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Fixed Point and Common Fixed Point Theorems on (α,F)-Contractive Multi-Valued Mappings in Uniform Spaces

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Abstract. In this paper, we introduce the notion of (α,F)-contractive multi-valued mappings in the setting of uniform spaces. Some fixed point and common fixed point theorems for (α,F)-contractive mappings endowed with the uniform spaces are established. The existence and uniqueness of fixed points by using the structure of uniform spaces is discussed in detail. We set up a non-trivial example for the elaboration of these novel results. eventually, an application is also provided to elaborate the applicability of our results.

1. Introduction

There are many extensions of metric spaces out of which $u.f.s.$ is a widely known. Some important fixed point results in $u.f.s.$ are established by [5] and [38]. The famous Nadler's [28] set valued contraction mappings results provided the researcher a strong platform to start the investigation of fixed point in the context of multi-valued mappings. By using this platform, many results are developed for existence and uniqueness of fixed point and common fixed point. (see for examples; [3, 4, 6, 11–13, 15, 17, 22–25, 31, 32, 35, 36]). Since then uniform spaces has become a focus of interest for many researchers for the establishment of fixed point theorems by using various contractions [10, 20, 22, 26, 27, 38–40].

Fixed point theory provides a strong tool to develope the iterative schemes for the solution of various differential and integral equations. Recently, Olatinwo and Omidire [30] focuses on the convergence of generalized pseudo-contractive operators with regard to a unique fixed point. In this article a new Jungck–Kirk–Mann type fixed point iterative algorithm as well as the general Kirk–Mann type iterative algorithms are used to find a unique common fixed point. Similarly one can find a self-adaptive projection method for finding a common element in the solution set of variational inequalities and fixed point set for relatively non-expansive mappings in 2-uniformly convex and uniformly smooth real Banach spaces [29].

The idea of F-contraction was initiated by Wardowski [41]. Many mathematicians are intrigued by this new notion and proved existence and uniqueness of the fixed point of single valued and multivalued mappings. One can find a lot of literature on existence of fixed point by using the concept of F-mappings in different contractions. Some of the interesting results can be found in [9, 12, 13, 23, 33]. Ali *et al*. [7] introduced the

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notion of (α,F)-admissible type mappings in the setting of $u.f.s.$ By using the notions of (α,F)-admissible type mappings many motivating and important results are established on the platform of metric spaces [7, 12, 13, 21]. Some results are proved on the notion of (α,F)-admissible type multi-valued mappings by Ali *et al.* [8], Hussain *et al.* [21] and Rasham *et al.* [34].

In this article, some new fixed point theorems for a multi-valued mapping from a complete $u.f.s.$ to its hyperspace are established. For this purpose, we used the notion of (α,F)-contractive mappings accompanied with $u.f.s.$ For the comprehensive understanding of $u.f.s.$ and (α,F)-contractive mappings, the readers are referred to see ([1–3, 6, 7, 9, 12, 13, 16, 20, 21, 25–27, 34, 38–40]).

The rest of the article is organized as: In Section 2, we collected some basic definitions and results that are related to our main work. Section 3 consists of our main contributions and in Section 4, an application of the main result is presented to show the existence of a solution of a nonlinear integral equation.

2. Preliminaries

In this section, we recall some fundamental definitions which will help in understanding the rest of the sections.

Definition 2.1. *[2] Let (X, v) be a uniform space and p be an A-distance on X.*

(i) *X is $\mathcal{S}$-complete if every p-Cauchy sequence $\{x_n\}$, there exists x in X with $\lim_{n\to\infty} p(x_n, x) = 0$.*

(ii) *X is p-Cauchy complete if every p-Cauchy sequence $\{x_n\}$, there exists x in X with $\lim_{n\to\infty} x_n = x$ with respect to $\tau(v)$.*

(iii) *$T : X \to X$ is p-continuous if $\lim_{n\to\infty} p(x_n, x) = 0$ implies $\lim_{n\to\infty} p(Tx_n, Tx) = 0$.*

Following definition is crucial for our next discussion. From now on, $\mathbb{R}$ is set of real numbers and $\mathbb{R}^+$ is set of positive reals.

Definition 2.2. *A mapping $F\colon \mathbb{R}^+ \to \mathbb{R}$ is said to be an F-mapping [41], if the following conditions are observed:*

(i) *F is strictly increasing function, that is, as for all $z_1, z_2 \in \mathbb{R}^+$, if $z_1 < z_2$ then $F(z_1) < F(z_2)$.*

(ii) *For each sequence $\{z_n\}$ of the positive real numbers $\mathbb{R}^+$,*

$$\lim_{n\to\infty}(z_n) = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} F(z_n) = -\infty.$$

(iii) *There is a real number $c \in (0, 1)$ such as*

$$\lim_{z\to 0^+} z^c F(z) = 0.$$

The family of F-functions will be denoted by ▲ throughout this research.

Theorem 2.3. *[8] Consider an $\mathcal{S}$-complete Hausdorff u.f.s., (Y, v) and p an E-distance on the nonempty set Y. Let the self-mapping T on Y be an (α,F)-contractive which fulfills the following assertions:*

(i) *the T is an α-admissible;*

(ii) *for some $x_0 \in Y$, $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$;*

(iii) *T is p-continuous mapping,*

then T has a fixed point.

