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On Some Properties of Riemann-Liouville Fractional Operator in Orlicz Spaces and Applications to Quadratic Integral Equations

Mohamed M. A. Metwali[a]

[a]*Department of Mathematics, Faculty of Sciences, Damanhour University, Egypt*

Abstract. This article demonstrates some properties of the Riemann-Liouville (R-L) fractional integral operator like acting, continuity, and boundedness in Orlicz spaces L_φ. We apply these results to examine the solvability of the quadratic integral equation of fractional order in L_φ. Because of the distinctive continuity and boundedness conditions of the operators in Orlicz spaces, we look for our concern in three situations when the generating N-functions fulfill Δ', Δ_2, or Δ_3-conditions. We utilize the analysis of the measure of noncompactness with the fixed point hypothesis. Our hypothesis can be effectively applied to various fractional problems.

1. Introduction.

The studies of fractional integrations and the convolution theorems in Orlicz spaces have been begun by O'Neil [30] and afterward, this point has interesting premium of studies (cf. [3, 11, 29]). In the literature, the fractional integral equations are mostly discussed in $C(I)$ [19, 36], in Banach algebras [15, 20] and in Lebesgue spaces with polynomial growth [2, 26, 28, 34]. The outcomes introduced in the former literature are oftentimes difficult to apply for fractional problems in Orlicz spaces.

To fulfill this gap, we focus in this article on studying some properties of the Riemann-Liouville (R-L) fractional integral operator like acting, continuity, and boundedness in Orlicz spaces and applying these properties in inspecting the fractional integral equation

$$x(t) \;=\; g(t) + \; G(x)(t) \cdot \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))\, ds, \;\; t \in [0, d],\; 0 < \alpha < 1, \tag{1}$$

in Orlicz spaces L_φ, where G is a general operator.

We discuss equation (1) in three situations when the generating N-function fulfills Δ', Δ_2, or Δ_3-conditions, independently.

We weight on presumptions that grant us to examine integral operators with singular kernels or operators with strong nonlinearity (for instance, of exponential growth), then discontinuous solutions are demands. So, we look for the solutions of the considered problem not in Lebesgue spaces, but in certain Orlicz spaces. These are important issues because they allow us to study the corresponding problems of

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Email address: `metwali@sci.dmu.edu.eg` (Mohamed M. A. Metwali)

equivalent differential equations in Orlicz spaces or Sobolev-Orlicz spaces (cf. [6, 7, 23]). Further, these are induced by the mathematical phenomena in physics and statistical physics (cf. [10, 21, 22]). For example, the integral equation with exponential nonlinearities

$$x(t) + \int_I k(t,s) \cdot e^{x(s)}\, ds = 0,$$

which has applications in thermodynamics (cf. [37]).

However, The integral equations have been examined in Orlicz spaces L_φ (cf. [31–33]) and in generalized Orlicz spaces (cf. [5, 32]). Some extra properties of solutions in Orlicz spaces like constant-sign solutions were also examined in [1]. Moreover, the quadratic integral equations were inspected in Orlicz spaces in [13, 14, 16] by using the methods of fixed point theorems and a proper measure of noncompactness under a different set of assumptions.

Our approach covers the cases of Lebesgue spaces $L_p,\ p > 1$, as a particular cases of Orlicz spaces L_φ with N-function $\varphi = \frac{t^p}{p}$ satisfies Δ_2-condition (cf. [12, 25, 27]).

This article is propelled by demonstrating some properties of the Riemann-Liouville (R-L) fractional integral operator and applying these outcomes in exhibiting the existence of the solution of the quadratic integral equation of fractional order (1) in Orlicz spaces under a general set of assumptions. We use Darbo's fixed point hypothesis and the measure of noncompactness to get our outcomes.

