1, [n/2]+1, \cdots, n-1\}$.*

Proof. Let $i \in \{1, \cdots, [n/2]-1, [n/2]+1, \cdots, n-1\}$ and take any $\mathscr{H}_i \in \mathscr{M}_i$. Note that

$$\begin{aligned} \mathscr{L}_{[n/2]}(\mathscr{H}_i) &= \mathscr{L}_{[n/2]}(\mathscr{P}_{\mathtt{N}}(\mathscr{H}_i, \mathscr{Q}_i, \cdots, \mathscr{Q}_i)) \\ &= \mathscr{P}_{\mathtt{N}}(\mathscr{L}_{[n/2]}(\mathscr{H}_i), \mathscr{Q}_i, \cdots, \mathscr{Q}_i) + \mathscr{P}_{\mathtt{N}}(\mathscr{H}_i, \mathscr{L}_{[n/2]}(\mathscr{Q}_i), \cdots, \mathscr{Q}_i)) \\ &\quad + \cdots + \mathscr{P}_{\mathtt{N}}(\mathscr{H}_i, \mathscr{Q}_i, \cdots, \mathscr{L}_{[n/2]}(\mathscr{Q}_i)) \\ &= \mathtt{P}_i\mathscr{L}_{[n/2]}(\mathscr{H}_i)\mathscr{Q}_i + (\mathtt{N}-1)[\mathscr{H}_i, \mathscr{L}_{[n/2]}(\mathscr{Q}_i)] \in \mathscr{M}_i. \end{aligned}$$

Multiplying by $\mathtt{P}_i$ from left and $\mathscr{Q}_i$ from right side, we find that $(\mathtt{N}-1)[\mathscr{H}_i, \mathscr{L}_{[n/2]}(\mathscr{Q}_i)] = 0$ and hence $[\mathscr{H}_i, \mathscr{L}_{[n/2]}(\mathscr{Q}_i)] = 0$. It follows that $\mathscr{L}_{[n/2]}(\mathscr{H}_i) = \mathtt{P}_i\mathscr{L}_{[n/2]}(\mathscr{H}_i)\mathscr{Q}_i$. Thus, by (8), we have

$$\mathscr{L}_{[n/2]}(\mathscr{H}_i) = \tau_i(\mathscr{H}_i) - [\mathscr{H}_i, \mathscr{L}_{[n/2]}(\mathscr{Q}_i)] = \tau_i(\mathscr{H}_i).$$

Moreover, the additivity of $\mathscr{L}_{[n/2]}$ on $\mathscr{M}_i$ can be obtained by Lemma 3.7 for τ_i. □

Henceforth, fix any $\mathscr{X} = \mathtt{A}_{[n/2]} + \mathscr{H}_{[n/2]} + \mathtt{B}_{[n/2]} \in \mathscr{T}$. Then (7) holds. Consider the map τ_i $(i = 1, \cdots, [n/2]-1, [n/2]+1, \cdots, n-1)$. $\mathscr{X}$ can also be written as $\mathscr{X} = \mathtt{A}_i + \mathscr{H}_i + \mathtt{B}_i$. So we have

$$\begin{aligned} \mathtt{K}_i &= \tau_i(\mathscr{X}) - \tau_i(\mathtt{A}_i) - \tau_i(\mathscr{H}_i) - \tau_i(\mathtt{B}_i) \\ &= \begin{bmatrix} s^i_{1,1} & \cdots & s^i_{1,i} & 0 & \cdots & 0 \\ & \ddots & \vdots & \vdots & \ddots & \vdots \\ & & s^i_{i,i} & 0 & \cdots & 0 \\ & & & s^i_{i+1,i+1} & \cdots & s^i_{i+1,n} \\ & & & & \ddots & \vdots \\ & & & & & s^i_{n,n} \end{bmatrix}. \end{aligned} \tag{9}$$

Lemma 3.11. *For $\mathscr{X} = A_{[n/2]} + \mathscr{H}_{[n/2]} + B_{[n/2]} = A_i + \mathscr{H}_i + B_i \in \mathscr{T}$ $(i = 1, \cdots, [n/2]-1, [n/2]+1, \cdots, n-1)$, the following statements hold.*

1. *For $i \in \{1, \cdots, [n/2]-1\}$, we have $\mathscr{E}_h\tau_i(A_{[n/2]})\mathscr{E}_j = \mathscr{E}_h\tau_i(\mathscr{H}_i)\mathscr{E}_j$ for $1 \leqslant h \leqslant i$ and $i+1 \leqslant j \leqslant [n/2]$;*
2. *For $i \in \{[n/2]+1, \cdots, n-1\}$, we have $\mathscr{E}_h\tau_i(B_{[n/2]})\mathscr{E}_j = \mathscr{E}_h\tau_i(\mathscr{H}_i)\mathscr{E}_j$ for $[n/2]+1 \leqslant h \leqslant i$ and $i+1 \leqslant j \leqslant n$.*

Proof. For any $\mathscr{X} \in \mathscr{T}$, we have

$$\begin{aligned}\mathscr{L}_{[n/2]}(\mathtt{P}_i\mathscr{X}\mathscr{Q}_i) &= \mathscr{L}_{[n/2]}(\mathscr{P}_{\mathtt{N}}(\mathscr{X},\mathscr{Q}_i,\cdots,\mathscr{Q}_i))\\ &= \mathscr{P}_{\mathtt{N}}(\mathscr{L}_{[n/2]}(\mathscr{X}),\mathscr{Q}_i,\cdots,\mathscr{Q}_i)+\mathscr{P}_{\mathtt{N}}(\mathscr{X},\mathscr{L}_{[n/2]}(\mathscr{Q}_i),\cdots,\mathscr{Q}_i)\\ &\quad+\cdots+\mathscr{P}_{\mathtt{N}}(\mathscr{X},\mathscr{Q}_i,\cdots,\mathscr{L}_{[n/2]}(\mathscr{Q}_i))\\ &= \mathtt{P}_i\mathscr{L}_{[n/2]}(\mathscr{X})\mathscr{Q}_i+\mathtt{P}_i[\mathscr{X},\mathscr{L}_{[n/2]}(\mathscr{Q}_i)]\mathscr{Q}_i+(\mathtt{N}-2)[\mathtt{P}_i\mathscr{X}\mathscr{Q}_i,\mathscr{L}_{[n/2]}(\mathscr{Q}_i)].\end{aligned}$$

Multiplying by $\mathtt{P}_i$ from left and by $\mathscr{Q}_i$ from right and then replacing $\mathscr{X}$ by $\mathtt{P}_i\mathscr{X}\mathscr{Q}_i$, find that $(\mathtt{N}-1)[\mathtt{P}_i\mathscr{X}\mathscr{Q}_i,\mathscr{L}_{[n/2]}(\mathscr{Q}_i)]=0$ it follows that $[\mathtt{P}_i\mathscr{X}\mathscr{Q}_i,\mathscr{L}_{[n/2]}(\mathscr{Q}_i)]=0$. Therefore,

$$\mathscr{L}_{[n/2]}(\mathtt{P}_i\mathscr{X}\mathscr{Q}_i)=\mathtt{P}_i\mathscr{L}_{[n/2]}(\mathscr{X})\mathscr{Q}_i+\mathtt{P}_i[\mathscr{X},\mathscr{L}_{[n/2]}(\mathscr{Q}_i)]\mathscr{Q}_i.$$

First assume that $i\in\{1,\cdots,[n/2]-1\}$. Since

$$\begin{aligned}\tau_i(\mathtt{A}_{[n/2]}) &= \mathscr{L}_{[n/2]}(\mathtt{A}_{[n/2]})+[\mathtt{A}_{[n/2]},\mathscr{L}_{[n/2]}(\mathscr{Q}_i)]\\ &= \mathtt{P}_i\mathscr{L}_{[n/2]}(\mathtt{A}_{[n/2]})\mathtt{P}_i+\mathtt{P}_i[\mathtt{A}_{[n/2]},\mathscr{L}_{[n/2]}(\mathscr{Q}_i)]\mathtt{P}_i+\mathtt{P}_i\mathscr{L}_{[n/2]}(\mathtt{A}_{[n/2]})\mathscr{Q}_i\\ &\quad+\mathtt{P}_i[\mathtt{A}_{[n/2]},\mathscr{L}_{[n/2]}(\mathscr{Q}_i)]\mathscr{Q}_i+\mathscr{Q}_i\mathscr{L}_{[n/2]}(\mathtt{A}_{[n/2]})\mathscr{Q}_i+\mathscr{Q}_i[\mathtt{A}_{[n/2]},\mathscr{L}_{[n/2]}(\mathscr{Q}_i)]\mathscr{Q}_i\\ &= \mathtt{P}_i\mathscr{L}_{[n/2]}(\mathtt{A}_{[n/2]})\mathtt{P}_i+\mathtt{P}_i[\mathtt{A}_{[n/2]},\mathscr{L}_{[n/2]}(\mathscr{Q}_i)]\mathtt{P}_i+\mathscr{L}_{[n/2]}(\mathtt{P}_i\mathtt{A}_{[n/2]}\mathscr{Q}_i)\\ &\quad+\mathscr{Q}_i\mathscr{L}_{[n/2]}(\mathtt{A}_{[n/2]})\mathscr{Q}_i+\mathscr{Q}_i[\mathtt{A}_{[n/2]},\mathscr{L}_{[n/2]}(\mathscr{Q}_i)]\mathscr{Q}_i,\end{aligned}$$

then we get

$$\mathtt{P}_i\tau_i(\mathtt{A}_{[n/2]})\mathscr{Q}_i=\mathtt{P}_i\mathscr{L}_{[n/2]}(\mathtt{P}_i\mathtt{A}_{[n/2]}\mathscr{Q}_i)\mathscr{Q}_i. \tag{10}$$

However, writing $\mathscr{H}_i=\mathtt{P}_i\mathtt{A}_{[n/2]}\mathscr{Q}_i+\mathtt{P}_i\mathscr{H}_{[n/2]}\mathscr{Q}_i\in\mathscr{M}_i$ by Lemma 3.10, we have

$\mathscr{M}_i\ni\tau_i(\mathscr{H}_i)=\mathtt{L}_{[n/2]}(\mathtt{P}_i\mathtt{A}_{[n/2]}\mathscr{Q}_i+\mathtt{P}_i\mathscr{H}_{[n/2]}\mathscr{Q}_i)=\mathscr{L}_{[n/2]}(\mathtt{P}_i\mathtt{A}_{[n/2]}\mathscr{Q}_i)+\mathscr{L}_{[n/2]}(\mathtt{P}_i\mathscr{H}_{[n/2]}\mathscr{Q}_i)$ which and (10) imply $\mathtt{P}_i\tau_i(\mathtt{A}_{[n/2]})\mathscr{Q}_i=\tau_i(\mathscr{H}_i)-\mathtt{P}_i\mathscr{L}_{[n/2]}(\mathtt{P}_i\mathscr{H}_{[n/2]}\mathscr{Q}_i)\mathscr{Q}_i$. Note that $\mathscr{L}_{[n/2]}(\mathtt{P}_i\mathscr{H}_{[n/2]}\mathscr{Q}_i)\in\mathscr{M}_{[n/2]}\cap\mathscr{M}_i$. Hence the last expression yields $\mathscr{E}_h\tau_i(\mathtt{A}_{[n/2]})\mathscr{E}_j=\mathscr{E}_h\tau_i(\mathscr{H}_i)\mathscr{E}_j$, where $1\leqslant h\leqslant i$ and $i+1\leqslant j\leqslant[n/2]$. For the case $i\in\{[n/2]+1,\cdots,n-1\}$ the proof being similar is omitted □

Lemma 3.12. *In* (7), $r_{ij}=0$ *for all* $1\leqslant i<j\leqslant[n/2]$ *and* $[n/2]+1\leqslant i<j\leqslant n$.

