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On the Basis Properties of a System of Eigenfunctions of a Spectral Problem for a Second-Order Discontinuous Differential Operator in the Weighted Grand-Lebesgue Space with a General Weight

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Abstract. The question of the basis property of a system of eigenfunctions of one spectral problem for a discontinuous second-order differential operator with a spectral parameter under discontinuity conditions is considered in the weighted grand-Lebesgue spaces $L_{p),\rho}(0,1), 1 < p < +\infty$, with a general weight $\rho(\cdot)$. These spaces are non-separable and therefore it is necessary to define its subspace associated with differential equation. In this paper, using the shift operator, a subspace $G_{p),\rho}(0,1)$ is considered, in which the basis property of exponentials and trigonometric systems of sines and cosines is established when the weight function $\rho(\cdot)$ satisfies the Muckenhoupt condition. It is proved that the system of eigenfunctions and associated functions of the discontinuous differential operator corresponding to the given problem forms a basis in the weighted space $G_{p),\rho}(0,1) \oplus \mathbb{C}, 1 < p < +\infty$ with the weight $\rho(\cdot)$ satisfying the Muckenhoupt condition. The question of the defect basis property of the system of eigenfunctions and associated functions of the given problem in the weighted spaces $G_{p),\rho}(0,1), 1 < p < +\infty$, is considered.

1. Introduction

Consider the following discontinuous spectral problem in weighted grand-Lebesgue spaces $L_{p),\rho}(0,1), 1 < p < +\infty$, with a general weight $\rho(\cdot)$

$$y''(x) + \lambda y(x) = 0, x \in (0,1)\,, \tag{1}$$

$$\left.\begin{array}{l} y(0) = y(1) = 0, \\ y(\frac{1}{3} - 0) = y(\frac{1}{3} + 0), \\ y'(\frac{1}{3} - 0) - y'(\frac{1}{3} + 0) = \lambda m y(\frac{1}{3}), m \neq 0, \end{array}\right\} \tag{2}$$

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where λ is a spectral parameter, m is a nonzero complex number. Note that discontinuous spectral problems of the form (1), (2) arise in the study of problems of oscillation of a loaded string, one or both ends of which are fixed. These problems have important applications in mathematics, mechanics, physics, and other fields of science. More details about such problems can be found in monographs [1, 2]. The basis properties of the system of eigenfunctions of problem (1), (2) in Lebesgue spaces were studied by various methods in [3–7]. In [3], using the method of the theory of close bases, the basis property of the system of eigenfunctions and associated functions of problem (1), (2) in the spaces $L_p(0,1)$ is proved.

In [4, 5] the authors showed that, the problem (1), (2) has two series of eigenvalues $\lambda_{1,n} = (\rho_{1,n})^2, n \in \mathbb{N}$, and $\lambda_{2,n} = (\rho_{2,n})^2, n \in \mathbb{N} \cup \{0\}$, where

$$\rho_{1,n} = 3\pi n, \qquad \rho_{2,n} = \frac{3\pi n}{2} + \frac{2+(-1)^n}{\pi m n} + O\left(\frac{1}{n^2}\right),$$

the corresponding eigenfunctions are expressed by the formulas

$$y_{1,n}(x) = \sin 3\pi n x, \qquad x \in [0,1], \qquad n \in \mathbb{N}, \tag{3}$$

$$y_{2,n}(x) = \begin{cases} \sin \rho_{2,n}(x - \frac{1}{3}) + \sin \rho_{2,n}(x + \frac{1}{3}), x \in [0, \frac{1}{3}] \\ \sin \rho_{2,n}(1-x), x \in [\frac{1}{3}, 1] \end{cases}. \tag{4}$$

Problem (1), (2) in weighted Lebesgue spaces with power weights was studied in [7]. Spectral problems with a discontinuity point and with a spectral parameter in the boundary conditions were also studied in [8–12].

Recently, in connection with important applications in various areas of mathematics, such as the theory of partial differential equations, approximation theory, interpolation theory, harmonic analysis, etc., interest in non-standard function spaces has increased greatly. Among such spaces, we can mention the Lebesgue space with variable summability exponent, Morrey space, grand-Lebesgue space, etc. Many classical facts of harmonic analysis, namely, the questions of the boundedness of a singular operator with the Cauchy kernel, maximal function, Hilbert transforms are studied in non-standard spaces. Numerous articles, survey papers and monographs of various mathematicians are known in this direction (see [13–26]). Taking into account the non-separability of the grand-Lebesgue space, when studying problems of differential equations in them, one has to consider their appropriate subspaces (see [20]) dictated by differential equations. This idea was applied to problem (1) and (2) in Morrey-type spaces in [13, 15, 34–36].

The presented work is devoted to the study of the question of the basis property of the system of eigenfunctions of the spectral problem (1) and (2) in weighted spaces $L_{p),\rho}(0,1), 1 < p < +\infty$, with a general weight $\rho(\cdot)$. Section 2 provides the necessary information from the theory of bases and the grand-Lebesgue space. In Section 3, we prove the basis property of the system of exponentials and the system of eigenfunctions of the spectral problem (1) and (2) in the weighted spaces $G_{p),\rho}(0,1) \oplus \mathbb{C}, 1 < p < +\infty$, when the weight function $\rho(\cdot)$ satisfies the Muckenhoupt condition. Finally, in Section 4, we study the basis property of the system of eigenfunctions and associated functions of problem (1), (2) with a finite defect in the spaces $G_{p),\rho}(0,1), 1 < p < +\infty$.

2. Preliminaries and Auxiliary Facts

Throughout the paper, $\mathbb{N}$ is the set of natural numbers, $Z_+ = \{0\} \cup \mathbb{N}$ is the set of non-negative integers, $\mathbb{R}_+$ is the set of non-negative real numbers, $\mathbb{C}$ is the set of complex numbers, δ_{nk} is the Kronecker symbol, $|I|$ is the Lebesgue measure on the line of the set I.

Let us present some concepts and facts concerning the theory of grand Lebesgue spaces. By $L_{p)}(a,b), 1 < p < +\infty$, we denote the grand-Lebesgue space of measurable on $[a,b]$ functions f satisfying the condition

$$\left\|f\right\|_{L_{p)}(a,b)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{b-a}\int_a^b |f(t)|^{p-\varepsilon} dt\right)^{\frac{1}{p-\varepsilon}} < +\infty.$$

The space $L_{p)}(a,b)$ is a non-separable Banach space with the norm $\|f\|_{L_{p)}(a,b)}$ (see [26]).

We define a separable subspace of the space $L_{p)}(a,b)$ as follows. Consider for $\forall\delta > 0$ the shift operator

$$T_\delta f(x) = \begin{cases} f(x+\delta), & x+\delta \in [a,b], \\ 0, & x+\delta \in \mathbb{R}\backslash[a,b], \end{cases} \qquad f \in L_{p)}(a,b),$$

and denote by $\tilde{G}_{p)}(a,b)$ the linear manifold of functions $f \in L_{p)}(a,b)$ satisfying the condition

$$\|T_\delta f - f\|_{p)} \to 0, \quad \delta \to 0.$$

Let $G_{p)}(a,b)$ be the closure of $\tilde{G}_{p)}(a,b)$ in $L_{p)}(a,b)$. There is a continuous embedding $L_p(a,b) \subset G_{p)}(a,b)$, and the inclusion is strict, that is, $G_{p)}(a,b)\backslash L_p(a,b) \neq \emptyset$. The space $G_{p)}(a,b)$ is separable in which the set of infinitely differentiable functions with compact support on the interval $[a,b]$ is dense (see [27, 28]).

