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New Inertial Approximation Schemes for General Quasi Variational Inclusions

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Dedicated to our respected Parents and Teachers

Abstract. In this article, we introduce and consider some new classes of general quasi variational inclusions, which provide us with unified, natural, novel and simple framework to consider a wide class of unrelated problems arising in pure and applied sciences. We prove that the general quasi variational inclusions are equivalent to the fixed point problems. This alternative formulation is used to discuss the existence of a solution as well as to propose some iterative methods. Convergence analysis is investigated under certain mild conditions. Since the general quasi variational inclusions include quasi variational inequalities, variational inequalities, and related optimization problems as special cases, our results continue to hold for these problems. It is an interesting problem to compare these methods with other technique for solving quasi variational inclusions for further research activities.

1. Introduction

Quasi variational inclusions, which were introduced and studied by Noor and Noor [35], are useful and important extension of the variational principles with a wide range of applications in industry, physical, regional, social, pure and applied sciences. Quasi variational inclusions provide us with a unified, natural, novel, innovative and general technique to study a wide class of problems arising in different branches of mathematical and engineering sciences, see, for example, [26, 28, 30, 35, 40, 43] and the references therein. One of the most difficult and important problems in variational inequalities is the development of efficient numerical methods. Several numerical methods have been developed for solving the variational inclusions and their variant forms. These methods have been extended and modified in numerous ways. Noor [30, 35] proved that the quasi variational inclusions are equivalent to the fixed point problem. This alternative formulation has allowed to consider the existence of a solution, iterative schemes, sensitivity analysis, merit functions and other aspects of the quasi variational inclusions.

It is very important to develop some efficient iterative methods for solving the quasi variational inequalities. Alvarez et al [1] and Alvarez et al.[2–4] used the inertial type projection methods for solving variational inequalities. The origin of which can be traced back to Polyak [44]. Noor [32] suggested and investigated inertial type projection methods for solving general variational inequalities. These inertial type methods

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have been modified in various directions for solving variational inequalities and related optimization problems. Jabeen et al [11–13] analyzed some inertial projection methods for some classes of general quasi variational inequalities. Convergence analysis of these inertial type methods has been considered under some mild conditions. For more details and applications of the inertial methods,[1–4, 9–13, 32, 36, 42–45, 49] and reference therein.

Motivated and inspired by the recent research activities, we consider and study some new classes of variational inclusions involving three arbitrary operators, which are called the general quasi variational inclusions. Some special important cases are also discussed. We have shown that the general quasi variational inclusions are equivalent to the fixed point problems. This equivalence is used to study the existence of a solution as well as to propose some new inertial type methods for solving general quasi variational inclusion. The convergence of the proposed inertial methods is analyzed under some suitable conditions. We have only considered theoretical aspects of the suggested methods. It is an interesting problem to implement these methods and to illustrate the efficiency. Comparison with other methods need further research efforts. The ideas and techniques of this paper may be extended for other classes of quasi variational inclusions and related optimization problems.

2. Basic Definitions and Results

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively. Let $\mathcal{A}(\cdot,\cdot) : H \times H \longrightarrow H$ be a maximal monotone operator with respect to the first argument. For given nonlinear single valued operator $\mathcal{T} : H \longrightarrow H$ and an arbitrary nonlinear operator $g : H \longrightarrow H$ consider the problem of finding $\mu \in H$ such that

$$0 \in \mathcal{T}(\mu) + \mathcal{A}(g(\mu), g(\mu)), \tag{2.1}$$

which is called the general quasi variational inclusion. A number of problems arising in structural analysis, mechanics and economics can be studied in the framework of the general quasi variational inclusions in a unified manner.
We now discuss some interesting and important problems, which can obtained from the problem (2.1).

Special Cases.

I. If $g = I$, the identity operator, then problem (2.1) reduces to finding $\mu \in H$ such that

$$0 \in \mathcal{T}(\mu) + \mathcal{A}(\mu, \mu), \tag{2.2}$$

which is called the quasi variational inclusion, introduced and studied by Noor and Noor [35].
II. If $\mathcal{A}(\cdot, \mu) = \partial\varphi(\cdot, \mu) : H \times H \longrightarrow R \cup \{+\infty\}$, the subdifferential of a convex, proper and lower semicontinuous function $\varphi(\cdot, \mu)$ with respect to the first argument, then problem (2.1) is equivalent to finding $\mu \in H$ such that

$$\langle \mathcal{T}(\mu), g(\nu) - g(\mu)\rangle + \varphi(g(\nu), g(\mu)) - \varphi(g(\mu), g(\mu)) \geq 0, \quad \forall \quad \nu \in H, \tag{2.3}$$

which is called the mixed general quasi variational inequality.
III. If $\mathcal{A}(g(\mu, \nu)) \equiv \mathcal{A}(g(\mu))$, for all $\nu \in H$, then problem (2.1) is equivalent to finding $\mu \in H$ such that

$$0 \in \mathcal{T}(\mu) + \mathcal{A}(g(\mu)), \tag{2.4}$$

a problem considered and studied by Noor [28, 30] using the resolvent equations technique.
IV. If $\mathcal{A}(g(\mu)) \equiv \partial\varphi(g(\mu))$ is the subdifferential of a proper, convex and lower, semicontinuous function $\varphi : H \longrightarrow R \cup \{+\infty\}$. then problem (2.4) reduces to: find $\mu \in H$ such that

$$\langle \mathcal{T}(\mu), g(\nu) - g(\mu)\rangle + \varphi(g(\nu)) - \varphi(g(\mu)) \geq 0. \tag{2.5}$$

Problem (2.5) is known as the mixed general variational inequality.
V. If the function $\varphi(\cdot,\cdot)$ is the indicator function of a closed convex-valued set $\mathcal{K}(\mu)$ in H, that is,

$$\varphi(\mu,\mu) = \mathcal{K}_{(\mu)}(\mu) = \begin{cases} 0, & \text{if} \quad \mu \in \mathcal{K}(\mu) \\ +\infty, & \text{otherwise}, \end{cases}$$

then problem (2.3) is equivalent to finding $\mu \in H$, $\mu \in \mathcal{T}(\mu)$, $g(\mu) \in \mathcal{K}(u)$ such that

$$\langle \mathcal{T}(\mu), g(\nu) - g(\mu)\rangle \geq 0, \quad \forall \quad \nu \in \mathcal{K}(u), \tag{2.6}$$

a problem considered and studied by Noor et al [42] using the projection method and the implicit Wiener-Hopf equations technique. They have also considered and analyzed some inertial type iterative methods for solving general quasi variational inequalities (2.6). For formulation, motivation, numerical methods, sensitivity analysis, dynamical systems and other aspects of quasi variational inequalities, see [5, 9, 11–14, 16, 19, 22, 23, 25, 26, 31, 33, 35, 37–39, 43, 45] and the references therein
VI. If $\mathcal{K}(\mu) = \mathcal{K}$, a convex set and $g = I$, the identity operator, then then problem (2.6) reduces to finding $\mu \in \mathcal{K}$ such that

$$\langle \mathcal{T}(\mu), \nu - \mu\rangle \geq 0, \quad \forall \quad \nu \in \mathcal{K}, \tag{2.7}$$