2. Notation and auxiliary facts

Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^+ = [0, \infty)$ and $I = [0, a] \subset \mathbb{R}^+$. A function $M : \mathbb{R}^+ \to \mathbb{R}^+$ refers to a Young function if

$$M(u) = \int_0^u a(s)ds, \quad \text{for } u \geq 0,$$

where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing, left-continuous function which is neither identically zero nor identically infinite on $\mathbb{R}^+$. The functions M and N are pointed to the complementary Young functions if $N(x) = \sup_{y\geq 0}(xy - M(x))$. In particular, if M is finite-valued, where $\lim_{u\to 0} \frac{M(u)}{u} = 0$, $\lim_{u\to\infty} \frac{M(u)}{u} = \infty$ and $M(u) > 0$ if $u > 0$ ($M(u) = 0 \iff u = 0$), then M is called a N-function.

Denote by $L_M = L_M(I)$ the *Orlicz space* of all measurable functions $x : I \to \mathbb{R}$ s.t.

$$\|x\|_M = \inf_{\epsilon>0} \left\{ \int_I M\left(\frac{x(s)}{\epsilon}\right) ds \leq 1 \right\}.$$

Let $E_M(I)$ be the closure in $L_M(I)$ of the set of all bounded functions and have absolutely continuous norms.

Moreover, we have $E_M = L_M$ if M fulfills the Δ_2-condition, i.e.

(Δ_2) there exist $\omega,\ t_0 \geq 0$ s.t. $M(2t) \leq \omega M(t),\ t \geq t_0$.

The N-function M is said to fulfill Δ'-condition if $\exists\ K,\ t_0 \geq 0$ s.t. for $t, s \geq t_0$, we have $M(ts) \leq KM(t)M(s)$.

Moreover, the N-function M is said to fulfill Δ_3-condition if $\exists\ K,\ t_0 \geq 0$ s.t. for $t \geq t_0$, we have $tM(t) \leq M(Kt)$.

Proposition 2.1. *[17] Assume that, M be a Young function, then we have*

(a) For fixed $\alpha_1 \in (0,1)$ and that $\int_0^t M(s^{-\alpha_1})\, ds$ is finite for any $t > 0$. If $\alpha_2 < \alpha_1$, then the integral

$$\int_0^t M(s^{-\alpha_2})\, ds$$

is finite as well.

(b) For any $t \in \mathbb{R}^+$ and $\alpha \in (0,1)$, the set

$$\mathbb{M}(t) = \left\{ k > 0 : \int_0^{tk^{\frac{1}{1-\alpha}}} M(s^{\alpha-1})\, ds \le k^{\frac{1}{1-\alpha}} \right\}$$

is increasing and continuous functions with $\mathbb{M}(0) = 0$.

Definition 2.2. *[26] The Riemann-Liouville (R-L) fractional integral of a well defined function x of order α is defined as*

$$J^\alpha x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\, ds, \quad \alpha > 0, \quad t > 0,$$

where $\Gamma(\alpha) = \int_0^\infty e^{-s} s^{\alpha-1}\, ds$.

Proposition 2.3. *[26] For $\alpha \in \mathbb{R}^+$, the operator J^α takes the nonnegative and a.e. nondecreasing functions into functions of the same type.*

Definition 2.4. *[21] Suppose that a function $f : I \times \mathbb{R} \to \mathbb{R}$ fulfills Carathéodory conditions i.e. it is continuous in x for almost all $t \in I$ and measurable in t for any $x \in \mathbb{R}$. Then, we denote the superposition operator F_f by*

$$F_f(x)(t) = f(t, x(t)), \quad t \in I$$

for every measurable function $x(t)$ on I.

Lemma 2.5. *[21, Theorem 17.6] Assume that $f : I \times \mathbb{R} \to \mathbb{R}$ fulfills Carathéodory conditions. Then*

$$|f(s,x)| \le a(s) + bM_2^{-1}\left[M_1\left(\frac{x}{r}\right)\right], \; t \in I \text{ and } x \in \mathbb{R},$$

where $b, r \ge 0$ and $a \in L_{M_2}(I)$. If the N-function M_2 fulfills Δ_2-condition, then the operator $F_f : B_r(E_{M_1}(I)) \to E_{M_2}(I)$ and is continuous.