Let us present some concepts and facts from the theory of bases in Banach spaces.

Definition 2.1. *A system $\{\phi_i\}_{i\in\mathbb{N}}$ of a Banach space X is called a basis with parentheses in X if there is a sequence of integers $\{n_k\}_{k\in\mathbb{Z}_+}, n_0 = 0, n_k < n_{k+1}, k \in \mathbb{Z}_+$, such that for $\forall x \in X$ there is a unique decomposition*

$$x = \sum_{k=0}^{+\infty} \sum_{i=n_k+1}^{n_{k+1}} c_i \phi_i.$$

Obviously, a basis with parentheses for which $n_k = k, k \in \mathbb{Z}_+$ is an ordinary basis.

We need the following theorem.

Theorem 2.2. *([3]) Let the system $\{u_{kn}\}_{k=\overline{(1,m)},n\in\mathbb{N}}$ form a basis in the space X with the biorthogonal system $\{v_{kn}\}_{k=\overline{1,m},n\in\mathbb{N}}$ and assume $A_n = \left(a_{ik}^n\right)_{i,k=\overline{1,m},n\in\mathbb{N}}$, $n \in \mathbb{N}$, is a matrix of scalars such that*

$$\Delta_n = \det A_n \neq 0.$$

Then the system $\{\phi_{kn}\}_{k=\overline{1,m},n\in\mathbb{N}}$ given by the equality

$$\phi_{kn} = \sum_i^m a_{ik}^{(n)} u_{in}, \qquad k = \overline{1,m}, \qquad n \in \mathbb{N},$$

forms a basis with parentheses in X. In addition, if the conditions

$$\sup_n \{\|A_n\|, \|A_n^{-1}\|\} < +\infty, \qquad \sup_n \{\|u_{kn}\|_X, \|v_{kn}\|_{X^*}\} < +\infty,$$

then the system $\{\phi_{kn}\}_{k=\overline{1,m},n\in\mathbb{N}}$ is an ordinary basis in X, $\|A_n\|$ is some norm of the matrix A_n.

Let the system $\{\hat{u}_n\}_{n\in\mathbb{N}}$ form a basis in the space $X \oplus \mathbb{C}^m$ with the biorthogonal system $\{\hat{v}_n\}_{n\in\mathbb{N}}$ and

$$\hat{u}_n = \left(u_n; \alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_m^{(n)}\right), \qquad \hat{v}_n = \left(v_n; \beta_1^{(n)}, \beta_2^{(n)}, \ldots, \beta_m^{(n)}\right). \tag{5}$$

In the following statement, for an arbitrary set $J = \{n_1, n_2, ..., n_m\}$ of m numbers, we study the basis property of the system $\{u_n\}_{n\in\mathbb{N}\backslash J}$ in X obtained from the system $\{u_n\}_{n\in\mathbb{N}}$ excluding vectors $u_{n_1}, u_{n_2}, \ldots, u_{n_m}$.

Theorem 2.3. *([29]) Let the system $\{\hat{u}_n\}_{n\in\mathbb{N}}$) form a basis in the space $X \oplus \mathbb{C}^m$ with the biorthogonal system $\{\hat{v}_n\}_{n\in\mathbb{N}}$, and equalities (5) hold. Let $J = \{n_1, n_2, ..., n_m\}$ be an arbitrary collection of m natural numbers. Then the system $\{u_n\}_{n\in\mathbb{N}\backslash J}$ forms a basis in X if and only if the condition*

$$\delta = \delta(J) = \det\left(\beta_i^{n_k}\right)_i \neq 0, k = \overline{1,m}. \tag{6}$$

Moreover, the system $\{u_n\}_{n\in\mathbb{N}\setminus J}$ has the following biorthogonal system

$$v_n^* = \frac{1}{\delta}\begin{vmatrix} v_n & v_{n_1} & \dots & v_{n_m} \\ \beta_1^{(n)} & \beta_1^{(n_1)} & \dots & \beta_1^{(n_m)} \\ \vdots & \vdots & \dots & \vdots \\ \beta_m^{(n)} & \beta_m^{(n_1)} & \dots & \beta_m^{(n_m)} \end{vmatrix}.$$

For $\delta = 0$, the system $\{u_n\}_{n\in\mathbb{N}\setminus J}$ is not complete and not minimal in X.

It is known that the closeness in a certain sense of a system to a basis under certain conditions ensures its isomorphic basis property. Let us give the concepts of basis and $p-$close systems.

Definition 2.4. *Systems $\{\phi_i\}_{i\in\mathbb{N}}$ and $\{\psi_i\}_{i\in\mathbb{N}}$ of vectors of the space X are called p-close if the condition*

$$\sum_{i=1}^{+\infty} \left\|\phi_i - \psi_i\right\|_X^p < +\infty.$$

Definition 2.5. *A basis $\{u_i\}_{i\in\mathbb{N}}$ of a space X is called a p -basis in X if there exists a number $M > 0$ such that $\forall x \in X$ satisfies the relation*

$$\left(\sum_{i=1}^{+\infty} |< x, v_i >|^p\right)^{\frac{1}{p}} \le M\,\|x\|_X\,,$$

where $\{v_i\}_{i\in\mathbb{N}}$ is a biorthogonal system to $\{u_i\}_{i\in\mathbb{N}}$.

The next statement gives equivalent conditions when a system $p-$close to a basis forms a basis isomorphic to it.

Theorem 2.6. *Let the system $\{\phi_i\}_{i\in\mathbb{N}}$ form a $p-$basis in X, the system $\{\psi_i\}_{i\in\mathbb{N}} \subset X$ is q-close to $\{\phi_i\}_{i\in\mathbb{N}}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then the following properties are equivalent:*

1. *the system $\{\psi_i\}_{i\in\mathbb{N}}$ is complete in X;*
2. *the system $\{\psi_i\}_{i\in\mathbb{N}}$ is minimal in X;*
3. *the system $\{\psi_i\}_{i\in\mathbb{N}}$ forms a basis isomorphic to $\{\phi_i\}_{i\in\mathbb{N}}$.*

More details about these and other facts can be found in [30].