which is called the variational inequality, introduced and studied by Stampacchia [46] in 1964. Variational inequalities can be viewed as novel extensions of the variational principles. Variational inequality has influenced several branches of mathematical, engineering, economics, transforation, regional and medical sciences and continue to inspire researchers to find its applications. For more details, see [5–7, 9–17, 19, 20, 24–36, 39–50].
VII. The problem (2.6) can be rewritten equivalently to finding $\mu \in \mathcal{H}$, such that

$$0 \in \mathcal{T}(\mu) + g(\mu) - \mu + \mathcal{A}(g(\mu), g(\mu)), \tag{2.8}$$

which is also called the general quasi variational inclusion, studied and considered by Noor et al.[43].
VIII. If

$$\mathcal{K}^*(\mu) = \{\mu \in H, \langle \mu, \nu\rangle \geq 0, \quad \forall \nu \in \mathcal{K}(\mu)\}$$

is a polar cone of the convex-valued cone $\mathcal{K}(\mu)$ in H, then problem (2.6) is equivalent to finding $\mu \in H$, such that

$$g(\mu) \in \mathcal{K}(\mu), \quad \mathcal{T}(\mu) \in \mathcal{K}^*(\mu), \qquad \langle \mathcal{T}(\mu), g(\mu)\rangle = 0, \tag{2.9}$$

which is called the generalized quasi complementarity problem.
IX. If $\mathcal{K}^*(\mu) = \mathcal{K}^*$ then problem (2.9) reduces to finding $\mu \in \mathcal{H}$ such that

$$g(\mu) \in \mathcal{K}, \quad \mathcal{T}(\mu) \in \mathcal{K}^*, \qquad \langle \mathcal{T}(\mu), g(\mu)\rangle = 0, \tag{2.10}$$

which is known as the general complementarity problem, see Noor [23, 24, 32]. For $g = I$, the identity operator, the problem (2.10) was introduced by Karamardian [15] and is known as nonlinear complementarity problem. Karamardian [15] proved that the nonlinear complementarity problems are equivalent to the variational inequalities. This result enabled us to use the techniques of complementarity problems to solve the variational inequalities and vice versa. For the applications, formulations and generalizations of the complementarity problems, see [7, 15, 32, 39].

Remark 2.1. *For special choices of the operators $\mathcal{T}$, g, $\mathcal{A}(.,.)$ and the convex set $\mathcal{K}$, one can obtain a large number of implicit (quasi) complementarity problems and variational inequality problems, which are very special cases of problem (2.1). Thus it is clear that problem (2.1) is general and unifying one and has numerous applications in pure and applied sciences.*

We now recall some well known results and notions.

Definition 2.1 [2]. If $\mathcal{T}$ is a maximal monotone operator on H, then, for a constant $\rho > 0$, the resolvent operator associated with $\mathcal{T}$ is defined by

$$\mathcal{J}_{\mathcal{T}}(\mu) = (I + \rho\mathcal{T})^{-1}(\mu), \quad \forall \mu \in H,$$

where I is the identity operator. Also the resolvent operator $\mathcal{J}_{\mathcal{T}}$ is single-valued and nonexpansive.

Remark 2.1. Since the operator $\mathcal{A}(.,.)$ is a maximal monotone operator with respect to the first argument, we denote by

$$\mathcal{J}_{\mathcal{A}(\mu)} \equiv (I + \rho\mathcal{A}(\mu))^{-1}(\mu), \quad \forall \mu \in H,$$

the resolvent operator associated with $\mathcal{A}(., \mu) \equiv \mathcal{A}(\mu)$. For example, if $\mathcal{A}(., \mu) = \partial\varphi(., \mu)$, for all $\mu \in H$, and $\varphi(.,.) : H \times H \longrightarrow R \cup \{\infty\}$ is a proper, convex and lower semicontinuous with respect to the first argument, then it is well known that $\partial\varphi(., \mu)$ is a maximal monotone operator with respect to the first argument. In this case, the resolvent operator $\mathcal{J}_{\mathcal{A}(\mu)} = \mathcal{J}_{\varphi(\mu)}$ is defined as

$$\mathcal{J}_{\varphi(\mu)} = (I + \rho\partial\varphi(., \mu))^{-1}(\mu) = (I + \rho\partial\varphi(\mu))^{-1}, \quad \forall \mu \in H,$$

which is defined everywhere on the whole space H, where $\partial\varphi(\mu) \equiv \partial\varphi(., \mu)$.
We need the following well-known definitions and results in obtaining our results.

Definition 2.2. *Let* $\mathtt{T} : \mathtt{H} \longrightarrow \mathtt{H}$ *be a given mapping.*

i. *The mapping* $\mathtt{T}$ *is called r-strongly monotone* ($\mathtt{r} \geq 0$)*, if*

$$\left\langle \mathtt{T}\mu - \mathtt{T}\nu, \mu - \nu \right\rangle \geq \mathtt{r}\,\|\mu - \nu\|^2, \qquad \forall\, \mu, \nu \in \mathtt{H}.$$

ii. *The mapping* $\mathtt{T}$ *is called* $\xi-$ *cocoercive* ($\xi > 0$)*, if*

$$\left\langle \mathtt{T}\mu - \mathtt{T}\nu, \mu - \nu \right\rangle \geq \xi\,\|\mathtt{T}\mu - \mathtt{T}\nu\|^2, \qquad \forall\, \mu, \nu \in \mathtt{H}.$$

iii. *The mapping* $\mathtt{T}$ *is called relaxed* $(\xi, \mathtt{r})$*-cocoercive* ($\mathtt{r} > 0$, $\xi > 0$)*, if*

$$\left\langle \mathtt{T}\mu - \mathtt{T}\nu, \mu - \nu \right\rangle \geq -\xi\,\|\mathtt{T}\mu - \mathtt{T}\nu\|^2 + \mathtt{r}\,\|\mu - \nu\|^2, \qquad \forall\, \mu, \nu \in \mathtt{H}.$$

For $\xi = 0$, $\mathtt{T}$ *is* $\mathtt{r}$*-strongly monotone. The class of relaxed* $(\xi, \mathtt{r})-$*cocoercive mapping is the generalized class than the* $r-$*strongly monotone mapping and* $\xi-$*cocoercive.*

iv. *The mapping* $\mathtt{T}$ *is called* $\eta-$*Lipschitz continuous* ($\eta_1 > 0$)*, if*

$$\|\mathtt{T}\mu - \mathtt{T}\nu\| \leq \eta_1\,\|\mu - \nu\|, \qquad \forall\, \mu, \nu \in \mathtt{H}.$$

The implicit resolvent operator $\mathcal{J}_{\mathtt{A}}(\mu)$ is nonexpansive and has the following characterization.

Assumption [35] The implicit resolvent operator $\mathcal{J}_{\mathtt{A}(\mu)}$, satisfies the condition

$$\|\mathcal{J}_{\mathtt{A}(\mu)}[\omega] - \mathcal{J}_{\mathtt{A}(\nu)}[\omega]\| \leq \eta\,\|\mu - \nu\|, \qquad \forall \mu, \nu, \omega \in \mathtt{H}, \tag{2.11}$$

where $\eta > 0$ is a constant.