Lemma 2.6. *[24, Theorem 10.2] Let φ_1, φ_2 and φ are arbitrary N-functions. The next conditions are equivalent:*

1. *For every functions $u \in L_{\varphi_1}(I)$ and $w \in L_{\varphi_2}$, $u \cdot w \in L_\varphi(I)$.*
2. *There exists a constant $k > 0$ s.t. for all measurable u, w on I we have $\|uw\|_\varphi \le k\|u\|_{\varphi_1}\|w\|_{\varphi_2}$.*
3. *There exists numbers $C > 0$, $u_0 \ge 0$ s.t. for all $s, t \ge u_0$, $\varphi\left(\frac{st}{C}\right) \le \varphi_1(s) + \varphi_2(t)$.*
4. $\limsup_{t\to\infty} \frac{\varphi_1^{-1}(t)\varphi_2^{-1}(t)}{\varphi(t)} < \infty$.

Let $S = S(I)$ point to the set of Lebesgue measurable functions on I and let "*meas*" refer to the Lebesgue measure in $\mathbb{R}$. The set S related with the metric

$$d(x,y) = \inf_{\epsilon>0}[\epsilon + meas\{s : |x(s) - y(s)| \ge \epsilon\}]$$

be a complete space. The convergence in measure on I is equivalent to convergence with respect to d (cf. Proposition 2.14 in [35]). The compactness in that space is said to be a "compactness in measure".

Lemma 2.7. *[14] Let $X \subset L_M(I)$ be bounded set. Suppose that, there is a family $(\Omega_c)_{0\le c\le a} \subset I$ s.t. meas $\Omega_c = c$ for every $c \in [0,a]$, and for every $x \in X$,*

$$x(t_1) \ge x(t_2), \quad (t_1 \in \Omega_c, \; t_2 \notin \Omega_c).$$

Then X is compact in measure in $L_M(I)$.

Next, assume that $(E, \|\cdot\|_E)$ be an arbitrary Banach space with zero element θ. Denote by $B_r = \{x \in E : \|x\|_E \leq r\}$ and the symbol $B_r(E)$ is to point out the space. If $X \subset E$, then $\bar{X}$ and $convX$ point to the closure and convex closure of X, respectively. The symbols $\mathcal{M}_E$ and $\mathcal{N}_E$ refer to the family of all nonempty and bounded subsets and the subfamily of all relatively compact subsets of E, respectively.

Definition 2.8. [4] *A mapping* $\mu : \mathcal{M}_E \to [0, \infty)$ *refers to a measure of noncompactness in E if it fulfills:*
(i) $\mu(Y) = 0 \iff Y \in \mathcal{N}_E$.
(ii) $Y \subset X \implies \mu(Y) \leq \mu(X)$.
(iii)$\mu(\bar{Y}) = \mu(convY) = \mu(Y)$.
(iv) $\mu(\lambda Y) = |\lambda| \, \mu(Y), \, for \, \lambda \in \mathbb{R}$.
(v) $\mu(Y + X) \leq \mu(Y) + \mu(X)$.
(vi) $\mu(Y \bigcup X) = \max\{\mu(Y), \mu(X)\}$.
(vii) If Y_n *is a sequence of nonempty, bounded, closed subsets of E such that* $Y_{n+1} \subset Y_n$, $n = 1, 2, 3, \cdots$, *and* $\lim_{n\to\infty} \mu(Y_n) = 0$, *then the set* $Y_\infty = \bigcap_{n=1}^{\infty} Y_n$ *is nonempty.*

Definition 2.9. *[4] Let* $X \subset E$ *be a bounded and nonempty set. The Hausdorff measure of noncompactness* $\beta_H(X)$ *is given by*

$$\beta_H(X) = \inf\{r > 0 : \textit{there exists a finite subset } Y \textit{ of } E \textit{ such that } x \subset Y + B_r\}.$$

For any $\epsilon > 0$, let c be a measure of equiintegrability of the set $X \in L_M(I)$ (cf. Definition 3.9 in [35] or [18]):

$$c(X) = \lim_{\epsilon\to 0} \sup_{mesD\leq\epsilon} \sup_{x\in X} \|x \cdot \chi_D\|_{L_M(I)},$$

where χ_D denotes the characteristic function of a measurable subset $D \subset I$.