We also need the following theorem on the relation between the Lebesgue space function $L_p(a,b)$ and its Fourier coefficients with respect to the system of functions $\phi_i(t), i \in \mathbb{N}$, orthonormal and uniformly bounded on the interval (a,b):

$$|\phi_i(t)| \le M, \qquad t \in (a,b), \qquad i \in \mathbb{N}.$$

Theorem 2.7. ***(F.Riesz)**([31]) Let $1 < p \le 2, \frac{1}{p} + \frac{1}{q} = 1$. Then*

1. *if $f \in L_p(a,b)$, then the Fourier coefficients $c_i = \int_a^b f(t)\overline{\phi_i}(t)dt$ of the function f satisfy the inequality*

$$\left(\sum_{i=1}^{+\infty} |c_i|^q\right)^{\frac{1}{q}} \le M^{\frac{2-p}{p}} \left\|f\right\|_{L_p(a,b)};$$

2. *if $\sum_{i=1}^{+\infty} |c_i|^p < +\infty$ holds for a sequence of numbers c_i, then there exists a function $f \in L_q(a,b)$, such that $c_i = \int_a^b f(t)\overline{\phi_i}(t)dt$ and*

$$\left\|f\right\|_{L_q(a,b)} \le M^{\frac{2-p}{p}} \left(\sum_{i=1}^{+\infty} |c_i|^p\right)^{\frac{1}{p}}.$$

3. On the basis property of a system of exponentials and trigonometric systems in weighted grand Lebesgue spaces

Let $\rho : [a, b] \to \mathbb{R}_+$ be some weight function. $A_p(a, b), 1 < p < +\infty$, denotes the Muckenhoupt class, that is, the class of weight functions $\rho(t)$ satisfying the condition

$$\sup_{I\subset[a,b]} \frac{1}{|I|}\int_I \rho^p(t)dt\left(\frac{1}{|I|}\int_I \rho(t)^{-\frac{p}{p-1}}dt\right)^{p-1} < +\infty.$$

Let $L_{p),\rho}(a, b)$ be the weighted grand-Lebesgue Banach space of measurable functions f on $[a, b]$ with finite norm

$$\left\|f\right\|_{L_{p),\rho}(a,b)} = \left\|f\rho\right\|_{L_{p)}(a,b)},$$

and $W^2_{p),\rho}(a, b), 1 < p < +\infty$ be the weighted grand-Sobolev space of measurable functions on $[a, b]$ with finite norm

$$\|f\|_{W^2_{p),\rho}(a,b)} = \|\rho f\|_{W^2_{p)}(a,b)},$$

where $W^2_{p)}(a, b)$ shows grand-Sobolev space of the functions $f \in L_{p)}(a, b)$ such that $f'' \in L_{p)}(a, b)$.

We need the following

Lemma 3.1. *Let the weight function ρ belong to the class $A_p(0, 1), 1 < p < +\infty$. Then there is a number $r_0 \in (1, +\infty)$ such that for $\forall r \in (1, r_0)$ there is a continuous embedding $L_{p),\rho}(0, 1) \subset L_r(0, 1)$.*

Proof. To begin with, we prove the theorem for the case of the space $L_{p,\rho}(0, 1)$. From the known result on the class $A_p(0, 1)$, there is a number $0 < \varepsilon < p - 1$, which is true for the inclusion $\rho \in A_{p-\varepsilon}(0, 1)$.

Now we choose $r_0 \in (1, p) : \frac{1}{r_0} = 1 + \frac{1}{p} - \frac{1}{p-\varepsilon}$. Then

$$\frac{p-\varepsilon}{p-\varepsilon-1} = \frac{pr_0}{p-r_0}. \tag{7}$$

From (7) we obtain

$$\frac{p-\varepsilon}{p-\varepsilon-1} > \frac{pr}{p-r} \quad \forall r \in (1, r_0). \tag{8}$$

Since the inclusion $\rho^{-1} \in \left(L_{p-\varepsilon}(0, 1)\right)^*$ is valid, it follows from (8) that the inclusion $\rho^{-r} \in \left(L_{\frac{p}{r}}(0, 1)\right)^*$ also holds for $\forall r \in (1, r_0)$. Then for $\forall f \in L_{p,\rho}(0, 1)$ from $|f(t)|^r = |f(t)\rho(t)|^r\rho^{-r}(t)$, belonging to $|f(t)\rho(t)|^r \in L_{\frac{p}{r}}(0, 1)$ and $\rho^{-r} \in \left(L_{\frac{p}{r}}(0, 1)\right)^*$ it follows that $f \in L_r(0, 1)$, for $\forall r \in (1, r_0)$. Indeed, using Hölder's inequality with exponent $\frac{p}{r}$, we obtain

$$\left(\int_0^1 |f(t)|^r dt\right)^{\frac{1}{r}} = \left(\int_0^1 |f(t)\rho(t)|^r\rho^{-r}(t)dt\right)^{\frac{1}{r}} \le c_{p,r}(\rho)\left(\int_0^1 |f(t)\rho(t)|^p dt\right)^{\frac{1}{p}} < +\infty, \tag{9}$$

where $c_{p,r}(\rho) = \left(\int_0^1 \rho(t)^{-\frac{pr}{p-r}}dt\right)^{\frac{p-r}{pr}} < +\infty$.

Further, let the number $0 < \varepsilon < p - 1$, be such that the inclusion $\rho \in A_{p-\varepsilon}(0, 1)$ is true. Moreover, let a number such that the inclusion is true. By what was proved, there exists $r_0 \in (1, +\infty)$ such that for $\forall r \in (1, r_0)$, the embedding $L_{p-\varepsilon,\rho}(0, 1) \subset L_r(0, 1)$ takes place. Thus, according to the embedding $L_{p),\rho}(0, 1) \subset L_{p-\varepsilon,\rho}(0, 1)$, we obtain the embedding $L_{p),\rho}(0, 1) \subset L_r(0, 1)$. Finally, taking into account (9), we obtain

$$\left\|f\right\|_{L_r(0,1)} \le c_{p,r}(\rho)\left\|f\right\|_{L_{p-\varepsilon,\rho}(0,1)} \le c_{p,r}(\rho)\varepsilon^{-\frac{1}{p-\varepsilon}}\left\|f\right\|_{L_{p),\rho}(0,1)},$$

that is, the embedding is continuous. □

Let $G_{p),\rho}(a,b)$ denote the subspace of the space $L_{p),\rho}(a,b)$ of functions f such that $\rho f \in G_{p)}(a,b)$. Let us prove that the classical system of exponentials is a basis in the spaces $G_{p),\rho}(-1,1)$.

Theorem 3.2. *Let the weight function ρ belong to the class $A_p(-1,1)$. Then the system of exponentials $\{e^{in\pi x}\}_{n\in\mathbb{Z}}$ forms a basis in the space $G_{p),\rho}(-1,1), 1 < p < +\infty$.*

Proof. It is easy to show that the system of functionals $\{v_n\}_{n\in\mathbb{Z}}$ given by the equality

$$< f, v_n >= \frac{1}{2}\int_{-1}^{1} f(x)e^{-in\pi x}dx, \qquad n \in \mathbb{Z},$$

is a biorthogonal system of the system $\{e^{in\pi x}\}_{n\in\mathbb{Z}}$. It is known ([32, 33]) that if $\rho \in A_p(-1,1)$ belongs, the system of exponentials $\{e^{in\pi x}\}_{n\in\mathbb{Z}}$ forms a basis in the space $L_{p,\rho}(-1,1)$. Then from the density of $L_{p,\rho}(-1,1)$ in $G_{p),\rho}(-1,1)$, taking into account the continuous embedding of $L_{p,\rho}(-1,1) \subset G_{p),\rho}(-1,1)$, we obtain that the system $\{e^{in\pi x}\}_{n\in\mathbb{Z}}$ is complete in $G_{p),\rho}(-1,1)$. Indeed, let $f \in G_{p),\rho}(-1,1)$ and $\delta > 0$. Then from the density $L_{p,\rho}(-1,1)$ in $G_{p),\rho}(-1,1)$ there exists $g \in L_{p,\rho}(-1,1)$ such that