Lemma 2.3. *[44] Consider a sequence of non negative real numbers* $\{\varrho_n\}$*, satisfying*

$$\varrho_{n+1} \leq (1 - \Upsilon_n)\varrho_n + \Upsilon_n\,\sigma_n + \varsigma_n, \quad \forall\, n \geq 1,$$

where

i. $\{\Upsilon_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} \Upsilon_n = \infty$;

ii. $\limsup \sigma_n \leq 0$;

iii. $\varsigma_n \geq 0$ $(n \geq 1)$, $\sum_{n=1}^{\infty} \varsigma_n < \infty$.

Then, $\varrho_n \longrightarrow 0$ *as* $n \longrightarrow \infty$.

3. Main results

In this section, we prove the equivalence between the problem (2.1) and the fixed point problems. This alternative formulation is used to discuss the existence of a solution as well as to suggest some new inertial-type approximation schemes for solving the general quasi variational inclusion (2.1).

Lemma 3.1. *The function $\mu \in \mathcal{H}$ is a solution of the general quasi variational inclusion (2.1), if and only if, $\mu \in \mathcal{H}$ satisfies the relation*

$$g(\mu) = \mathcal{J}_{\mathcal{A}(\mu)}[g(\mu) - \rho\mathcal{T}\mu], \tag{3.1}$$

where $\mathcal{J}_{\mathcal{A}(\mu)}$ is the resolvent operator and $\rho > 0$ is a constant.

Proof. Let $\mu \in \mathcal{H}$ be a solution of (2.1), then, for a constant ρ,

$$\begin{aligned} \rho\mathcal{T}\mu \quad &+ \quad \rho\mathcal{A}(g(\mu), g(\mu)) \ni 0, \\ &\Longleftrightarrow \\ -g(\mu) + \rho\mathcal{T}\mu \quad &+ \quad g(\mu) + \rho\mathcal{A}(g(\mu), g(\mu)) \ni 0 \\ &\Longleftrightarrow \\ g(\mu) = \quad &\mathcal{J}_{\mathcal{A}}[g(\mu) - \rho\mathcal{T}\mu]. \end{aligned}$$

the required (3.1). □

Lemma 3.1 implies that the general quasi variational inclusion (2.1) is equivalent to the fixed point problem (3.1).
Clearly, from (3.1), we have

$$\mu = \mu - g(\mu) + \mathcal{J}_{\mathcal{A}(\mu)}[g(\mu) - \rho\mathcal{T}\mu], \tag{3.2}$$

which play an important part to derive the main results.
Using (3.2), we define the mapping Φ associated with (3.1) as:

$$\Phi(\mu) = \mu - g(\mu) + \mathcal{J}_{\mathcal{A}(\mu)}[g(\mu) - \rho\mathcal{T}\mu], \tag{3.3}$$

To prove the existence of the solution of problem (2.1), it is enough that the mapping Φ defined by (3.3) is a contraction mapping.

Theorem 3.2. *Let the operators $\mathcal{T}, g$ be strongly monotone with constants $\alpha > 0, \quad \delta > 0$ and Lipschitz continuous with constant $\beta > 0, \quad \sigma > 0$ respectively. If there exists a constant $\rho > 0$, such that*

$$\|\rho - \frac{\alpha}{\beta^2}\| < \frac{\sqrt{\alpha^2 - \beta^2 k(2-k)}}{\beta^2}, \quad \alpha > \beta\sqrt{k(2-k)}, \quad k < 1, \tag{3.4}$$

where

$$k = 2\sqrt{(1 - 2\delta + \sigma^2)} + \eta, \tag{3.5}$$

then there exists a solution $\mu \in \mathcal{H}$ satisfying problem (2.1).

Proof. Let $u \neq v \in H$ be two solutions of problem (2.1). Then, from problem (3.3), we have

$$\begin{aligned} \|\Phi(v) - \Phi(u)\| &\leq \|\nu - \mu - (g(\nu) - g(\mu))\| \\ &+ \|\mathcal{J}_{\mathcal{A}(\nu)}[g(\nu) - \rho\mathcal{T}\nu] - \mathcal{J}_{\mathcal{A}(\mu)}[g(\mu) - \rho(\mathcal{T}\mu]\| \\ &= \|\nu - \mu - (g(\nu) - g(\mu))\| + \|\mathcal{J}_{\mathcal{A}(\nu)}[g(\nu) - \rho\mathcal{T}\nu] - \mathcal{J}_{\mathcal{A}(\mu)}[g(\nu) - \rho\mathcal{T}\nu]\| \\ &+ \|\mathcal{J}_{\mathcal{A}(\mu)}[g(\nu) - \rho\mathcal{T}\nu] - \mathcal{J}_{\mathcal{A}(\mu)}[g(\mu) - \rho(\mathcal{T}\mu]\| \\ &\leq \|\nu - \mu - (g(\nu) - g(\mu))\| + \|g(\nu) - g(\mu) - \rho(\mathcal{T}\nu - \mathcal{T}\nu)\| + \eta\|\nu - \mu\| \\ &\leq 2\|\nu - \mu - (g(\nu) - g(\mu))\| + \| + \|\nu - \mu - \rho(\mathcal{T}\nu - \mathcal{T}\mu)\| \\ &\quad +\eta\|\nu - \mu\|. \end{aligned} \tag{3.6}$$

Since the operator $\mathcal{T}$ is strongly monotonicity with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, so

$$\begin{aligned} \|\mu - \nu - \rho(\mathcal{T} - \mathcal{T}\nu)\|^2 &= \|\mu - \nu\|^2 - \rho\langle \mathcal{T}\mu - \mathcal{T}\nu, \mu - \nu\rangle \\ &\quad + \rho^2\|\mathcal{T}\mu - \mathcal{T}\nu\|^2, \\ &\leq (1 - 2\alpha\rho + \beta^2\rho^2)\|\mu - \nu\|^2. \end{aligned} \tag{3.7}$$

Similarly, using the strongly monotonicity with constant δ and Lipschitz continuity with constant σ of the operator g, we have

$$\begin{aligned} \|\mu - \nu - (g(\mu) - g(\nu))\|^2 &= \|\mu - \nu\|^2 - \langle g(\mu) - g(\nu), \mu - \nu\rangle \\ &\quad + \|g(\mu) - g(\nu)\|^2, \\ &\leq (1 - 2\delta + \sigma^2)\|\mu - \nu\|^2. \end{aligned} \tag{3.8}$$

Combining (3.8), (3.7) and (3.6), we have

$$\begin{aligned} \|\Phi(v) - \Phi(u)\| &\leq \{\sqrt{(1 - 2\alpha\rho + \beta^2\rho^2)} + 2\sqrt{(1 - 2\delta + \sigma^2)} + \eta\}\|\mu - \nu\| \\ &= \theta\|\mu - \nu\|, \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} \theta &= \{\sqrt{(1 - 2\alpha\rho + \beta^2\rho^2)} + 2\sqrt{(1 - 2\delta + \sigma^2)} + \eta\} \\ &= \{\sqrt{(1 - 2\alpha\rho + \beta^2\rho^2)} + k\} \end{aligned} \tag{3.10}$$

and k is defined by (3.5). From (3.4). it follows that $\theta < 1$. Thus it follows that the mapping $\Phi(\mu)$ defined by (3.3) is a contraction mapping and consequently, the mapping $\Phi(\mu)$ has a fixed point $\Phi(\mu) = \mu \in \mathcal{H}$ satisfying (2.1), the required result. □

Using the result (3.2), we can propose some iterative approximation schemes for solving the general quasi variational inclusion (2.1).