Proof. Primarily we prove that $r_{12}=\cdots=r_{1,[n/2]}=0$. Undoubtedly, for $\mathscr{X}=\mathtt{A}_{[n/2]}+\mathscr{H}_{[n/2]}+\mathtt{B}_{[n/2]}=\mathtt{A}_1+\mathscr{H}_1+\mathtt{B}_1$, we have

$$\begin{aligned}\mathtt{H}_1 &= \mathtt{H}_0-\mathtt{K}_1\\ &= \mathscr{L}_{[n/2]}(\mathscr{X})-(\tau_1(\mathtt{A}_{[n/2]})+[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),\mathtt{A}_{[n/2]}])-(\tau_1(\mathscr{H}_{[n/2]})+[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),\mathscr{H}_{[n/2]}])\\ &\quad-(\tau_1(\mathtt{B}_{[n/2]})+[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),\mathtt{B}_{[n/2]}])-(\mathscr{L}_{[n/2]}(\mathscr{X})-[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),\mathscr{X}])+(\mathscr{L}_{[n/2]}(\mathtt{A}_1)\\ &\quad-[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),\mathtt{A}_1])+(\mathscr{L}_{[n/2]}(\mathscr{H}_1)-[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),\mathscr{H}_1])+(\mathscr{L}_{[n/2]}(\mathtt{B}_1)-[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),\mathtt{B}_1])\\ &= -\tau_1(\mathtt{A}_{[n/2]})-\tau_1(\mathscr{H}_{[n/2]})-\tau_1(\mathtt{B}_{[n/2]})+\mathscr{L}_{[n/2]}(\mathtt{A}_1)-[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),\mathtt{A}_1]\\ &\quad+\mathscr{L}_{[n/2]}(\mathscr{H}_1)-[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),\mathscr{H}_1]+\mathscr{L}_{[n/2]}(\mathtt{B}_1)-[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),\mathtt{B}_1]\\ &= -\tau_1(\mathtt{A}_{[n/2]})-\tau_1(\mathscr{H}_{[n/2]})-\tau_1(\mathtt{B}_{[n/2]})+\mathscr{L}_{[n/2]}(\mathtt{A}_1)-\mathtt{P}_1[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),\mathtt{A}_1]\mathtt{P}_1\\ &\quad-\mathtt{P}_1[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),\mathtt{A}_1]\mathscr{Q}_1-\mathscr{Q}_1[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),\mathtt{A}_1]\mathscr{Q}_1+\mathscr{L}_{[n/2]}(\mathscr{H}_1)-\mathtt{P}_1[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),\mathscr{H}_1]\mathtt{P}_1\\ &\quad-\mathtt{P}_1[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),\mathscr{H}_1]\mathscr{Q}_1+\mathscr{L}_{[n/2]}(\mathtt{B}_1)-\mathscr{Q}_1[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),\mathscr{H}_1]\mathscr{Q}_1-\mathtt{P}_1[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),\mathtt{B}_1]\mathtt{P}_1\\ &\quad-\mathtt{P}_1[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),\mathtt{B}_1]\mathscr{Q}_1-\mathscr{Q}_1[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),\mathtt{B}_1]\mathscr{Q}_1.\end{aligned} \tag{11}$$

Note that

$$\begin{aligned}\mathtt{P}_1[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),\mathtt{A}_1]\mathscr{Q}_1 &= \mathscr{L}_{[n/2]}(\mathscr{P}_{\mathtt{N}}(\mathtt{A}_1,\mathscr{Q}_1,\cdots,\mathscr{Q}_1))+\mathtt{P}_1[\mathscr{L}_{[n/2]}(\mathtt{A}_1),\mathscr{Q}_1]\mathscr{Q}_1=\mathtt{P}_1\mathscr{L}_{[n/2]}(\mathtt{A}_1)\mathscr{Q}_1;\\ \mathtt{P}_1[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),\mathtt{B}_1]\mathscr{Q}_1 &= \mathscr{L}_{[n/2]}(\mathscr{P}_{\mathtt{N}}(\mathtt{B}_1,\mathscr{Q}_1,\cdots,\mathscr{Q}_1))+\mathtt{P}_1[\mathscr{L}_{[n/2]}(\mathtt{B}_1),\mathscr{Q}_1]\mathscr{Q}_1=\mathtt{P}_1\mathscr{L}_{[n/2]}(\mathtt{B}_1)\mathscr{Q}_1;\end{aligned}$$

and by Lemma 3.10 it follows that

$$\begin{aligned}\tau_1(\mathscr{H}_1)=\mathscr{L}_{[n/2]}(\mathscr{H}_1) &= P_1\mathscr{L}_{[n/2]}(\mathscr{H}_1)\mathscr{Q}_1+P_1[\mathscr{H}_1,\mathscr{L}_{[n/2]}(\mathscr{Q}_1)]\mathscr{Q}_1\\ &= \mathscr{L}_{[n/2]}(\mathscr{H}_1)-P_1[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),\mathscr{H}_1]\mathscr{Q}_1.\end{aligned}$$

So, (11) changes to

$$\begin{aligned}H_1 &= -\tau_1(A_{[n/2]})-\tau_1(\mathscr{H}_{[n/2]})-\tau_1(B_{[n/2]})+P_1\mathscr{L}_{[n/2]}(A_1)P_1+\mathscr{Q}_1\mathscr{L}_{[n/2]}(A_1)\mathscr{Q}_1+\tau_1(\mathscr{H}_1)\\ &\quad +P_1\mathscr{L}_{[n/2]}(B_1)P_1+\mathscr{Q}_1\mathscr{L}_{[n/2]}(B_1)\mathscr{Q}_1-P_1[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),A_1]P_1-\mathscr{Q}_1[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),A_1]\mathscr{Q}_1\\ &\quad -P_1[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),B_1]P_1-\mathscr{Q}_1[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),B_1]\mathscr{Q}_1-P_1[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),\mathscr{H}_1]P_1-\mathscr{Q}_1[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),\mathscr{H}_1]\mathscr{Q}_1\end{aligned}$$

which and Lemma 3.11 imply

$$\begin{aligned}\mathscr{E}_1H_1\mathscr{E}_j &= -\mathscr{E}_1\tau_1(A_{[n/2]})\mathscr{E}_j-\mathscr{E}_1\tau_1(\mathscr{H}_{[n/2]})\mathscr{E}_j-\mathscr{E}_1\tau_1(B_{[n/2]})\mathscr{E}_j+\mathscr{E}_1\tau_1(\mathscr{H}_1)\mathscr{E}_j\\ &= -\mathscr{E}_1\tau_1(\mathscr{H}_{[n/2]})\mathscr{E}_j-\mathscr{E}_1\tau_1(B_{[n/2]})\mathscr{E}_j,\end{aligned}$$

where $j=2,\cdots,[n/2]$. As $B_{[n/2]}\in B_1$, by Lemma 3.3 for τ_1, we get $\mathscr{E}_1\tau_1(B_{[n/2]})\mathscr{E}_j=0$ for $j=2,\cdots,n$. Additionally, $\tau_1(\mathscr{H}_{[n/2]})=\mathscr{L}_{[n/2]}(\mathscr{H}_{[n/2]})-[\mathscr{L}_{[n/2]}(\mathscr{Q}_1),\mathscr{H}_{[n/2]}]\in\mathscr{M}_{[n/2]}$, which implies $\mathscr{E}_1\tau_1(\mathscr{H}_{[n/2]})\mathscr{E}_j=0$ for $j=2,\cdots,[n/2]$. Hence $\mathscr{E}_1H_1\mathscr{E}_j=0$ for $j=2,\cdots,[n/2]$, that is, $r_{12}=\cdots=r_{1,[n/2]}=0$.

Similarly, by considering the maps τ_i for $i=2,\cdots,n-1$, respectively, we can show $r_{ij}=0$ for $2\leqslant i<j\leqslant[n/2]$ and $[n/2]+1\leqslant i<j\leqslant n$. The lemma holds. □

Lemma 3.13. $H_0=\bigoplus_{i=1}^{n}r_{ii}\in\mathscr{Z}(\mathscr{T})$.

Proof. By Lemma 3.12, we get $H_0=\bigoplus_{i=1}^{n}r_{ii}$ in (7). To prove $H_0\in\mathscr{Z}(\mathscr{T})$, we have to review that

$$r_{ii}m_{ij}=m_{ij}r_{jj}\text{ holds for all }m_{ij}\in\mathscr{M}_{ij},1\leqslant i<j\leqslant n. \tag{12}$$

Primarily, take any $\mathscr{H}'_{[n/2]}=[m_{ij}]\in\mathscr{M}_{[n/2]}$. By Lemma 3.9 for the map $\mathscr{L}_{[n/2]}$, we have $[H_0,\mathscr{H}'_{[n/2]}]=[\bigoplus_{i=1}^{n}r_{ii},\mathscr{H}'_{[n/2]}]=0$. A direct calculation provides the following

$$r_{ii}m_{ij}=m_{ij}r_{jj}\text{ for }i=1,2,\cdots,[n/2]\text{ and }j=[n/2]+1,\cdots,n. \tag{13}$$

Note that, by Lemma 3.9 for τ_i ($i\in\{1,\cdots,[n/2]-1,[n/2]+1,\cdots,n-1\}$) and (13), we know that $[K_i,\mathscr{H}_i']=0$ holds for all $\mathscr{H}_i'\in\mathscr{M}_i$ implying

$$s^i_{ii}m_{ij}=m_{ij}s^i_{jj}\text{ for all }m_{ij}\in\mathscr{M}_{ij},1\leqslant i<j\leqslant n. \tag{14}$$

Our aim is to reveal

$$(r_{ii}-s^i_{ii})m_{ij}=m_{ij}(r_{jj}-s^i_{jj})\text{ for all }m_{ij}\in\mathscr{M}_{ij},1\leqslant i<j\leqslant[n/2],[n/2]+1\leqslant i<j\leqslant n, \tag{15}$$

which and (13)–(14) lead to (12). Hence, finding the proof of (15) would be our aim in the upcoming part of this manuscript.

