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Product-Type Operators Acting Between Dirichlet and Zygmund-Type Spaces

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Abstract. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. By $H(\mathbb{D})$, denote the space of all holomorphic functions on $\mathbb{D}$. For an analytic self map φ on $\mathbb{D}$ and $u, v \in H(\mathbb{D})$, we have a product type operator $T_{u,v,\varphi}$ defined by

$$T_{u,v,\varphi} f(z) = u(z)f(\varphi(z)) + v(z)f'(\varphi(z)), \quad f \in H(\mathbb{D}),\ z \in \mathbb{D},$$

This operator is basically a combination of three other operators namely composition operator, multiplication operator and differentiation operator. We study the boundedness and compactness of this operator from Dirichlet-type spaces to Zygmund-type spaces.

1. Introduction and Preliminaries

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. By $H(\mathbb{D})$ and $S(\mathbb{D})$, respectively, we denote the class of all analytic functions on $\mathbb{D}$ and the space of all analytic self-maps of $\mathbb{D}$. Let H^∞ be the space of all bounded holomorphic functions on $\mathbb{D}$. For $\beta > 0$, the *weighted Zygmund space* $\mathcal{Z}_\beta$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_\beta = \sup_{z\in\mathbb{D}} (1 - |z|^2)^\beta |f''(z)| < \infty.$$

If $\beta = 1$, we get the *Zygmund space* which is denoted by $\mathcal{Z}$. A continuous function $\omega : \mathbb{D} \to (0, \infty)$ is termed as a *weight*. Weight ω is called to be a *standard weight*, if for $z \in \mathbb{D}$, we have $\lim_{|z|\to 1^-} \omega(z) = 0$. Further, for $z \in \mathbb{D}$, we call a weight ω to be *radial*, if $\omega(z) = \omega(|z|)$. For a weight ω the *Zygmund-type spaces* $\mathcal{Z}_\omega$ is the class of all $f \in H(\mathbb{D})$ for which

$$\sup_{z\in\mathbb{D}} \omega(z)|f''(z)| < \infty.$$

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The space $\mathcal{Z}_\omega$ forms a Banach space with the following norm

$$\|f\|_{\mathcal{Z}_\omega} = |f(0)| + |f'(0)| + \|f\|_\omega.$$

To know more about these spaces and operators acting on them one may refer [7, 11, 13–15, 17–21, 23, 30–32] and the related references therein.

The *Dirichlet space* is the class of all those analytic functions on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty,$$

where $dA(z)$ is the normalized Lebesgue area measure defined on $\mathbb{D}$. The space forms a Hilbert space under the following norm

$$\|f\|^2_{\mathfrak{D}} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z).$$

Let $K : [0,\infty) \to [0,\infty)$ be a function with the property that it is right continuous and increasing. These functions have been studied in various papers, see, for example [5, 28, 29]. By treating function K as a weight, we can obtain the space $\mathfrak{D}_K$ termed as the *Dirichlet-type space* which consists of all those analytic functions on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f'(z)|^2 K\Big(1-|z|^2\Big) dA(z) < \infty.$$

Further, we can check that the space $\mathfrak{D}_K$ forms a Banach space under the norm $\|\cdot\|_{\mathfrak{D}_K}$ given as follows

$$\|f\|^2_{\mathfrak{D}_K} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 K\Big(1-|z|^2\Big) dA(z).$$

For $K(t) = t^p$, where $0 \le p < \infty$, the space $\mathfrak{D}_K$ gives the usual Dirichlet type space $\mathfrak{D}_p$. Further, by taking $p = 0$, we obtain the classical Dirichlet space $\mathfrak{D}$ and for $p = 1$, we gain the Hardy space H^2. These spaces have been studied widely in various papers. For details one can see [1–3, 5, 7, 16, 22, 25, 27] and the references therein.

Let φ be an analytic self-map of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. Then, the *composition, multiplication,* and *weighted composition operator* on $H(\mathbb{D})$ are respectively defined as

$$\begin{aligned} &C_\varphi f(z) = (f\circ\varphi)(z) = f(\varphi(z)),\\ &M_\psi f(z) = \psi(z) f(z)\\ \text{and}\quad &\mathcal{W}_{\psi,\varphi} f(z) = (M_\psi C_\varphi) f(z) = \psi(z) f(\varphi(z)), \quad z\in\mathbb{D},\ f\in H(\mathbb{D}).\end{aligned}$$

$\mathcal{W}_{\psi,\varphi}$ is a product-type operator as $\mathcal{W}_{\psi,\varphi} = M_\psi C_\varphi$. More results on weighted composition operators on class of holomorphic functions can be found in [6, 8, 9, 11] and the references therein. Further, for $f \in H(\mathbb{D})$, the differentiation operator denoted by D is defined as $Df = f'$. The product-type operators $\mathcal{W}_{\psi,\varphi}D$ and $D\mathcal{W}_{\psi,\varphi}$ were respectively, considered in [12] and [13]. For $u, v \in H(\mathbb{D})$, the composition operator together with multiplication operator and differentiation operator give rise to a new product-type operator denoted by $T_{u,v,\varphi}$ and defined by

$$T_{u,v,\varphi} f(z) = u(z) f(\varphi(z)) + v(z) f'(\varphi(z)), \quad f\in H(\mathbb{D}),\ z\in\mathbb{D}.$$

This operator is basically a product of composition, multiplication and differentiation operators. Clearly, by fixing u, v in $T_{u,v,\varphi}$, all possible products of above defined operators can be obtained. In particular, by setting $v(z) \equiv 0$ and $u \equiv \psi$, the operator $T_{u,v,\varphi}$ get reduced to weighted composition operator $\mathcal{W}_{\psi,\varphi} = \psi\cdot f(\varphi)$.

Similarly, for $u(z) \equiv 0$ and $v \equiv \psi$, the operator $T_{u,v,\varphi}$ get reduced to weighted differentiation composition operator $\mathcal{W}_{\psi,\varphi}D = \psi \cdot f'(\varphi)$. For more information about these operators (see [4, 10, 14, 19–21, 24, 26, 30]) and references therein. We call a linear operator to be bounded if the bounded sets map to bounded sets. Further, a linear operator is called to be compact if the images of bounded sets are such sets whose closure is compact. In [8], we studied the boundedness as well as compactness of operator $\mathcal{W}_{\psi,\varphi}$ acting from $\mathfrak{D}_K$ to Bloch and Bers-type spaces, and compute their essential norm in [9]. Continuing our study, here we have considered the operator $T_{u,v,\varphi}$ acting between $\mathfrak{D}_K$ and Zygmund-type spaces. This paper is represented in a systematic manner. Introduction and literature part is kept in Section 1 and some auxiliary results which are used to derive the main results are considered in Section 2. In Section 3, we investigate the boundedness of operator $T_{u,v,\varphi}$ from $\mathfrak{D}_K$ to $\mathcal{Z}_\omega$ and in Section 4, the compactness of operators $T_{u,v,\varphi}$ from $\mathfrak{D}_K$ to $\mathcal{Z}_\omega$ is given. Throughout the paper, for any two positive quantities a and b, the notation $a \lesssim b$ means that $a \leq Cb$, for some constant $C > 0$. The constant C may differ at each occurrence. Further, if both $a \gtrsim b$ and $b \gtrsim a$ hold, then we simply write $a \asymp b$.

2. Auxiliary Results

To arrive at the main results we use certain lemmas. The first lemma can be easily obtained from [5].

Lemma 2.1. *Let K be a weight function. Then for any $w, z \in \mathbb{D}$ and $\rho > 0$, we have*

$$f_w(z) = \frac{(1-|w|^2)^{\rho/2}}{\sqrt{K(1-|w|^2)}(1-z\overline{w})^{1+\rho/2}}$$

is in $\mathfrak{D}_K$. Moreover,

$$\sup_{w\in\mathbb{D}} \|f_w\|_{\mathfrak{D}_K} \lesssim 1,$$

and f_w converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|w| \to 1^-$.

Lemma 2.2. *Let K be a weight function. Then for every $f \in \mathfrak{D}_K$ we have*

$$|f(z)| \lesssim \frac{\|f\|_{\mathfrak{D}_K}}{\sqrt{K(1-|z|^2)}(1-|z|^2)}, \quad z \in \mathbb{D}.$$

and for a positive integer n, we have

$$|f^{(n)}(z)| \lesssim \frac{\|f\|_{\mathfrak{D}_K}}{\sqrt{K(1-|z|^2)}(1-|z|^2)^{n+1}}, \quad z \in \mathbb{D}.$$

The following criterion characterize the compactness. Its proof can be easily follows from Proposition 3.11 in [7].