Algorithm 3.1. *For given* $\mu_0 \in H$, *compute* μ_{n+1} *by the recurrence relation*

$$g(\mu_{n+1}) = (1 - \alpha_n)\mu_n + \alpha_n\{g(\mu_n) - \mathcal{J}_{A(\mu_n)}[g(\mu_n) - \rho T\mu_n]\}, \quad n = 1, 2, \ldots,$$

where $\alpha_n \in [0, 1]$. Algorithm 3.1 is called the Mann iterative resolvent method.

Algorithm 3.2. *For given* $\mu_0 \in H$, *compute* μ_{n+1} *by the recurrence relation*

$$g(\mu_{n+1}) = (1 - \alpha_n)\mu_n + \alpha_n\{g(\mu_n) - \mathcal{J}_{A(\mu_n)}[g(\mu_n) - \rho T\mu_{n+1}]\}, \quad \alpha_n \in [0, 1]$$

which is known as the extraresolvent method in the sense of Koperlevich [17] and is equivalent to the following two-step method.

Algorithm 3.3. *For given* $\mu_0 \in H$, *compute* μ_{n+1} *by the recurrence relation*

$$\begin{aligned} g(\omega_n) &= \mathcal{J}_{A(\mu_n)}[g(\mu_n) - \rho T\mu_n] \\ g(\mu_{n+1}) &= (1 - \alpha_n)\mu_n + \alpha_n\{g(\mu_n) - \mathcal{J}_{A(\mu_n)}[g(\mu_n) - \rho T\omega_n]\}, \quad \alpha_n \in [0, 1], \end{aligned}$$

which is called the predictor-corrector resolvent method.
Using the equation(3.1), we suggest the following double resolvent method:

Algorithm 3.4. *For given* $\mu_0 \in H$, *compute* μ_{n+1} *by the recurrence relation*

$$g(\mu_{n+1}) = \mathcal{J}_{A(\mu_n)}[g(\mu_{n+1}) - \rho T\mu_{n+1}], \quad \alpha_n \in [0, 1],$$

Using the predictor-corrector technique, Algorithm 3.4 be written in the following form

Algorithm 3.5. *For given $\mu_0 \in H$, compute μ_{n+1} by the recurrence relation*

$$\begin{aligned} g(\omega_n) &= \mathcal{J}_{A(\mu_n)}[g(\mu_n) - \rho T\mu_n] \\ g(\mu_{n+1}) &= (1-\alpha_n)\mu_n + \alpha_n\{g(\mu_n) - \mathcal{J}_{A(\mu_n)}[g(\omega_n) - \rho T\omega_n]\}, \quad \alpha_n \in [0,1]. \end{aligned}$$

Algorithm 3.5 appeared to be a new two-step method for solving general quasi variational inclusion (2.1). We can rewrite (3.1 as

$$g(\mu) = \mathcal{J}_{A(\mu)}\{g((1-\Theta_n)\mu + \Theta_n\mu) - \rho T\mu\},$$

where $\Theta_n \in [0,1]$, $\forall n \geq 1$.
This fixed point formulation is used to suggest the following two-step method for solving general quasi variational inclusion (2.1) using the ideas of Alvarez [5], Alvarez et al. [6] and Noor [4].

Algorithm 3.6. *For given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the recurrence relation*

$$\begin{aligned} \omega_n &= \mu_n - \Theta_n(\mu_n - \mu_{n-1}) \\ g(\mu_{n+1}) &= 1-\alpha_n)\mu_n + \alpha_n\{g(\mu_n) - \mathcal{J}_{A(\mu_n)}[g(\omega_n) - \rho T\mu_n]\}, \quad n = 1,2,\ldots, \end{aligned}$$

where $\Theta_n \in [0,1]$, $\forall\, n \geq 1$.

Such type of inertial projection methods for solving general quasi variational inequalities have been considered by Noor [4] and Noor et al[29].
We can rewrite (3.1 as

$$g(\mu) = \mathcal{J}_{A((1-\Theta_n)\mu+\Theta_n\mu))}\{g((1-\Theta_n)\mu + \Theta_n\mu)) - \rho T((1-\Theta_n)\mu + \Theta_n\mu))\},$$

where $\Theta_n \in [0,1]$, $\forall n \geq 1$.
Using this fixed point formulation, we can suggest the following inertial type methods for solving general quasi variational inclusion (2.1).

Algorithm 3.7. *For given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the recurrence relation*

$$\begin{aligned} \omega_n &= \mu_n - \Theta_n(\mu_n - \mu_{n-1}) \\ g(\mu_{n+1}) &= 1-\alpha_n)\mu_n + \alpha_n\{g(\mu_n) - \mathcal{J}_{A(\omega_n)}[g(\omega_n) - \rho T\omega_n], \quad n = 1,2,\ldots, \end{aligned}$$

where $\Theta_n \in [0,1]$, $\forall n \geq 1$.

Algorithm 3.7 is known as modified inertial resolvent method for solving general quasi variational inclusion (2.1).

Algorithm 3.8. *For given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the recurrence relation*

$$\begin{aligned} \omega_n &= \mu_n - \Theta_n(\mu_n - \mu_{n-1}) \\ g(y_n) &= \mathcal{J}_{A(\omega_n)}[g(\omega_n) - \rho T\omega_n], \\ g(\mu_{n+1}) &= \mathcal{J}_{A(y_n)}[g(y_n) - \rho T y_n], \quad n = 1,2,\ldots, \end{aligned}$$

where $\Theta_n \in [0,1], \forall n \geq 1$.

Algorithm 3.8 is a three-step modified inertial method for solving general quasi variational inclusion(2.1). We now suggest a four-step inertial method for solving the general quasi variational inclusion (2.1.

Algorithm 3.9. *For given $\mu_0, \mu_1 \in \mathtt{H}$, compute μ_{n+1} by the recurrence relation*

$$\omega_n = \mu_n - \Theta_n(\mu_n - \mu_{n-1}), \tag{3.11}$$

$$\mathtt{x}_n = (1-\gamma_n)\mu_n + \gamma_n\Big\{\omega_n - \mathtt{g}(\omega_n) + \mathcal{J}_{\mathtt{A}(\omega_n)}[\mathtt{g}(\omega_n) - \rho\mathtt{T}\omega_n]\Big\}, \tag{3.12}$$

$$\mathtt{y}_n = (1-\beta_n)\mu_n + \beta_n\Big\{\mathtt{x}_n - \mathtt{g}(\mathtt{x}_n) + \mathcal{J}_{\mathtt{A}(\mathtt{x}_n)}[\mathtt{g}(\mathtt{x}_n) - \rho\mathtt{T}\mathtt{x}_n]\Big\}, \tag{3.13}$$

$$\mu_{n+1} = (1-\alpha_n)\mu_n + \alpha_n\Big\{\mathtt{y}_n - \mathtt{g}(\mathtt{y}_n) + \mathcal{J}_{\mathtt{A}(\mathtt{y}_n)}[\mathtt{g}(\mathtt{y}_n) - \rho\mathtt{T}\mathtt{y}_n]\Big\}, \quad n = 1, 2, \ldots, \tag{3.14}$$

where $\alpha_n, \beta_n, \gamma_n, \Theta_n \in [0,1], \quad \forall n \geq 1$.