For any $Y\in\mathscr{T}$ and any fixed $i\in\{1,\cdots,[n/2]-1,[n/2]+1,\cdots,n-1\}$, by Lemma 3.10, one has

$$\begin{aligned}\tau_i(Y) &= \mathscr{L}_{[n/2]}(Y)-[\mathscr{L}_{[n/2]}(\mathscr{Q}_i),Y]\\ &= \mathscr{L}_{[n/2]}(Y)-P_i[\mathscr{L}_{[n/2]}(\mathscr{Q}_i),Y]\mathscr{Q}_i-P_i[\mathscr{L}_{[n/2]}(\mathscr{Q}_i),Y]P_i-\mathscr{Q}_i[\mathscr{L}_{[n/2]}(\mathscr{Q}_i),Y]\mathscr{Q}_i\\ &= \mathscr{L}_{[n/2]}(Y)+\mathscr{L}_{[n/2]}(P_iY\mathscr{Q}_i)-P_i\mathscr{L}_{[n/2]}(Y)\mathscr{Q}_i-P_i[\mathscr{L}_{[n/2]}(\mathscr{Q}_i),Y]P_i-\mathscr{Q}_i[\mathscr{L}_{[n/2]}(\mathscr{Q}_i),Y]\mathscr{Q}_i\\ &= \mathscr{L}_{[n/2]}(Y)+\tau_i(P_iY\mathscr{Q}_i)-P_i\mathscr{L}_{[n/2]}(Y)\mathscr{Q}_i-P_i[\mathscr{L}_{[n/2]}(\mathscr{Q}_i),Y]P_i-\mathscr{Q}_i[\mathscr{L}_{[n/2]}(\mathscr{Q}_i),Y]\mathscr{Q}_i.\end{aligned}$$

As $\tau_i(P_iY\mathscr{Q}_i)-P_i\mathscr{L}_{[n/2]}(Y)\mathscr{Q}_i\in\mathscr{M}_i$, the above equation implies $\tau_i(Y)-\mathscr{L}_{[n/2]}(Y)+P_i[\mathscr{L}_{[n/2]}(\mathscr{Q}_i),Y]P_i+\mathscr{Q}_i[\mathscr{L}_{[n/2]}(\mathscr{Q}_i),Y]\mathscr{Q}_i\in\mathscr{M}_i$ and hence $P_i(\tau_i(Y)-\mathscr{L}_{[n/2]}(Y))\mathscr{Q}_i\in\mathscr{M}_i$ for all $Y\in\mathscr{T}$, and so $\mathscr{E}_k(\tau_i(Y)-$

$\mathscr{L}_{[n/2]}(\mathtt{Y}))\mathscr{E}_k = 0$ holds for all $\mathtt{Y} \in \mathscr{T}, k = 1, 2, \cdots, n$. Also noting that $\mathscr{L}_{[n/2]}(\mathscr{H}_{[n/2]}) \in \mathscr{M}_{[n/2]}$ and $\tau_i(\mathscr{H}_i) \in \mathscr{M}_i$, we get

$$\begin{aligned}
\mathscr{E}_k(\mathtt{H}_0 - \mathtt{K}_i)\mathscr{E}_k &= \mathscr{E}_k(\mathscr{L}_{[n/2]}(\mathscr{X}) - \mathscr{L}_{[n/2]}(\mathtt{A}_{[n/2]}) - \mathscr{L}_{[n/2]}(\mathscr{H}_{[n/2]}) - \mathscr{L}_{[n/2]}(\mathtt{B}_{[n/2]}))\mathscr{E}_k \\
&\quad -\mathscr{E}_k(\tau_i(\mathscr{X}) - \tau_i(\mathtt{A}_i) - \tau_i(\mathscr{H}_i) - \tau_i(\mathtt{B}_i))\mathscr{E}_k \\
&= \mathscr{E}_k\mathscr{L}_{[n/2]}(\mathscr{X})\mathscr{E}_k - \mathscr{E}_k\mathscr{L}_{[n/2]}(\mathtt{A}_{[n/2]})\mathscr{E}_k - \mathscr{E}_k\mathscr{L}_{[n/2]}(\mathtt{B}_{[n/2]})\mathscr{E}_k \\
&\quad -\mathscr{E}_k\tau_i(\mathscr{X})\mathscr{E}_k + \mathscr{E}_k\tau_i(\mathtt{A}_i)\mathscr{E}_k + \mathscr{E}_k\tau_i(\mathtt{B}_i)\mathscr{E}_k \\
&= \mathscr{E}_k(\mathscr{L}_{[n/2]}(\mathtt{A}_i) + \mathscr{L}_{[n/2]}(\mathtt{B}_i) - \mathscr{L}_{[n/2]}(\mathtt{A}_{[n/2]}) - \mathscr{L}_{[n/2]}(\mathtt{B}_{[n/2]}))\mathscr{E}_k, \qquad (16)
\end{aligned}$$

where $i \in \{1, \cdots, [n/2]-1, [n/2]+1, \cdots, n-1\}$ and $k = 1, 2, \cdots, n$.

Now, we consider two cases.

<u>Case 1.</u> $1 \leqslant i < j \leqslant [n/2]$.

Take any $\mathscr{H}_{ij} \in \mathscr{M}_{ij}$. Since $\mathscr{H}_{ij} \in \mathrm{A}_{[n/2]} \cap \mathscr{M}_i$, by Lemma 3.10, we get $\mathscr{L}_{[n/2]}(\mathscr{H}_{ij}) = \tau_i(\mathscr{H}_{ij}) \in \mathrm{A}_{[n/2]} \cap \mathscr{M}_i$. Additionally, it can be easily checked that $[\mathtt{A}_i + \mathtt{B}_i - \mathtt{A}_{[n/2]}, \mathscr{H}_{ij}] \in \mathscr{M}_{[n/2]}$ and $[\mathtt{A}_i + \mathtt{B}_i - \mathtt{A}_{[n/2]}, \mathscr{L}_{[n/2]}(\mathscr{H}_{ij})] \in \mathscr{M}_{[n/2]}$. So, by Lemma 3.3 for $i = [n/2]$, we have

$$\begin{aligned}
[\mathscr{L}_{[n/2]}(\mathtt{A}_i + \mathtt{B}_i - \mathtt{A}_{[n/2]}), \mathscr{H}_{ij}] &= \mathscr{P}_{\mathtt{N}}(\mathscr{L}_{[n/2]}(\mathtt{A}_i + \mathtt{B}_i - \mathtt{A}_{[n/2]}), \mathscr{H}_{ij}, \mathscr{Q}_j, \cdots, \mathscr{Q}_j) \\
&= \mathscr{L}_{[n/2]}(\mathscr{P}_{\mathtt{N}}(\mathtt{A}_i + \mathtt{B}_i - \mathtt{A}_{[n/2]}, \mathscr{H}_{ij}, \mathscr{Q}_j, \cdots, \mathscr{Q}_j)) \\
&\quad -\mathscr{P}_{\mathtt{N}}(\mathtt{A}_i + \mathtt{B}_i - \mathtt{A}_{[n/2]}, \mathscr{L}_{[n/2]}(\mathscr{H}_{ij}), \mathscr{Q}_j, \cdots, \mathscr{Q}_j) \\
&\quad - \cdots - \mathscr{P}_{\mathtt{N}}(\mathtt{A}_i + \mathtt{B}_i - \mathtt{A}_{[n/2]}, \mathscr{H}_{ij}, \mathscr{Q}_j, \cdots, \mathscr{L}_{[n/2]}(\mathscr{Q}_j)) \\
&= \mathscr{L}_{[n/2]}([\mathtt{A}_i + \mathtt{B}_i - \mathtt{A}_{[n/2]}, \mathscr{H}_{ij}]) \\
&\quad -[\mathtt{A}_i + \mathtt{B}_i - \mathtt{A}_{[n/2]}, \mathscr{L}_{[n/2]}(\mathscr{H}_{ij})] \in \mathscr{M}_{[n/2]}.
\end{aligned}$$

Implying

$$\mathtt{P}_{[n/2]}[\mathscr{L}_{[n/2]}(\mathtt{A}_i + \mathtt{B}_i - \mathtt{A}_{[n/2]}), \mathscr{H}_{ij}]\mathtt{P}_{[n/2]} = 0. \qquad (17)$$

Nevertheless, note that $[\mathtt{A}_i, \mathscr{H}_{ij}], [\mathtt{B}_i, \mathscr{H}_{ij}], [\mathtt{A}_{[n/2]}, \mathscr{H}_{ij}] \in \mathscr{M}_i$. By Lemma 3.10, we acquire

$$\begin{aligned}
&\mathscr{L}_{[n/2]}([\mathtt{A}_i + \mathtt{B}_i - \mathtt{A}_{[n/2]}, \mathscr{H}_{ij}]) \\
&= \mathscr{L}_{[n/2]}([\mathtt{A}_i, \mathscr{H}_{ij}]) + \mathscr{L}_{[n/2]}([\mathtt{B}_i, \mathscr{H}_{ij}]) - \mathscr{L}_{[n/2]}([\mathtt{A}_{[n/2]}, \mathscr{H}_{ij}]) \\
&= \mathscr{L}_{[n/2]}(\mathscr{P}_{\mathtt{N}}(\mathtt{A}_i, \mathscr{H}_{ij}, \mathscr{Q}_j, \cdots, \mathscr{Q}_j)) + \mathscr{L}_{[n/2]}(\mathscr{P}_{\mathtt{N}}(\mathtt{B}_i, \mathscr{H}_{ij}, \mathscr{Q}_j, \cdots, \mathscr{Q}_j)) \\
&\quad -\mathscr{L}_{[n/2]}(\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]}, \mathscr{H}_{ij}, \mathscr{Q}_j, \cdots, \mathscr{Q}_j)) \\
&= \mathscr{P}_{\mathtt{N}}(\mathscr{L}_{[n/2]}(\mathtt{A}_i), \mathscr{H}_{ij}, \mathscr{Q}_j, \cdots, \mathscr{Q}_j) + \mathscr{P}_{\mathtt{N}}(\mathtt{A}_i, \mathscr{L}_{[n/2]}(\mathscr{H}_{ij}), \mathscr{Q}_j, \cdots, \mathscr{Q}_j) \\
&\quad + \cdots + \mathscr{P}_{\mathtt{N}}(\mathtt{A}_i, \mathscr{H}_{ij}, \mathscr{Q}_j, \cdots, \mathscr{L}_{[n/2]}(\mathscr{Q}_j)) + \mathscr{P}_{\mathtt{N}}(\mathscr{L}_{[n/2]}(\mathtt{B}_i), \mathscr{H}_{ij}, \mathscr{Q}_j, \cdots, \mathscr{Q}_j) \\
&\quad +\mathscr{P}_{\mathtt{N}}(\mathtt{B}_i, \mathscr{L}_{[n/2]}(\mathscr{H}_{ij}), \mathscr{Q}_j, \cdots, \mathscr{Q}_j) + \cdots + \mathscr{P}_{\mathtt{N}}(\mathtt{B}_i, \mathscr{H}_{ij}, \mathscr{Q}_j, \cdots, \mathscr{L}_{[n/2]}(\mathscr{Q}_j)) \\
&\quad -\mathscr{P}_{\mathtt{N}}(\mathscr{L}_{[n/2]}(\mathtt{A}_{[n/2]}), \mathscr{H}_{ij}, \mathscr{Q}_j, \cdots, \mathscr{Q}_j) - \mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]}, \mathscr{L}_{[n/2]}(\mathscr{H}_{ij}), \mathscr{Q}_j, \cdots, \mathscr{Q}_j) \\
&\quad - \cdots - \mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]}, \mathscr{H}_{ij}, \mathscr{Q}_j, \cdots, \mathscr{L}_{[n/2]}(\mathscr{Q}_j)) \\
&= [\mathscr{L}_{[n/2]}(\mathtt{A}_i), \mathscr{H}_{ij}] + [\mathtt{A}_i, \mathscr{L}_{[n/2]}(\mathscr{H}_{ij})] + [\mathscr{L}_{[n/2]}(\mathtt{B}_i), \mathscr{H}_{ij}] \\
&\quad +[\mathtt{B}_i, \mathscr{L}_{[n/2]}(\mathscr{H}_{ij})] - [\mathscr{L}_{[n/2]}(\mathtt{A}_{[n/2]}), \mathscr{H}_{ij}] - [\mathtt{A}_{[n/2]}, \mathscr{L}_{[n/2]}(\mathscr{H}_{ij})].
\end{aligned}$$

So

$$\begin{aligned}
&[\mathscr{L}_{[n/2]}(\mathtt{A}_i + \mathtt{B}_i - \mathtt{A}_{[n/2]}), \mathscr{H}_{ij}] \\
&= \mathscr{L}_{[n/2]}([\mathtt{A}_i + \mathtt{B}_i - \mathtt{A}_{[n/2]}, \mathscr{H}_{ij}]) - [\mathtt{A}_i + \mathtt{B}_i - \mathtt{A}_{[n/2]}, \mathscr{L}_{[n/2]}(\mathscr{H}_{ij})] \\
&= [\mathscr{L}_{[n/2]}(\mathtt{A}_i) + \mathscr{L}_{[n/2]}(\mathtt{B}_i) - \mathscr{L}_{[n/2]}(\mathtt{A}_{[n/2]}), \mathscr{H}_{ij}]
\end{aligned}$$

which and (17) imply

$$P_{[n/2]}[\mathscr{L}_{[n/2]}(A_i)+\mathscr{L}_{[n/2]}(B_i)-\mathscr{L}_{[n/2]}(A_{[n/2]}),\mathscr{H}_{ij}]P_{[n/2]}=0.$$