In a *u.f.s.*, (Y, v), denote $Q = \{p_i : i \in I\}$ by a family of pseudo-metrics on Y where I is an an indexing set. This family is known as an associated family for the uniformity v of the family.
$\varrho = \{V(i, r) : i \in I, r > 0\}$, where $V(i, r) = \{(x_1, x_2) : x_1, x_2 \in Y; p_i(x_1, x_2) < r\}$, is a sub-base for the uniformity v. We can consider ϱ itself to be a base by including the finite intersections of the members of ϱ, if needed. The related family of pseudo-metrics is called an augmented associated family for v. This family for v will be denoted by Q^*. For the more details, one can consider [38]. Now, we will denote Y the *u.f.s.*, (Y, v) defined by Q^*.
For a nonempty subset M of a *u.f.s.*, define

$$\delta^*(M) = \sup\{p_i(x_1, x_2) : x_1, x_2 \in Y, i \in I\};$$

where $\{p_i(x_1, x_2) : i \in I\} = Q^*$. Then we shall say that $\delta^*(M)$ is an augmented diameter of M. Furthermore, M is called Q^*-bounded if $\delta^*(M) < \infty$ (see [26]). Let

$$2^Y = \{M : \text{M is a nonempty } \ Q^* - \text{bounded subset of Y}\}.$$

For two nonempty $M, N \subseteq Y$, define

$$p_i(x, M) = \inf\{p_i(x, m) : m \in M, i \in I\},$$

and

$$d_i^\circ(M, N) = \sup\{p_i(m, n) : m \in M, n \in N, i \in I\},$$

and

$$H_i^\circ(M, N) = \max\{\sup_{m \in M} p_i(m, N), \sup_{n \in N} p_i(M, n)\}.$$

For a *u.f.s.*, (Y, v), consider $V \in v$ as an arbitrary entourage. The uniformity 2^v on 2^Y can be generated by the base $2^\varrho = \{V^{*\!*} : V \in v\}$, where

$$V^{*\!*} = \{(M, N) \in 2^Y \times 2^Y : M \times N \subseteq V\} \cup \delta,$$

with δ the diagonal of $Y \times Y$. $Q^{*\!*}$ induces a uniformity $v^{*\!*}$ on 2^Y defined by the base

$$\varrho^* = \{V^*(i, r) : i \in I, r > 0\},$$

where

$$V^*(i, r) = \{(M, N) \in 2^Y \times 2^Y : d_i(M, N) < \epsilon\} \cup \delta.$$

The uniformities $2^v$and v^* on 2^Y are uniformly isomorphic. The space $(2^Y, v^*)$ is thus a *u.f.s.* and it is said to be a hyperspace of (Y, v). The uniformities on Y and 2^Y can be generated by any other basis as well. [(see, for details: [14, 32])].
Some necessary definitions and notations are taken from ([18, 19]).
A sequence $M_n \subseteq 2^Y$ is said to convergent to the subset M of Y if the following two conditions are satisfied:

(i) $\forall\, m \in M$ there is a sequence $m_n \subseteq M_n$ for all n and $m_n \to m$.

(ii) $\forall\ \epsilon > 0,\ \exists\ N\ \in \mathbb{N}$, such that $M_n \subseteq M_\epsilon$ for $n \geqslant N$ where $M_\epsilon = \cup_{x \in M} \cup (x) = \{y \in Y : p_i(x, y) < \epsilon \text{ for some } x \in M, i \in I\}$.
Here M is called the limit of the sequence M_n or symbolically one can write $\lim\limits_{n \to \infty} M_n = M$.

(iii) $T : Y \to 2^Y$ is said to be continuous at $x_0 \in Y$ if for any sequence $x_n \in Y$ which converges to x, the sequence $Tx_n \in 2^Y$ converges to Tx in 2^Y. The mapping T is a continuous on Y if it is continuous at each point $x \in Y$.

3. Main Results

Definition 3.1. *Consider a u.f.s., (Y, v) with p an E-distance on Y. A mapping $T\colon Y \to 2^Y$ is said to be an (α,F)-contractive mapping if there exists a function $\alpha\colon Y \times Y \to [0, \infty)$, $F \in \blacktriangle$ and $\tau > 0$ such that for every $x_1, x_2 \in Y$, we have*

$$\tau + F\left(\alpha(x_1, x_2)H(Tx_1, Tx_2)\right) \leq F(M(x_1, x_2)), \tag{1}$$

where

$$\min\{\alpha(x_1, x_2)H(Tx_1, Tx_2)), M(x_1, x_2)\} > 0,$$

and

$$M(x_1, x_2) = \max\left\{p(x_1, x_2), \frac{p(x_1, Tx_1) + p(x_2, Tx_2)}{2}, \frac{p(x_1, Tx_2) + p(x_2, Tx_1)}{2}\right\}.$$

Theorem 3.2. *Let p be an E-distance on $\mathcal{S}$-complete Hausdorff u.f.s., (Y, v). Let $T\colon Y \to 2^Y$ be an (α,F)-contractive multi-valued mapping and it satisfies the following assertions:*

(a) *T is an α-admissible,*

(b) *There exists an $x_0 \in Y$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$,*

(c) *the T is p-continuous,*

then T has a fixed point $\eta \in Y$.

Proof. From assertion (**b**) there exists $x_0 \in Y$ such that $\alpha(x_0, Tx_0) \geq 1$. Now define a sequence $\{x_n\}$ in Y by $\phi \neq Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If $x_{n_0} \in x_{n_0+1}$ for some n_0, then T has a fixed point. Therefore, we can assume that $x_n \notin T(x_n)$ for all n. Since T is α-admissible, we have $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geq 1$. So, we conclude inductively the nth iteration

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \forall\, n \in \mathbb{N} \cup \{0\}. \tag{2}$$

(1) and (2) implies

$$\begin{aligned}
&\tau + F(\alpha(x_n, x_{n+1})H(Tx_n, Tx_{n+1}))\\
&\quad \leq F\left(\max\left\{p(x_n, x_{n+1}), \frac{p(x_n, Tx_n) + p(x_{n+1}, Tx_{n+1})}{2}, \frac{p(x_n, Tx_{n+1}) + p(x_{n+1}, Tx_n)}{2}\right\}\right)\\
&\quad \leq F(\max\{p(x_n, x_{n+1}), p(x_{n+1}, Tx_{n+1})\})\\
&\quad \leq F(p(x_n, Tx_n)) \leq F(p(x_n, x_{n+1})) \;\; \forall\, x_n, x_{n+1} \in Y.
\end{aligned}$$