If $g = I$, the identity, then Algorithm (3.9) reduces to:

Algorithm 3.10. *For given $\mu_0, \mu_1 \in \mathtt{H}$, compute μ_{n+1} by the recurrence relation*

$$\begin{aligned}
\omega_n &= \mu_n - \Theta_n(\mu_n - \mu_{n-1}), \\
\mathtt{x}_n &= (1-\gamma_n)\mu_n + \gamma_n\Big\{\omega_n - (\omega_n) + \mathcal{J}_{\mathtt{A}(\omega_n)}[(\omega_n) - \rho\mathtt{T}\omega_n]\Big\}, \\
\mathtt{y}_n &= (1-\beta_n)\mu_n + \beta_n\Big\{\mathtt{x}_n - (\mathtt{x}_n) + \mathcal{J}_{\mathtt{A}(\mathtt{x}_n)}[(\mathtt{x}_n) - \rho\mathtt{T}\mathtt{x}_n]\Big\}, \\
\mu_{n+1} &= (1-\alpha_n)\mu_n + \alpha_n\Big\{\mathtt{y}_n - (\mathtt{y}_n) + \mathcal{J}_{\mathtt{A}(\mathtt{y}_n)}[(\mathtt{y}_n) - \rho\mathtt{T}\mathtt{y}_n]\Big\}, \quad n = 1, 2, \ldots,
\end{aligned}$$

where $\alpha_n, \beta_n, \gamma_n, \Theta_n \in [0,1], \quad \forall n \geq 1$.

For a different and suitable choice of operators and spaces in Algorithm (3.9), one can obtain numerous new and previous iterative schemes for solving general quasi variational inclusion (2.1) and related problems. This shows that the Algorithm (3.9) is quite flexible and unifying ones. We now estimate convergence analysis for Algorithm 3.9 under some mild and appropriate conditions.

Theorem 3.3. *Let the following assumptions be fulfilled:*

i. *The operators $\mathtt{T}, \mathtt{g} : \mathtt{H} \longrightarrow \mathtt{H}$ are relaxed $(\xi_1, \mathtt{r}_1), (\xi_2, \mathtt{r}_2)$, $-$cocoercive and η_1, η_2-Lipschitz continuous, respectively.*

ii. *Assumption 2 holds.*

iii. *The parameter $\rho > 0$ satisfies the condition*

$$\left|\rho - \frac{(\mathtt{r}_1 - \xi_1\eta_1^2)}{\eta_1^2}\right| < \frac{\sqrt{(\mathtt{r}_1 - \xi_1\eta_1^2)^2 - \eta_1^2\mathtt{k}(2-\mathtt{k})}}{\eta_1^2},$$
$$\mathtt{r}_1 > \xi_1\eta_1^2 + \eta_1\sqrt{\mathtt{k}(2-\mathtt{k})}, \ \mathtt{k} < 1, \tag{3.15}$$

where

$$\mathtt{k} = 2\sqrt{1 - 2(\mathtt{r}_2 - \xi_2\eta_2^2) + \eta_2^2} + \upsilon. \tag{3.16}$$

iv. *Let $\alpha_n, \beta_n, \gamma_n, \Theta_n \in [0,1]$, for all $n \geq 1$ such that*

$$\sum_{n=1}^{\infty}\alpha_n = \infty, \quad \sum_{n=1}^{\infty}\Theta_n \parallel \mu_n - \mu_{n-1} \parallel < \infty.$$

Then, for every initial approximation μ_n, the sequence $\{\mu_n\}$ obtained from the iterative scheme defined in Algorithm 3.9 converges to unique solution $\mu \in \mathtt{H}$ satisfying the general quasi variational inclusion (2.1) as $n \longrightarrow \infty$.

Proof. Let $\mu \in \mathtt{H} : \mathtt{g}(\mu) \in \mathtt{K}(\mu)$ be a solution of (2.1). Then

$$\mu = (1-\alpha_n)\mu + \alpha_n\Big\{\mu - \mathtt{g}(\mu) + \mathcal{J}_{\mathtt{A}(\mu)}\left[\mathtt{g}(\mu) - \rho\mathtt{T}\mu\right]\Big\}, \tag{3.17}$$

$$= (1-\beta_n)\mu + \beta_n\Big\{\mu - \mathtt{g}(\mu) + \mathcal{J}_{\mathtt{A}(\mu)}\left[\mathtt{g}(\mu) - \rho\mathtt{T}\mu\right]\Big\}, \tag{3.18}$$

$$= (1-\gamma_n)\mu + \gamma_n\Big\{\mu - \mathtt{g}(\mu) + \mathcal{J}_{\mathtt{A}(\mu)}\left[\mathtt{g}(\mu) - \rho\mathtt{T}\mu\right]\Big\}, \tag{3.19}$$

where $0 \le \alpha_n, \beta_n, \gamma_n \le 1, \forall n \ge 1$, are constants.