Lemma 2.10. *[14, 18] Let* $X \subset E_M(I)$ *be a bounded, nonempty, and compact in measure set. Then*

$$\beta_H(X) = c(X).$$

Theorem 2.11. *[4] Let* $Q \subset E$ *be a bounded, nonempty, convex, and closed set and let* $V : Q \to Q$ *be a continuous transformation that is a contraction with respect to the measure of noncompactness* μ*, i.e. there exists* $k \in [0, 1)$ *s.t.*

$$\mu(V(X)) \leq k\mu(X),$$

for any nonempty $X \subset E$*. Then V has at least one fixed point in Q.*

3. Main results.

Let $\mathbb{I} = [0, d]$ and rewrite equation (1) in operator form as following

$$x = B(x) = g + U(x),$$

where

$$U(x) = G(x) \cdot A(x), \quad A(x)(t) = J^\alpha F_f(x),$$

such that F_f is the superposition operator and J^α is as in Definition 2.2. We will describe three cases, which allows us to use general growth conditions.

3.1. The case of Δ'-condition.

Assume, that $\varphi, \varphi_1, \varphi_2$ are N-functions and that M and N are complementary N-functions. Moreover, write the assumptions:

(G1) There exists a constant $k_1 > 0$ s.t. for every $v \in L_{\varphi_1}(\mathbb{I})$ and $w \in L_{\varphi_2}(\mathbb{I})$ we have $\|vw\|_\varphi \le k_1\|v\|_{\varphi_1}\|w\|_{\varphi_2}$,

(G2) $G : L_\varphi(\mathbb{I}) \to L_{\varphi_1}(\mathbb{I})$, takes continuously $E_\varphi(\mathbb{I}) \to E_{\varphi_1}(\mathbb{I})$ and there exists a constant $b_0 > 0$ s.t. $|G(x)| \le b_0\|x\|_\varphi$ and that G takes the set of all a.e. nondecreasing functions into itself. Moreover, assume that for any $x \in E_\varphi(\mathbb{I})$, we have $G(x) \in E_{\varphi_1}(\mathbb{I})$.

(C1) $g \in E_\varphi(\mathbb{I})$ is nondecreasing a.e. on $\mathbb{I}$,

(C2) $f(t,x) : \mathbb{I}\times\mathbb{R} \to \mathbb{R}$ fulfills Carathéodory conditions. Further, $f(t,x)$ is assumed to be nondecreasing with respect to each variable t and x separately,

(C3) $|f(t,x)| \le b(t) + R(|x|)$ for $t \in \mathbb{I}$ and $x \in \mathbb{R}$, where $b \in E_N(\mathbb{I})$ and R is nonnegative, continuous, nondecreasing function on $\mathbb{R}^+$.

(C4) Let N fulfills the Δ'-condition and suppose that there exist $\omega,\ \gamma,\ u_0\ \ge 0$ for which

$$N(\omega(R(u))) \le \gamma\varphi(u) \le \gamma M(u) \ \text{ for } \ u \ge u_0.$$

(K1) Assume that $k(t) = \frac{1}{\epsilon^{\frac{1}{1-\alpha}}}\int_0^{t\epsilon^{\frac{1}{1-\alpha}}} M(s^{\alpha-1})\,ds \in E_{\varphi_2}(\mathbb{I})$ for a.e. $s \in \mathbb{I}$ and $\epsilon > 0$.