It follows that

$$\mathscr{E}_i(\mathscr{L}_{[n/2]}(A_i)+\mathscr{L}_{[n/2]}(B_i)-\mathscr{L}_{[n/2]}(A_{[n/2]}))\mathscr{E}_i\mathscr{H}_{ij}=\mathscr{H}_{ij}\mathscr{E}_j(\mathscr{L}_{[n/2]}(A_i)+\mathscr{L}_{[n/2]}(B_i)-\mathscr{L}_{[n/2]}(A_{[n/2]}))\mathscr{E}_j. \quad (18)$$

Combining (16) and (18), keeping in mind that $P_{[n/2]}\mathscr{L}_{[n/2]}(B_{[n/2]})P_{[n/2]}\in\mathscr{Z}(A_{[n/2]})$ (Lemma 3.4 for $\mathscr{L}_{[n/2]}$), we have

$$\begin{aligned}\mathscr{E}_i(H_0-K_i)\mathscr{E}_i\mathscr{H}_{ij} &= \mathscr{H}_{ij}\mathscr{E}_j(\mathscr{L}_{[n/2]}(A_i)+\mathscr{L}_{[n/2]}(B_i)-\mathscr{L}_{[n/2]}(A_{[n/2]}))\mathscr{E}_j-\mathscr{E}_i\mathscr{L}_{[n/2]}(B_{[n/2]})\mathscr{E}_i\mathscr{H}_{ij}\\ &= \mathscr{H}_{ij}\mathscr{E}_j(H_0-K_i)\mathscr{E}_j+\mathscr{H}_{ij}\mathscr{E}_j\mathscr{L}_{[n/2]}(B_{[n/2]})\mathscr{E}_j-\mathscr{E}_i\mathscr{L}_{[n/2]}(B_{[n/2]})\mathscr{E}_i\mathscr{H}_{ij}\\ &= \mathscr{H}_{ij}\mathscr{E}_j(H_0-K_i)\mathscr{E}_j,\end{aligned}$$

that is, $(r_{ii}-s^i_{ii})m_{ij}=m_{ij}(r_{jj}-s^i_{jj})$ for $1\leqslant i<j\leqslant[n/2]$.

Case 2. $[n/2]+1\leqslant i<j\leqslant n$.

In this case, for any $\mathscr{H}_{ij}\in\mathscr{M}_{ij}$, we have $\mathscr{H}_{ij}\in B_{[n/2]}\cap\mathscr{M}_i$ and so $\mathscr{L}_{[n/2]}(\mathscr{H}_{ij})=\tau_i(\mathscr{H}_{ij})\in B_{[n/2]}\cap\mathscr{M}_i$. Moreover, $[A_i+B_i-B_{[n/2]},\mathscr{H}_{ij}]\in\mathscr{M}_{[n/2]}$ and $[A_i+B_i-B_{[n/2]},\mathscr{L}_{[n/2]}(\mathscr{H}_{ij})]\in\mathscr{M}_{[n/2]}$. Now, for $\mathscr{L}_{[n/2]}([A_i+B_i-B_{[n/2]},\mathscr{H}_{ij}])$, by a congruent discussion to that of Case 1, it follows that

$$(r_{ii}-s^i_{ii})m_{ij}=m_{ij}(r_{jj}-s^i_{jj}) \text{ holds for } [n/2]+1\leqslant i<j\leqslant n.$$

Again, the combination of Cases 1 and 2, (15) holds proving the lemma. □

Until now, by Lemmas 3.13 and 3.3–3.4, we prove that, for any $\mathscr{X}=A_{[n/2]}+\mathscr{H}_{[n/2]}+B_{[n/2]}\in\mathscr{T}$,

$$\begin{aligned}&\mathscr{L}_{[n/2]}(\mathscr{X})-\mathscr{L}_{[n/2]}(A_{[n/2]})-\mathscr{L}_{[n/2]}(\mathscr{H}_{[n/2]})-\mathscr{L}_{[n/2]}(B_{[n/2]})=H_0\in\mathscr{Z}(\mathscr{T}),\\ &\mathscr{L}_{[n/2]}(A_{[n/2]})\subseteq A_{[n/2]}+\mathscr{Z}(B_{[n/2]}),\mathscr{L}_{[n/2]}(B_{[n/2]})\subseteq B_{[n/2]}+\mathscr{Z}(A_{[n/2]}),\\ &\mathscr{L}_{[n/2]}(\mathscr{M}_{[n/2]})\subseteq\mathscr{M}_{[n/2]}.\end{aligned}$$

Let $\psi:\psi_{A_{[n/2]}}(\mathscr{Z}(\mathscr{T}))\to\psi_{B_{[n/2]}}(\mathscr{Z}(\mathscr{T}))$ be the unique ring isomorphism so that $z\oplus\psi(z)\in\mathscr{Z}(\mathscr{T})$ (that is, Lemma 3.3). Following the hypotheses on $\mathscr{T}$ in Theorem 3.1, we have $\psi_{A_{[n/2]}}(\mathscr{Z}(\mathscr{T}))=\mathscr{Z}(A_{[n/2]})$ and $\psi_{B_{[n/2]}}(\mathscr{Z}(\mathscr{T}))=\mathscr{Z}(B_{[n/2]})$. Thus, for each $A\in\mathscr{Z}(A_{[n/2]})$, $A\mathscr{H}=\mathscr{H}\psi(A)$ holds for all $\mathscr{H}\in\mathscr{M}_{[n/2]}$. Define two maps $\mho,\zeta:\mathscr{T}\to\mathscr{T}$ respectively by

$$\begin{aligned}\mho(\mathscr{X}) &= P_{[n/2]}\mathscr{L}_{[n/2]}(A_{[n/2]})P_{[n/2]}-\psi^{-1}(\mathscr{Q}_{[n/2]}\mathscr{L}_{[n/2]}(A_{[n/2]})\mathscr{Q}_{[n/2]})\\ &\quad+\mathscr{Q}_{[n/2]}\mathscr{L}_{[n/2]}(B_{[n/2]})\mathscr{Q}_{[n/2]}-\psi(P_{[n/2]}\mathscr{L}_{[n/2]}(B_{[n/2]})P_{[n/2]})+\mathscr{L}_{[n/2]}(\mathscr{H}_{[n/2]})\end{aligned}$$

and

$$\zeta(\mathscr{X})=\mathscr{L}_{[n/2]}(\mathscr{X})-\mho(\mathscr{X}),\ \forall\ \mathscr{X}=A_{[n/2]}+\mathscr{H}_{[n/2]}+B_{[n/2]}\in\mathscr{T}.$$

It can be inferred that

$$\mho(A_{[n/2]})\subseteq A_{[n/2]},\mho(B_{[n/2]})\subseteq B_{[n/2]},\mho(\mathscr{M}_{[n/2]})=\mathscr{L}_{[n/2]}(\mathscr{M}_{[n/2]})\subseteq\mathscr{M}_{[n/2]}, \quad (19)$$

$$\mho(\mathscr{X})=\mho(A_{[n/2]})+\mho(B_{[n/2]})+\mho(\mathscr{H}_{[n/2]}). \quad (20)$$

Additionally,

$$\begin{aligned}\zeta(\mathscr{X}) &= \mathscr{Q}_{[n/2]}\mathscr{L}_{[n/2]}(A_{[n/2]})\mathscr{Q}_{[n/2]}+\psi^{-1}(\mathscr{Q}_{[n/2]}\mathscr{L}_{[n/2]}(A_{[n/2]})\mathscr{Q}_{[n/2]})\\ &\quad+P_{[n/2]}\mathscr{L}_{[n/2]}(B_{[n/2]})P_{[n/2]}+\psi(P_{[n/2]}\mathscr{L}_{[n/2]}(B_{[n/2]})P_{[n/2]})+H_0\in\mathscr{Z}(\mathscr{T}).\end{aligned} \quad (21)$$

Lemma 3.14. *For any $\mathscr{H}_i\in\mathscr{M}_i=P_i\mathscr{T}\mathscr{Q}_i$, we have $\mho(\mathscr{H}_i)=\mathscr{L}_{[n/2]}(\mathscr{H}_i)\in\mathscr{M}_i$; and therefore, $\mho$ is additive on $\mathscr{M}_i$ $(i=1,\cdots,n-1)$.*

Proof. If $i = [n/2]$, it is evidently true. Assume that $i \in \{1, \cdots, [n/2]-1\}$. Keep in mind that

$$\mathtt{P}_i \mathscr{X} \mathscr{Q}_i = \mathtt{P}_{[n/2]} \mathtt{P}_i \mathscr{X} \mathscr{Q}_i \mathtt{P}_{[n/2]} + \mathtt{P}_{[n/2]} \mathtt{P}_i \mathscr{X} \mathscr{Q}_i \mathscr{Q}_{[n/2]}. \tag{22}$$

Since $\mathscr{L}_{[n/2]}(\mathscr{H}_i) \in \mathscr{M}_i$ (Lemma 3.10), we have

$$\mathscr{Q}_{[n/2]} \mathscr{L}_{[n/2]}(\mathtt{P}_{[n/2]} \mathtt{P}_i \mathscr{X} \mathscr{Q}_i \mathtt{P}_{[n/2]}) \mathscr{Q}_{[n/2]} = 0. \tag{23}$$

However, by the definition of $\mho$ and (22)-(23), we get

$$\begin{aligned} \mho(\mathtt{P}_i \mathscr{X} \mathscr{Q}_i) &= \mathtt{P}_{[n/2]} \mathscr{L}_{[n/2]}(\mathtt{P}_{[n/2]} \mathtt{P}_i \mathscr{X} \mathscr{Q}_i \mathtt{P}_{[n/2]}) \mathtt{P}_{[n/2]} \\ &\quad - \psi^{-1}(\mathscr{Q}_{[n/2]} \mathscr{L}_{[n/2]}(\mathtt{P}_{[n/2]} \mathtt{P}_i \mathscr{X} \mathscr{Q}_i \mathtt{P}_{[n/2]}) \mathscr{Q}_{[n/2]}) + \mathscr{L}_{[n/2]}(\mathtt{P}_{[n/2]} \mathtt{P}_i \mathscr{X} \mathscr{Q}_i \mathscr{Q}_{[n/2]}) \\ &= \mathtt{P}_{[n/2]} \mathscr{L}_{[n/2]}(\mathtt{P}_{[n/2]} \mathtt{P}_i \mathscr{X} \mathscr{Q}_i \mathtt{P}_{[n/2]}) \mathtt{P}_{[n/2]} + \mathscr{L}_{[n/2]}(\mathtt{P}_{[n/2]} \mathtt{P}_i \mathscr{X} \mathscr{Q}_i \mathscr{Q}_{[n/2]}). \end{aligned} \tag{24}$$

While, by (23) and the additivity of $\mathscr{L}_{[n/2]}$ on $\mathscr{M}_i$ (Lemma 3.10), we have

$$\begin{aligned} \mathscr{L}_{[n/2]}(\mathtt{P}_i \mathscr{X} \mathscr{Q}_i) &= \mathscr{L}_{[n/2]}(\mathtt{P}_{[n/2]} \mathtt{P}_i \mathscr{X} \mathscr{Q}_i \mathtt{P}_{[n/2]}) + \mathscr{L}_{[n/2]}(\mathtt{P}_{[n/2]} \mathtt{P}_i \mathscr{X} \mathscr{Q}_i \mathscr{Q}_{[n/2]}) \\ &= \mathtt{P}_{[n/2]} \mathscr{L}_{[n/2]}(\mathtt{P}_{[n/2]} \mathtt{P}_i \mathscr{X} \mathscr{Q}_i \mathtt{P}_{[n/2]}) \mathtt{P}_{[n/2]} + \mathscr{Q}_{[n/2]} \mathscr{L}_{[n/2]}(\mathtt{P}_{[n/2]} \mathtt{P}_i \mathscr{X} \mathscr{Q}_i \mathtt{P}_{[n/2]}) \mathscr{Q}_{[n/2]} \\ &\quad + \mathscr{L}_{[n/2]}(\mathtt{P}_{[n/2]} \mathtt{P}_i \mathscr{X} \mathscr{Q}_i \mathscr{Q}_{[n/2]}) \\ &= \mathtt{P}_{[n/2]} \mathscr{L}_{[n/2]}(\mathtt{P}_{[n/2]} \mathtt{P}_i \mathscr{X} \mathscr{Q}_i \mathtt{P}_{[n/2]}) \mathtt{P}_{[n/2]} + \mathscr{L}_{[n/2]}(\mathtt{P}_{[n/2]} \mathtt{P}_i \mathscr{X} \mathscr{Q}_i \mathscr{Q}_{[n/2]}). \end{aligned} \tag{25}$$

Combining (24)-(25) yields $\mho(\mathtt{P}_i \mathscr{X} \mathscr{Q}_i) = \mathscr{L}_{[n/2]}(\mathtt{P}_i \mathscr{X} \mathscr{Q}_i)$ for $i \in \{1, \cdots, [n/2]-1\}$.