Using Assumption (2), from (3.15), and (3.17), we have

$$\begin{aligned}
||\mu_{n+1} - \mu|| &= ||(1-\alpha_n)\mu_n + \alpha_n\Big\{\mathtt{y}_n - \mathtt{g}(\mathtt{y}_n) + \mathcal{J}_{\mathtt{A}(\mathtt{y}_n)}\left[\mathtt{g}(\mathtt{y}_n) - \rho\mathtt{T}\mathtt{y}_n\right]\Big\} \\
&\quad - (1-\alpha_n)\mu - \alpha_n\Big\{\mu - \mathtt{g}(\mu) + \mathcal{J}_{\mathtt{A}(\mu)}\left[\mathtt{g}(\mu) - \rho\mathtt{T}\mu\right]\Big\}|| \\
&\le (1-\alpha_n)||\mu_n - \mu|| + \alpha_n||\mathtt{y}_n - \mu - [\mathtt{g}(\mathtt{y}_n) - \mathtt{g}(\mu)]|| \\
&\quad + \alpha_n||\mathcal{J}_{\mathtt{A}(\mathtt{y}_n)}\left[\mathtt{g}(\mathtt{y}_n) - \rho\mathtt{T}\mathtt{y}_n\right] - \mathcal{J}_{\mathtt{A}(\mu)}\left[\mathtt{g}(\mu) - \rho\mathtt{T}\mu\right]|| \\
&\le (1-\alpha_n)||\mu_n - \mu|| + \alpha_n||\mathtt{y}_n - \mu - [\mathtt{g}(\mathtt{y}_n) - \mathtt{g}(\mu)]|| \\
&\quad + \alpha_n||\mathcal{J}_{\mathtt{A}(\mathtt{y}_n)}\left[\mathtt{g}(\mathtt{y}_n) - \rho\mathtt{T}\mathtt{y}_n\right] - \mathcal{J}_{\mathtt{A}(\mathtt{y}_n)}\left[\mathtt{g}(\mu) - \rho\mathtt{T}\mu\right]|| \\
&\quad + \alpha_n||\mathcal{J}_{\mathtt{A}(\mathtt{y}_n)}\left[\mathtt{g}(\mu) - \rho\mathtt{T}\mu\right] - \mathcal{J}_{\mathtt{A}(\mu)}\left[\mathtt{g}(\mu) - \rho\mathtt{T}\mu\right]|| \\
&\le (1-\alpha_n)||\mu_n - \mu|| + \alpha_n||\mathtt{y}_n - \mu - [\mathtt{g}(\mathtt{y}_n) - \mathtt{g}(\mu)]|| \\
&\quad + \alpha_n||\left[\mathtt{g}(\mathtt{y}_n) - \mathtt{g}(\mu)\right] - \rho\left[\mathtt{T}\mathtt{y}_n - \mathtt{T}\mu\right]|| + \alpha_n\upsilon||\mathtt{y}_n - \mu|| \\
&= (1-\alpha_n)||\mu_n - \mu|| + \alpha_n||\mathtt{y}_n - \mu - [\mathtt{g}(\mathtt{y}_n) - \mathtt{g}(\mu)]|| \\
&\quad + \alpha_n|| - (\mathtt{y}_n - \mu) + [\mathtt{g}(\mathtt{y}_n) - \mathtt{g}(\mu)] + (\mathtt{y}_n - \mu) - \rho\left[\mathtt{T}\mathtt{y}_n - \mathtt{T}\mu\right]|| \\
&\quad + \alpha_n\upsilon||\mathtt{y}_n - \mu|| \\
&\le (1-\alpha_n)||\mu_n - \mu|| + \alpha_n||\mathtt{y}_n - \mu - [\mathtt{g}(\mathtt{y}_n) - \mathtt{g}(\mu)]|| \\
&\quad + \alpha_n||\mathtt{y}_n - \mu - [\mathtt{g}(\mathtt{y}_n) - \mathtt{g}(\mu)]|| + \alpha_n||\mathtt{y}_n - \mu - \rho\left[\mathtt{T}\mathtt{y}_n - \mathtt{T}\mu\right]|| \\
&\quad + \alpha_n\upsilon||\mathtt{y}_n - \mu||.
\end{aligned} \tag{3.20}$$

From the relaxed $(\xi_1, \mathtt{r}_1)$−cocoercive and η_1−Lipschitzian definition for operator $\mathtt{T}$, we have

$$\begin{aligned}
&||\mathtt{y}_n - \mu - \rho\left[\mathtt{T}\mathtt{y}_n - \mathtt{T}\mu\right]||^2 \\
&= ||\mathtt{y}_n - \mu||^2 - 2\rho\Big\langle \mathtt{T}\mathtt{y}_n - \mathtt{T}\mu, \mathtt{y}_n - \mu\Big\rangle + \rho^2||[\mathtt{T}\mathtt{y}_n - \mathtt{T}\mu||^2 \\
&\le ||\mathtt{y}_n - \mu||^2 + 2\rho\xi_1||\mathtt{T}\mathtt{y}_n - \mathtt{T}\mu||^2 - 2\rho\mathtt{r}_1||\mathtt{y}_n - \mu||^2 + \rho^2||\mathtt{T}\mathtt{y}_n - \mathtt{T}\mu||^2 \\
&\le ||\mathtt{y}_n - \mu||^2 + 2\rho\xi_1\eta_1^2||\mathtt{y}_n - \mu||^2 - 2\rho\mathtt{r}_1||\mathtt{y}_n - \mu||^2 + \rho^2\eta_1^2||\mathtt{y}_n - \mu||^2 \\
&= \Big(1 - 2\rho(\mathtt{r}_1 - \xi_1\eta_1^2) + \rho^2\eta_1^2\Big)||\mathtt{y}_n - \mu||^2.
\end{aligned} \tag{3.21}$$

Similarly, from the relaxed $(\xi_2, \mathtt{r}_2),)$−cocoercive and $\eta_2,$−Lipschitzian definition for operators $\mathtt{g}$, respectively, we have

$$||\mathtt{y}_n - \mu - [\mathtt{g}(\mathtt{y}_n) - \mathtt{g}(\mu)]||^2 \le \Big(1 - 2(\mathtt{r}_2 - \xi_2\eta_2^2) + \eta_2^2\Big)||\mathtt{y}_n - \mu||^2, \tag{3.22}$$

From (3.20), (3.21) and (3.22), , we have

$$\begin{aligned}
&||\mu_{n+1} - \mu|| \\
&\le (1-\alpha_n)||\mu_n - \mu|| + \alpha_n\Big(\sqrt{1 - 2(\mathtt{r}_2 - \xi_2\eta_2^2)} \\
&\quad + \sqrt{1 - 2\rho(\mathtt{r}_1 - \xi_1\eta_1^2) + \rho^2\eta_1^2} + \upsilon\Big)||\mathtt{y}_n - \mu|| \\
&= (1-\alpha_n)||\mu_n - \mu|| + \alpha_n\Big(\mathtt{k} + \mathtt{t}(\rho)\Big)||\mathtt{y}_n - \mu|| \\
&= (1-\alpha_n)||\mu_n - \mu|| + \alpha_n\vartheta||\mathtt{y}_n - \mu||,
\end{aligned} \tag{3.23}$$

where

$$\vartheta = \mathtt{k} + \mathtt{t}(\rho) < 1, \quad \text{from condition (3.15)},$$

$$\mathtt{t}(\rho) = \sqrt{1 - 2\rho(\mathtt{r}_1 - \xi_1\eta_1^2) + \rho^2\eta_1^2}, \text{ and } \mathtt{k} \text{ is defined by (3.16).}$$

Similarly, from (3.13) and (3.18), we have

$$\begin{aligned}
\|\mathtt{y}_n - \mu\| &= \|(1-\beta_n)\mu_n + \beta_n\Big\{\mathtt{x}_n - \mathtt{g}(\mathtt{x}_n) + \mathcal{J}_{\mathtt{A}(\mathtt{x}_n)}[\mathtt{g}(\mathtt{x}_n) - \rho\mathtt{T}\mathtt{x}_n]\Big\} \\
&\quad - (1-\beta_n)\mu - \beta_n\Big\{\mu - \mathtt{g}(\mu) + \mathcal{J}_{\mathtt{A}(\mu)}\| \\
&\leq (1-\beta_n)\|\mu_n - \mu\| + \beta_n\vartheta\|\mathtt{x}_n - \mu\|.
\end{aligned} \tag{3.24}$$

In a similar way, from (3.15) and (3.24), we have

$$\begin{aligned}
\|\mathtt{x}_n - \mu\| &= \|(1-\gamma_n)\mu_n + \gamma_n\Big\{\omega_n - \mathtt{g}(\omega_n) + \mathcal{J}_{\mathtt{A}(\omega_n)}[\mathtt{g}(\omega_n) - \rho\mathtt{T}\omega_n]\Big\} \\
&\quad - (1-\gamma_n)\mu - \gamma_n\Big\{\mu - \mathtt{g}(\mu) + \mathcal{J}_{\mathtt{A}(\mu)}[\mathtt{g}(\mu) - \rho\mathtt{T}\mu]\Big\}\| \\
&\leq (1-\gamma_n)\|\mu_n - \mu\| + \gamma_n\vartheta\|\omega_n - \mu\|.
\end{aligned} \tag{3.25}$$