Lemma 2.3. *Let ω be the standard weight and the operator $T_{u,v,\varphi} : \mathfrak{D}_K \to \mathcal{Z}_\omega$ is bounded. Then $T_{u,v,\varphi} : \mathfrak{D}_K \to \mathcal{Z}_\omega$ is compact if and only if for any bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathfrak{D}_K$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, we have*

$$\lim_{n\to\infty} \|T_{u,v,\varphi} f_n\|_{\mathcal{Z}_\omega} = 0.$$

3. Boundedness of the operator $T_{u,v,\varphi}$ from $\mathfrak{D}_K$ spaces to Zygmund type spaces

Theorem 3.1. *Let ω and K be two weight functions, $u, v \in H(\mathbb{D})$ and φ be an analytic self-map on $\mathbb{D}$. Then, operator $T_{u,v,\varphi} : \mathfrak{D}_K \to \mathcal{Z}_\omega$ is bounded if and only if the functions u, v and φ satisfy the following conditions :*

$$(i)\ P_1 = \sup_{z\in\mathbb{D}} \frac{\omega(z)|u''(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)} < \infty,$$

$$(ii)\ P_2 = \sup_{z\in\mathbb{D}} \frac{\omega(z)|2u'(z)\varphi'(z)+u(z)\varphi''(z)+v''(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^2} < \infty,$$

$$(iii)\ P_3 = \sup_{z\in\mathbb{D}} \frac{\omega(z)|u(z)(\varphi'(z))^2+2v'(z)\varphi'(z)+v(z)\varphi''(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^3} < \infty \ \ \textit{and}$$

$$(iv)\ P_4 = \sup_{z\in\mathbb{D}} \frac{\omega(z)|v(z)||\varphi'(z)|^2}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^4} < \infty.$$

Further,

$$P_1+P_2+P_3+P_4 \lesssim \|T_{u,v,\varphi}\|_{\mathfrak{D}_K\to\mathcal{Z}_\omega} \lesssim P+P_1+P_2+P_3+P_4, \tag{3.1}$$

$$\textit{where}\quad P = \frac{|u(0)|+|u'(0)|}{\sqrt{K(1-|\varphi(0)|^2)}(1-|\varphi(0)|^2)} + \frac{|v(0)|+|u(0)\varphi'(0)+v'(0)|}{\sqrt{K(1-|\varphi(0)|^2)}(1-|\varphi(0)|^2)^2} + \frac{|v(0)||\varphi'(0)|}{\sqrt{K(1-|\varphi(0)|^2)}(1-|\varphi(0)|^2)^3}.$$

Proof. First suppose that conditions (i), (ii), (iii) and (iv) hold. Since

$$\left(T_{u,v,\varphi}f\right)(z) = u(z)f(\varphi(z)) + v(z)f'(\varphi(z)),$$

this implies

$$\left(T_{u,v,\varphi}f\right)'(z) = u'(z)f(\varphi(z)) + \left(u(z)\varphi'(z)+v'(z)\right)f'(\varphi(z)) + v(z)\varphi'(z)f''(\varphi(z))$$

and

$$\begin{aligned}\left(T_{u,v,\varphi}f\right)''(z) = {} & u''(z)f(\varphi(z)) + \left(u(z)\varphi''(z)+2u'(z)\varphi'(z)+v''(z)\right)f'(\varphi(z)) \\ & + \left(u(z)(\varphi'(z))^2+2v'(z)\varphi'(z)+v(z)\varphi''(z)\right)f''(\varphi(z)) \\ & + v(z)(\varphi'(z))^2 f'''(\varphi(z)).\end{aligned}$$

Thus, for $z = 0$, we have

$$\left(T_{u,v,\varphi}f\right)(0) = u(0)f(\varphi(0)) + v(0)f'(\varphi(0))$$

and

$$\left(T_{u,v,\varphi}f\right)'(0) = u'(0)f(\varphi(0)) + \left(u(0)\varphi'(0)+v'(0)\right)f'(\varphi(0)) + v(0)\varphi'(0)f''(\varphi(0)).$$

Now for $f \in \mathfrak{D}_K$, an arbitrary $z \in \mathbb{D}$ and by Lemma 2.1, we get

$$\begin{aligned}
&\|T_{u,v,\varphi} f\|_{\mathfrak{D}_K \to \mathcal{Z}_\omega} \\
&= |(T_{u,v,\varphi} f)(0)| + |(T_{u,v,\varphi} f)'(0)| + \sup_{z\in\mathbb{D}} \omega(z)|(T_{u,v,\varphi} f)''(z)| \\
&\leq \Big(\Big(|u(0)| + |u'(0)|\Big)|f(\varphi(0))| + \Big(|v(0)| + |u(0)\varphi'(0) + v'(0)|\Big)|f'(\varphi(0))| \\
&\quad + \Big(|v(0)||\varphi'(0)|\Big)|f''(\varphi(0))|\Big) + \sup_{z\in\mathbb{D}} \omega(z)|u''(z)||f(\varphi(z))| \\
&\quad + \sup_{z\in\mathbb{D}} \omega(z)\Big|2u'(z)\varphi'(z) + u(z)\varphi''(z) + v''(z)\Big||f'(\varphi(z))| \\
&\quad + \sup_{z\in\mathbb{D}} \omega(z)\Big|u(z)(\varphi'(z))^2 + 2v'(z)\varphi'(z) + v(z)\varphi''(z)\Big||f''(\varphi(z))| \\
&\quad + \sup_{z\in\mathbb{D}} \omega(z)|v(z)||\varphi'(z)|^2|f'''(\varphi(z))| \\
&\lesssim \Bigg(\frac{|u(0)| + |u'(0)|}{\sqrt{K(1-|\varphi(0)|^2)}(1-|\varphi(0)|^2)} + \frac{|v(0)| + |u(0)\varphi'(0) + v'(0)|}{\sqrt{K(1-|\varphi(0)|^2)}(1-|\varphi(0)|^2)^2} \\
&\quad + \frac{|v(0)||\varphi'(0)|}{\sqrt{K(1-|\varphi(0)|^2)}(1-|\varphi(0)|^2)^3} + \sup_{z\in\mathbb{D}} \frac{\omega(z)|u''(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)} \\
&\quad + \sup_{z\in\mathbb{D}} \frac{\omega(z)|2u'(z)\varphi'(z) + u(z)\varphi''(z) + v''(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^2} \\
&\quad + \sup_{z\in\mathbb{D}} \frac{\omega(z)|u(z)(\varphi'(z))^2 + 2v'(z)\varphi'(z) + v(z)\varphi''(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^3} \\
&\quad + \sup_{z\in\mathbb{D}} \frac{\omega(z)|v(z)||\varphi'(z)|^2}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^4}\Bigg)\|f\|_{\mathfrak{D}_K} \\
&\lesssim \Big(P + P_1 + P_2 + P_3 + P_4\Big)\|f\|_{\mathfrak{D}_K}. \qquad (3.2)
\end{aligned}$$

From (3.2), we conclude that the operator $T_{u,v,\varphi} : \mathfrak{D}_K \to \mathcal{Z}_\omega$ is bounded and

$$\|T_{u,v,\varphi}\|_{\mathfrak{D}_K \to \mathcal{Z}_\omega} \lesssim P + P_1 + P_2 + P_3 + P_4. \qquad (3.3)$$

Conversely, assume that $T_{u,v,\varphi} : \mathfrak{D}_K \to \mathcal{Z}_\omega$ is bounded. At first we shall prove that $P_1 < \infty$. For this take a function $p_0(z) \equiv 1 \in \mathfrak{D}_K$. Since the operator $T_{u,v,\varphi} : \mathfrak{D}_K \to \mathcal{Z}_\omega$ is bounded, we get

$$\sup_{z\in\mathbb{D}} \omega(z)|u''(z)| \lesssim \Big\|T_{u,v,\varphi}\Big\|_{\mathfrak{D}_K \to \mathcal{Z}_\omega}. \qquad (3.4)$$

For $w \in \mathbb{D}$, set

$$\begin{aligned}
f_w(z) &= a_1 \frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}}}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+1}} + b_1 \frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+1}}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+2}} \\
&\quad + c_1 \frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+2}}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+3}} + d_1 \frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+3}}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+4}},
\end{aligned}$$

where

$$\begin{aligned}
&a_1 = \Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big)\Big(\frac{\rho}{2}+4\Big), \quad b_1 = -3\Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+3\Big)\Big(\frac{\rho}{2}+4\Big), \\
&c_1 = 3\Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+4\Big), \quad \text{and} \quad d_1 = -\Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big).
\end{aligned}$$