Inductively, we conclude its nth iteration

$$\begin{aligned}
&n\tau + F(p(x_n, x_{n+1})) \leq F(p(x_0, x_1))\\
&F(p(x_n, x_{n+1})) \leq F(p(x_0, x_1)) - n\tau.
\end{aligned} \tag{3}$$

Taking limit $n \to \infty$ on both sides

$$\begin{aligned}
&\lim_{n\to\infty} F(p(x_n, x_{n+1})) \leq \lim_{n\to\infty}[F(p(x_0, x_1)) - n\tau]\\
&\qquad \Rightarrow \lim_{n\to\infty} F(p(x_n, x_{n+1})) = -\infty.
\end{aligned}$$

By using Definition 2.2, we have

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{4}$$

Suppose $p_n = p(x_n, x_{n+1})$.
From Definition 2.2, there exists $k \in (0,1)$ such that (4) becomes

$$\lim_{n\to\infty} p_n^k F(p_n) = 0. \tag{5}$$

Therefore (3) reduces to

$$\begin{aligned}
&F(p_n) - F(p_0) \le -n\tau \\
&p_n^k F(p_n) - p_n^k F(p_0) \le p_n^k (F(p_0) - n\tau) - p_n^k F(p_0) = -np_n^k \tau \le 0 \\
&\lim_{n\to\infty} [p_n^k F(p_n) - p_n^k F(p_0)] \le \lim_{n\to\infty} -p_n^k n\tau \\
&\lim_{n\to\infty} -np_n^k \tau \ge 0 \\
&\lim_{n\to\infty} np_n^k = 0 \quad as \quad \tau > 0.
\end{aligned}$$

There exists $n_0 \in \mathbb{N}$ such that $np_n^k \le 1$ for all $n \ge n_0$,

$$\Rightarrow \ p_n^k \le \frac{1}{n} \Rightarrow p_n \le \frac{1}{n^{\frac{1}{k}}}.$$

For $m > n$, consider

$$p(x_n, x_m) \le p(x_n, p_{n+1}) + p(x_{n+1}, x_{n+2}) + \ldots + p(x_{m-1}, x_m) \le \sum_{i=n}^{\infty} p_i$$

$$\le \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \tag{6}$$

Taking limit $n \to \infty$ on both sides and from (6), we get

$$\lim_{n\to\infty} p(x_n, x_m) \le \lim_{n\to\infty} \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}} = 0.$$

Since p, doesn't posses symmetry hence by repeating the process, we can obtain

$$\lim_{n\to\infty} p(x_m, x_n) = 0.$$

Therefore, $\{x_n\}$ is a p-Cauchy sequence.
Thus by completeness, there exists an $\eta \in Y$ such that

$$\lim_{n\to\infty} p(x_n, \eta) = 0.$$

Applying assertion **(c)** we have

$$\lim_{n\to\infty} p(Tx_n, T\eta) \le \lim_{n\to\infty} H(Tx_n, T\eta) = 0$$

and

$$\lim_{n\to\infty} p(x_{n+1}, T\eta) = 0,$$

which implies $\lim_{n\to\infty} p(x_n, \eta) = 0$ and $\lim_{n\to\infty} p(x_n, T\eta) = 0$.
Consequently T has a fixed point. □

In the next theorem, p-continuity is replaced with a suitable limiting condition on the iterative sequence.

Theorem 3.3. *Let p be an E-distance on $\mathcal{S}$-complete Hausdorff u.f.s., (Y, v). Further consider a multi-valued (α,F)-contractive mapping $T\colon Y \to 2^Y$. If T satisfies the following assertions:*

(a) *T is α-admissible,*

(b) *There exists $x_0 \in Y$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$,*

(c) *For any sequence $\{x_n\}$ in Y with $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all, $n \in \mathbb{N} \cup \{0\}$, such that $\alpha(x_n, x) \geq 1$, for all, $n \in \mathbb{N} \cup \{0\}$,*

then mapping T has a fixed point.

Proof. The proof of the first part of the result is analogous to Theorem 3.2. Hence one can prove in the same way that $\{x_n\}$ is a p-Cauchy in the $\mathcal{S}$-complete space Y. Thus, there exists $\eta \in Y$ such that

$$\lim_{n\to\infty} p(x_n, \eta) = 0,$$

implies that $\lim_{n\to\infty} x_n = \eta$. Using assertion **(c)** and the assumption that T is (α,F) contractive we obtain

$$\begin{aligned} p(x_n, T\eta) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, T\eta) \\ &\leq p(x_n, x_{n+1}) + (\tau + F(\alpha(x_n, \eta)H(Tx_n, T\eta))) \\ &\leq p(x_n, x_{n+1}) + F(M(x_n, \eta)). \end{aligned}$$

Applying limit $n \to \infty$, we get

$$\lim_{n\to\infty} p(x_n, T\eta) = 0.$$

Hence, it follows

$$\lim_{n\to\infty} p(x_n, \eta) = 0$$

and

$$\lim_{n\to\infty} p(x_n, T\eta) = 0.$$

Consequently T has a fixed point. □

Example 3.4. *Let $Y = \left\{\frac{1}{2^n} : n \in \mathbb{N}\right\} \cup \{0, 1\}$ along with the usual metric p. Define $v = \{\cup_\epsilon : \epsilon > 0\}$. It is simple to see that (Y, v) is a u.f.s. Now define $T : Y \to 2^Y$ on u.f.s. as*

$$Ta = \begin{cases} \left\{\frac{1}{2^n}, 1\right\}, & \text{if } a \in \{\frac{1}{2^{n-1}} : n \in \mathbb{N}\}, \\ \{0\}, & \text{if } a = 0, \end{cases}$$

and $\alpha : Y \times Y \to [0, \infty)$ as

$$\alpha(a, b) = \begin{cases} 1, & \text{if } a, b \in \{\frac{1}{2^{n-1}} : n \in \mathbb{N}\}, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$\min\{\alpha(a, b)H(Ta, Tb)), M(a, b)\} > 0.$$