From (3.15), we have

$$\begin{aligned}
\|\omega_n - \mu\| &= \|\mu_n - \mu - \Theta_n(\mu_n - \mu_{n-1})\| \\
&\leq \|\mu_n - \mu\| + \Theta_n\|\mu_n - \mu_{n-1}\|.
\end{aligned} \tag{3.26}$$

From (3.25) and (3.26), we have

$$\begin{aligned}
\|\,\mathtt{x}_n - \mu\,\| &\leq (1-\gamma_n)\|\mu_n - \mu\| + \gamma_n\vartheta\Big[\|\mu_n - \mu\| + \Theta_n\|\mu_n - \mu_{n-1}\|\Big] \\
&\leq (1-\gamma_n)\|\mu_n - \mu\| + \gamma_n\vartheta\|\mu_n - \mu\| + \Theta_n\|\mu_n - \mu_{n-1}\| \\
&= \Big[1 - \gamma_n(1-\vartheta)\Big]\|\mu_n - \mu\| + \Theta_n\|\mu_n - \mu_{n-1}\| \\
&\leq \|\mu_n - \mu\| + \Theta_n\|\mu_n - \mu_{n-1}\|.
\end{aligned} \tag{3.27}$$

From (3.24) and (3.27), we have

$$\begin{aligned}
\|\,\mathtt{y}_n - \mu\,\| &\leq (1-\beta_n)\|\mu_n - \mu\| + \beta_n\vartheta\Big[\|\mu_n - \mu\| + \Theta_n\|\mu_n - \mu_{n-1}\|\Big] \\
&\leq (1-\beta_n)\|\mu_n - \mu\| + \beta_n\vartheta\|\mu_n - \mu\| + \Theta_n\|\mu_n - \mu_{n-1}\| \\
&= \Big[1 - \beta_n(1-\vartheta)\Big]\|\mu_n - \mu\| + \Theta_n\|\mu_n - \mu_{n-1}\| \\
&\leq \|\mu_n - \mu\| + \Theta_n\|\mu_n - \mu_{n-1}\|.
\end{aligned} \tag{3.28}$$

From (3.23) and (3.28), we have

$$\begin{aligned}
\|\,\mu_{n+1} - \mu\,\| &\leq (1-\alpha_n)\|\mu_n - \mu\| + \alpha_n\vartheta\Big[\|\mu_n - \mu\| + \Theta_n\|\mu_n - \mu_{n-1}\|\Big] \\
&\leq (1-\alpha_n)\|\mu_n - \mu\| + \alpha_n\vartheta\|\mu_n - \mu\| + \Theta_n\|\mu_n - \mu_{n-1}\| \\
&= \Big[1 - \alpha_n(1-\vartheta)\Big]\|\mu_n - \mu\| + \Theta_n\|\mu_n - \mu_{n-1}\|.
\end{aligned}$$

From condition (3.15), we have $\vartheta < 1$. Since $\sum_{n=1}^{\infty} \alpha_n = \infty$, setting $\sigma_n = 0$ and $\varsigma_n = \sum_{n=1}^{\infty} \Theta_n \|\,\mu_n - \mu_{n-1}\,\| < \infty$, using Lemma 2.3, we have $\mu_n \longrightarrow \mu$, $n \longrightarrow \infty$. Hence the sequence $\{\mu_n\}$ obtained from Algorithm 3.9 converges to a unique solution $\mu \in \mathtt{H}$ satisfying the inequality (2.1), the required result. □

Similarly convergence analysis for other inertial iterative methods can be estimated.

(I.) If $\mathtt{J}(\mu) = \mathtt{J}$, then the following result can be obtained from Theorem 3.3.

Theorem 3.4. *Let the following assumptions be fulfilled:*

i. *The operators* $\mathtt{T}, \mathtt{g} : \mathtt{H} \longrightarrow \mathtt{H}$ *are relaxed* $(\xi_1, \mathtt{r}_1), (\xi_2, \mathtt{r}_2).)-$*cocoercive and* $\eta_1, \eta_2, -$*Lipschitz continuous, respectively.*

ii. *The parameter* $\rho > 0$ *satisfies the condition*

$$\left|\rho - \frac{(\mathtt{r}_1 - \xi_1\eta_1^2)}{\eta_1^2}\right| < \frac{\sqrt{(\mathtt{r}_1 - \xi_1\eta_1^2)^2 - \eta_1^2\mathtt{k}(2-\mathtt{k})}}{\eta_1^2},$$

$$\mathtt{r}_1 > \xi_1\eta_1^2 + \eta_1\sqrt{\mathtt{k}(2-\mathtt{k})},\ \mathtt{k} < 1,$$

where

$$\mathtt{k} = 2\sqrt{1 - 2(\mathtt{r}_2 - \xi_2\eta_2^2) + \eta_2^2}.$$

iii. *Let* $\alpha_n, \beta_n, \gamma_n, \Theta_n \in [0,1]$, *for all* $n \geq 1$ *such that*

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} \Theta_n \parallel \mu_n - \mu_{n-1} \parallel < \infty.$$

Then, for every initial approximation μ_n, *the sequence* $\{\mu_n\}$ *obtained from*

$$\begin{aligned}
\omega_n &= \mu_n - \Theta_n(\mu_n - \mu_{n-1}), \\
\mathtt{x}_n &= (1-\gamma_n)\mu_n + \gamma_n\Big\{\omega_n - \mathtt{g}(\omega_n) + \mathcal{J}_{\mathtt{A}}[\mathtt{g}(\omega_n) - \rho\mathtt{T}\omega_n]\Big\}, \\
\mathtt{y}_n &= (1-\beta_n)\mu_n + \beta_n\Big\{\mathtt{x}_n - \mathtt{g}(\mathtt{x}_n) + \mathcal{J}_{\mathtt{A}}[\mathtt{g}(\mathtt{x}_n) - \rho\mathtt{T}\mathtt{x}_n]\Big\}, \\
\mu_{n+1} &= (1-\alpha_n)\mu_n + \alpha_n\Big\{\mathtt{y}_n - \mathtt{g}(\mathtt{y}_n) + \mathcal{J}_{\mathtt{A}}[\mathtt{g}(\mathtt{y}_n) - \rho\mathtt{T}\mathtt{y}_n]\Big\}, \quad n = 1, 2, \ldots,
\end{aligned}$$

converges to unique solution $\mu \in \mathtt{H}$ *satisfying the general variational inclusion (2.2) as* $n \longrightarrow \infty$.

(II.) If $\mathtt{J}(\mu) = \mathtt{J}$ and $\mathtt{g} = \mathtt{I}$, then we get the following result from Theorem 3.3.