$$\left\|f-g\right\|_{L_{p),\rho}(-1,1)} < \frac{\delta}{2}. \tag{10}$$

Since the system $\{e^{in\pi x}\}_{n\in\mathbb{Z}}$ is complete in $L_{p,\rho}(-1,1)$, there exists an $h \in span\{e^{in\pi x}\}_{n\in\mathbb{Z}}$ such that

$$\left\|g-h\right\|_{L_{p,\rho}(-1,1)} < \frac{\delta}{2c}, \tag{11}$$

where the number $c > 0$ is such that $\left\|f\right\|_{L_{p),\rho}(-1,1)} \le c\left\|f\right\|_{L_{p,\rho}(-1,1)}, f \in L_{p,\rho}(-1,1)$. Then using (10) and (11) we obtain

$$\left\|f-h\right\|_{L_{p),\rho}(-1,1)} \le \left\|f-g\right\|_{L_{p),\rho}(-1,1)} + \left\|g-h\right\|_{L_{p),\rho}(-1,1)} < \delta,$$

i.e. $\{e^{in\pi x}\}_{n\in\mathbb{Z}}$ is complete in $G_{p),\rho}(-1,1)$. It remains to show that the sequence of projectors

$$S_m(f)(x) = \sum_{n=-m}^{m} < f, v_n > e^{in\pi x}, \forall f \in G_{p),\rho}(-1,1), m \in \mathbb{Z}_+,$$

is uniformly bounded. Since $\{e^{in\pi x}\}_{n\in\mathbb{Z}}$ is basis in the space $L_{p,\rho}(-1,1)$, there exists a $c_p > 0$ such that

$$\left\|S_m(f)\right\|_{L_{p,\rho}(-1,1)} \le c_p\left\|f\right\|_{L_{p,\rho}(-1,1)}. \tag{12}$$

Since $\rho \in A_p$, there exists an $\varepsilon_0 \in (0, p-1)$ such that $\rho \in A_{p-\varepsilon_0}$. Then there is $c_{p-\varepsilon_0} > 0$ such that

$$\left\|S_m(f)\right\|_{L_{p-\varepsilon_0,\rho}(-1,1)} \le c_{p-\varepsilon_0}\left\|f\right\|_{L_{p-\varepsilon_0,\rho}(-1,1)}. \tag{13}$$

Using (12) and (13) according to the Riesz-Thorin theorem for $\varepsilon : \varepsilon_0 \le \varepsilon < p-1$ we get

$$\left\|S_m(f)\right\|_{L_{p-\varepsilon,\rho}(-1,1)} \le c_1(p,\varepsilon_0)\left\|f\right\|_{L_{p-\varepsilon,\rho}(-1,1)}. \tag{14}$$

In other case for $\varepsilon : 0 < \varepsilon < \varepsilon_0$, using Hölder's inequality and (12), we have

$$\left\|S_m(f)\right\|_{L_{p-\varepsilon,\rho}(-1,1)} \le c_2(p,\varepsilon_0)\left\|f\right\|_{L_{p-\varepsilon_0,\rho}(-1,1)}. \tag{15}$$

Therefore, taking into account (14) and (15), we obtain

$$\left\|S_m(f)\right\|_{L_{p),\rho}(-1,1)} \le \sup_{\varepsilon_0\le\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}}\left\|S_m(f)\right\|_{L_{p-\varepsilon,\rho}(-1,1)} + \sup_{0<\varepsilon<\varepsilon_0} \varepsilon^{\frac{1}{p-\varepsilon}}\left\|S_m(f)\right\|_{L_{p-\varepsilon,\rho}(-1,1)} \le$$

$$\le c_1(p,\varepsilon_0)\left\|f\right\|_{L_{p),\rho}(-1,1)} + c_3(p,\varepsilon_0)\left\|f\right\|_{L_{p),\rho}(-1,1)} = c(p,\varepsilon_0)\left\|f\right\|_{L_{p),\rho}(-1,1)},$$

i.e. $\|S_m\| \le c(p,\varepsilon_0)$. The theorem is proved. □

Now let us establish the basicity of trigonometric systems of sines and cosines in the weighted grand-Lebesgue spaces.

Theorem 3.3. *Let the weight function ρ belong to the class $A_p(0,1)$. Then the system of sines $\{\sin \pi nx\}_{n\in\mathbb{N}}$ and cosines $\{\cos \pi nx\}_{n\in\mathbb{Z}_+}$ form a basis in the space $G_{p),\rho}(0,1), 1 < p < +\infty$.*

Proof. Let $\theta(x)$ be an even extension of the function $\rho(x)$ to $[-1,1]$, that is,

$$\theta(x) = \begin{cases} \rho(t), t \in [0,1] \\ \rho(-t), t \in [-1,0] \end{cases}.$$

Let us show that from $\rho \in A_p(0,1)$ it follows that $\theta \in A_p(-1,1)$. For an arbitrary interval $I \subset [-1,1], I = I_1 \cup I_2$, where $I_1 = I \cap [0,1]$ and $I_2 = I \cap [-1,0]$, consider the set

$$I_+ = \begin{cases} I_1, |I_2| \le |I_1| \\ I_2, |I_1| \le |I_2| \end{cases}.$$

Let $J = I_+ \cup I_-$, where I_- is the set of numbers opposite to the numbers in the set I_+. It is obvious that $I \subset J \subset [-1,1]$. We have

$$\sup_{I\subset[-1,1]} \frac{1}{|I|}\int_I \theta^p(t)dt\left(\frac{1}{|I|}\int_I \theta(t)^{-\frac{p}{p-1}}dt\right)^{p-1} \le \sup_{I\subset[-1,1]} \frac{1}{|I_+|}\int_J \theta^p(t)dt\left(\frac{1}{|I_+|}\int_J \theta(t)^{-\frac{p}{p-1}}dt\right)^{p-1} =$$

$$= \sup_{I\subset[-1,1]} \frac{2}{|I_+|}\int_{I_+} \theta^p(t)dt\left(\frac{2}{|I_+|}\int_{I_+} \theta(t)^{-\frac{p}{p-1}}dt\right)^{p-1} = 2^p \sup_{I\subset[0,1]} \frac{1}{|I|}\int_I \rho^p(t)dt\left(\frac{1}{|I|}\int_I \rho(t)^{-\frac{p}{p-1}}dt\right)^{p-1} < +\infty.$$

Let us take an arbitrary function $f \in G_{p),\rho}(0,1)$ and extend it in an odd way, namely, let

$$F(t) = \begin{cases} f(t), t \in [0,1] \\ -f(t), t \in [-1,0] \end{cases}.$$

We have

$$\|F\|_{L_{p),\theta}(-1,1)} = \sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon}{2}\int_{-1}^{1} |F(t)\theta(t)|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}} = \sup_{0<\varepsilon<p-1}\left(\varepsilon\int_0^1 |f(t)\rho(t)|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}} = \left\|f\right\|_{L_{p),\rho}(0,1)} < +\infty$$

i.e. $F \in L_{p),\theta}(-1,1)$. It follows immediately that $F \in G_{p),\theta}(-1,1)$ (see Lemma 2.3, [27]). Then F can be expanded in the basis $\{e^{in\pi x}\}_{n\in\mathbb{Z}}$ in the form

$$F(x) = \sum_{n=-\infty}^{+\infty} c_n e^{in\pi x}.$$

For the coefficients c_n we have

$$c_n = \frac{1}{2}\int_{-1}^{1} F(t)e^{-in\pi t}dt = \frac{1}{2}\int_0^1 f(t)e^{-in\pi t}dt - \frac{1}{2}\int_0^1 f(t)e^{in\pi t}dt =$$