Lemma 3.1. *Assume, that φ_2 is N-function and that M and N are complementary N-functions. Moreover, assume that assumption (K1) is fulfilled, then the operator $J^\alpha : L_N(\mathbb{I}) \to L_{\varphi_2}(\mathbb{I})$ and is continuous.*

Proof. Suppose that

$$K(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} & \text{if } s \in [0,t],\ t>0,\\ 0 & \text{otherwise.}\end{cases}$$

Then, for $x \in L_N(\mathbb{I})$ and by Hölder inequality, we have

$$\begin{aligned} |J^\alpha x(t)| &= \left|\int_0^\infty K(t,s)x(s)\,ds\right| \\ &\le 2\|K(t,\cdot)\|_M\|x\|_N \\ &\le \frac{2}{\Gamma(\alpha)}\|(t-\cdot)^{\alpha-1}\|_M\|x\|_N \\ &\le \frac{2}{\Gamma(\alpha)}\inf_{\epsilon>0}\left\{\int_{\mathbb{I}} M\left(\frac{(t-s)^{\alpha-1}}{\epsilon}\right)\,ds \le 1\right\}\|x\|_N. \end{aligned}$$

Put $u = \frac{t-s}{\epsilon^{\frac{1}{\alpha-1}}}$ and by using (K1), we have

$$\begin{aligned} \|J^\alpha x\|_{\varphi_2} &\le \frac{2}{\Gamma(\alpha)}\left\|\inf_{\epsilon>0}\left\{\frac{1}{\epsilon^{\frac{1}{1-\alpha}}}\int_0^{t\epsilon^{\frac{1}{1-\alpha}}} M(u^{\alpha-1})\,du \le 1\right\}\right\|_{\varphi_2}\|x\|_N \\ &\le \frac{2}{\Gamma(\alpha)}\|k\|_{\varphi_2}\|x\|_N. \end{aligned}$$

Then by using Proposition 2.1 and [21, Lemma 16.3], we have $J^\alpha : L_N(\mathbb{I}) \to L_{\varphi_2}(\mathbb{I})$ and is continuous. □

Remark 3.2. *By using Lemma 3.1 and assumption (C3), then for arbitrary measurable subset T of $\mathbb{I}$ and $x \in E_\varphi(\mathbb{I})$, we have*

$$\|A(x)\chi_T\|_{\varphi_2} \leq \frac{2}{\Gamma(\alpha)}\|k\cdot\chi_T\|_{\varphi_2}\cdot(\|b\|_N + \|R(|x(\cdot)|\|_N).$$

Then, by using assumptions (C4) (cf. [21, Theorem 19.1]), there exist $\omega,\ \gamma,\ u_0 > 0$, s.t.

$$\|R(|x(\cdot)|)\|_N \leq \frac{1}{\omega}\left(1+\int_0^d N(\omega R(|x(t)|))dt\right) \leq \frac{1}{\omega}\left(1+N(\omega R(u_0))+\gamma\int_0^d \varphi(|x(t)|)dt\right).$$

Theorem 3.3. *Let the assumptions (G1), (G2), (C1) - (C4), and (K1) be fulfilled. If*

$$\|g\|_\varphi + \frac{2k_1\cdot b_0}{\Gamma(\alpha)}\|k\|_{\varphi_2}\Big(\|b\|_N + R(1)\Big) < 1,$$

then there exists a solution $x \in E_\varphi(I)$ of (1) which is a.e. nondecreasing on $I = [0,a] \subset \mathbb{I}$.

Proof. **Step I.** Firstly, Lemma 3.1 gives that the operator $J^\alpha : L_N(\mathbb{I}) \to L_{\varphi_2}(\mathbb{I})$ and is continuous and by (C2) the operator F_f is continuous mappings from the unit ball $B_1(E_\varphi(\mathbb{I}))$ into $Ł_N(\mathbb{I})$. Then the operator $A = J^\alpha F_f : B_1(E_\varphi(\mathbb{I})) \to E_{\varphi_2}(\mathbb{I})$ is continuous. By assumptions (G2) and (N1) the operator $U : B_1(E_\varphi(\mathbb{I})) \to E_\varphi(\mathbb{I})$ is continuous. Finally, by (C1), we can deduce that the operator $B : B_1(E_\varphi(\mathbb{I})) \to E_\varphi(\mathbb{I})$ is continuous.