Theorem 3.5. *Let the following assumptions be fulfilled:*

i. *The operator* $\mathtt{T} : \mathtt{H} \longrightarrow \mathtt{H}$ *be relaxed* $(\xi_1, \mathtt{r}_1)-$*cocoercive and* η_1-*Lipschitz continuous, respectively.*

ii. *The parameter* $\rho > 0$ *satisfies the condition*

$$\left|\rho - \frac{(\mathtt{r}_1 - \xi_1\eta_1^2)}{\eta_1^2}\right| < \frac{\sqrt{(\mathtt{r}_1 - \xi_1\eta_1^2)^2}}{\eta_1^2},$$

$$\mathtt{r}_1 > \xi_1\eta_1^2$$

iii. *Let* $\alpha_n, \beta_n, \gamma_n, \Theta_n \in [0,1]$, *for all* $n \geq 1$ *such that*

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} \Theta_n \parallel \mu_n - \mu_{n-1} \parallel < \infty.$$

Then, for every initial approximation μ_n*, the sequence* $\{\mu_n\}$ *obtained from*

$$\begin{aligned}\omega_n &= \mu_n - \Theta_n(\mu_n - \mu_{n-1}),\\ x_n &= (1-\gamma_n)\mu_n + \gamma_n\Big\{\omega_n - (\omega_n) + \mathcal{J}_A[\omega_n - \rho T\omega_n]\Big\},\\ y_n &= (1-\beta_n)\mu_n + \beta_n\Big\{x_n - (x_n) + \mathcal{J}_A[x_n - \rho T x_n]\Big\},\\ \mu_{n+1} &= (1-\alpha_n)\mu_n + \alpha_n\Big\{y_n - (y_n) + \mathcal{J}_A[y_n - \rho T y_n]\Big\}, \quad n = 1, 2, \ldots,\end{aligned}$$

converges to unique solution $\mu \in H$ *satisfying the general variational inclusion (2.4) as* $n \longrightarrow \infty$*.*

We again suggest some iterative methods by rearranging the equation (3.1). To be more precise, taking $z = g(\mu) - \rho\mathcal{T}(\mu)$ in (3.1), we have

$$g(\mu) = \mathcal{J}_{\mathcal{A}(\mu)}(z) \tag{3.29}$$

$$z = g(\mu) - \rho\mathcal{T}(\mu) \tag{3.30}$$

$$= \mathcal{J}_{\mathcal{A}(\mu)}(z) - \rho\mathcal{T}(\mu) \tag{3.31}$$

$$= \mathcal{J}_{\mathcal{A}(\mu)}(z) - \rho\mathcal{T}g^{-1}\mathcal{J}_{\mathcal{A}(\mu)}(z). \tag{3.32}$$

Combining (3.30) and (3.31), we obtain

$$g(\mu) = \rho\mathcal{T}(\mu) + \mathcal{J}_{\mathcal{A}(\mu)}(z) - \rho\mathcal{T}g^{-1}\mathcal{J}_{\mathcal{A}(\mu)}(z),$$

which can be written as

$$\mu = (1-\alpha_n)\mu + \alpha_n\{g(\mu) - \rho\mathcal{T}(\mu) - \mathcal{J}_{\mathcal{A}(\mu)}(z) + \rho\mathcal{T}g^{-1}\mathcal{J}_{\mathcal{A}(\mu)}(z)\}, \tag{3.33}$$

We use the fixed point formulation (3.33) to suggest the following predictor-corrector method for solving the problem (2.1).

Algorithm 3.11. *For given* $\mu_0, \mu_1 \in H$*, compute* μ_{n+1} *by the recurrence relation*

$$\begin{aligned}g(\omega_n) &= \mathcal{J}_{\mathcal{A}(\mu)}[\mu_n - \rho\mathcal{T}(\mu_n)]\\ \mu_{n+1} &= (1-\alpha_n)\mu_n + \alpha_n\{g(\omega_n) - \rho\mathcal{T}(\omega_n) - g(\nu_n) + \rho\mathcal{T}\nu_n\},\end{aligned}$$

where $\alpha_n \in [0,1]$ *is a constant.*

The Algorithm 3.11 is known as the forward-backward iterative method.

Now, we again use fixed point formulation (3.33)to suggest the following inertial iterative methods for solving the general quasi variational inclusion (2.1):

Algorithm 3.12. *For given* $\mu_0, \mu_1 \in H$*, compute* μ_{n+1} *by the recurrence relation*

$$\begin{aligned}g(\omega_n) &= (1-\beta_n)\mu_n + \beta_n\mu_{n-1}\\ \mu_{n+1} &= (1-\alpha_n)\mu_n + \alpha_n\{g(\mu_n) - \rho\mathcal{T}(\mu_n) - g(\omega_n) + \rho\mathcal{T}\omega_n\},\end{aligned}$$

where $\alpha_n \in [0,1]$ *is a constant.*

Algorithm 3.12 is called the forward-backward inertial iterative method for solving the general quasi variational inclusions and this method does not involve any resolvent operator.

From (3.30 and (3.32), we have

$$g(\mu) = \mathcal{J}_{\mathcal{A}(\mu)}[\mathcal{J}_{\mathcal{A}(\mu)}[g(\mu) - \rho\mathcal{T}(\mu)] - \rho\mathcal{T}(\mu),$$

which can be written as

$$\mu = (1-\alpha_n)\mu + \alpha_n\{g(\mu) - \mathcal{J}_{\mathcal{A}(\mu)}[\mathcal{J}_{\mathcal{A}(\mu)}[g(\mu) - \rho\mathcal{T}(\mu)] + \rho\mathcal{T}(\mu)\}, \tag{3.34}$$

where $\alpha_n \in [0,1]$ is a constant.
This fixed point formulation is used to propose a new iterative method for solving the general quasi variational inequalities (2.1).

Algorithm 3.13. *For given $\mu_0, \mu_1 \in \mathtt{H}$, compute μ_{n+1} by the recurrence relation*

$$\mu_{n+1} = (1-\alpha_n)\mu_n + \alpha_n\{g(\mu_n) - \mathcal{J}_{\mathcal{A}(\mu_n)}[\mathcal{J}_{\mathcal{A}(\mu_n)}[g(\mu_n) - \rho\mathcal{T}(\mu_n)] + \rho\mathcal{T}(\mu_n)\},$$

where $\alpha_n \in [0,1]$ is a constant.

Algorithm 3.13 is called the forward-backward explicit iterative method for solving the general quasi variational inclusions.

Conclusion: In this paper, several new inertial projection methods have been suggested and analyzed for solving general quasi variational inclusions. We have proved that the general quasi variational inclusions are equivalent to the implicit fixed point problems. This alternative formulation is used to discuss the existence of a solution of the general quasi variational inclusions. A wide class of three step inertial type iterative methods for solving general quasi variational inclusions are suggested and analyzed. Several important special cases are discussed. Convergence analysis of these proposed inertial resolvent methods is investigated. It is worth mentioning that Algorithm 3.12 does not involve the calculation of the resolvent operator. We would like to point out that very few numerical examples are available for classical quasi variational inclusions due to their complex nature. In spite of these activities, further efforts are needed to develop numerical implementable methods. It is an interesting problem to compare the efficiency of the proposed methods with other known methods. We expect that the ideas and techniques of this paper will motivate and inspire the interested readers to explore its applications in various fields.

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