Let $F(a) = \ln a$, for all, $a > 0$. Now through the following way, T can be easily seen as an (α,F)-contractive and α-admissible mapping. Let $a = \frac{1}{2^n}$ and $b = \frac{1}{2^m}$, such that $m > n \geq 1$. Then, by Definition 3.1,

$$\begin{aligned} F(\alpha(a,b)H(Ta,Tb)) - F(p(a,b)) &= \ln\left|\frac{2^{m-n}-1}{2^{m+1}}\right| - \ln\left|\frac{2^{m-n}-1}{2^m}\right| \\ &= \ln\frac{1}{2} < -\frac{1}{2} \ \ \forall\, a,b \in Y. \end{aligned}$$

Therefore, T is a multi-valued (α,F)-contractive mapping with $\tau = \frac{1}{2}$. Hence all the conditions of Theorem 3.2 are satisfied and $0, 1$ are fixed points of T.

For the uniqueness of the fixed point, we include the following condition:
(∇) : For all $\zeta, \eta \in Fix(T)$, there exists $b \in Y$ such that $\alpha(b,\zeta) \geq 1$ and $\alpha(b,\eta) \geq 1$, where $Fix(T)$ is set of all fixed points of T.

Theorem 3.5. *Assume that all hypothesis of Theorem 3.3 are true with an addition of condition (∇), then T has a unique fixed point.*

Proof. Suppose that there exists two fixed point u and v for T. From the imposed condition (∇), there exists $\eta \in Y$ such that

$$\alpha(\eta,u) \geq 1 \textit{ and } \alpha(\eta,v) \geq 1. \tag{7}$$

Since T is α-admissible mapping, therefore from (7),

$$\alpha(T^n\eta,u) \geq 1 \textit{ and } \alpha(T^n\eta,v) \geq 1, \text{ for all, } n \in \mathbb{N}\cup\{0\}. \tag{8}$$

Now define a sequence $\eta_n \in Y$ by $\eta_{n+1} = T\eta_n = T^n\eta_0$, for all, $n \in \mathbb{N}\cup\{0\}$.
From above both inequalities (1) and (7),

$$p(\eta_{n+1},u) = p(T\eta_n,Tu) \leq \tau + F(\alpha(\eta_n,u)H(T\eta_n,Tu)) \tag{9}$$
$$\leq F(M(\eta_n,u)) \leq F(p(\eta_n,u)), \text{ for all, } n \in \mathbb{N}\cup\{0\}. \tag{10}$$

Inductively, its nth iteration is written as

$$p(\eta_n,u) \leq n\tau + F(p(\eta_0,u)), \text{ for all, } n \in \mathbb{N}\cup\{0\}.$$

$$\Rightarrow \lim_{n\to\infty} p(\eta_n,u) = 0. \tag{11}$$

Likewise,

$$\lim_{n\to\infty} p(\eta_n,v) = 0, \Rightarrow u = v. \tag{12}$$

It follows that $u = v$.
□

Definition 3.6. *[4] A pair of self mappings (T,S) on Y is said to be α-admissible if for any $x_1, x_2 \in Y$ with $\alpha(x_1,x_2) \geq 1$, we have $\alpha(Tx_1,Sx_2) \geq 1$ and $\alpha(Sx_1,Tx_2) \geq 1$.*

Definition 3.7. *Let p be an E-distance on $\mathcal{S}$-complete Hausdorff u.f.s., (Y, v). A pair of multi-valued mappings $T, S\colon Y \to 2^Y$ is said to be (α,F)-contractive, if there exists a function $\alpha\colon Y \times Y \to [0, \infty)$, $F \in \blacktriangle$ and $\tau > 0$ such that*

$$\tau + F(\alpha(x, y)\max\{H(Tx, Sy), H(Sx, Ty)\}) \leq F(M(x, y)), \tag{13}$$

for all $x, y \in Y$ with $\max\{\alpha(x, y)\max\{H(Tx, Sy), H(Sx, Ty)\}, M(x, y)\} > 0$ and

$$M(x, y) = \max\left\{p(x, y), \frac{p(x, Tx) + p(y, Sy)}{2}, \frac{p(x, Sy) + p(y, Tx)}{2}\right\}.$$

Theorem 3.8. *Let p be an E-distance on $\mathcal{S}$-complete Hausdorff u.f.s., (Y, v). Consider a pair of multi-valued mappings $T, S\colon Y \to 2^Y$ which are (α,F)-contractive and if this pair satisfies the following conditions:*

(a) *The pair (T, S) is α-admissible,*

(b) *There exists $x_0 \in Y$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$,*

(c) *For any sequence $\{x_n\}$ in Y with $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$, then $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$,*

then (T, S) pair has a common fixed point.

Proof. From assertion (**b**), let $x_0 \in Y$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$. As (T, S) is an α-admissible, we can construct a sequence such that $Tx_{2n} = x_{2n+1}$, $Sx_{2n+1} = x_{2n+2}$ and $\alpha(x_n, x_{n+1}) \geq 1$ and for $n \in \mathbb{N} \cup \{0\}$ we have $\alpha(x_{n+1}, x_n) \geq 1$. From (13) and $n \in \mathbb{N} \cup \{0\}$, we will get