If $i \in \{[n/2]+1, \cdots, n\}$, noting that $\mathtt{P}_i \mathscr{X} \mathscr{Q}_i = \mathtt{P}_{[n/2]} \mathtt{P}_i \mathscr{X} \mathscr{Q}_i \mathscr{Q}_{[n/2]} + \mathscr{Q}_{[n/2]} \mathtt{P}_i \mathscr{X} \mathscr{Q}_i \mathscr{Q}_{[n/2]}$, by a congruent discussion to the above, one can also prove that $\mho(\mathtt{P}_i \mathscr{X} \mathscr{Q}_i) = \mathscr{L}_{[n/2]}(\mathtt{P}_i \mathscr{X} \mathscr{Q}_i)$, and so $\mho(\mathscr{H}_i) \in \mathscr{M}_i$ by Lemma 3.10. Lastly, the additivity of $\mho$ on $\mathscr{M}_i$ can be obtained by the one of $\mathscr{L}_{[n/2]}$. □

Lemma 3.15. $\mho$ *is additive on* $\mathscr{T}$.

Proof. We will now show this lemma by various steps.

Step 1. $\mho$ is additive on $\mathscr{M}_{[n/2]}$. By Lemma 3.14, this is true.

Step 2. $\mho$ is additive on $\mathrm{A}_{[n/2]}$. Take any $\mathtt{A}_{[n/2]}, \mathtt{A}'_{[n/2]} \in \mathrm{A}_{[n/2]}$ and any $\mathscr{H}_{[n/2]} \in \mathscr{M}_{[n/2]}$. By the definition of $\mho$ and (19)-(21), we get

$$\begin{aligned}
\mho((\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]})\mathscr{H}_{[n/2]}) &= \mathscr{L}_{[n/2]}(\mathtt{A}_{[n/2]}\mathscr{H}_{[n/2]})+\mathscr{L}_{[n/2]}(\mathtt{A}'_{[n/2]}\mathscr{H}_{[n/2]})\\
&= \mathscr{L}_{[n/2]}(\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]},\mathscr{H}_{[n/2]},\mathscr{Q}_{[n/2]},\cdots,\mathscr{Q}_{[n/2]}))\\
&\quad +\mathscr{L}_{[n/2]}(\mathscr{P}_{\mathtt{N}}(\mathtt{A}'_{[n/2]},\mathscr{H}_{[n/2]},\mathscr{Q}_{[n/2]},\cdots,\mathscr{Q}_{[n/2]}))\\
&= \mathscr{P}_{\mathtt{N}}(\mathscr{L}_{[n/2]}(\mathtt{A}_{[n/2]}),\mathscr{H}_{[n/2]},\mathscr{Q}_{[n/2]},\cdots,\mathscr{Q}_{[n/2]})\\
&\quad +\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]},\mathscr{L}_{[n/2]}(\mathscr{H}_{[n/2]}),\mathscr{Q}_{[n/2]},\cdots,\mathscr{Q}_{[n/2]})\\
&\quad +\cdots+\mathscr{P}_{n}(\mathtt{A}_{[n/2]},\mathscr{H}_{[n/2]},\mathscr{Q}_{[n/2]},\cdots,\mathscr{L}_{[n/2]}(\mathscr{Q}_{[n/2]}))\\
&\quad +\mathscr{P}_{\mathtt{N}}(\mathscr{L}_{[n/2]}(\mathtt{A}'_{[n/2]}),\mathscr{H}_{[n/2]},\mathscr{Q}_{[n/2]},\cdots,\mathscr{Q}_{[n/2]})\\
&\quad +\mathscr{P}_{\mathtt{N}}(\mathtt{A}'_{[n/2]},\mathscr{L}_{[n/2]}(\mathscr{H}_{[n/2]}),\mathscr{Q}_{[n/2]},\cdots,\mathscr{Q}_{[n/2]})\\
&\quad +\cdots+\mathscr{P}_{\mathtt{N}}(\mathtt{A}'_{[n/2]},\mathscr{H}_{[n/2]},\mathscr{Q}_{[n/2]},\cdots,\mathscr{L}_{[n/2]}(\mathscr{Q}_{[n/2]}))\\
&= \mathscr{P}_{\mathtt{N}}(\mho(\mathtt{A}_{[n/2]})+\zeta(\mathtt{A}_{[n/2]}),\mathscr{H}_{[n/2]},\mathscr{Q}_{[n/2]},\cdots,\mathscr{Q}_{[n/2]})\\
&\quad +\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]},\mho(\mathscr{H}_{[n/2]})+\zeta(\mathscr{H}_{[n/2]}),\mathscr{Q}_{[n/2]},\cdots,\mathscr{Q}_{[n/2]})\\
&\quad +\cdots+\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]},\mathscr{H}_{[n/2]},\mathscr{Q}_{[n/2]},\cdots,\mho(\mathscr{Q}_{[n/2]})+\zeta(\mathscr{Q}_{[n/2]}))\\
&\quad +\mathscr{P}_{\mathtt{N}}(\mho(\mathtt{A}'_{[n/2]})+\zeta(\mathscr{H}_{[n/2]}),\mathscr{H}_{[n/2]},\mathscr{Q}_{[n/2]},\cdots,\mathscr{Q}_{[n/2]})\\
&\quad +\mathscr{P}_{\mathtt{N}}(\mathtt{A}'_{[n/2]},\mho(\mathscr{H}_{[n/2]})+\zeta(\mathscr{H}_{[n/2]})\,\mathscr{Q}_{[n/2]},\cdots,\mathscr{Q}_{[n/2]})\\
&\quad +\cdots+\mathscr{P}_{\mathtt{N}}(\mathtt{A}'_{[n/2]},\mathscr{H}_{[n/2]},\mathscr{Q}_{[n/2]},\cdots,\mho(\mathscr{Q}_{[n/2]})+\zeta(\mathscr{Q}_{[n/2]}))\\
&= \mathscr{P}_{\mathtt{N}}(\mho(\mathtt{A}_{[n/2]}),\mathscr{H}_{[n/2]},\mathscr{Q}_{[n/2]},\cdots,\mathscr{Q}_{[n/2]})\\
&\quad +\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]},\mho(\mathscr{H}_{[n/2]}),\mathscr{Q}_{[n/2]},\cdots,\mathscr{Q}_{[n/2]})\\
&\quad +\cdots+\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]},\mathscr{H}_{[n/2]},\mathscr{Q}_{[n/2]},\cdots,\mho(\mathscr{Q}_{[n/2]}))\\
&\quad +\mathscr{P}_{\mathtt{N}}(\mho(\mathtt{A}'_{[n/2]}),\mathscr{H}_{[n/2]},\mathscr{Q}_{[n/2]},\cdots,\mathscr{Q}_{[n/2]})\\
&\quad +\mathscr{P}_{\mathtt{N}}(\mathtt{A}'_{[n/2]},\mho(\mathscr{H}_{[n/2]}),\mathscr{Q}_{[n/2]},\cdots,\mathscr{Q}_{[n/2]})\\
&\quad +\cdots+\mathscr{P}_{\mathtt{N}}(\mathtt{A}'_{[n/2]},\mathscr{H}_{[n/2]},\mathscr{Q}_{[n/2]},\cdots,\mho(\mathscr{Q}_{[n/2]}))
\end{aligned}$$

and

$$\begin{aligned}
&\mho((\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]})\mathscr{H}_{[n/2]})\\
&\quad= \mathscr{L}_{[n/2]}(\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]},\mathscr{H}_{[n/2]},\mathscr{Q}_{[n/2]},\cdots,\mathscr{Q}_{[n/2]}))\\
&\quad= \mathscr{P}_{\mathtt{N}}(\mathscr{L}_{[n/2]}(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]}),\mathscr{H}_{[n/2]},\mathscr{Q}_{[n/2]},\cdots,\mathscr{Q}_{[n/2]})\\
&\qquad +\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]},\mathscr{L}_{[n/2]}(\mathscr{H}_{[n/2]}),\mathscr{Q}_{[n/2]},\cdots,\mathscr{Q}_{[n/2]})\\
&\qquad +\cdots+\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]},\mathscr{H}_{[n/2]},\mathscr{Q}_{[n/2]},\cdots,\mathscr{L}_{[n/2]}(\mathscr{Q}_{[n/2]}))\\
&\quad= \mathscr{P}_{\mathtt{N}}(\mho(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]})+\zeta(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]}),\mathscr{H}_{[n/2]},\mathscr{Q}_{[n/2]},\cdots,\mathscr{Q}_{[n/2]})\\
&\qquad +\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]},\mho(\mathscr{H}_{[n/2]})+\zeta(\mathscr{H}_{[n/2]}),\mathscr{Q}_{[n/2]},\cdots,\mathscr{Q}_{[n/2]})\\
&\qquad +\cdots+\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]},\mathscr{H}_{[n/2]},\mathscr{Q}_{[n/2]},\cdots,\mho(\mathscr{Q}_{[n/2]})+\zeta(\mathscr{Q}_{[n/2]}))\\
&\quad= \mathscr{P}_{\mathtt{N}}(\mho(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]}),\mathscr{H}_{[n/2]},\mathscr{Q}_{[n/2]},\cdots,\mathscr{Q}_{[n/2]})\\
&\qquad +\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]},\mho(\mathscr{H}_{[n/2]}),\mathscr{Q}_{[n/2]},\cdots,\mathscr{Q}_{[n/2]})\\
&\qquad +\cdots+\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]},\mathscr{H}_{[n/2]},\mathscr{Q}_{[n/2]},\cdots,\mho(\mathscr{Q}_{[n/2]})).
\end{aligned}$$

From the above two equations, we get $(\mho(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]})-\mho(\mathtt{A}_{[n/2]})-\mho(\mathtt{A}'_{[n/2]}))\mathscr{H}_{[n/2]}=0$ for all $\mathscr{H}_{[n/2]}\in\mathscr{M}_{[n/2]}$. Since $\mathscr{M}_{ij}$ is a faithful left $\mathscr{R}_i$-module, the above equation implies

$$\mathscr{E}_i(\mho(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]})-\mho(\mathtt{A}_{[n/2]})-\mho(\mathtt{A}'_{[n/2]}))\mathscr{E}_i=0 \ \text{ for } i=1,\cdots,[n/2]. \tag{26}$$