$$= -\frac{1}{2}\int_0^1 f(t)\left(e^{in\pi t} - e^{-in\pi t}\right)dt = \frac{1}{i}\int_0^1 f(t)\sin \pi nt dt = \frac{1}{2i} < f, g_n >, n \in \mathbb{N}$$

where $< f, g_n >= 2\int_0^1 f(x)\sin \pi nx dx$. It is clear that the system $\{g_n\}_{n\in\mathbb{N}}$ is a biorthogonal system to the system $\{\sin \pi nt\}_{n\in\mathbb{N}}$. Taking into account the equality $c_{-n} = -c_n$, $n \in \mathbb{N}$, and $c_0 = 0$ for $\forall m \in \mathbb{N}$, we obtain

$$\sum_{n=-m}^{m} c_n e^{i\pi nt} = \sum_{n=1}^{m} c_n e^{i\pi nt} - \sum_{n=1}^{m} c_n e^{-i\pi nt} = \sum_{n=1}^{m} c_n\left(e^{i\pi nt} - e^{-i\pi nt}\right) =$$

$$= 2i\sum_{n=1}^{m} c_n \sin \pi nt = \sum_{n=1}^{m} < f, g_n > \sin \pi nt.$$

It is easy to show that $F(t) - \sum_{n=-m}^{m} c_n e^{int}$ is an odd extension of $f(t) - \sum_{n=1}^{m} < f, g_n > \sin \pi nt$ to $[-1,1]$. Therefore, as $m \to \infty$, we obtain

$$\left\| f - \sum_{n=1}^{m} < f, g_n > \sin \pi nt \right\|_{L_{p),\rho}(0,1)} = \left\| F - \sum_{n=-m}^{m} c_n e^{i\pi nt} \right\|_{L_{p),\theta}(-1,1)} \to 0,$$

that is, the system $\{\sin \pi nt\}_{n\in\mathbb{N}}$ forms a basis in the space $G_{p),\rho}(0,1)$.

The system of cosines $\{\cos \pi nt\}_{n\in\mathbb{Z}_+}$ in the space $G_{p),\rho}(0,1)$ is proved in a similar way. The theorem is proved. □

Remark 3.4. *i) Let the weight function ρ belong to the class $A_p(-1,1)$. Then there exists a number $r > 2$ such that the system of exponentials $\{e^{in\pi x}\}_{n\in\mathbb{Z}}$ forms an r-basis in the space $G_{p),\rho}(-1,1), 1 < p < +\infty$.*

ii) Let the weight function ρ belong to the class $A_p(0,1)$. Then there is a number $r > 2$ such that the trigonometric systems $\{\sin \pi nx\}_{n\in\mathbb{N}}$ and $\{\cos \pi nx\}_{n\in\mathbb{Z}_+}$ form an $r-$basis in the space $G_{p),\rho}(0,1), 1 < p < +\infty$.

Indeed, by Lemma 3.1, there exists a number $r > 2$ such that there is a continuous embedding $G_{p),\rho}(0,1) \subset L_{r'}(0,1), r' = \frac{r}{r-1}$.

Therefore, by the Hausdorff-Young Theorem, for $\forall f \in G_{p),\rho}(0,1)$, the following inequality holds:

$$\left(\sum_{n\in\mathbb{Z}} |c_n|^r\right)^{\frac{1}{r}} \le c \left\|f\right\|_{L_{r'}(0,1)} \le M \left\|f\right\|_{L_{p),\rho}(0,1)},$$

where $c_n = \frac{1}{2}\int_{-1}^{1} f(x)e^{-in\pi x}dx$. Therefore, the system $\{e^{in\pi x}\}_{n\in\mathbb{Z}}$ is an $r-$basis in $G_{p),\rho}(0,1)$. The $r-$basicity of systems of sines and cosines in $G_{p),\rho}(0,1)$ is established similarly.

4. On the basis property of the system of eigenfunctions of the differential operator of the corresponding problem (1),(2)

Let $GW^2_{p)}(a,b), 1 < p < +\infty$ denote the grand-Sobolev subspace of the space $W^2_{p)}(a,b)$ (see [29]) of the functions $f \in W^2_{p)}(a,b)$ for which $f'' \in G_{p)}(a,b)$. Consider the following direct sum

$$GW^2_{p)}\left(\left(0,\frac{1}{3}\right)\cup\left(\frac{1}{3},1\right)\right) = GW^2_{p)}\left(0,\frac{1}{3}\right) \oplus GW^2_{p)}\left(\frac{1}{3},1\right).$$

Consider the operator L in the space $G_{p)}(0,1) \oplus \mathbb{C}$ by the formula

$$L(\hat{y}) = \left(-y''; y'\left(\frac{1}{3}-0\right) - y'\left(\frac{1}{3}+0\right)\right), \tag{16}$$

which domain $D_{p)}(L)$ consists of

$$\hat{y} = \left(y; my\left(\frac{1}{3}\right)\right) \in GW^2_{p)}\left(\left(0,\frac{1}{3}\right)\cup\left(\frac{1}{3},1\right)\right) \oplus \mathbb{C}$$

satisfying the conditions

$$y(0) = y(1) = 0, \qquad y\left(\frac{1}{3}-0\right) = y\left(\frac{1}{3}+0\right).$$

It follows from the results of [3, 4] that the operator L defined by equality (16) with the domain $D_p(L)$ consisting of $\hat{y} = \left(y, my\left(\frac{1}{3}\right)\right)$ such that

$$y \in W_p^2\left(\left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, 1\right)\right), \qquad y(0) = y(1) = 0, y\left(\frac{1}{3} - 0\right) = y\left(\frac{1}{3} + 0\right)$$

is a closed densely defined operator in the space $L_p(0,1) \oplus \mathbb{C}$, with a compact resolvent. Moreover, the eigenvalues of the operator L and problem (1), (2) coincide, and the system $\{\hat{y}_n\}_{n\in\mathbb{N}}$ of eigenfunctions and associated functions of the operator L are expressed by the system of eigenfunctions and associated functions $\{y_n\}_{n\in\mathbb{Z}_+}$ of problem (1), (2) in the form

$$\hat{y}_n = \left(y_n; my_n\left(\frac{1}{3}\right)\right). \tag{17}$$

These facts also hold for the operator L, with the domain of definition $D_{p)}(L)$ in the space $G_{p)}(0,1) \oplus \mathbb{C}$. The following takes place.