$$\begin{aligned}
&\tau + F(p(x_{2n+1}, x_{2n+2})) = \tau + F(p(Tx_{2n}, Sx_{2n+1}))\\
&\leq \tau + F(\alpha(x_{2n}, x_{2n+1}) \times \max\{H(Tx_{2n}, Sx_{2n+1}), H(Sx_{2n}, Tx_{2n+1})\})\\
&\leq \tau + F(\alpha(x_{2n}, x_{2n+1}) \times \max\{H(Tx_{2n}, Sx_{2n+1}), H(Sx_{2n}, Tx_{2n+1})\})\\
&\leq F\Bigg(\max\Bigg\{p(x_{2n+1}, x_{2n+2}), \frac{p(x_{2n+1}, Tx_{2n+1}) + p(x_{2n+2}, Sx_{2n+2})}{2},\\
&\frac{p(x_{2n+1}, Sx_{2n+2}) + p(x_{2n+2}, Tx_{2n+1})}{2}\Bigg\}\Bigg)\\
&\leq F(p(x_{2n}, x_{2n+1})).
\end{aligned}$$

$$\Rightarrow \tau + F(p(Tx_{2n}, Tx_{2n+1})) \leq F(p(x_{2n}, x_{2n+1})).$$

Further, we can write it as:

$$\tau + F(p(x_{2n+1}, x_{2n+2})) \leq F(p(x_{2n}, x_{2n+1})). \tag{14}$$

Likewise, we can get that

$$\begin{aligned}
\tau + F(p(x_{2n+2}, x_{2n+3})) &= \tau + F(p(Sx_{2n+1}, Tx_{2n+2}))\\
&\leq \tau + F(\alpha(x_{2n+1}, x_{2n+2}) \times \max\{H(Tx_{2n+1}, Sx_{2n+2}), H(Sx_{2n+1}, Tx_{2n+2})\})\\
&\leq \tau + F(\alpha(x_{2n+1}, x_{2n+2}) \times \max\{H(Tx_{2n+1}, Sx_{2n+2}), H(Sx_{2n+1}, Tx_{2n+2})\})\\
&\leq F(p(x_{2n+1}, x_{2n+2})).
\end{aligned}$$

It follows that

$$\tau + F(p(Tx_{2n+1}, Tx_{2n+2})) \leq F(p(x_{2n+1}, x_{2n+2})).$$

Similarly, we can obtain:

$$\tau + F(p(x_{2n+2}, x_{2n+3})) \leq F(p(x_{2n+2}, x_{2n+3})). \tag{15}$$

Thus from (14) and (15), and by running the iteration, we get

$$n\tau + F(p(x_n, x_{n+1})) \le F(p(x_0, x_1))$$

$$F(p(x_n, x_{n+1})) \le F(p(x_0, x_1)) - n\tau \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{16}$$

Taking limit $n \to \infty$ on both sides

$$\begin{aligned}\lim_{n\to\infty} F(p(x_n, x_{n+1})) &\le \lim_{n\to\infty}[F(p(x_0, x_1)) - n\tau]\\ \Rightarrow \lim_{n\to\infty} F(p(x_n, x_{n+1})) &= -\infty.\end{aligned}$$

By using Definition 2.2, we have

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{17}$$

Let us denote $p_n = p(x_n, x_{n+1})$ and using Definition 2.2, $\exists\, k \in (0, 1)$ such that

$$\lim_{n\to\infty} p_n^k F(p_n) = 0. \tag{18}$$

From (16) we have

$$\begin{aligned}F(p_n) - F(p_0) &\le -n\tau\\ \Rightarrow\ p_n^k F(p_n) - p_n^k F(p_0) &\le p_n^k(F(p_0) - n\tau) - p_n^k F(p_0)\\ &= -np_n^k\tau \le 0\\ \Rightarrow\ \lim_{n\to\infty}[p_n^k F(p_n) - p_n^k F(p_0)] &\le \lim_{n\to\infty} -np_n^k\tau\\ \Rightarrow\ \lim_{n\to\infty} -np_n^k\tau &\ge 0\\ \Rightarrow\ \lim_{n\to\infty} np_n^k &= 0, \text{ since } \tau > 0.\end{aligned}$$

Therefore $\exists\, n_0 \in \mathbb{N}$ such that

$$\begin{aligned}&np_n^k \le 1, \quad \forall\, n \ge n_0\\ &\Rightarrow p_n^k \le \frac{1}{n}\\ &\Rightarrow p_n \le \frac{1}{n^{1/k}}.\end{aligned}$$

To prove that $\{x_n\}$ is a p-Cauchy sequence consider $m > n$

$$\begin{aligned}p(x_n, x_m) &\le p(x_n, p_{n+1}) + p(x_{n+1}, x_{n+2}) + \ldots + p(x_{m-1}, x_m)\\ &\le \sum_{i=n}^{\infty} p_i\\ &\le \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}.\end{aligned}$$

Taking limit $n \to \infty$ on both sides, we get

$$\begin{aligned}&\lim_{n\to\infty} p(x_n, x_m) \le \lim_{n\to\infty} \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}\\ &\Rightarrow \lim_{n\to\infty} p(x_n, x_m) = 0.\end{aligned}$$

By repeating the same process for the other pair since p is not symmetric, we obtain $\lim_{n\to\infty} p(x_m, x_n) = 0$. Therefore, $\{x_n\}$ is a p-Cauchy sequence. Thus, $\exists,\ \eta \in Y$ such that

$$\lim_{n\to\infty} p(x_n, \eta) = 0;$$

which implies $\lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Sx_{2n+1} = \eta$. From (13) and from condition (**c**) , we get

$$\begin{aligned}
p(x_n, T\eta) \leq& p(x_n, x_{2n+2}) + p(x_{2n+2}, T\eta)\\
&= p(x_n, x_{2n+2}) + p(Sx_{2n+1}, T\eta)\\
&\leq p(x_n, x_{2n+2}) + (\tau + F(\alpha(x_{2n+1}, \eta) \max\{H(Tx_{2n+1}, S\eta), H(Sx_{2n+1}, T\eta)\}))\\
&\leq p(x_n, x_{2n+2}) + (\tau + F(p(x_{2n+1}, \eta))\\
&\leq p(x_n, x_{2n+2}) + F(p(x_{2n+1}, \eta).
\end{aligned}$$

Taking limit on the both sides we obtain $\lim_{n\to\infty} p(x_n, T\eta) = 0$.

$$\Rightarrow \lim_{n\to\infty} p(x_n, \eta) = 0$$

and

$$\lim_{n\to\infty} p(x_n, T\eta) = 0.$$

Thus, we have $\eta \in T\eta$. Analogously, we can prove that $\eta \in S\eta$. Hence, $\eta \in T\eta \cap S\eta$. Consequently T and S have a common fixed point. □

Remark 3.9. *Theorem 3.8 is valid if we replace assertion (**b**) in following way:*
$\exists,\ x_0 \in Y$ *such that* $\alpha(Tx_0, x_0) \geq 1$ *and* $\alpha(x_0, Tx_0) \geq 1$.