Contrarily, let $i \in \{1, 2, \cdots, [n/2]-1\}$. Note that, for any $\mathscr{X} \in \mathscr{T}$, by Lemma 3.14, it is true that

$$\begin{aligned}
&\mho((\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]})\mathscr{Q}_i)\\
&= \mathscr{L}_{[n/2]}(\mathtt{A}_{[n/2]}\mathscr{Q}_i)+\mathscr{L}_{[n/2]}(\mathtt{A}'_{[n/2]}\mathscr{Q}_i)\\
&= \mathscr{L}_{[n/2]}(\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]},\mathscr{Q}_i,\cdots,\mathscr{Q}_i))+\mathscr{L}_{[n/2]}(\mathscr{P}_{\mathtt{N}}(\mathtt{A}'_{[n/2]},\mathscr{Q}_i,\cdots,\mathscr{Q}_i))\\
&= \mathscr{P}_{\mathtt{N}}(\mathscr{L}_{[n/2]}(\mathtt{A}_{[n/2]}),\mathscr{Q}_i,\cdots,\mathscr{Q}_i)+\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]},\mathscr{L}_{[n/2]}(\mathscr{Q}_i),\cdots,\mathscr{Q}_i)\\
&\quad+\cdots+\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]},\mathscr{Q}_i,\cdots,\mathscr{L}_{[n/2]}(\mathscr{Q}_i))+\mathscr{P}_{\mathtt{N}}(\mathscr{L}_{[n/2]}(\mathtt{A}'_{[n/2]}),\mathscr{Q}_i,\cdots,\mathscr{Q}_i)\\
&\quad+\mathscr{P}_{\mathtt{N}}(\mathtt{A}'_{[n/2]},\mathscr{L}_{[n/2]}(\mathscr{Q}_i),\cdots,\mathscr{Q}_i)+\cdots+\mathscr{P}_{\mathtt{N}}(\mathtt{A}'_{[n/2]},\mathscr{Q}_i,\cdots,\mathscr{L}_{[n/2]}(\mathscr{Q}_i))\\
&= \mathscr{P}_{\mathtt{N}}(\mho(\mathtt{A}_{[n/2]})+\zeta(\mathtt{A}_{[n/2]}),\mathscr{Q}_i,\cdots,\mathscr{Q}_i)+\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]},\mho(\mathscr{Q}_i)+\zeta(\mathscr{Q}_i),\cdots,\mathscr{Q}_i)\\
&\quad+\cdots+\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]},\mathscr{Q}_i,\cdots,\mho(\mathscr{Q}_i)+\zeta(\mathscr{Q}_i))+\mathscr{P}_{\mathtt{N}}(\mho(\mathtt{A}'_{[n/2]})+\zeta(\mathtt{A}'_{[n/2]}),\cdots,\mathscr{Q}_i)\\
&\quad+\mathscr{P}_{\mathtt{N}}(\mathtt{A}'_{[n/2]},\mho(\mathscr{Q}_i)+\zeta(\mathscr{Q}_i),\cdots,\mathscr{Q}_i)+\cdots+\mathscr{P}_{\mathtt{N}}(\mathtt{A}'_{[n/2]},\mathscr{Q}_i,\cdots,\mho(\mathscr{Q}_i)+\zeta(\mathscr{Q}_i))\\
&= \mathscr{P}_{\mathtt{N}}(\mho(\mathtt{A}_{[n/2]}),\mathscr{Q}_i,\cdots,\mathscr{Q}_i)+\mathscr{P}_{n}(\mathtt{A}_{[n/2]},\mho(\mathscr{Q}_i),\cdots,\mathscr{Q}_i)+\cdots+\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]},\mathscr{Q}_i,\cdots,\mho(\mathscr{Q}_i))\\
&\quad+\mathscr{P}_{\mathtt{N}}(\mho(\mathtt{A}'_{[n/2]}),\mathscr{Q}_i,\cdots,\mathscr{Q}_i)+\mathscr{P}_{\mathtt{N}}(\mathtt{A}'_{[n/2]},\mho(\mathscr{Q}_i),\cdots,\mathscr{Q}_i)+\cdots+\mathscr{P}_{\mathtt{N}}(\mathtt{A}'_{[n/2]},\mathscr{Q}_i,\cdots,\mho(\mathscr{Q}_i))
\end{aligned}$$

and

$$\begin{aligned}
&\mho((\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]})\mathscr{Q}_i)\\
&= \mathscr{L}_{[n/2]}(\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]},\mathscr{Q}_i,\cdots,\mathscr{Q}_i))\\
&= \mathscr{P}_{\mathtt{N}}(\mathscr{L}_{[n/2]}(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]}),\mathscr{Q}_i,\cdots,\mathscr{Q}_i)+\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]},\mathscr{L}_{[n/2]}(\mathscr{Q}_i),\cdots,\mathscr{Q}_i)\\
&\quad+\cdots+\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]},\mathscr{Q}_i,\cdots,\mathscr{L}_{[n/2]}(\mathscr{Q}_i))\\
&= \mathscr{P}_{\mathtt{N}}(\mho(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]})+\zeta(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]}),\mathscr{Q}_i,\cdots,\mathscr{Q}_i)+\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]},\mho(\mathscr{Q}_i)+\zeta(\mathscr{Q}_i),\cdots,\mathscr{Q}_i)\\
&\quad+\cdots+\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]},\mathscr{Q}_i,\cdots,\mho(\mathscr{Q}_i)+\zeta(\mathscr{Q}_i))\\
&= \mathscr{P}_{\mathtt{N}}(\mho(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]}),\mathscr{Q}_i,\cdots,\mathscr{Q}_i)+\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]},\mho(\mathscr{Q}_i),\cdots,\mathscr{Q}_i)\\
&\quad+\cdots+\mathscr{P}_{\mathtt{N}}(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]},\mathscr{Q}_i,\cdots,\mho(\mathscr{Q}_i))
\end{aligned}$$

implying that

$$\mathtt{P}_i(\mho(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]})-\mho(\mathtt{A}_{[n/2]})-\mho(\mathtt{A}'_{[n/2]}))\mathscr{Q}_i=0 \text{ for } i=1,\cdots,[n/2]-1. \tag{27}$$

Now, combining (26)-(27) yields $\mho(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]})-\mho(\mathtt{A}_{[n/2]})-\mho(\mathtt{A}'_{[n/2]})=0$, that is, $\mho$ is additive on $A_{[n/2]}$.

Step 3. $\mho$ is additive on $B_{[n/2]}$. The proof being similar to that of Step 2 is omitted here.

Step 4. $\mho$ is additive on $\mathscr{T}$. For any $\mathscr{X}_1=\mathtt{A}_{[n/2]}+\mathscr{H}_{[n/2]}+\mathtt{B}_{[n/2]}$, $\mathscr{X}_2=\mathtt{A}'_{[n/2]}+\mathscr{H}'_{[n/2]}+\mathtt{B}'_{[n/2]}\in\mathscr{T}$, by (20) and Steps 1-3, we get

$$\begin{aligned}
\mho(\mathscr{X}_1+\mathscr{X}_2) &= \mho(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]}+\mathtt{B}_{[n/2]}+\mathtt{B}'_{[n/2]}+\mathscr{H}_{[n/2]}+\mathscr{H}'_{[n/2]})\\
&= \mho(\mathtt{A}_{[n/2]}+\mathtt{A}'_{[n/2]})+\mho(\mathscr{H}_{[n/2]}+\mathscr{H}'_{[n/2]})+\mho(\mathtt{B}_{[n/2]}+\mathtt{B}'_{[n/2]})\\
&= \mho(\mathtt{A}_{[n/2]})+\mho(\mathtt{B}_{[n/2]})+\mho(\mathscr{H}_{[n/2]})+\mho(\mathtt{A}'_{[n/2]})+\mho(\mathtt{B}'_{[n/2]})+\mho(\mathscr{H}'_{[n/2]})\\
&= \mho(\mathscr{X}_1)+\mho(\mathscr{X}_2).
\end{aligned}$$

That is, $\mho$ is additive on $\mathscr{T}$. □

Lemma 3.16. *$\mho$ is a derivation on $\mathscr{T}$.*

Proof. We will prove it by various steps.

Step 1. For any $\mathscr{H}_{[n/2]},\mathscr{H}'_{[n/2]}\in\mathscr{M}_{[n/2]}$, we have

$$\mho(\mathscr{H}_{[n/2]}\mathscr{H}'_{[n/2]})=\mho(\mathscr{H}_{[n/2]})\mathscr{H}'_{[n/2]}+\mathscr{H}_{[n/2]}\mho(\mathscr{H}'_{[n/2]})=0.$$

Note that $\mho(\mathscr{H}_{[n/2]}) \subseteq \mathscr{M}_{[n/2]}$ by (19). This step is obvious.

Step 2. For any $\mathtt{A}_{[n/2]}, \mathtt{A}'_{[n/2]} \in \mathrm{A}_{[n/2]}$ and any $\mathscr{H}_{[n/2]} \in \mathscr{M}_{[n/2]}$, we have

$$\begin{aligned}\mho(\mathtt{A}_{[n/2]}\mathscr{H}_{[n/2]}) &= \mho(\mathtt{A}_{[n/2]})\mathscr{H}_{[n/2]} + \mathtt{A}_{[n/2]}\mho(\mathscr{H}_{[n/2]}),\\ \mho(\mathtt{A}_{[n/2]}\mathtt{A}'_{[n/2]}) &= \mho(\mathtt{A}_{[n/2]})\mathtt{A}'_{[n/2]} + \mathtt{A}_{[n/2]}\mho(\mathtt{A}'_{[n/2]}).\end{aligned}$$

Take any $\mathtt{A}_{[n/2]}, \mathtt{A}'_{[n/2]} \in \mathrm{A}_{[n/2]}$ and any $\mathscr{H}_{[n/2]} \in \mathscr{M}_{[n/2]}$. By the definition of $\mho$ and (19)-(21), we have

$$\begin{aligned}\mho(\mathtt{A}_{[n/2]}\mathtt{A}'_{[n/2]}\mathscr{H}_{[n/2]}) &= \mathscr{L}_{[n/2]}(\mathscr{P}_{N-1}([\mathtt{A}_{[n/2]}\mathtt{A}'_{[n/2]}, \mathscr{H}_{[n/2]}], \mathscr{Q}_{[n/2]}, \cdots, \mathscr{Q}_{[n/2]}))\\ &= \mathscr{P}_{N-1}([\mho(\mathtt{A}_{[n/2]}\mathtt{A}'_{[n/2]}), \mathscr{H}_{[n/2]}], \mathscr{Q}_{[n/2]}, \cdots, \mathscr{Q}_{[n/2]})\\ &\quad + \mathscr{P}_{N-1}([\mathtt{A}_{[n/2]}\mathtt{A}'_{[n/2]}, \mho(\mathscr{H}_{[n/2]})], \mathscr{Q}_{[n/2]}, \cdots, \mathscr{Q}_{[n/2]})\\ &\quad + \cdots + \mathscr{P}_{N-1}([\mathtt{A}_{[n/2]}\mathtt{A}'_{[n/2]}, \mathscr{H}_{[n/2]}], \mathscr{Q}_{[n/2]}, \cdots, \mho(\mathscr{Q}_{[n/2]})), \qquad (28)\end{aligned}$$

implying to $\mho(\mathtt{A}_{[n/2]}\mathscr{H}_{[n/2]}) = \mho(\mathtt{A}_{[n/2]})\mathscr{H}_{[n/2]} + \mathtt{A}_{[n/2]}\mho(\mathscr{H}_{[n/2]})$ for all $\mathtt{A}_{[n/2]} \in \mathrm{A}_{[n/2]}, \mathscr{H}_{[n/2]} \in \mathscr{M}_{[n/2]}$. Moreover,

$$\begin{aligned}\mho(\mathtt{A}_{[n/2]}\mathtt{A}'_{[n/2]}\mathscr{H}_{[n/2]}) &= \mathscr{L}_{[n/2]}(\mathscr{P}_{N-1}([\mathtt{A}_{[n/2]}, \mathtt{A}'_{[n/2]}\mathscr{H}_{[n/2]}], \mathscr{Q}_{[n/2]}, \cdots, \mathscr{Q}_{[n/2]}))\\ &= \mathscr{P}_{N-1}([\mho(\mathtt{A}_{[n/2]}), \mathtt{A}'_{[n/2]}\mathscr{H}_{[n/2]}], \mathscr{Q}_{[n/2]}, \cdots, \mathscr{Q}_{[n/2]})\\ &\quad + \mathscr{P}_{N-1}([\mathtt{A}_{[n/2]}, \mho(\mathtt{A}'_{[n/2]}\mathscr{H}_{[n/2]})], \mathscr{Q}_{[n/2]}, \cdots, \mathscr{Q}_{[n/2]})\\ &\quad + \cdots + \mathscr{P}_{N-1}([\mathtt{A}_{[n/2]}, \mathtt{A}'_{[n/2]}\mathscr{H}_{[n/2]}], \mathscr{Q}_{[n/2]}, \cdots, \mho(\mathscr{Q}_{[n/2]})). \qquad (29)\end{aligned}$$