Using Lemma 2.1, it can be seen that for every $w \in \mathbb{D}$, $f_w \in \mathfrak{D}_K$ and $\sup_{w\in\mathbb{D}} \|f_w\|_{\mathfrak{D}_K} \lesssim 1$. Further, we can check that

$$\begin{aligned} f_w'(z) = a_1\Big(\frac{\rho}{2}+1\Big)&\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}}(\overline{\varphi(w)})}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+2}} \\ &+ b_1\Big(\frac{\rho}{2}+2\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+1}(\overline{\varphi(w)})}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+3}} \\ &+ c_1\Big(\frac{\rho}{2}+3\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+2}(\overline{\varphi(w)})}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+4}} \\ &+ d_1\Big(\frac{\rho}{2}+4\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+3}(\overline{\varphi(w)})}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+5}}, \end{aligned}$$

$$\begin{aligned} f_w''(z) = a_1\Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)&\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}}(\overline{\varphi(w)})^2}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+3}} \\ &+ b_1\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+1}(\overline{\varphi(w)})^2}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+4}} \\ &+ c_1\Big(\frac{\rho}{2}+3\Big)\Big(\frac{\rho}{2}+4\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+2}(\overline{\varphi(w)})^2}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+5}} \\ &+ d_1\Big(\frac{\rho}{2}+4\Big)\Big(\frac{\rho}{2}+5\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+3}(\overline{\varphi(w)})^2}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+6}}, \end{aligned}$$

and

$$\begin{aligned} f_w'''(z) = a_1\Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big)&\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}}(\overline{\varphi(w)})^3}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+4}} \\ &+ b_1\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big)\Big(\frac{\rho}{2}+4\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+1}(\overline{\varphi(w)})^3}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+5}} \\ &+ c_1\Big(\frac{\rho}{2}+3\Big)\Big(\frac{\rho}{2}+4\Big)\Big(\frac{\rho}{2}+5\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+2}(\overline{\varphi(w)})^3}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+6}} \\ &+ d_1\Big(\frac{\rho}{2}+4\Big)\Big(\frac{\rho}{2}+5\Big)\Big(\frac{\rho}{2}+6\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+3}(\overline{\varphi(w)})^3}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+7}}. \end{aligned}$$

Hence,

$$f_w'(\varphi(w)) = f_w''(\varphi(w)) = f_w'''(\varphi(w)) = 0 \quad \text{and} \tag{3.5}$$

$$f_w(\varphi(w)) = 6\frac{1}{\sqrt{K(1-|\varphi(w)|^2)}(1-|\varphi(w)|^2)}. \tag{3.6}$$

Since the operator $T_{u,v,\varphi} : \mathfrak{D}_K \to \mathcal{Z}_\omega$ is bounded, thus we get

$$\begin{aligned}\|T_{u,v,\varphi}\|_{\mathfrak{D}_K\to\mathcal{Z}_\omega} &\gtrsim \|T_{u,v,\varphi} f_w\|_{\mathcal{Z}_\omega}\\ &\gtrsim \omega(w)\Big|u''(w) f_w(\varphi(w))\\ &\quad + \Big(u(w)\varphi''(w) + 2u'(w)\varphi'(w) + v''(w)\Big) f'_w(\varphi(w))\\ &\quad + \Big(u(w)(\varphi'(w))^2 + 2v'(w)\varphi'(w) + v(w)\varphi''(w)\Big) f''_w(\varphi(w))\\ &\quad + v(w)(\varphi'(w))^2 f'''_w(\varphi(w))\Big|\\ &= 6\frac{\omega(w)\Big|u''(w)\Big|}{\sqrt{K(1-|\varphi(w)|^2)}(1-|\varphi(w)|^2)}. \end{aligned} \tag{3.7}$$

Thus, inequalities (3.4) and (3.7) implies that

$$\sup_{w\in\mathbb{D}} \frac{\omega(w)|u''(w)|}{\sqrt{K(1-|\varphi(w)|^2)}(1-|\varphi(w)|^2)} \lesssim \|T_{u,v,\varphi}\|_{\mathfrak{D}_K\to\mathcal{Z}_\omega}.$$

This implies that (*i*) holds and

$$P_1 \lesssim \|T_{u,v,\varphi}\|_{\mathfrak{D}_K\to\mathcal{Z}_\omega}. \tag{3.8}$$

Next, for $p_1(z) = z \in \mathfrak{D}_K$, we obtain

$$\sup_{z\in\mathbb{D}} \omega(z)|u''(z)\varphi(z) + 2u'(z)\varphi'(z) + u(z)\varphi''(z) + v''(z)| \lesssim \left\|T_{u,v,\varphi}\right\|_{\mathfrak{D}_K\to\mathcal{Z}_\omega}. \tag{3.9}$$

Using (3.4) with the fact that $|\varphi(z)| < 1$, from (3.9) we get

$$\sup_{z\in\mathbb{D}} \omega(z)|2u'(z)\varphi'(z) + u(z)\varphi''(z) + v''(z)| \lesssim \left\|T_{u,v,\varphi}\right\|_{\mathfrak{D}_K\to\mathcal{Z}_\omega}. \tag{3.10}$$

For $w \in \mathbb{D}$, define a family of functions

$$\begin{aligned} g_w(z) &= a_2 \frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}}}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+1}} + b_2 \frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+1}}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+2}}\\ &\quad + c_2 \frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+2}}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+3}} + d_2 \frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+3}}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+4}}, \end{aligned}$$

where

$$\begin{aligned} a_2 &= \Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big)\Big(\frac{\rho}{2}+3\Big)\Big(\frac{\rho}{2}+4\Big),\\ b_2 &= -\Big(3\Big(\frac{\rho}{2}\Big)+7\Big)\Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big)\Big(\frac{\rho}{2}+4\Big),\\ c_2 &= \Big(3\Big(\frac{\rho}{2}\Big)+11\Big)\Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big),\\ \text{and}\quad d_2 &= -\Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big)\Big(\frac{\rho}{2}+3\Big). \end{aligned}$$

Using Lemma 2.1, it can be seen that for every $w \in \mathbb{D}$, $g_w \in \mathfrak{D}_K$ and $\sup_{w\in\mathbb{D}} \|g_w\|_{\mathfrak{D}_K} \lesssim 1$. Further, we can check that

$$\begin{aligned} g_w'(z) &= a_2\Big(\frac{\rho}{2}+1\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}}(\overline{\varphi(w)})}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+2}} \\ &+ b_2\Big(\frac{\rho}{2}+2\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+1}(\overline{\varphi(w)})}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+3}} \\ &+ c_2\Big(\frac{\rho}{2}+3\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+2}(\overline{\varphi(w)})}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+4}} \\ &+ d_2\Big(\frac{\rho}{2}+4\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+3}(\overline{\varphi(w)})}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+5}}, \end{aligned}$$

$$\begin{aligned} g_w''(z) &= a_2\Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}}(\overline{\varphi(w)})^2}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+3}} \\ &+ b_2\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+1}(\overline{\varphi(w)})^2}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+4}} \\ &+ c_2\Big(\frac{\rho}{2}+3\Big)\Big(\frac{\rho}{2}+4\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+2}(\overline{\varphi(w)})^2}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+5}} \\ &+ d_2\Big(\frac{\rho}{2}+4\Big)\Big(\frac{\rho}{2}+5\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+3}(\overline{\varphi(w)})^2}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+6}}, \end{aligned}$$

and

$$\begin{aligned} g_w'''(z) &= a_2\Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}}(\overline{\varphi(w)})^3}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+4}} \\ &+ b_2\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big)\Big(\frac{\rho}{2}+4\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+1}(\overline{\varphi(w)})^3}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+5}} \\ &+ c_2\Big(\frac{\rho}{2}+3\Big)\Big(\frac{\rho}{2}+4\Big)\Big(\frac{\rho}{2}+5\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+2}(\overline{\varphi(w)})^3}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+6}} \\ &+ d_2\Big(\frac{\rho}{2}+4\Big)\Big(\frac{\rho}{2}+5\Big)\Big(\frac{\rho}{2}+6\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+3}(\overline{\varphi(w)})^3}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+7}}. \end{aligned}$$

Hence,

$$\begin{aligned} & g_w(\varphi(w)) = g_w''(\varphi(w)) = g_w'''(\varphi(w)) = 0 \\ \text{and}\quad & g_w'(\varphi(w)) = -2\Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big)\frac{(\overline{\varphi(w)})}{\sqrt{K(1-|\varphi(w)|^2)}(1-|\varphi(w)|^2)^2}. \end{aligned}$$