Theorem 4.1. *Let L be an operator defined in $G_{p)}(0,1) \oplus \mathbb{C}, 1 < p < +\infty$ by formula* (16) *with domain $D_{p)}(L)$. Then the operator L is a closed densely defined operator in the space $G_{p)}(0,1) \oplus \mathbb{C}$, with a compact resolvent. The eigenvalues of the operator L and problem* (1),(2) *coincide, and the corresponding eigenvectors are*

$$\begin{aligned} \hat{y}_{1,n}(x) &= \left(y_{1,n}(x); my_{1,n}\left(\frac{1}{3}\right)\right), \quad n \in \mathbb{N} \\ \hat{y}_{2,n}(x) &= \left(y_{2,2n}(x); my_{2,2n}\left(\frac{1}{3}\right)\right), \quad n \in \mathbb{Z}_+ \\ \hat{y}_{3,n}(x) &= \left(y_{2,2n-1}(x); my_{2,2n-1}\left(\frac{1}{3}\right)\right), \quad n \in \mathbb{N} \end{aligned} \tag{18}$$

where the systems $\{y_{1,n}\}_{n\in\mathbb{N}}$ and $\{y_{2,n}\}_{n\in\mathbb{Z}_+}$ are expressed by formulas (3) *and* (4), *respectively.*

Proof. Obviously, there is a continuous embedding

$$D_p(L) \subset D_{p)}(L) \subset D_{p-\varepsilon}(L), \qquad \varepsilon \in (0, p-1)$$

Take arbitrary $\hat{y} \in G_{p)}(0,1) \oplus \mathbb{C}$ and a positive number $\delta > 0$. Due to the density $L_p(0,1) \oplus \mathbb{C}$ in $G_{p)}(0,1) \oplus \mathbb{C}$, there exists $\hat{u} \in L_p(0,1) \oplus \mathbb{C}$ such that

$$\left\|\hat{y} - \hat{u}\right\|_{L_{p)}(0,1)\oplus\mathbb{C}} < \delta. \tag{19}$$

Since $D_p(L)$ is dense in $L_p(0,1) \oplus \mathbb{C}$, there exists $\hat{v} \in D_p(L)$ such that

$$\|\hat{u} - \hat{v}\|_{L_p(0,1)\oplus\mathbb{C}} < \delta.$$

The continuity of the embedding $L_p(0,1) \subset L_{p)}(0,1)$ implies that there exists a number $c > 0$ such that

$$\|\hat{u} - \hat{v}\|_{L_{p)}(0,1)\oplus\mathbb{C}} \le c\,\|\hat{u} - \hat{v}\|_{L_p(0,1)\oplus\mathbb{C}} < c\delta. \tag{20}$$

Therefore, from (19) and (20) we obtain

$$\left\|\hat{y} - \hat{v}\right\|_{L_{p)}(0,1)\oplus\mathbb{C}} \le \left\|\hat{y} - \hat{u}\right\|_{L_{p)}(0,1)\oplus\mathbb{C}} + \|\hat{u} - \hat{v}\|_{L_{p)}(0,1)\oplus\mathbb{C}} < \delta + c\delta = (1 + c)\delta.$$

Hence, taking into account $D_p(L) \subset D_{p)}(L)$, we conclude that $\hat{y} \in \overline{D_{p)}(L)}$.

Let us establish the closedness of the operator L in $G_{p)}(0,1) \oplus \mathbb{C}$. Let the sequences $\hat{x}_n \in D_{p)}(L)$ and $L(\hat{x}_n) = \hat{z}_n$ converge in $G_{p)}(0,1) \oplus \mathbb{C}$ to $\hat{x} \in G_{p)}(0,1) \oplus \mathbb{C}$ and $\hat{z} \in G_{p)}(0,1) \oplus \mathbb{C}$, respectively. Let us fix a number $\varepsilon \in (0, p-1)$. From $\hat{x}_n \in D_{p-\varepsilon}(L)$ and the continuity of the embedding $D_{p)}(L) \subset D_{p-\varepsilon}(L)$ we obtain that $\hat{x}_n$ converges to $\hat{x}$ in $L_{p-\varepsilon}(0,1) \oplus \mathbb{C}$. Since the operator L is closed (see [4]) in $L_{p-\varepsilon}(0,1) \oplus \mathbb{C}$, we obtain that $\hat{x} \in D_{p-\varepsilon}(L)$ and $L(\hat{x}) = \hat{z}$. It follows from $\hat{z} \in G_{p)}(0,1) \oplus \mathbb{C}$ and $L(\hat{x}) = \hat{z}$ that $x'' \in G_{p)}(0,1)$, and therefore $\hat{x} \in D_{p)}(L)$, that is, the operator L is closed in $G_{p)}(0,1) \oplus \mathbb{C}$. The rest of the proof is clear. The theorem is proved. □

As is known ([4]), the system of eigenvectors and associated vectors of the operator L with domain $D_p(L)$ forms a basis in the space $L_p(0,1)\oplus\mathbb{C}$ and its biorthogonally conjugate system has the form

$$\hat{z}_n=\left(z_n;\overline{m}z_n\left(\frac{1}{3}\right)\right), \tag{21}$$

where $\{z_n\}_{n\in\mathbb{Z}_+}$ is the system of eigenfunctions of the corresponding adjoint spectral problem

$$z''(x)+\lambda z(x)=0, \qquad x\in\left(0,\frac{1}{3}\right)\cup\left(\frac{1}{3},1\right),$$

$$z(0)=z(1)=0,$$

$$z\left(\frac{1}{3}-0\right)=z\left(\frac{1}{3}+0\right),$$

$$z'\left(\frac{1}{3}-0\right)-z'\left(\frac{1}{3}+0\right)=\lambda\overline{m}z\left(\frac{1}{3}\right).$$

The system $\{z_n\}_{n\in\mathbb{Z}_+}$ is determined by the formulas

$$z_{1,n}(x)=2\sin 3\pi nx \quad x\in[0,1], \quad n\in\mathbb{N}. \tag{22}$$

$$z_{2,n}(x)=\begin{cases} c_{2,n}\left(\sin\frac{3\pi n}{2}\left(x-\frac{1}{3}\right)+\sin\frac{3\pi n}{2}\left(x+\frac{1}{3}\right)\right)+O\left(\frac{1}{n}\right), & x\in\left[0,\frac{1}{3}\right]\\ c_{2,n}\sin\frac{3\pi n}{2}(1-x)+O\left(\frac{1}{n}\right), & x\in\left[\frac{1}{3},1\right]\end{cases}, n\in\mathbb{Z}_+ \tag{23}$$

where $c_{2,n}$ are normalizing numbers for which the asymptotic relations are valid

$$c_{2,n}=\frac{2+(-1)^n}{3}+O\left(\frac{1}{n}\right), \quad n\in\mathbb{Z}_+.$$

Let us prove that a similar fact holds for the operator L with domain $D_{p)}(L)$ in the space $G_{p),\rho}(0,1)\oplus\mathbb{C}$.

Theorem 4.2. *Let the weight function ρ belong to the class $A_p(0,1)$. Then the system $\{\hat{y}_n\}_{n\in\mathbb{Z}_+}$ of eigenvectors and associated vectors of the operator L forms a basis in the space $G_{p),\rho}(0,1)\oplus\mathbb{C}, 1<p<+\infty$.*

Proof. It follows from Theorem 4.1 that the system $\{\hat{y}_n\}_{n\in\mathbb{Z}_+}$ of eigenvectors and associated vectors of the operator L is defined by formulas (17) and (18). It is not difficult to show that the system $\{\hat{z}_n\}_{n\in\mathbb{Z}_+}$ is also a biorthogonal system for $\{\hat{y}_n\}_{n\in\mathbb{Z}_+}$ in $G_{p)}(0,1)\oplus\mathbb{C}$. Consider the following system of functions

$$u_{1,n}(x)=\sin 3\pi nx, \quad x\in[0,1], \quad n\in\mathbb{N},$$

$$u_{2,n}(x)=\begin{cases} 2(-1)^n\sin 3\pi nx, & x\in\left[0,\frac{1}{3}\right]\\ \sin 3\pi nx, & x\in\left[\frac{1}{3},1\right]\end{cases}, \quad n\in\mathbb{N}, \tag{24}$$

$$u_{3,n}(x)=\begin{cases} 0, & x\in\left[0,\frac{1}{3}\right]\\ -\cos 3\pi\left(n-\frac{1}{2}\right)x, & x\in\left[\frac{1}{3},1\right]\end{cases}, \quad n\in\mathbb{N}.$$