Comparing (28)-(29) yields $(\mho(\mathtt{A}_{[n/2]}\mathtt{A}'_{[n/2]}) - \mho(\mathtt{A}_{[n/2]})\mathtt{A}'_{[n/2]} - \mathtt{A}_{[n/2]}\mho(\mathtt{A}'_{[n/2]}))\mathscr{H}_{[n/2]} = 0$ for all $\mathscr{H}_{[n/2]} \in \mathscr{M}_{[n/2]}$, implying

$$\mathscr{E}_i(\mho(\mathtt{A}_{[n/2]}\mathtt{A}'_{[n/2]}) - \mho(\mathtt{A}_{[n/2]})\mathtt{A}'_{[n/2]} - \mathtt{A}_{[n/2]}\mho(\mathtt{A}'_{[n/2]}))\mathscr{E}_i = 0 \ \text{ for } \ i = 1, \cdots, [n/2]. \qquad (30)$$

Especially, by taking $\mathtt{A}_{[n/2]} = \mathtt{A}_{kk}$ and $\mathtt{A}'_{[n/2]} = \mathscr{E}_k$ with $k = i$ in Equation (30) and by Lemma 3.14, we get

$$0 = \mathscr{E}_i(\mho(\mathtt{A}_{kk}\mathscr{E}_k) - \mho(\mathtt{A}_{kk})\mathscr{E}_k - \mathtt{A}_{kk}\mho(\mathscr{E}_k))\mathscr{E}_i = \mathscr{E}_i\mho(\mathtt{A}_{kk})\mathscr{E}_i - \mathscr{E}_i\mathtt{A}_{kk}\mho(\mathscr{E}_k)\mathscr{E}_i = \mathscr{E}_i\mho(\mathtt{A}_{kk})\mathscr{E}_i,$$

that is,

$$\mathscr{E}_i\mho(\mathtt{A}_{kk})\mathscr{E}_i = 0, i = 1, \cdots, [n/2] \text{ with } i = k \leqslant [n/2]. \qquad (31)$$

In the conclusion, let $i \in \{1, \cdots, [n/2]\}$, $\mathtt{A}_{[n/2]} = (a_{kl})_{[n/2]\times[n/2]}$ and $\mathtt{A}'_{[n/2]} = (a'_{st})_{[n/2]\times[n/2]}$, then

$$\mathtt{A}_{[n/2]}\mathtt{A}'_{[n/2]} = \sum_{1\leqslant k\leqslant l\leqslant s\leqslant t\leqslant [n/2]} \mathtt{A}_{kl}\mathtt{A}'_{st} = \sum_{1\leqslant k\leqslant l\leqslant t\leqslant [n/2]} \mathtt{A}_{kl}\mathtt{A}'_{lt},$$

where $\mathtt{A}_{kl}$ is the matrix with (k, l) position $a_{k,l}$ and other positions 0.

To show that $\mho$ is a derivation on $\mathtt{A}_{[n/2]}$, that is, $\mho(\mathtt{A}_{[n/2]}\mathtt{A}'_{[n/2]}) = \mho(\mathtt{A}_{[n/2]})\mathtt{A}'_{[n/2]} + \mathtt{A}_{[n/2]}\mho(\mathtt{A}'_{[n/2]})$, as $\mho$ is additive by Lemma 3.15, one only needs to check that $\mho$ satisfies the derivable condition on every element of $\mathtt{A}_{[n/2]}$, that is, to analyse that

$$\begin{aligned}&\mho(\mathtt{A}_{kl}\mathtt{A}'_{lt}) = \mho(\mathtt{A}_{kl})\mathtt{A}'_{lt} + \mathtt{A}_{kl}\mho(\mathtt{A}'_{lt}) \text{ for all } 1 \leqslant k \leqslant l \leqslant t \leqslant [n/2],\\ &\mho(\mathtt{A}_{kl})\mathtt{A}'_{st} + \mathtt{A}_{kl}\mho(\mathtt{A}'_{st}) = 0 \text{ for all } 1 \leqslant k \leqslant l \leqslant [n/2], 1 \leqslant s \leqslant t \leqslant [n/2] \text{ with } l = s. \qquad (32)\end{aligned}$$

We will prove this by the following steps.

Step 2.1. For any $\mathtt{A}_{kk}, \mathtt{A}'_{ss} \in \mathtt{A}_{[n/2]}$ with $k < s$, we have

$$0 = \mho(\mathtt{A}_{kk})\mathtt{A}'_{ss} + \mathtt{A}_{kk}\mho(\mathtt{A}'_{ss}) \ \text{ and } \ \mho(\mathtt{A}'_{ss})\mathtt{A}_{kk} = \mathtt{A}'_{ss}\mho(\mathtt{A}_{kk}) = 0.$$

For A_{kk} and A'_{ss} with $k < l$, by the definition of $\mho$ and the fact $\zeta(\cdot) \in \mathscr{Z}(\mathscr{T})$, we have

$$\begin{aligned} 0 &= \mho(\mathscr{P}_N(A_{kk}, A'_{ss}, \mathscr{Q}_i, \cdots, \mathscr{Q}_i)) \\ &= \mathscr{L}_{[n/2]}(\mathscr{P}_N(A_{kk}, A'_{ss}, \mathscr{Q}_i, \cdots, \mathscr{Q}_i)) \\ &= \mathscr{P}_{N-1}([\mathscr{L}_{[n/2]}(A_{kk}), A'_{ss}], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) + \mathscr{P}_{N-1}([A_{kk}, \mathscr{L}_{[n/2]}(A'_{ss})], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) \\ &= \mathscr{P}_{N-1}([\mho(A_{kk}) + \zeta(A_{kk}), A'_{ss}], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) + \mathscr{P}_{N-1}([A_{kk}, \mho(A'_{ss}) + \zeta(A'_{ss})], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) \\ &= \mathscr{P}_{N-1}([\mho(A_{kk}), A'_{ss}] + [A_{kk}, \mho(A'_{ss})], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) \\ &= P_i\left(\mho(A_{kk})A'_{ss} - A'_{ss}\mho(A_{kk}) + A_{kk}\mho(A'_{ss}) - \mho(A'_{ss})A_{kk}\right)\mathscr{Q}_i. \end{aligned} \tag{33}$$

Note that, by (31), we have

$$\begin{aligned} &\mho(A_{kk})A'_{ss} \in \sum_{j=1}^{s-1} T_{js},\ A'_{ss}\mho(A_{kk}) \in \sum_{j=s+1}^{[n/2]} T_{sj}, \\ &A_{kk}\mho(A'_{ss}) \in \sum_{j=k+1}^{[n/2]} T_{kj},\ \mho(A'_{ss})A_{kk} \in \sum_{j=1}^{k-1} T_{jk}. \end{aligned}$$

These and (30),(31),(33) imply to $\mho(A_{kk})A'_{ss} + A_{kk}\mho(A'_{ss}) = 0 = \mho(A_{kk}A'_{ss})$ and $A'_{ss}\mho(A_{kk}) = \mho(A'_{ss})A_{kk} = 0$, completing the proof. Note that, by Step 2.1, we can easily check that

$$\mho(A_{kk}) \in \mathscr{T}_{1k} + \cdots + \mathscr{T}_{(k-1)k} + \mathscr{T}_{kk} + \mathscr{T}_{k(k+1)} + \cdots + \mathscr{T}_{k[n/2]},\ \ k = 1, 2, \cdots, [n/2]. \tag{34}$$

Step 2.2. For any $A_{kl}, A'_{st} \in A_{[n/2]}$ for $s \leqslant t$ and $k \leqslant l$, we have $\mho(A_{kk})A'_{st} = 0$ for $k > s$; $A_{kl}\mho(A'_{ss}) = 0$ for $l > s$. It is obvious by (34).

Step 2.3. For any $A_{kl}, A'_{st} \in A_{[n/2]}$ with $k \leqslant l$ and $s \leqslant t$, we have

(i) $\mho(A_{kl})A'_{st} = A_{kl}\mho(A'_{st}) = 0$ if $k > s$ or $k = s, k < l$;
(ii) $A'_{st}\mho(A_{kl}) = \mho(A'_{st})A_{kl} = 0$ if $k < s$ or $k = s, s < t$.

Note that, by Lemma 3.14, we know that $\mho(S_{kl}) \subseteq \mathscr{M}_k \cap A_{[n/2]}$ holds for all $S_{kl} \in A_{[n/2]}$ with $k < l$. So, the step is true.

Step 2.4. For any $A_{kk}, A'_{kk} \in A_{[n/2]}$, we have $\mho(A_{kk}A'_{kk}) = \mho(A_{kk})A'_{kk} + A_{kk}\mho(A'_{kk})$. For $A_{kk} \in A_{[n/2]}$, by Lemma 3.15, we get

$$\begin{aligned} 0 &= \mho(A_{kk}\mathscr{E}_k) - \mho(\mathscr{E}_k A_{kk}) \\ &= \mho(\mathscr{P}_{N-1}([A_{kk}, \mathscr{E}_k], \mathscr{Q}_i, \cdots, \mathscr{Q}_i)) \\ &= \mathscr{L}_{[n/2]}(\mathscr{P}_{N-1}([A_{kk}, \mathscr{E}_k], \mathscr{Q}_i, \cdots, \mathscr{Q}_i)) - \zeta(\mathscr{P}_{N-1}([A_{kk}, \mathscr{E}_k], \mathscr{Q}_i, \cdots, \mathscr{Q}_i)) \\ &= \mathscr{P}_{N-1}([\mathscr{L}_{[n/2]}(A_{kk}), \mathscr{E}_k], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) + \mathscr{P}_{N-1}([A_{kk}, \mathscr{L}_{[n/2]}(\mathscr{E}_k)], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) \\ &= \mathscr{P}_{N-1}([\mho(A_{kk}), \mathscr{E}_k], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) + \mathscr{P}_{N-1}([A_{kk}, \mho(\mathscr{E}_k)], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) \\ &= P_i(\mho(A_{kk})\mathscr{E}_k - \mathscr{E}_k\mho(A_{kk}) + A_{kk}\mho(\mathscr{E}_k) - \mho(\mathscr{E}_k)A_{kk})\mathscr{Q}_i. \end{aligned}$$