Since the operator $T_{u,v,\varphi} : \mathfrak{D}_K \to \mathcal{Z}_\omega$ is bounded, thus we get

$$\begin{aligned}
&\|T_{u,v,\varphi}\|_{\mathfrak{D}_K\to\mathcal{Z}_\omega}\\
&\gtrsim \|T_{u,v,\varphi}g_w\|_{\mathcal{Z}_\omega}\\
&\gtrsim \omega(w)\Big|u''(w)g_w(\varphi(w))\\
&\quad + \Big(u(w)\varphi''(w) + 2u'(w)\varphi'(w) + v''(w)\Big)g_w'(\varphi(w))\\
&\quad + \Big(u(w)(\varphi'(w))^2 + 2v'(w)\varphi'(w) + v(w)\varphi''(w)\Big)g_w''(\varphi(w))\\
&\quad + v(w)(\varphi'(w))^2 g_w'''(\varphi(w))\Big|\\
&= 2\Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big)\frac{\omega(w)\big|u(w)\varphi''(w) + 2u'(w)\varphi'(w) + v''(w)\big||\varphi(w)|}{\sqrt{K(1-|\varphi(w)|^2)}(1-|\varphi(w)|^2)^2}. \qquad (3.11)
\end{aligned}$$

For fixed $\eta \in (0,1)$, inequalities (3.10) and (3.11) implies that

$$\begin{aligned}
&\sup_{w\in\mathbb{D}} \frac{\omega(w)|u(w)\varphi''(w) + 2u'(w)\varphi'(w) + v''(w)|}{\sqrt{K(1-|\varphi(w)|^2)}(1-|\varphi(w)|^2)^2}\\
&\leq \sup_{|\varphi(w)|\leq\eta} \frac{\omega(w)|u(w)\varphi''(w) + 2u'(w)\varphi'(w) + v''(w)|}{\sqrt{K(1-|\varphi(w)|^2)}(1-|\varphi(w)|^2)^2}\\
&\quad + \sup_{|\varphi(w)|>\eta} \frac{\omega(w)|u(w)\varphi''(w) + 2u'(w)\varphi'(w) + v''(w)|}{\sqrt{K(1-|\varphi(w)|^2)}(1-|\varphi(w)|^2)^2}\\
&\leq \frac{1}{(1-\eta^2)^2}\sup_{|\varphi(w)|\leq\eta} \frac{\omega(w)|u(w)\varphi''(w) + 2u'(w)\varphi'(w) + v''(w)|}{\sqrt{K(1-\eta^2)}}\\
&\quad + \frac{1}{\eta}\sup_{|\varphi(w)|>\eta} \frac{\omega(w)|u(w)\varphi''(w) + 2u'(w)\varphi'(w) + v''(w)||\varphi(w)|}{\sqrt{K(1-|\varphi(w)|^2)}(1-|\varphi(w)|^2)^2}\\
&\lesssim \Big(\frac{1}{(1-\eta^2)^2} + \frac{1}{\eta}\Big)\|T_{u,v,\varphi}\|_{\mathfrak{D}_K\to\mathcal{Z}_\omega}.
\end{aligned}$$

This implies that (ii) holds and

$$P_2 \lesssim \|T_{u,v,\varphi}\|_{\mathfrak{D}_K\to\mathcal{Z}_\omega}. \qquad (3.12)$$

Taking $p_2(z) = \frac{z^2}{2!} \in \mathfrak{D}_K$, we get

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} \omega(z)\Big|\frac{1}{2}u''(z)(\varphi(z))^2 + 2u'(z)\varphi(z)\varphi'(z) + u(z)(\varphi'(z))^2\\
&\quad + u(z)\varphi(z)\varphi''(z) + v''(z)\varphi(z) + 2v'(z)\varphi'(z) + v(z)\varphi''(z)\Big|\\
&\lesssim \big\|T_{u,v,\varphi}\big\|_{\mathfrak{D}_K\to\mathcal{Z}_\omega}, \qquad (3.13)
\end{aligned}$$

which together with (3.4), (3.10) and the fact that $|\varphi(z)| < 1$ implies that

$$\sup_{z\in\mathbb{D}} \omega(z)|u(w)(\varphi'(w))^2 + 2v'(w)\varphi'(z) + v(w)\varphi''(w)| \lesssim \big\|T_{u,v,\varphi}\big\|_{\mathfrak{D}_K\to\mathcal{Z}_\omega}. \qquad (3.14)$$

For $w \in \mathbb{D}$, set

$$\begin{aligned}
h_w(z) = {}& a_3\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}}}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+1}} + b_3\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+1}}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+2}}\\
&+ c_3\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+2}}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+3}} + d_3\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+3}}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+4}},
\end{aligned}$$

where

$$\begin{aligned} a_3 &= \Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big)\Big(\frac{\rho}{2}+4\Big), \\ b_3 &= -\Big(3\Big(\frac{\rho}{2}\Big)+11\Big)\Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big), \\ c_3 &= \Big(3\Big(\frac{\rho}{2}\Big)+10\Big)\Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big), \\ \text{and} \quad d_3 &= -\Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big)\Big(\frac{\rho}{2}+3\Big). \end{aligned}$$

Using Lemma 2.1, it can be seen that for every $w \in \mathbb{D}$, $h_w \in \mathfrak{D}_K$ and $\sup_{w\in\mathbb{D}} \|h_w\|_{\mathfrak{D}_K} \lesssim 1$. Further, we can check that

$$\begin{aligned} h_w'(z) &= a_3\Big(\frac{\rho}{2}+1\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}}(\overline{\varphi(w)})}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+2}} \\ &+ b_3\Big(\frac{\rho}{2}+2\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+1}(\overline{\varphi(w)})}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+3}} \\ &+ c_3\Big(\frac{\rho}{2}+3\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+2}(\overline{\varphi(w)})}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+4}} \\ &+ d_3\Big(\frac{\rho}{2}+4\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+3}(\overline{\varphi(w)})}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+5}}, \end{aligned}$$

$$\begin{aligned} h_w''(z) &= a_3\Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}}(\overline{\varphi(w)})^2}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+3}} \\ &+ b_3\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+1}(\overline{\varphi(w)})^2}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+4}} \\ &+ c_3\Big(\frac{\rho}{2}+3\Big)\Big(\frac{\rho}{2}+4\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+2}(\overline{\varphi(w)})^2}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+5}} \\ &+ d_3\Big(\frac{\rho}{2}+4\Big)\Big(\frac{\rho}{2}+5\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+3}(\overline{\varphi(w)})^2}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+6}}, \end{aligned}$$

and

$$\begin{aligned} h_w'''(z) &= a_3\Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}}(\overline{\varphi(w)})^3}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+4}} \\ &+ b_3\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big)\Big(\frac{\rho}{2}+4\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+1}(\overline{\varphi(w)})^3}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+5}} \\ &+ c_3\Big(\frac{\rho}{2}+3\Big)\Big(\frac{\rho}{2}+4\Big)\Big(\frac{\rho}{2}+5\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+2}(\overline{\varphi(w)})^3}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+6}} \\ &+ d_3\Big(\frac{\rho}{2}+4\Big)\Big(\frac{\rho}{2}+5\Big)\Big(\frac{\rho}{2}+6\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+3}(\overline{\varphi(w)})^3}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+7}}. \end{aligned}$$

Hence,

$$h_w(\varphi(w)) = h'_w(\varphi(w)) = h'''_w(\varphi(w)) = 0$$

$$\text{and} \quad h''_w(\varphi(w)) = 2\Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big)\frac{(\overline{\varphi(w)})^2}{\sqrt{K(1-|\varphi(w)|^2)}(1-|\varphi(w)|^2)^3}.$$

Since the operator $T_{u,v,\varphi}:\mathfrak{D}_K\to\mathcal{Z}_\omega$ is bounded. Thus, we get

$$\begin{aligned}
&\|T_{u,v,\varphi}\|_{\mathfrak{D}_K\to\mathcal{Z}_\omega}\\
&\gtrsim \|T_{u,v,\varphi}h_w\|_{\mathcal{Z}_\omega}\\
&\gtrsim \omega(w)\Big|u''(w)h_w(\varphi(w))\\
&\quad + \Big(u(w)\varphi''(w)+2u'(w)\varphi'(w)+v''(w)\Big)h'_w(\varphi(w))\\
&\quad + \Big(u(w)(\varphi'(w))^2+2v'(w)\varphi'(w)+v(w)\varphi''(w)\Big)h''_w(\varphi(w))\\
&\quad + v(w)(\varphi'(w))^2h'''_w(\varphi(w))\Big|\\
&= 2\Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big)\frac{\omega(w)\Big|u(w)(\varphi'(w))^2+2v'(w)\varphi'(w)+v(w)\varphi''(w)\Big|\Big|(\varphi(w))^2\Big|}{\sqrt{K(1-|\varphi(w)|^2)}(1-|\varphi(w)|^2)^3}.
\end{aligned} \tag{3.15}$$