Step II. We will construct the ball $B_1(E_\varphi(I)) = \{x \in L_\varphi(I) : \|x\|_\varphi \leq 1\}$.
Let $x \in B_1(E_\varphi(I))$ and by recalling Lemma 3.1, we have

$$\begin{aligned}
\|B(x)\|_\varphi &\leq \|g\|_\varphi + \|Ux\|_\varphi \\
&\leq \|g\|_\varphi + \|G(x)\cdot A(x)\|_\varphi \\
&\leq \|g\|_\varphi + k_1\|G(x)\|_{\varphi_1}\cdot\|A(x)\|_{\varphi_2} \\
&\leq \|g\|_\varphi + k_1\cdot b_0\cdot\|x\|_\varphi\cdot\|J^\alpha F_f(x)\|_{\varphi_2} \\
&\leq \|g\|_\varphi + k_1 b_0\|x\|_\varphi\frac{2}{\Gamma(\alpha)}\|k\|_{\varphi_2}\|F_f(x)\|_N \\
&\leq \|g\|_\varphi + \frac{2k_1 b_0}{\Gamma(\alpha)}\|x\|_\varphi\|k\|_{\varphi_2}\Big(\|b\|_N + \|R(|x(\cdot)|)\|_N\Big) \\
&\leq \|g\|_\varphi + \frac{2k_1 b_0}{\Gamma(\alpha)}\|k\|_{\varphi_2}\Big(\|b\|_N + R(1)\Big) \leq 1,
\end{aligned}$$

whenever $\|x\|_\varphi \leq 1$. Then $B : B_1(E_\varphi(I)) \to E_\varphi(I)$ is continuous, where $I = [0,a] \subset \mathbb{I}$.

Step III. Let $Q_1 \subset B_1(E_\varphi(I))$ consisting of all functions that are a.e. nondecreasing on I. This set is nonempty, convex, bounded and closed set in $L_\varphi(I)$ see [14]. Moreover, the set Q_1 is compact in measure due to Lemma 2.7.

Step IV. Now, we will show that B preserves the monotonicity of functions. Take $x \in Q_1$, then x is a.e. nondecreasing on I and consequently $A(x)$ is a.e. nondecreasing on I thanks for the assumption (C2) and Proposition 2.3. By (G2), the operator $U = G(x)\cdot A(x)$ is a.e. nondecreasing on I. Finally, assumption (C1) gives that $B : Q_1 \to Q_1$ is continuous.

Step V. We will prove that B is a contraction concerning the measure of noncompactness. Assume that $X \subset Q_1$ is nonempty set and let $\epsilon > 0$ be arbitrary. Then for $x \in X$ and for a set $D \subset I$, meas$D \leq \epsilon$, and assumption (G2) implies

$$\|G(x)\cdot\chi_D\|_{\varphi_1} \leq \|G(x\cdot\chi_D)\|_{\varphi_1} \leq b_0\|x\cdot\chi_D\|_\varphi,$$

then we have

$$\begin{aligned}
\|B(x)\cdot\chi_D\|_\varphi &\le \|g\cdot\chi_D\|_\varphi+\|U(x)\cdot\chi_D\|_\varphi\\
&\le \|g\cdot\chi_D\|_\varphi+\|G(x)\cdot A(x)\cdot\chi_D\|_\varphi\\
&\le \|g\cdot\chi_D\|_\varphi+k_1\cdot\|G(x)\cdot\chi_D\|_{\varphi_1}\cdot\|A(x)\cdot\chi_D\|_{\varphi_2}\\
&\le \|g\cdot\chi_D\|_\varphi+k_1\cdot\|G(x\cdot\chi_D)\|_{\varphi_1}\cdot\|A(x)\|_{\varphi_2}\\
&\le \|g\cdot\chi_D\|_\varphi+\frac{2k_1\cdot b_0}{\Gamma(\alpha)}\|x\cdot\chi_D\|_\varphi\|k\|_{\varphi_2}\|F_f(x)\|_N\\
&\le \|g\cdot\chi_D\|_\varphi+\frac{2k_1\cdot b_0}{\Gamma(\alpha)}\|x\cdot\chi_D\|_\varphi\|k\|_{\varphi_2}\Big(\|b\|_N+R(1)\Big).
\end{aligned}$$