Hence by (34), we obtain

$$\begin{aligned} &(a)\ P_i\mho(A_{kk})\mathscr{Q}_i = P_i\mho(A_{kk})\mathscr{E}_k\mathscr{Q}_i = P_i\mho(\mathscr{E}_k)A_{kk}\mathscr{Q}_i \text{ for } 1 \leqslant i < k; \\ &(b)\ P_i\mho(A_{kk})\mathscr{Q}_i = P_i\mathscr{E}_k\mho(A_{kk})\mathscr{Q}_i = P_iA_{kk}\mho(\mathscr{E}_k)\mathscr{Q}_i \text{ for } k \leqslant i \leqslant [n/2]. \end{aligned} \tag{35}$$

Therefore, for any $A'_{kk} \in A_{[n/2]}$, if $1 \leqslant i < k$, (35)(a) implies

$$\begin{aligned} &P_i(\mho(A_{kk}A'_{kk}) - \mho(A_{kk})A'_{kk} - A_{kk}\mho(A'_{kk}))\mathscr{Q}_i \\ &= P_i\mho(A_{kk}A'_{kk})\mathscr{Q}_i - P_i\mho(A_{kk})A'_{kk}\mathscr{Q}_i = P_i\mho(A_{kk}A'_{kk})\mathscr{Q}_i - P_i\mho(A_{kk})\mathscr{Q}_iA'_{kk}\mathscr{Q}_i \\ &= P_i\mho(\mathscr{E}_k)A_{kk}A'_{kk}\mathscr{Q}_i - P_i\mho(\mathscr{E}_k)A_{kk}\mathscr{Q}_iA'_{kk}\mathscr{Q}_i = 0; \end{aligned}$$

if $k \leqslant i \leqslant [n/2]$, (35)b) also implies $P_i(\mho(A_{kk}A'_{kk}) - \mho(A_{kk})A'_{kk} - A_{kk}\mho(A'_{kk}))\mathscr{Q}_i = 0$. So until now we prove that

$$P_i(\mho(A_{kk}A'_{kk}) - \mho(A_{kk})A'_{kk} - A_{kk}\mho(A'_{kk}))\mathscr{Q}_i = 0 \text{ for } i \in \{1, \cdots, [n/2]\}.$$

Combining the above equation and (30) gives $\mho(A_{kk}A'_{kk}) - \mho(A_{kk})A'_{kk} - A_{kk}\mho(A'_{kk}) = 0$.

Step 2.5. For any $A_{kk}, A'_{kl} \in A_{[n/2]}$ with $k < l$, we have $\mho(A_{kk}A'_{kl}) = \mho(A_{kk})A'_{kl} + A_{kk}\mho(A'_{kl})$. Let $k < l$. Then, by Lemma 3.14 and Steps 2.2-2.3, we have

$$\begin{aligned} \mho(A_{kk}A'_{kl}) &= \mho(\mathscr{P}_N(A_{kk}, A'_{kl}, \mathscr{Q}_i, \cdots, \mathscr{Q}_i)) \\ &= \mathscr{L}_{[n/2]}(\mathscr{P}_N(A_{kk}, A'_{kl}, \mathscr{Q}_i, \cdots, \mathscr{Q}_i)) \\ &= \mathscr{P}_{N-1}([\mathscr{L}_{[n/2]}(A_{kk}), A'_{kl}], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) + \mathscr{P}_{N-1}([A_{kk}, \mathscr{L}_{[n/2]}(A'_{kl})], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) \\ &\quad + \cdots + \mathscr{P}_{N-1}([A_{kk}, A'_{kl}], \mathscr{Q}_i, \cdots, \mathscr{L}_{[n/2]}(\mathscr{Q}_i)) \\ &= \mathscr{P}_{N-1}([\mho(A_{kk}) + \zeta(A_{kk}), A'_{kl}] + [A_{kk}, \mho(A'_{kl})], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) \\ &\quad + \cdots + \mathscr{P}_{N-1}([A_{kk}, A'_{kl}], \mathscr{Q}_i, \cdots, \mho(\mathscr{Q}_i)) \\ &= \mathscr{P}_{N-1}([\mho(A_{kk}), A'_{kl}] + [A_{kk}, \mho(A'_{kl})], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) \\ &= P_i(\mho(A_{kk})A'_{kl} - A'_{kl}\mho(A_{kk}) + A_{kk}\mho(A'_{kl}) - \mho(A'_{kl})A_{kk})\mathscr{Q}_i \\ &= P_i(\mho(A_{kk})A'_{kl} + A_{kk}\mho(A'_{kl}))\mathscr{Q}_i. \end{aligned}$$

Multiplying by P_i and $\mathscr{Q}_i$ from left and right respectively, we have $P_i(\mho(A_{kk}A'_{kl}) - \mho(A_{kk})A'_{kl} - A_{kk}\mho(A'_{kl}))\mathscr{Q}_i = 0$ which together with (30) leads to the required outcome.

By a similar argument to that of Step 2.5 and by using Steps 2.2-2.3 again, we can show the following Steps 2.6-2.7.

Step 2.6. For any $A_{kl}, A'_{ll} \in A_{[n/2]}$ with $k < l$, we have $\mho(A_{kl}A'_{ll}) = \mho(A_{kl})A'_{ll} + A_{kl}\mho(A'_{ll})$.

Step 2.7. For any $A_{kl}, A'_{lt} \in A_{[n/2]}$ with $k < l < t$, we have $\mho(A_{kl}A'_{lt}) = \mho(A_{kl})A'_{lt} + A_{kl}\mho(A'_{lt})$.

Step 2.8. For any $A_{kl}, A'_{st} \in A_{[n/2]}$ with $k \leqslant l, s \leqslant t$ and $l \neq s$, we have $\mho(A_{kl})A'_{st} + A_{kl}\mho(A'_{st}) = 0$. If $k \leqslant l < s \leqslant t$, by Step 2.1, (34) and Step 2.3(ii), we have

$$\begin{aligned} 0 &= \mathscr{L}_{[n/2]}(\mathscr{P}_N(A_{kl}, A'_{st}, \mathscr{Q}_i, \cdots, \mathscr{Q}_i)) \\ &= \mathscr{P}_{N-1}([\mathscr{L}_{[n/2]}(A_{kl}), A'_{st}], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) + \mathscr{P}_{N-1}([A_{kl}, \mathscr{L}_{[n/2]}(A'_{st})], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) \\ &= \mathscr{P}_{N-1}([\mho(A_{kl}), A'_{st}], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) + \mathscr{P}_{N-1}([A_{kl}, \mho(A'_{st})], \mathscr{Q}_i, \cdots, \mathscr{Q}_i) \\ &= P_i(\mho(A_{kl})A'_{st} - A'_{st}\mho(A_{kl}) + A_{kl}\mho(A'_{st}) - \mho(A'_{st})A_{kl})\mathscr{Q}_i \\ &= P_i(\mho(A_{kl})A'_{st} + A_{kl}\mho(A'_{st}))\mathscr{Q}_i. \end{aligned} \tag{36}$$

This with (30) gives $\mho(A_{kl})A'_{st} + A_{kl}\mho(A'_{st}) = 0$.

If $k \leqslant s < l \leqslant t, k \leqslant s < t \leqslant l$ or $k < s \leqslant t < l$, the by similar argument as above gives

$$\mho(A_{kl})A'_{st} + A_{kl}\mho(A'_{st}) = A'_{st}\mho(A_{kl}) = \mho(A'_{st})A_{kl} = 0;$$

if $k = s = t < l$, we have

$$\begin{aligned} -\mho(A'_{kk}A_{kl}) &= \mho_{[n/2]}(\mathscr{P}_{N-1}([A_{kl}, A'_{kk}], \mathscr{Q}_i, \cdots, \mathscr{Q}_i)) \\ &= \mathscr{L}_{[n/2]}(\mathscr{P}_{N-1}([A_{kl}, A'_{kk}], \mathscr{Q}_i, \cdots, \mathscr{Q}_i)) \\ &= P_i(\mho(A_{kl})A'_{kk} - A'_{kk}\mho(A_{kl}) + A_{kl}\mho(A'_{kk}) - \mho(A'_{kk})A_{kl})\mathscr{Q}_i \end{aligned}$$

Multiplying by P_i and $\mathscr{Q}_i$ from left and right respectively, we have $P_i(\mho(A'_{kk}A_{kl})-\mho(A_{kl})A'_{kk}+A'_{kk}\mho(A_{kl})-A_{kl}\mho(A'_{kk})+\mho(A'_{kk})A_{kl})\mathscr{Q}_i=0$ which together with (30) and Step 2.5 gives $\mho(A_{kl})A'_{kk}+A_{kl}\mho(A'_{kk})=0$.

Similarly, if $s<k$, by considering subcases $s<k<t<l, s<k<l<t$, and $s<t<k<l$, one can give $\mho(A_{kl})A'_{st}+A_{kl}\mho(A'_{st})=0$. The substep is true. Now, combining Steps 2.1-2.8, and by an easy and direct calculation, we can show that

$$\begin{aligned}\mho(A_{[n/2]}A'_{[n/2]}) &= \sum_{1\leqslant k\leqslant l\leqslant t\leqslant [n/2]}\mho(A_{kl}A'_{lt})\\ &= \sum_{1\leqslant k\leqslant l\leqslant t\leqslant [n/2]}(\mho(A_{kl})A'_{lt}+A_{kl}\mho(A'_{lt}))\\ &= \mho(A_{[n/2]})A'_{[n/2]}+A_{[n/2]}\mho(A'_{[n/2]}).\end{aligned}$$

Hence, the step holds.

Step 3. For any $B_{[n/2]}, B'_{[n/2]}\in B_{[n/2]}$ and any $\mathscr{H}_{[n/2]}\in\mathscr{M}_{[n/2]}$, we have

$$\begin{aligned}\mho(\mathscr{H}_{[n/2]}B_{[n/2]}) &= \mho(\mathscr{H}_{[n/2]})B_{[n/2]}+\mathscr{H}_{[n/2]}\mho(B_{[n/2]}),\\ \mho(B_{[n/2]}B'_{[n/2]}) &= \mho(B_{[n/2]})B'_{[n/2]}+B_{[n/2]}\mho(B'_{[n/2]}).\end{aligned}$$

The proof is similar to that of Step 2. Now, combining Lemma 3.15 and Steps 1-3, one can prove that $\mho$ is a derivation. □

Lemma 3.17. $\zeta(\mathscr{P}_{N-1}(\mathscr{X}_1,\mathscr{X}_2,\cdots,\mathscr{X}_n))=0$ *holds for all* $\mathscr{X}_1,\mathscr{X}_2,\cdots,\mathscr{X}_n\in\mathscr{T}$.

Proof. Note that $\zeta=\mathscr{L}_{[n/2]}-\mho$. Since $\mathscr{L}_{[n/2]}$ is a multiplicative Lie type derivation, $\mho$ is an additive derivation and ζ is a central-valued map, it is a direct calculation that $\zeta(\mathscr{P}_{N-1}(\mathscr{X}_1,\mathscr{X}_2,\cdots,\mathscr{X}_n))=0$ holds for all $\mathscr{X}_1,\mathscr{X}_2,\cdots,\mathscr{X}_n\in\mathscr{T}$. □

Proof. **[Proof of Theorem 3.1]** Note that $\mathscr{L}=\mathscr{L}_{[n/2]}+\mathscr{D}_{[n/2]}=\mathscr{L}_{[n/2]}+[\mathscr{L}(\mathscr{Q}_{[n/2]}),\cdot]$ and $\mathscr{L}_{[n/2]}=\mho+\zeta$. Let $\mathscr{D}=\mathscr{D}_{[n/2]}+\mho$. Then $\mathscr{L}=\mathscr{D}+\zeta$; moreover, $\mathscr{D}$ is an additive derivation as Lemmas 3.15-3.17 and the definition of $\mathscr{D}_{[n/2]}$, and ζ is a central-valued map annihilating all commutators. Thereby, finishing the proof of the theorem. □

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