For fixed $\eta\in(0,1)$, inequalities (3.14) and (3.15) implies that

$$\begin{aligned}
&\sup_{w\in\mathbb{D}}\frac{\omega(w)|u(w)(\varphi'(w))^2+2v'(w)\varphi'(w)+v(w)\varphi''(w)|}{\sqrt{K(1-|\varphi(w)|^2)}(1-|\varphi(w)|^2)^3}\\
&\le \sup_{|\varphi(w)|\le\eta}\frac{\omega(w)|u(w)(\varphi'(w))^2+2v'(w)\varphi'(w)+v(w)\varphi''(w)|}{\sqrt{K(1-|\varphi(w)|^2)}(1-|\varphi(w)|^2)^3}\\
&\quad + \sup_{|\varphi(w)|>\eta}\frac{\omega(w)|u(w)(\varphi'(w))^2+2v'(w)\varphi'(w)+v(w)\varphi''(w)|}{\sqrt{K(1-|\varphi(w)|^2)}(1-|\varphi(w)|^2)^3}\\
&\le \frac{1}{(1-\eta^2)^3}\sup_{|\varphi(w)|\le\eta}\frac{\omega(w)|u(w)(\varphi'(w))^2+2v'(w)\varphi'(w)+v(w)\varphi''(w)|}{\sqrt{K(1-\eta^2)}}\\
&\quad + \frac{1}{\eta^2}\sup_{|\varphi(w)|>\eta}\frac{\omega(w)|u(w)(\varphi'(w))^2+2v'(w)\varphi'(w)+v(w)\varphi''(w)||(\varphi(w))^2|}{(1-|\varphi(w)|^2)^3}\\
&\lesssim \Big(\frac{1}{(1-\eta^2)^3}+\frac{1}{\eta^2}\Big)\|T_{u,v,\varphi}\|_{\mathfrak{D}_K\to\mathcal{Z}_\omega}.
\end{aligned}$$

This implies that (*iii*) holds and

$$P_3 \lesssim \|T_{u,v,\varphi}\|_{\mathfrak{D}_K\to\mathcal{Z}_\omega}. \tag{3.16}$$

Taking $p_3(z)=\frac{z^3}{3!}\in\mathfrak{D}_K$, we get

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}\omega(z)\Big|\frac{1}{6}u''(z)(\varphi(z))^3+u'(z)(\varphi(z))^2\varphi'(z)+u(z)\varphi(z)(\varphi'(z))^2\\
&\quad + \frac{1}{2}u(z)(\varphi(z))^2\varphi''(z)+\frac{1}{2}v''(z)(\varphi(z))^2+2v'(z)\varphi(z)\varphi'(z)\\
&\quad + v(z)(\varphi'(z))^2+v(z)\varphi(z)\varphi''(z)\Big|\\
&\lesssim \|T_{u,v,\varphi}\|_{\mathfrak{D}_K\to\mathcal{Z}_\omega},
\end{aligned} \tag{3.17}$$

which along with (3.4), (3.10), (3.14) and the fact that $|\varphi(z)| < 1$ implies that

$$\sup_{z\in\mathbb{D}} \omega(z)|v(z)||\varphi'(z)|^2 \lesssim \left\|T_{u,v,\varphi}\right\|_{\mathfrak{D}_K\to\mathcal{Z}_\omega}. \tag{3.18}$$

For $w \in \mathbb{D}$, set

$$\begin{aligned} k_w(z) = {} & a_4 \frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}}}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+1}} + b_4 \frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+1}}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+2}} \\ & + c_4 \frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+2}}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+3}} + d_4 \frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+3}}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+4}}, \end{aligned}$$

where

$$\begin{aligned} & a_4 = \Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big), \quad b_4 = -3\Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big), \\ & c_4 = 3\Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big), \quad \text{and} \quad d_4 = -\Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big). \end{aligned}$$

Using Lemma 2.1, it can be seen that for every $w \in \mathbb{D}$, $k_w \in \mathfrak{D}_K$ and $\sup_{w\in\mathbb{D}} \|k_w\|_{\mathfrak{D}_K} \lesssim 1$. Further, we can check that

$$\begin{aligned} k_w'(z) = {} & a_4\Big(\frac{\rho}{2}+1\Big) \frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}}(\overline{\varphi(w)})}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+2}} \\ & + b_4\Big(\frac{\rho}{2}+2\Big) \frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+1}(\overline{\varphi(w)})}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+3}} \\ & + c_4\Big(\frac{\rho}{2}+3\Big) \frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+2}(\overline{\varphi(w)})}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+4}} \\ & + d_4\Big(\frac{\rho}{2}+4\Big) \frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+3}(\overline{\varphi(w)})}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+5}}, \end{aligned}$$

$$\begin{aligned} k_w''(z) = {} & a_4\Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big) \frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}}(\overline{\varphi(w)})^2}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+3}} \\ & + b_4\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big) \frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+1}(\overline{\varphi(w)})^2}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+4}} \\ & + c_4\Big(\frac{\rho}{2}+3\Big)\Big(\frac{\rho}{2}+4\Big) \frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+2}(\overline{\varphi(w)})^2}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+5}} \\ & + d_4\Big(\frac{\rho}{2}+4\Big)\Big(\frac{\rho}{2}+5\Big) \frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+3}(\overline{\varphi(w)})^2}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+6}}, \end{aligned}$$

and

$$k_w'''(z) = a_4\Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}}(\overline{\varphi(w)})^3}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+4}}$$
$$+ b_4\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big)\Big(\frac{\rho}{2}+4\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+1}(\overline{\varphi(w)})^3}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+5}}$$
$$+ c_4\Big(\frac{\rho}{2}+3\Big)\Big(\frac{\rho}{2}+4\Big)\Big(\frac{\rho}{2}+5\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+2}(\overline{\varphi(w)})^3}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+6}}$$
$$+ d_4\Big(\frac{\rho}{2}+4\Big)\Big(\frac{\rho}{2}+5\Big)\Big(\frac{\rho}{2}+6\Big)\frac{(1-|\varphi(w)|^2)^{\frac{\rho}{2}+3}(\overline{\varphi(w)})^3}{\sqrt{K(1-|\varphi(w)|^2)}(1-z\overline{\varphi(w)})^{\frac{\rho}{2}+7}}.$$

Hence,

$$k_w(\varphi(w)) = k_w'(\varphi(w)) = k_w''(\varphi(w)) = 0$$
$$\text{and}\quad k_w'''(\varphi(w)) = -6\Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big)\frac{(\overline{\varphi(w)})^3}{\sqrt{K(1-|\varphi(w)|^2)}(1-|\varphi(w)|^2)^4}.$$

Since the operator $T_{u,v,\varphi} : \mathfrak{D}_K \to \mathcal{Z}_\omega$ is bounded. Thus, we get

$$\begin{aligned}
&\|T_{u,v,\varphi}\|_{\mathfrak{D}_K\to\mathcal{Z}_\omega}\\
&\gtrsim \|T_{u,v,\varphi}k_w\|_{\mathcal{Z}_\omega}\\
&\gtrsim \omega(w)\Big|u''(w)k_w(\varphi(w))\\
&\quad + \Big(u(w)\varphi''(w) + 2u'(w)\varphi'(w) + v''(w)\Big)k_w'(\varphi(w))\\
&\quad + \Big(u(w)(\varphi'(w))^2 + 2v'(w)\varphi'(w) + v(w)\varphi''(w)\Big)k_w''(\varphi(w))\\
&\quad + v(w)(\varphi'(w))^2 k_w'''(\varphi(w))\Big|\\
&= 6\Big(\frac{\rho}{2}+1\Big)\Big(\frac{\rho}{2}+2\Big)\Big(\frac{\rho}{2}+3\Big)\frac{\omega(w)\Big|v(w)(\varphi'(w))^2(\varphi(w))^3\Big|}{\sqrt{K(1-|\varphi(w)|^2)}(1-|\varphi(w)|^2)^4}.
\end{aligned}\tag{3.19}$$