Comparing formulas (4) and (24), we obtain

$$y_{,n}(x)=u_{1,n}(x)+O\left(\frac{1}{n}\right), \quad n\in\mathbb{N},$$

$$y_{2,2n}(x)=u_{2,n}(x)+O\left(\frac{1}{n}\right), \quad n\in\mathbb{N}, \tag{25}$$

$$y_{2,2n-1}(x) = u_{3,n}(x) + O\left(\frac{1}{n}\right), \quad n \in \mathbb{N}.$$

We put

$$e_{1,n}(x) = \begin{cases} \sin 3\pi n x, & x \in \left[0, \frac{1}{3}\right] \\ 0, & x \in \left[\frac{1}{3}, 1\right] \end{cases},$$

$$e_{2,n}(x) = \begin{cases} 0, & x \in \left[0, \frac{1}{3}\right] \\ \sin 3\pi n x, & x \in \left[\frac{1}{3}, 1\right] \end{cases},$$

$$e_{3,n}(x) = \begin{cases} 0, & x \in \left[0, \frac{1}{3}\right] \\ -\cos 3\pi \left(n - \frac{1}{2}\right) x, & x \in \left[\frac{1}{3}, 1\right] \end{cases}.$$

Theorem 3.3 immediately implies that the system $\{\sin 3\pi n x\}_{n\in\mathbb{N}}$ forms a basis in $G_{p),\rho}\left(0, \frac{1}{3}\right)$. Changing the variable in the form $t = \frac{3x-1}{2}$, the basis property of the system $\{\sin 3\pi n x; -\cos 3\pi \left(n - \frac{1}{2}\right)\}_{n\in\mathbb{N}}$ in $G_{p),\rho}\left(\frac{1}{3}, 1\right)$ reduces to the basis property of the system $\{\sin \pi n t\}_{n\in\mathbb{N}}$ in $G_{p),\rho}(0, 1)$. Therefore, the system $\{e_{i,n}\}_{i=\overline{1,3},n\in\mathbb{N}}$ forms a basis in $G_{p),\rho}(0, 1)$. According to (24), the system $\{u_{i,n}\}_{i=\overline{1,3},n\in\mathbb{N}}$ is transformed through the system $\{e_{i,n}\}_{i=\overline{1,3},n\in\mathbb{N}}$ by the formula

$$u_{i,n}(x) = \sum_{j=1}^{3} a_{ij}^{(n)} e_{j,n}(x), \quad i = 1, 2, 3, \quad n \in \mathbb{N}. \tag{26}$$

The matrix of transformation coefficients (26) has the form

$$A_n = \begin{pmatrix} 1 & 1 & 0 \\ 2(-1)^n & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and $\det A_n = 1 - 2(-1)^n \neq 0$. Hence, by Theorem 2.2, the system $\{u_{i,n}\}_{i=\overline{1,3},n\in\mathbb{N}}$ also forms a basis in $G_{p),\rho}(0, 1)$. Then it is obvious that the system $\{\hat{u}_0\} \cup \{\hat{u}_{i,n}\}_{i=\overline{1,3},n\in\mathbb{N}}$ forms a basis in $G_{p),\rho}(0, 1)$, where

$$\hat{u}_0(x) = (0; 1), \quad \hat{u}_{i,n}(x) = (u_{i,n}(x); 0), \quad i = \overline{1,3}, \quad n \in \mathbb{N}.$$

Further, take an arbitrary vector $\hat{f} = (f; \alpha) \in L_{p),\rho}(0, 1) \oplus \mathbb{C}$. By Lemma 3.1, there exists a number $r \in (1, 2)$ such that $L_{p),\rho}(0, 1)$ is continuously embedded in $L_r(0, 1)$. Then $f \in L_r(0, 1)$ and, according to the Hausdorff-Young inequality and the continuity of the embedding $L_{p),\rho}(0, 1) \subset L_r(0, 1)$, we have

$$\left(\sum_{i=1}^{3} \sum_{n=1}^{+\infty} | < f, e_{i,n} > |^{r'}\right)^{\frac{1}{r'}} \leq M_1 \|f\|_{L_{p),\rho}(0,1)}, \tag{27}$$

where $M_1 > 0$ is some number, $\frac{1}{r} + \frac{1}{r'} = 1$. It is easy to show that the system $\{v_{i,n}\}_{i=\overline{1,3},n\in\mathbb{N}}$ transforms through the system $\{e_{i,n}\}_{i=\overline{1,3},n\in\mathbb{N}}$ in the form

$$v_{i,n}(x) = \sum_{j=1}^{3} b_{ij}^{(n)} e_{j,n}(x), \quad i = 1, 2, 3, \quad n \in \mathbb{N},$$

where the transformation matrix is $B_n = \left(b_{i,j}^{(n)}\right) = \left(A_n^{-1}\right)^*$. Then

$$| < f, v_{i,n} > |^{r'} \leq \left(\sum_{j=1}^{3} |b_{ij}^{(n)}|^r\right)^{\frac{r'}{r}} \sum_{j=1}^{3} | < f, e_{j,n} > |^{r'}.$$

Hence, taking into accou(27), we obtain

$$\left(\sum_{i=1}^{3}\sum_{n=1}^{+\infty}|<f,v_{i,n}>|^{r'}\right)^{\frac{1}{r'}} \le \sup_n\left(\sum_{j=1}^{3}|b_{ij}^{(n)}|^{r}\right)^{\frac{1}{r}}\left(\sum_{i=1}^{3}\sum_{n=1}^{+\infty}|<f,e_{i,n}>|^{r'}\right)^{\frac{1}{r'}} \le$$

$$\le M_1\sup_n\left(\sum_{j=1}^{3}|b_{ij}^{(n)}|^{r}\right)^{\frac{1}{r}}\left\|f\right\|_{L_{p),\rho}(0,1)} = M_2\left\|f\right\|_{L_{p),\rho}(0,1)}. \tag{28}$$

Thus, taking into account (28) and that $<f,v_0>=\alpha$, we obtain

$$\left(\sum_{i=1}^{3}\sum_{n=1}^{+\infty}|<f,v_{i,n}>|^{r'}+|<f,v_0>|^{r'}\right)^{\frac{1}{r'}} \le \left(\sum_{i=1}^{3}\sum_{n=1}^{+\infty}|<f,v_{i,n}>|^{r'}\right)^{\frac{1}{r'}}+|<f,v_0>|^{r'} \le$$

$$\le M_2\left\|f\right\|_{L_{p),\rho}(0,1)}+|\alpha| \le M_2\left(\left\|f\right\|_{L_{p),\rho}(0,1)}+|\alpha|\right) = M_2\left\|\hat{f}\right\|_{L_{p),\rho}(0,1)}.$$