By taking $S = T$ in Theorem 3.8 we obtain the following.

Corollary 3.10. *Let p be an E-distance on $\mathcal{S}$-complete Hausdorff u.f.s., (Y, v). Let $T\colon Y \to 2^Y$ be a multi-valued mapping such that*

$$\tau + F(\alpha(x, y) \max\{H(Tx, y), H(x, Ty)\}) \leq F(M(x, y)), \tag{19}$$

for any $x, y \in Y$ *and T is F-contraction with*
$\max\{\alpha(x, y) \max\{H(Tx, y), H(x, Ty)\}, M(x, y)\} > 0$,
which satisfying the following assertions:

(i) *T is α-admissible,*

(ii) $\exists,\ x_0 \in Y \ni, \alpha(Tx_0, x_0) \geq 1$, *and* $\alpha(x_0, Tx_0) \geq 1$.

(iii) *For any sequence $\{x_n\} \subseteq Y$ with $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$,such that $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$.*

Then T has a fixed point.

Example 3.11. *Let $Y = \left\{\frac{1}{n+1} : n \in \mathbb{N}\right\} \cup \{0, 1\}$, along with the usual metric p. Define $v = \{\cup_\epsilon : \epsilon > 0\}$. It is simple to see that (Y, v) is a u.f.s. Now define $T : Y \to 2^Y$ as:*

$$Ta = \begin{cases} 0, & if \ \ a = 0;\\ \dfrac{1}{2n+1}, & if \ \ a = \dfrac{1}{n+1}, n \geq 1;\\ 1, & if \ \ a = 1, \end{cases}$$

$$Sa = \begin{cases} 0, & if \;\; a = 0; \\ \dfrac{1}{2n+1}, & if \;\; a = \dfrac{1}{n+1}, n \geq 1; \\ 1, & if \;\; a = 1, \end{cases}$$

and $\alpha : Y \times Y \to [0, \infty)$ *as*

$$\alpha(a, b) = \begin{cases} 1, & if \;\; a, b \in Y - \{1\}, \\ 0, & otherwise. \end{cases}$$

where

$$\max\{\alpha(a, b) \max\{H(Ta, Sb), H(Sa, Tb)\}, M(a, b)\} > 0.$$

Let $F(a) = \ln a$ *for all* $a > 0$. *Now to prove that T and S are* (α,*F*)*-contractive and* α*-admissible mappings. Let* $a = \dfrac{1}{3}$ *and* $b = \dfrac{1}{4}$, *such that* $m > n \geq 1$. *Then, by using Definition 3.7.*

$$\begin{aligned} F(\alpha(a, b) \max\{H(Ta, Sb), H(Sa, Tb)\}) - F(p(a, b)) &= \ln\left|\frac{1}{5} - \frac{1}{7}\right| - \ln\left|\frac{1}{3} - \frac{1}{4}\right| \\ &= \ln\left|\frac{24}{35}\right| < -\frac{1}{4} \;\; \forall\, a, b \in Y, \end{aligned}$$

with $\max\{\alpha(a, b) \max\{H(Ta, Sb), H(Sa, Tb)\}, M(a, b)\} > 0$. *Therefore, T and S are multi-valued* (α,*F*)*-contractive mappings with* $\tau = \dfrac{1}{4}$. *Hence all the conditions of Theorem 3.8 are satisfied and* 0 *is a common fixed point of T and S.*

Theorem 3.12. *Suppose all the assumptions of Theorem 3.8 holds with an additional condition* (∇), *then T and S have a unique common fixed point.*

Proof. Assume $u, v \in Y$ are two different fixed points of T and S. From the above assertion (∇) and from previous Theorem 3.8, we have

$$\begin{aligned} p(u, v) &\leq \tau + F(\alpha(u, v) \max\{H(Tu, Sv), H(Su, Tv)\}) \\ &\leq \tau + F(M(u, v)) \leq F(M(u, v)) < p(u, v), \end{aligned}$$

which is not possible for $p(u, v) > 0$ and as a result, we get $p(u, v) = 0$. Similarly $p(v, u) = 0$. Therefore, we get $u = v$ which is contradiction to our supposition. Hence T and S have a unique common fixed point. □

Choosing $\alpha(x, y) = 1$ for all $x, y \in Y$ in Theorem 3.5, we obtain the following corollary.

Corollary 3.13. *Let* (Y, v) *be a S-complete Hausdorff u.f.s. such that p be an E-distance on Y. Let* $T \colon Y \to 2^Y$ *be a multi-valued F-contractive mapping that is*

$$\tau + F(H(Tx, Ty)) \leq F(M(x, y)),$$

$\forall\, x, y \in Y$, *with* $\min\{H(Tx, Ty), M(x, y)\} > 0$.
Then T has a unique fixed point.

If we replace $F(t) = kt$ such that $k \in (0, 1)$ in the Corollary 3.13, we obtain the following.

Corollary 3.14. *Consider a* $\mathcal{S}$*-complete Hausdorff u.f.s.*(Y, v) *such that p be an E-distance on Y . Let* $T \colon Y \to 2^Y$ *be a multi-valued mapping that satisfies the following condition:*

$$\tau + k(H(Tx, Ty)) \leq k(M(x, y)),$$

$\forall\, x, y \in Y$, *with* $\min\{H(Tx, Ty), M(x, y)\} > 0$.
Then T has a unique fixed point.