For fixed $\eta \in (0,1)$, inequalities (3.18) and (3.19) implies that

$$\begin{aligned}
&\sup_{w\in\mathbb{D}} \frac{\omega(w)|v(w)(\varphi'(w))^2|}{\sqrt{K(1-|\varphi(w)|^2)}(1-|\varphi(w)|^2)^4}\\
&\le \sup_{|\varphi(w)|\le\eta} \frac{\omega(w)|v(w)(\varphi'(w))^2|}{\sqrt{K(1-|\varphi(w)|^2)}(1-|\varphi(w)|^2)^4} + \sup_{|\varphi(w)|>\eta} \frac{\omega(w)|v(w)(\varphi'(w))^2|}{\sqrt{K(1-|\varphi(w)|^2)}(1-|\varphi(w)|^2)^4}\\
&\le \frac{1}{(1-\eta^2)^4}\sup_{|\varphi(w)|\le\eta} \frac{\omega(w)|v(w)(\varphi'(w))^2|}{\sqrt{K(1-\eta^2)}} + \frac{1}{\eta^3}\sup_{|\varphi(w)|>\eta} \frac{\omega(w)|v(w)(\varphi'(w))^2||(\varphi(w))^3|}{\sqrt{K(1-|\varphi(w)|^2)}(1-|\varphi(w)|^2)^4}\\
&\lesssim \Big(\frac{1}{(1-\eta^2)^4} + \frac{1}{\eta^3}\Big)\|T_{u,v,\varphi}\|_{\mathfrak{D}_K\to\mathcal{Z}_\omega},
\end{aligned}$$

which implies that (*iv*) holds and

$$P_4 \lesssim \|T_{u,v,\varphi}\|_{\mathfrak{D}_K\to\mathcal{Z}_\omega}. \tag{3.20}$$

Combining (3.8), (3.12), (3.16) and (3.20), we get that

$$P_1 + P_2 + P_3 + P_4 \lesssim \|T_{u,v,\varphi}\|_{\mathfrak{D}_K\to\mathcal{Z}_\omega}. \tag{3.21}$$

Thus, from (3.3) and (3.21), it follows that

$$P_1 + P_2 + P_3 + P_4 \lesssim \|T_{u,v,\varphi}\|_{\mathfrak{D}_K \to \mathcal{Z}_\omega} \lesssim P + P_1 + P_2 + P_3 + P_4. \tag{3.22}$$

Hence the theorem. □

In Theorem 3.1, if we take $u(z) = \psi(z)$ and $v(z) \equiv 0$, then the operator get reduced to the weighted composition operator $\mathcal{W}_{\psi,\varphi} : \mathfrak{D}_K \to \mathcal{Z}_\omega$. Thus, we get the following corollary for the boundedness of $\mathcal{W}_{\psi,\varphi} : \mathfrak{D}_K \to \mathcal{Z}_\omega$ as:

Corollary 3.2. *Let ω and K be two weight functions, $\psi \in H(\mathbb{D})$ and φ be an analytic self-map on $\mathbb{D}$. Then, operator $\mathcal{W}_{\psi,\varphi} : \mathfrak{D}_K \to \mathcal{Z}_\omega$ is bounded if and only if the functions ψ and φ satisfy the following conditions :*

$$(i)\ Q_1 = \sup_{z\in\mathbb{D}} \frac{\omega(z)|\psi''(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)} < \infty,$$
$$(ii)\ Q_2 = \sup_{z\in\mathbb{D}} \frac{\omega(z)|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^2} < \infty,$$
$$(iii)\ Q_3 = \sup_{z\in\mathbb{D}} \frac{\omega(z)|\psi(z)(\varphi'(z))^2|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^3} < \infty.$$

Further,

$$Q_1 + Q_2 + Q_3 \lesssim \|\mathcal{W}_{\psi,\varphi}\|_{\mathfrak{D}_K \to \mathcal{Z}_\omega} \lesssim Q + Q_1 + Q_2 + Q_3,$$

$$\text{where} \quad Q = \frac{|\psi(0)| + |\psi'(0)|}{\sqrt{K(1-|\varphi(0)|^2)}(1-|\varphi(0)|^2)} + \frac{|\psi(0)\varphi'(0)|}{\sqrt{K(1-|\varphi(0)|^2)}(1-|\varphi(0)|^2)^2}.$$

Again by taking $v(z) = \psi(z)$ and $u(z) \equiv 0$ in Theorem 3.1, we can obtain the boundedness of the weighted differentiation composition operator $\mathcal{W}_{\psi,\varphi}D : \mathfrak{D}_K \to \mathcal{Z}_\omega$ which can be given by the following corollary:

Corollary 3.3. *Let ω and K be two weight functions, $\psi \in H(\mathbb{D})$ and φ be an analytic self-map on $\mathbb{D}$. Then, operator $\mathcal{W}_{\psi,\varphi}D : \mathfrak{D}_K \to \mathcal{Z}_\omega$ is bounded if and only if the functions ψ and φ satisfy the following conditions :*

$$(i)\ R_1 = \sup_{z\in\mathbb{D}} \frac{\omega(z)|\psi''(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^2} < \infty,$$
$$(ii)\ R_2 = \sup_{z\in\mathbb{D}} \frac{\omega(z)|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^3} < \infty,$$
$$(iii)\ R_3 = \sup_{z\in\mathbb{D}} \frac{\omega(z)|\psi(z)(\varphi'(z))^2|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^4} < \infty.$$

Further,

$$R_1 + R_2 + R_3 \lesssim \|\mathcal{W}_{\psi,\varphi}D\|_{\mathfrak{D}_K \to \mathcal{Z}_\omega} \lesssim R + R_1 + R_2 + R_3,$$

$$\text{where} \quad R = \frac{|\psi(0)| + |\psi'(0)|}{\sqrt{K(1-|\varphi(0)|^2)}(1-|\varphi(0)|^2)^2} + \frac{|\psi(0)\varphi'(0)|}{\sqrt{K(1-|\varphi(0)|^2)}(1-|\varphi(0)|^2)^3}.$$

4. Compactness of the operator $T_{u,v,\varphi}$ from $\mathfrak{D}_K$ spaces to Zygmund type spaces

Theorem 4.1. *Let ω and K be two weight functions, $u, v \in H(\mathbb{D})$ and φ be a self analytic map on $\mathbb{D}$. Then, the following conditions are equivalent:*

(i) The operator $T_{u,v,\varphi} : \mathfrak{D}_K \to \mathcal{Z}_\omega$ is compact.

(ii) Functions u, v and φ are such that

$$\begin{aligned}
p_1 &= \sup_{z\in\mathbb{D}} \omega(z)|u''(z)| < \infty,\\
p_2 &= \sup_{z\in\mathbb{D}} \omega(z)|2u'(z)\varphi'(z) + u(z)\varphi''(z) + v''(z)| < \infty,\\
p_3 &= \sup_{z\in\mathbb{D}} \omega(z)|u(z)(\varphi'(z))^2 + 2v'(z)\varphi'(z) + v(z)\varphi''(z)| < \infty,\\
p_4 &= \sup_{z\in\mathbb{D}} \omega(z)|v(z)(\varphi'(z))^2| < \infty,
\end{aligned}$$

$$\begin{aligned}
&\lim_{|\varphi(z)|\to 1} \frac{\omega(z)|u''(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)} = 0,\\
&\lim_{|\varphi(z)|\to 1} \frac{\omega(z)|2u'(z)\varphi'(z) + u(z)\varphi''(z) + v''(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^2} = 0,\\
&\lim_{|\varphi(z)|\to 1} \frac{\omega(z)|u(z)(\varphi'(z))^2 + 2v'(z)\varphi'(z) + v(z)\varphi''(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^3} = 0,\\
&\lim_{|\varphi(z)|\to 1} \frac{\omega(z)|v(z)||\varphi'(z)|^2}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^4} = 0.
\end{aligned}$$