Hence, taking into account that $g\in E_\varphi$, we have

$$\lim_{\epsilon\to 0}\{\sup_{mes\, D\le\epsilon}[\sup_{x\in X}\{\|g\cdot\chi_D\|_\varphi\}]\}=0.$$

Thus from definition of $c(x)$, we get

$$c(B(X))\le\frac{2k_1\cdot b_0}{\Gamma(\alpha)}\|k\|_{\varphi_2}\Big(\|b\|_N+R(1)\Big)\cdot c(X).$$

Since $X\subset Q_1$ is a bounded, nonempty, and compact in measure subset of E_φ, we can apply Lemma 2.10 and have

$$\beta_H(B(X))\le\frac{2k_1\cdot b_0}{\Gamma(\alpha)}\|k\|_{\varphi_2}\Big(\|b\|_N+R(1)\Big)\cdot\beta_H(X).$$

Since $\frac{2k_1\cdot b_0}{\Gamma(\alpha)}\|k\|_{\varphi_2}\Big(\|b\|_N+R(1)\Big)<1$, then by Theorem 2.11, we have wrapped up. □

3.2. *The case of* Δ_3*-condition.*

Next, we consider the case of N-functions fulfilling Δ_3-condition with the growth essentially more rapid than a polynomial. Let $\vartheta=\sup\{\|x\|_1 : x\in B_1(L_\varphi(I))\}$ be the norm of the identity operator from $L_\varphi(I)$ into $L^1(I)$.

Write the next assumptions:

(C5)
1. N fulfills the Δ_3-condition.
2. There exist $\beta,\ u_0>0$ s.t.
$$R(u)\le\beta\frac{M(u)}{u},\quad\text{for}\quad u\ge u_0.$$
3. Assume that
$$\frac{2(2+a(1+\varphi_2(1)))k_1\cdot b_0\|k\|_{\varphi_2}}{\Gamma(\alpha)}\|x\cdot\chi_D\|_\varphi\Big(\|b\|_N+R(r_0|)\Big)<1,$$
where
$$r_0=\frac{1}{2\eta_0\vartheta}\left(\frac{\Gamma(\alpha)}{2\Big(2+a(1+\varphi(1))\Big)k_1\cdot b_0\|k\|_{\varphi_2})}-\|b\|_N\right).$$

Remark 3.4. (*a*) *Assumption (C5) implies that the intervals* $I=\mathbb{I}$.

(*b*) *Let us note, that the assumption (C5) 2. implies that there exist constants* $\omega,u_0>0$ *and* $\eta_0>1$ *s.t.* $N(\omega R(u))\le\eta_0 u$ *for* $u\ge u_0$. *Thus for* $x\in L_\varphi(I)$

$$\|R(|x(\cdot)|)\|_N\ \le\ \frac{1}{\omega}\left(1+\int_I N(\omega R(|x(s)|)\,ds\right)\le\frac{1}{\omega}\left(1+\eta_0u_0a+\eta_0\int_I|x(s)|\,ds\right).$$

By [21, Lemma 15.1 and Theorem 19.2] and the assumption (K1):

$$\begin{aligned}
\|A(x)\chi_T\|_{\varphi_2} &\leq \frac{2\cdot(2+a(1+\varphi_2(1)))}{\Gamma(\alpha)}\|k\cdot\chi_{T\times I}\|_{\varphi_2}\left(\|b\|_N+\|R(|x(\cdot)|)\|_N\right)\\
&\leq \frac{2\cdot(2+a(1+\varphi_2(1)))}{\Gamma(\alpha)}\|k\cdot\chi_{T\times I}\|_{\varphi_2}\\
&\quad\times\left(\|b\|_N+\frac{1}{\omega}\left(1+\eta_0u_0a+\eta_0\int_I|x(s)|\,ds\right)\right)
\end{aligned}$$

for arbitrary $x\in L_\varphi(I)$ and arbitrary measurable subset T of I.