Choosing $\alpha(x, y) = 1$ for all $x, y \in Y$ in Theorem 4.1, we obtain the following.

Corollary 3.15. *Let (Y, v) be a $\mathcal{S}$-complete Hausdorff u.f.s. such that p be an E-distance on Y. Let $T, S\colon Y \to 2^Y$ be multi-valued F-contractive mapping that is*

$$\tau + F(\max\{H(Tx, Sy), H(Sx, Ty)\}) \le F(M(x, y)),$$

$\forall\ x, y \in Y$,
with $\max\{\max\{H(Tx, Sy), H(Sx, Ty)\}, M(x, y)\} > 0$.
Then T and S have a unique common fixed point.

Corollary 3.16. *Let (Y, v) be a $\mathcal{S}$-complete Hausdorff u.f.s. such that p be an E-distance on Y. Let M_1, M_2 be non-empty closed subsets of Y with respect to the topological space $(Y, \tau(v))$. Let $T\colon N \to N$ be a mapping and $N = \cup_{i=1}^{2} M_i$ and suppose that it satisfies the following assertions:*

(i) *$T(M_1) \subseteq M_2$ and $T(M_2) \subseteq M_1$;*

(ii) *$\exists$ an F-mapping such that*

$$\tau + F(H(Tx, Ty)) \le F(M(x, y)), \text{ for all, } (x, y) \in M_1 \times M_2,$$

then T has a unique fixed point that belongs to $M_1 \cap M_2$.

Proof. As M_1 and M_2 are two closed subsets of Y and (N, d) is an S-complete Hausdorff $u.f.s.$ Define a mapping $\alpha : N \times N \to [0, \infty)$ as

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in (M_1 \times M_2) \cup (M_2 \times M_1) \\ 0 & \text{otherwise,} \end{cases}$$

By assertion (**ii**) and from the definition of α, we can express

$$\tau + F(\alpha(x, y)H(Tx, Ty)) \le F(M(x, y)), \ \ \forall\ x, y \in N,$$

then T is an (α,F)-contractive mapping.
Suppose $(x, y) \in N \times N$ such that $\alpha(x, y) \ge 1$. Now if $(x, y) \in M_1 \times M_2$, by assertion (**i**) $(Tx, Ty) \in M_2 \times M_1$, which implies that $\alpha(Tx, Ty) \ge 1$ and if $(x, y) \in M_2 \times M_1$, by assertion (**ii**) $(Tx, Ty) \in M_1 \times M_2$, which implies that $\alpha(Tx, Ty) \ge 1$. Thus by both ways, we have $\alpha(Tx, Ty) \ge 1$. This proves that T is α-admissible.
Further, by assertion (**i**), for any $z \in M_1$, we have $(z, Tz) \in M_1 \times M_2$, which implies that $\alpha(z, Tz) \ge 1$. Now for any sequence $\{x_n\}$ in Y such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n and $x_n \to x \in Y$ as $n \to \infty$. So, from the definition of α it follows that

$$(x_n, x_{n+1}) \in (M_1, M_2) \cup (M_2, M_1), \text{ for all } n \in \mathbb{N}.$$

As $(M_1, M_2) \cup (M_2, M_1)$ for all $n \in \mathbb{N}$, is a closed subset of Y with respect to the topological space $(Y, \tau(v))$, we obtained

$$(x, x) \in (M_1, M_2) \cup (M_2, M_1), \text{ for all } n \in \mathbb{N},$$

which shows that $x \in (M_1 \cap M_2)$. Thus, it can be easily obtained from the definition of α that $\alpha(x_n, x) \ge 1$ for all n. Lastly, let $x, y \in Fix(T)$. By assertion (**i**), it follows that $x, y \in (M_1 \cap M_2)$. Therefore, for any $a \in N$, we have $\alpha(a, x) \ge 1$ and $\alpha(a, y) \ge 1$. Thus condition (∇) is satisfied. Now, all the hypotheses of Theorem 3.5 are satisfied, T has a unique fixed point in $(M_1 \cap M_2)$. $\square$

4. Applications:

In this section established results are applied to prove the existence of the solution of integral equation. To provide an existence theorem for Volterra-type integral inclusion. Let $Y = C([0,1],\mathbb{R})$, the space of all continuous real valued functions on $[0,1]$. Then Y is $\mathcal{S}$-complete $u.f.s.$ with respect to $p(a,b) = \sup_{t\in[0,1]} |a(t)-b(t)|$.

Consider the Volterra-type integral inclusion as

$$a(t) = \int_0^t N(t,s,a(s))ds + f(t) \tag{20}$$

such that for all $t,s \in [0,1]$ along with the continuous functions $f\colon [0,1] \to \mathbb{R}_+$ and $N\colon [0,1]\times[0,1]\times\mathbb{R} \to \mathbb{R}_+$. For each $a \in C([0,1],\mathbb{R})$, the operator $N(t,s,a(s))$ is lower semi-continuous. For the above integral equation , we define a multi-valued operator $T\colon Y \to CL(Y)$ by as below:

$$T(a(t)) = \{u \in C([0,1],\mathbb{R}) : u \in \int_0^t N(t,s,a(s))ds + f(t), t \in [0,1]\}.$$

Let $a \in C([0,1],R)$, and denote $N_a = N(t,s,a(s))$ for each $t,s \in [0,1]$. Now for $N_a\colon [0,1]\times[0,1] \to P_{cv}(\mathbb{R}_+)$, by Michael's selection theorem, there exists a continuous operator $n_a\colon [0,1]\times[0,1] \to \mathbb{R}_+$ such that $n_a(t,s) \in N_a(t,s)$ for all $t,s \in [0,1]$. This shows that

$$\int_0^t n_a(t,s)ds + f(t) \in T(a(t)).$$

Thus the operator Ta is non-empty and closed.