Proof. First, suppose that the condition (*i*) holds, that is, operator $T_{u,v,\varphi} : \mathfrak{D}_K \to \mathcal{Z}_\omega$ is compact. This implies that $T_{u,v,\varphi} : \mathfrak{D}_K \to \mathcal{Z}_\omega$ is bounded. Thus, from Theorem 3.1, we obtain that p_1, p_2, p_3 and p_4 are finite. Consider a sequence $(u_m)_{m\in\mathbb{N}} \in \mathbb{D}$ such that $|\varphi(u_m)| \to 1$ as $m \to \infty$. Conditions (*ii*) hold obviously if such a sequence does not exists. By making use of $(u_m)_{m\in\mathbb{N}}$, define

$$\begin{aligned}
f_m(z) = {} & a_1 \frac{(1-|\varphi(u_m)|^2)^{\frac{\rho}{2}}}{\sqrt{K(1-|\varphi(u_m)|^2)}(1-z\overline{\varphi(u_m)})^{\frac{\rho}{2}+1}} + b_1 \frac{(1-|\varphi(u_m)|^2)^{\frac{\rho}{2}+1}}{\sqrt{K(1-|\varphi(u_m)|^2)}(1-z\overline{\varphi(u_m)})^{\frac{\rho}{2}+2}}\\
& + c_1 \frac{(1-|\varphi(u_m)|^2)^{\frac{\rho}{2}+2}}{\sqrt{K(1-|\varphi(u_m)|^2)}(1-z\overline{\varphi(u_m)})^{\frac{\rho}{2}+3}} + d_1 \frac{(1-|\varphi(u_m)|^2)^{\frac{\rho}{2}+3}}{\sqrt{K(1-|\varphi(u_m)|^2)}(1-z\overline{\varphi(u_m)})^{\frac{\rho}{2}+4}},
\end{aligned}$$

$$\begin{aligned}
g_m(z) = {} & a_2 \frac{(1-|\varphi(u_m)|^2)^{\frac{\rho}{2}}}{\sqrt{K(1-|\varphi(u_m)|^2)}(1-z\overline{\varphi(u_m)})^{\frac{\rho}{2}+1}} + b_2 \frac{(1-|\varphi(u_m)|^2)^{\frac{\rho}{2}+1}}{\sqrt{K(1-|\varphi(u_m)|^2)}(1-z\overline{\varphi(u_m)})^{\frac{\rho}{2}+2}}\\
& + c_2 \frac{(1-|\varphi(u_m)|^2)^{\frac{\rho}{2}+2}}{\sqrt{K(1-|\varphi(u_m)|^2)}(1-z\overline{\varphi(u_m)})^{\frac{\rho}{2}+3}} + d_2 \frac{(1-|\varphi(u_m)|^2)^{\frac{\rho}{2}+3}}{\sqrt{K(1-|\varphi(u_m)|^2)}(1-z\overline{\varphi(u_m)})^{\frac{\rho}{2}+4}},
\end{aligned}$$

$$\begin{aligned}
h_m(z) = {} & a_3 \frac{(1-|\varphi(u_m)|^2)^{\frac{\rho}{2}}}{\sqrt{K(1-|\varphi(u_m)|^2)}(1-z\overline{\varphi(u_m)})^{\frac{\rho}{2}+1}} + b_3 \frac{(1-|\varphi(u_m)|^2)^{\frac{\rho}{2}+1}}{\sqrt{K(1-|\varphi(u_m)|^2)}(1-z\overline{\varphi(u_m)})^{\frac{\rho}{2}+2}}\\
& + c_3 \frac{(1-|\varphi(u_m)|^2)^{\frac{\rho}{2}+2}}{\sqrt{K(1-|\varphi(u_m)|^2)}(1-z\overline{\varphi(u_m)})^{\frac{\rho}{2}+3}} + d_3 \frac{(1-|\varphi(u_m)|^2)^{\frac{\rho}{2}+3}}{\sqrt{K(1-|\varphi(u_m)|^2)}(1-z\overline{\varphi(u_m)})^{\frac{\rho}{2}+4}},
\end{aligned}$$

$$k_m(z) = a_4\frac{(1-|\varphi(u_m)|^2)^{\frac{\rho}{2}}}{\sqrt{K(1-|\varphi(u_m)|^2)}(1-z\overline{\varphi(u_m)})^{\frac{\rho}{2}+1}} + b_4\frac{(1-|\varphi(u_m)|^2)^{\frac{\rho}{2}+1}}{\sqrt{K(1-|\varphi(u_m)|^2)}(1-z\overline{\varphi(u_m)})^{\frac{\rho}{2}+2}}$$
$$+ c_4\frac{(1-|\varphi(u_m)|^2)^{\frac{\rho}{2}+2}}{\sqrt{K(1-|\varphi(u_m)|^2)}(1-z\overline{\varphi(u_m)})^{\frac{\rho}{2}+3}} + d_4\frac{(1-|\varphi(u_m)|^2)^{\frac{\rho}{2}+3}}{\sqrt{K(1-|\varphi(u_m)|^2)}(1-z\overline{\varphi(u_m)})^{\frac{\rho}{2}+4}},$$

where $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4, d_1, d_2, d_3$ and d_4 are defined in Theorem 3.1. From Theorem 3.1, it can be seen that the sequences (f_m), (g_m), (h_m) and (k_m) are norm bounded in $\mathfrak{D}_K$ and on compact subsets of $\mathbb{D}$ uniformly converge to zero as $m \to \infty$. Thus, by Lemma 2.3, we get

$$\lim_{m\to\infty} \|T_{u,v,\varphi} f_m\|_{\mathcal{Z}_\omega} = 0, \quad \lim_{m\to\infty} \|T_{u,v,\varphi} g_m\|_{\mathcal{Z}_\omega} = 0,$$
$$\lim_{m\to\infty} \|T_{u,v,\varphi} h_m\|_{\mathcal{Z}_\omega} = 0 \quad \text{and} \quad \lim_{m\to\infty} \|T_{u,v,\varphi} k_m\|_{\mathcal{Z}_\omega} = 0. \tag{4.1}$$

From (3.7), (3.11), (3.15) and (3.19), it follows that

$$\frac{\omega(u_m)\left|u''(u_m)\right|}{\sqrt{K(1-|\varphi(u_m)|^2)}(1-|\varphi(u_m)|^2)} \lesssim \|T_{u,v,\varphi} f_m\|_{\mathcal{Z}_\omega}, \tag{4.2}$$

$$\frac{\omega(u_m)\left|u(u_m)\varphi''(u_m)+2u'(u_m)\varphi'(u_m)+v''(u_m)\right|\left|\varphi(u_m)\right|}{\sqrt{K(1-|\varphi(u_m)|^2)}(1-|\varphi(u_m)|^2)^2} \lesssim \|T_{u,v,\varphi} g_m\|_{\mathcal{Z}_\omega}, \tag{4.3}$$

$$\frac{\omega(u_m)\left|u(u_m)(\varphi'(u_m))^2+2v'(u_m)\varphi'(u_m)+v(u_m)\varphi''(u_m)\right|\left|(\varphi(u_m))^2\right|}{\sqrt{K(1-|\varphi(u_m)|^2)}(1-|\varphi(u_m)|^2)^3}$$
$$\lesssim \|T_{u,v,\varphi} h_m\|_{\mathcal{Z}_\omega} \tag{4.4}$$

and

$$\frac{\omega(u_m)\left|v(u_m)(\varphi'(u_m))^2(\varphi(u_m))^3\right|}{\sqrt{K(1-|\varphi(u_m)|^2)}(1-|\varphi(u_m)|^2)^4} \lesssim \|T_{u,v,\varphi} k_m\|_{\mathcal{Z}_\omega}. \tag{4.5}$$

By taking $m \to \infty$ in (4.2), (4.3), (4.4), (4.5) and using (4.1), we obtain that conditions (*ii*) hold.

Conversely, suppose that condition (*ii*) holds. To prove the compactness of $T_{u,v,\varphi}$ we first show that $T_{u,v,\varphi}$ is bounded. Using condition (*ii*), we see that for every $\varepsilon > 0$, there is an $\eta \in (0,1)$ such that

$$L_1 = \frac{\omega(z)\left|u''(z)\right|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)} < \varepsilon, \tag{4.6}$$

$$L_2 = \frac{\omega(z)\left|u(z)\varphi''(z)+2u'(z)\varphi'(z)+v''(z)\right|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^2} < \varepsilon, \tag{4.7}$$

$$L_3 = \frac{\omega(z)\left|u(z)(\varphi'(z))^2+2v'(z)\varphi'(z)+v(z)\varphi''(z)\right|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^3} < \varepsilon \tag{4.8}$$

and

$$L_4 = \frac{\omega(z)\left|v(z)(\varphi'(z))^2\right|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^4} < \varepsilon, \tag{4.9}$$

for any $z \in A = \{z \in \mathbb{D} : |\varphi(z)| > \eta\}$. Now, by (4.6) and condition $p_1 < \infty$, we get

$$P_1 = \sup_{z\in\mathbb{D}} L_1(z) \le \sup_{z\in\mathbb{D}\setminus A} L_1(z) + \sup_{z\in A} L_1(z) \le \frac{p_1}{\sqrt{K(1-\eta^2)}(1-\eta^2)} + \varepsilon.$$

This implies that $P_1 < \infty$. Now by (4.7) and $p_2 < \infty$, we get

$$P_2 = \sup_{z\in\mathbb{D}} L_2(z) \le \sup_{z\in\mathbb{D}\setminus A} L_2(z) + \sup_{z\in A} L_2(z) \le \frac{p_2}{\sqrt{K(1-\eta^2)}(1-\eta^2)^2} + \varepsilon,$$

which implies that $P_2 < \infty$. Again, from (4.8) and $p_3 < \infty$, we get

$$P_3 = \sup_{z\in\mathbb{D}} L_3(z) \le \sup_{z\in\mathbb{D}\setminus A} L_3(z) + \sup_{z\in A} L_3(z) \le \frac{p_3}{\sqrt{K(1-\eta^2)}(1-\eta^2)^3} + \varepsilon.$$