This inequality implies that the system $\{\hat{u}_0\}\cup\{\hat{u}_{i,n}\}_{i=\overline{1,3},n\in\mathbb{N}}$ is an $r'-$basis in $G_{p),\rho}(0,1)\oplus\mathbb{C}$. We put

$$\hat{y}_0(x) = (y_0(x);1) = (0;1),$$

$$\hat{y}_{1,n}(x) = \left(y_{1,n}(x); my_{1,n}\left(\frac{1}{3}\right)\right),$$

$$\hat{y}_{2,n}(x) = \left(y_{2,2n}(x); my_{2,2n}\left(\frac{1}{3}\right)\right),$$

$$\hat{y}_{3,n}(x) = \left(y_{2,2n-1}(x); my_{2,2n-1}\left(\frac{1}{3}\right)\right).$$

It follows from (25) that for $\forall r: r>1$ the condition

$$\sum_{i=1}^{3}\sum_{n=1}^{+\infty}\left\|\hat{y}_{i,n}-\hat{u}_{i,n}\right\|^{r}_{L_{p),\rho}(0,1)\oplus\mathbb{C}} < +\infty,$$

and hence the systems $\{\hat{y}_0\}\cup\{\hat{y}_{i,n}\}_{i=\overline{1,3},n\in\mathbb{N}}$ and $\{\hat{u}_0\}\cup\{\hat{u}_{i,n}\}_{i=\overline{1,3},n\in\mathbb{N}}$ are $r-$close. It follows from the results of [4] that the system $\{\hat{y}_0\}\cup\{\hat{y}_{i,n}\}_{i=\overline{1,3},n\in\mathbb{N}}$ is minimal in $L_p(0,1)\oplus\mathbb{C}$ and its biorthogonal system $\{\hat{v}_0\}\cup\{\hat{v}_{i,n}\}_{i=\overline{1,3},n\in\mathbb{N}}$ has the form

$$\hat{v}_0(x) = (0;1), \qquad \hat{v}_{i,n}(x) = (v_{i,n}(x);0), \qquad i=1,2,3, \qquad n\in\mathbb{N},$$

where

$$v_{1,n}(x) = \begin{cases} \frac{6}{1-2(-1)^n}\sin 3\pi nx, & x\in\left[0,\frac{1}{3}\right] \\ -\frac{6(-1)^n}{1-2(-1)^n}\sin 3\pi nx, & x\in\left[\frac{1}{3},1\right] \end{cases},$$

$$v_{2,n}(x) = \begin{cases} -\frac{6}{1-2(-1)^n}\sin 3\pi nx, & x\in\left[0,\frac{1}{3}\right] \\ \frac{3}{1-2(-1)^n}\sin 3\pi nx, & x\in\left[\frac{1}{3},1\right] \end{cases},$$

$$v_{3,n}(x) = \begin{cases} 0, & x\in\left[0,\frac{1}{3}\right] \\ -3\cos 3\pi\left(n-\frac{1}{2}\right)x, & x\in\left[\frac{1}{3},1\right] \end{cases}.$$

Taking into account the embedding of $L_p(0,1)\subset L_{p)}(0,1)\subset L_{p-\varepsilon}(0,1), 0<\varepsilon<p-1$, we see that the system $\{\hat{v}_0\}\cup\{\hat{v}_{i,n}\}_{i=\overline{1,3},n\in\mathbb{N}}$ is a system of linear continuous functionals in $L_{p)}(0,1)\oplus\mathbb{C}$, that is, the system $\{\hat{y}_0\}\cup\{\hat{y}_{i,n}\}_{i=\overline{1,3},n\in\mathbb{N}}$ is minimal in $L_{p)}(0,1)\oplus\mathbb{C}$. Thus, since $\{\hat{u}_0\}\cup\{\hat{u}_{i,n}\}_{i=\overline{1,3},n\in\mathbb{N}}$ is an $r'-$ basis in $G_{p),\rho}(0,1)\oplus\mathbb{C}$ and the system $\{\hat{y}_0\}\cup\{\hat{y}_{i,n}\}_{i=\overline{1,3},n\in\mathbb{N}}$ is $r-$close to $\{\hat{u}_0\}\cup\{\hat{u}_{i,n}\}_{i=\overline{1,3},n\in\mathbb{N}}$, it follows from the minimality of the system $\{\hat{y}_0\}\cup\{\hat{y}_{i,n}\}_{i=\overline{1,3},n\in\mathbb{N}}$ by Theorem 2.6 that it forms a basis in $G_{p),\rho}(0,1)\oplus\mathbb{C}$ isomorphic to $\{\hat{u}_0\}\cup\{\hat{u}_{i,n}\}_{i=\overline{1,3},n\in\mathbb{N}}$. The theorem is proved. □

In the next theorem, we study the basic properties of the system of eigenvectors and associated vectors $\{y_0\} \cup \{y_{i,n}\}_{i=1,2,n\in\mathbb{N}}$ of problem (1), (2) in the space $G_{p),\rho}(0,1), 1 < p < +\infty$.

Theorem 4.3. *Let the weight function ρ belong to the class $A_p(0,1)$. The following statements are true:*

i) if from the system $\{y_0\} \cup \{y_{i,n}\}_{i=1,2,n\in\mathbb{N}}$ we discard an arbitrary function $y_{2,n_0}(x)$ corresponding to a simple eigenvalue, then the resulting system forms a basis in the space $G_{p),\rho}(0,1), 1 < p < +\infty$;

ii) if from the system $\{y_0\} \cup \{y_{i,n}\}_{i=1,2,n\in\mathbb{N}}$ we discard an arbitrary function $y_{1,n_0}(x)$, then the resulting system is not a basis in $G_{p),\rho}(0,1), 1 < p < +\infty$. Moreover, this system is incomplete and not minimal in $G_{p),\rho}(0,1), 1 < p < +\infty$.

Proof. By Theorem 4.2, the system $\{\hat{y}_0\} \cup \{\hat{y}_{i,n}\}_{i=1,2,n\in\mathbb{N}}$,

$$\hat{y}_0(x) = (0;1),$$

$$\hat{y}_{i,n}(x) = \left(y_{i,n}(x); m y_{i,n}\left(\frac{1}{3}\right)\right), \qquad i = 1,2,$$

forms a basis in $G_{p),\rho}(0,1) \oplus \mathbb{C}$ and has a biorthogonal system $\{\hat{z}_0\} \cup \{\hat{z}_{i,n}\}_{i=1,2,n\in\mathbb{N}}$ given by formulas (22) and (23). Let $y_{2,n_0}(x)$ be an arbitrary eigenfunction of problem (1), (2) corresponding to a simple eigenvalue. Since $\delta = \overline{m} z_{2,n_0}\left(\frac{1}{3}\right) \neq 0$ holds for the system $\{y_0\} \cup \{y_{i,n}\}_{i=1,2,n\in\mathbb{N}}$ without $y_{2,n_0}(x)$, by virtue of Theorem 2.3 it forms a basis in $G_{p),\rho}(0,1)$, i.e., assertion *i*) holds.

Next, take an arbitrary function $y_{1,n_0}(x)$ and consider the question of the basis property of the system $\{y_0\} \cup \{y_{i,n}\}_{i=1,2,n\in\mathbb{N}}$ with the discarded function $y_{1,n_0}(x)$. For this system, we have $\delta = \overline{m} z_{1,n_0}\left(\frac{1}{3}\right) = 0$. Then, by Theorem 2.3, the resulting system is not complete and not minimal in $G_{p),\rho}(0,1)$. □

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