Theorem 4.1. *Let $Y = C([0,1],\mathbb{R})$ be the space of all continuous real valued functions on $[0,1]$. Let $T\colon Y \to CL(Y)$ be a multi-valued operator defined by as below:*

$$T(a(t)) = \{u \in C([0,1],\mathbb{R}) : u \in \int_0^t N(t,s,a(s))ds + f(t), t \in [0,1]\}.$$

with the continuous functions $f\colon [0,1] \to \mathbb{R}_+$ and $N\colon [0,1]\times[0,1]\times\mathbb{R} \to P_{cv}(\mathbb{R}_+)$ is such that for each $a \in C([0,1],\mathbb{R})$, the operator $N(t,s,a(s))$ is lower semi-continuous along with the following assumptions

(i) *there exists*
tau > 0 and $\alpha\colon Y\times Y \to [0,\infty)$ for each $a,b \in Y$, we have

$$|N(t,s,a(s)) - N(t,s,b(s))| \le \frac{\tau M(a,b)}{(\alpha(a,b))^2(\sqrt{|M(a,b)|}+1)^2},$$

where

$$|a|_\tau = \sup_{t\in[0,1]} |a(t)e^{-\tau t}|$$

such that

$$|a-b|_\tau = \sup_{t\in[0,1]} |a(t) - b(t)e^{-\tau t}|$$

(ii) *there exists $a_0 \in Y$ and $a_1 \in Ta_0$ with $\alpha(a_0,a_1) \ge 1$;*

(iii) *if $a \in Y$ and $b \in \tau$ such that $\alpha(a,b) \ge 1$, then we have $\alpha(b,c) \ge 1$ for each $c \in Tb$;*

(iv) *for any sequence $a_n \in Y$ such that $a_n \to u$ as $n \to \infty$ and $\alpha(a_{n+1},a_n) \ge 1$ for each $n \in \mathbb{N}$, we have $\alpha(a_n,u) \ge 1$ for each $n \in \mathbb{N}$;*

then Volterra integral equation has a solution.

Proof. Define

$$T(a(t)) = \left\{ u \in C([0,1],\mathbb{R}) : u \in \int_0^t N(t,s,a(s))ds + f(t), t \in [0,1] \right\}.$$

We show that the operator T satisfies all conditions of Theorem 3.2. Let $a, b \in Y$ such that $u \in Ta$. On the other hand, from hypothesis **(iii)**, we follow as:

$$\begin{aligned}
|Ta - Tb| &= \int_0^t (|N(t,s,a(s)) - N(t,s,b(s))|)ds \\
&\leq \int_0^t (\alpha(a,b))^2 \frac{\tau M(a,b)}{(\tau\sqrt{M(a,b)_\tau} + 1)^2} ds \\
&\leq \int_0^t (\alpha(a,b))^2 \frac{\tau (M(a,b)e^{-\tau s})e^{\tau s} ds}{(\tau\sqrt{M(a,b)_\tau} + 1)^2} \\
&\leq (\alpha(a,b))^2 \frac{\tau (M(a,b)e^{-\tau s})}{(\tau\sqrt{M(a,b)_\tau} + 1)^2} \int_0^t e^{\tau s} ds \\
&\leq (\alpha(a,b))^2 \frac{M(a,b)_\tau e^{\tau t}}{(\tau\sqrt{M(a,b)_\tau} + 1)^2}
\end{aligned}$$

Furthermore, it follows:

$$\begin{aligned}
|Ta - Tb| &\leq \frac{(\alpha(a,b))^2 M(a,b)_\tau e^{\tau t}}{(\tau\sqrt{M(a,b)_\tau} + 1)^2} \\
|Ta - Tb|e^{-\tau t} &\leq \frac{(\alpha(a,b))^2 M(a,b)_\tau}{(\tau\sqrt{M(a,b)_\tau} + 1)^2} \\
M(Ta,Tb)_\tau &\leq \frac{(\alpha(a,b))^2 M(a,b)_\tau}{(\tau\sqrt{M(a,b)_\tau} + 1)^2} \\
\frac{(\tau\sqrt{M(a,b)_\tau} + 1)^2}{M(a,b)_\tau} &\leq (\alpha(a,b))^2 \frac{1}{M(Ta,Tb)_\tau} \\
\frac{(\tau\sqrt{M(a,b)_\tau} + 1)}{\sqrt{M(a,b)_\tau}} &\leq \alpha(a,b) \frac{1}{\sqrt{M(Ta,Tb)_\tau}} \\
(\tau + \frac{1}{\sqrt{M(a,b)_\tau}}) &\leq \alpha(a,b) \frac{1}{\sqrt{M(Ta,Tb)_\tau}} \\
\tau - \alpha(a,b) \frac{1}{\sqrt{M(aT,Tb)_\tau}} &\leq -\frac{1}{\sqrt{M(a,b)_\tau}}
\end{aligned}$$

So, we can conclude:

$$\tau + F(\alpha(a,b)H(Ta,Tb)) \leq F(M(a,b)) \text{ for all } a, b \in Y,$$

where as

$$\min\{\alpha(a,b)H(Ta,Tb), M(a,b)\} > 0.$$

So, T is an (α,F)-contractive mapping for $F(a) = -\frac{1}{\sqrt{a}}; a > 0$. Therefore all the conditions of Theorem 3.2 holds. Hence T has a fixed point and integral equation has a solution.

□

5. Conclusion

Following the approaches of ([8], [41]) a new notion of (α,F)-contractive multi-valued mappings has been introduced on S complete $u.f.s.$ in this article. Using new concept some fixed point and common fixed point results are established. Some interesting consequences of these results are presented as corollaries. The obtained results are elaborated by endorsing examples along with application that can be a good contributions towards fixed point theory.

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