Thus, $P_3 < \infty$. Again, from (4.9) and $p_4 < \infty$, we get

$$P_4 = \sup_{z\in\mathbb{D}} L_4(z) \le \sup_{z\in\mathbb{D}\setminus A} L_4(z) + \sup_{z\in A} L_4(z) \le \frac{p_4}{\sqrt{K(1-\eta^2)}(1-\eta^2)^4} + \varepsilon,$$

which implies that $P_4 < \infty$. From the above we obtain that P_1, P_2, P_3 and P_4 are finite. Therefore, by Theorem 3.1, we have that the operator $T_{u,v,\varphi} : \mathfrak{D}_K \to \mathcal{Z}_\omega$ is bounded.
Now, we prove that $T_{u,v,\varphi} : \mathfrak{D}_K \to \mathcal{Z}_\omega$ is compact. Consider a sequence $(g_m)_{m\in\mathbb{N}} \in \mathfrak{D}_K$ such that $g_m \to 0$ uniformly on compact subsets of $\mathbb{D}$ and $\|g_m\|_{\mathfrak{D}_K} \lesssim 1$. Then, g'_m, g''_m and g'''_m uniformly converges to zero on compact subsets of $\mathbb{D}$ as $m \to \infty$.
Thus, using Lemma 2.2, (3.1), condition (*ii*), for every $\varepsilon > 0$ and η, we obtain

$$\begin{aligned}
&\|T_{u,v,\varphi} g_m\|_{\mathcal{Z}_\omega} \\
&= |(T_{u,v,\varphi} g_m)(0)| + |(T_{u,v,\varphi} g_m)'(0)| + \sup_{z\in\mathbb{D}} \omega(z)|(T_{u,v,\varphi} g_m)''(z)| \\
&\lesssim P + \sup_{z\in\mathbb{D}} \omega(z)|u''(z) g_m(\varphi(z))| \\
&\quad + \sup_{z\in\mathbb{D}} \omega(z)\Big|\Big(2u'(z)\varphi'(z) + u(z)\varphi''(z) + v''(z)\Big) g'_m(\varphi(z))\Big| \\
&\quad + \sup_{z\in\mathbb{D}} \omega(z)\Big|\Big(u(z)(\varphi'(z))^2 + 2v'(z)\varphi'(z) + v(z)\varphi''(z)\Big) g''_m(\varphi(z))\Big| \\
&\quad + \sup_{z\in\mathbb{D}} \omega(z)|v(z)(\varphi'(z))^2 g'''_m(\varphi(z))| \\
&\le P + \sup_{z\in A} \omega(z)|u''(z)||g_m(\varphi(z))| + \sup_{z\in\mathbb{D}\setminus A} \omega(z)|u''(z)||g_m(\varphi(z))| \\
&\quad + \sup_{z\in A} \omega(z)\Big|2u'(z)\varphi'(z) + u(z)\varphi''(z) + v''(z)\Big||g'_m(\varphi(z))| \\
&\quad + \sup_{z\in\mathbb{D}\setminus A} \omega(z)\Big|2u'(z)\varphi'(z) + u(z)\varphi''(z) + v''(z)\Big||g'_m(\varphi(z))| \\
&\quad + \sup_{z\in A} \omega(z)\Big|u(z)(\varphi'(z))^2 + 2v'(z)\varphi'(z) + v(z)\varphi''(z)\Big||g''_m(\varphi(z))| \\
&\quad + \sup_{z\in\mathbb{D}\setminus A} \omega(z)\Big|u(z)(\varphi'(z))^2 + 2v'(z)\varphi'(z) + v(z)\varphi''(z)\Big||g''_m(\varphi(z))| \\
&\quad + \sup_{z\in A} \omega(z)|v(z)||\varphi'(z)|^2|g'''_m(\varphi(z))| + \sup_{z\in\mathbb{D}\setminus A} \omega(z)|v(z)||\varphi'(z)|^2|g'''_m(\varphi(z))| \\
&\le P + A_n + C\sup_{z\in A} L_1(z) + C\sup_{z\in A} L_2(z) + C\sup_{z\in A} L_3(z) + C\sup_{z\in A} L_4(z) \\
&\lesssim P + A_n + 4\varepsilon,
\end{aligned} \tag{4.10}$$

where

$$A_n = p_1 \sup_{\{z:|z|\le\eta\}} |g_m(z)| + p_2 \sup_{\{z:|z|\le\eta\}} |g'_m(z)| + p_3 \sup_{\{z:|z|\le\eta\}} |g''_m(z)| + p_4 \sup_{\{z:|z|\le\eta\}} |g'''_m(z)|.$$

We know that if $(g_m)_{m\in\mathbb{N}}$ converges to zero uniformly on any compact subset of $\mathbb{D}$ then $(g'_m)_{m\in\mathbb{N}}$, $(g''_m)_{m\in\mathbb{N}}$ and $(g'''_m)_{m\in\mathbb{N}}$ do the same as $m \to \infty$. Thus, $A_n \to 0$ as $m \to \infty$. Also, $\{\varphi(0)\}$ and $\{z : |z| \le \eta\}$ are compact subsets of $\mathbb{D}$, so by taking $m \to \infty$ in (4.10), we obtain

$$\lim_{m\to\infty} \|T_{u,v,\varphi} g_m\|_{\mathfrak{D}_K \to \mathcal{Z}_\omega} = 0.$$

Hence the operator $T_{u,v,\varphi} : \mathfrak{D}_K \to \mathcal{Z}_\omega$ is compact. □

Taking $u(z) = \psi(z)$ and $v(z) \equiv 0$ in Theorem 4.1, we can obtain the compactness of the $\mathcal{W}_{\psi,\varphi} : \mathfrak{D}_K \to \mathcal{Z}_\omega$ given by the following corollary:

Corollary 4.2. *Let ω and K be two weight functions, $\psi \in H(\mathbb{D})$ and φ be an analytic self-map on $\mathbb{D}$. Then, following conditions are equivalent:*

(i) The operator $\mathcal{W}_{\psi,\varphi} : \mathfrak{D}_K \to \mathcal{Z}_\omega$ is compact.

(ii) Functions ψ and φ are such that

$$q_1 = \sup_{z\in\mathbb{D}} \omega(z)|\psi''(z)| < \infty,$$
$$q_2 = \sup_{z\in\mathbb{D}} \omega(z)|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)| < \infty,$$
$$q_3 = \sup_{z\in\mathbb{D}} \omega(z)|\psi(z)(\varphi'(z))^2| < \infty,$$

$$\lim_{|\varphi(z)|\to 1} \frac{\omega(z)|\psi''(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)} = 0,$$
$$\lim_{|\varphi(z)|\to 1} \frac{\omega(z)|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^2} = 0,$$
$$\lim_{|\varphi(z)|\to 1} \frac{\omega(z)|\psi(z)(\varphi'(z))^2|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^3} = 0.$$

Again by taking $v(z) = \psi(z)$ and $u(z) \equiv 0$ in Theorem 4.1, we can obtain the compactness of the operator $\mathcal{W}_{\psi,\varphi}D : \mathfrak{D}_K \to \mathcal{Z}_\omega$ which can be given by the following corollary:

Corollary 4.3. *Let ω and K be two weight functions, $\psi \in H(\mathbb{D})$ and φ be an analytic self-map on $\mathbb{D}$. Then, following conditions are equivalent:*

(i) The operator $\mathcal{W}_{\psi,\varphi}D : \mathfrak{D}_K \to \mathcal{Z}_\omega$ is compact.

(ii) Functions ψ and φ are such that

$$r_1 = \sup_{z\in\mathbb{D}} \omega(z)|\psi''(z)| < \infty,$$
$$r_2 = \sup_{z\in\mathbb{D}} \omega(z)|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)| < \infty,$$
$$r_3 = \sup_{z\in\mathbb{D}} \omega(z)|\psi(z)(\varphi'(z))^2| < \infty,$$

$$\lim_{|\varphi(z)|\to 1} \frac{\omega(z)|\psi''(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^2} = 0,$$
$$\lim_{|\varphi(z)|\to 1} \frac{\omega(z)|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^3} = 0,$$
$$\lim_{|\varphi(z)|\to 1} \frac{\omega(z)|\psi(z)(\varphi'(z))^2|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^4} = 0.$$

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' contributions

All authors contributed equally to writing this paper. All authors read and approved the manuscript.

Data Availability

No data were used to support this study.

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