Theorem 3.5. *Assume, that $\varphi,\varphi_1,\varphi_2$ are N-functions and that M and N are complementary N-functions, and that (G1), (G2), (C1) - (C3), (C5), and (K1) hold, then there exists a solution $x\in E_\varphi(I)$ of (1) which is a.e. nondecreasing on I.*

Proof. **Step I′.** It is equivalent to **Step I** but on the whole $E_\varphi(I)$, i.e. the operator $B:E_\varphi(I)\to E_\varphi(I)$ is continuous.

Step II′. We will study the operator B on the ball $B_r(E_\varphi(I))$, where $r\geq 0$ is a number fulfilling

$$\|g\|_\varphi+\frac{2Ck_1b_0\|k\|_{\varphi_2}}{\Gamma(\alpha)}\cdot r\left(\|b\|_N+\frac{1}{\omega}\left(1+\eta_0u_0a+\eta_0\vartheta r\right)\right)\leq r, \tag{2}$$

where $C=\left(2+a(1+\varphi(1))\right)$. There are two numbers $0\leq r_1<r_2$ fulfilling (2) (see [14]).

The next assumption about the discriminant infers the presence of solution of (2)

$$\frac{2\eta_0\vartheta\Gamma(\alpha)\|g\|_\varphi}{Ck_1b_0\|k\|_\varphi} < \left(\|b\|_N+\frac{1}{\omega}(1+\eta_0u_0a)-\frac{\Gamma(\alpha)}{2Ck_1b_0\|k\|_{\varphi_2}}\right)^2.$$

For $x\in B_r(E_\varphi(I))$, and by using Lemma 3.4, we have

$$\begin{aligned}
\|B(x)\|_\varphi &\leq \|g\|_\varphi+\frac{2Ck_1b_0\|k\|_{\varphi_2}}{\Gamma(\alpha)}\|x\|_\varphi\left(\|b\|_N+\frac{1}{\omega}\left(1+N(\omega R(u_0))+\eta_0\int_I|x(s)|\,ds\right)\right)\\
&\leq \|g\|_\varphi+\frac{2Ck_1b_0\|k\|_{\varphi_2}}{\Gamma(\alpha)}\|x\|_\varphi\left(\|b\|_N+\frac{1}{\omega}\left(1+N(\omega R(u_0))+\eta_0\|x\|_1\right)\right)\\
&\leq \|g\|_\varphi+\frac{2Ck_1b_0\|k\|_{\varphi_2}}{\Gamma(\alpha)}\|x\|_\varphi\left(\|b\|_N+\frac{1}{\omega}\left(1+N(\omega R(u_0))+\eta_0\vartheta\|x\|_\varphi\right)\right)\\
&\leq \|g\|_\varphi+\frac{2Ck_1b_0\|k\|_{\varphi_2}}{\Gamma(\alpha)}\cdot r\left(\|b\|_N+\frac{1}{\omega}\left(1+\eta_0u_0a+\eta_0\vartheta r\right)\right)\leq r.
\end{aligned}$$

Then $B:B_r(E_\varphi(I))\to B_r(E_\varphi(I))$ is continuous.

Step III′ and **Step IV′** are equivalent to **Step III** and **Step IV** for a subset $Q_r\subset B_r(E_\varphi(I))$.

Step V′. Assume that $X\subset Q_r$ is nonempty set and let $\epsilon>0$ be arbitrary. Then for $x\in X$ and a set $D\subset I$, meas$D\leq\epsilon$, we obtain

$$\begin{aligned}
\|B(x)\cdot\chi_D\|_\varphi &\leq \|g\cdot\chi_D\|_\varphi + k_1\cdot\|G(x)\cdot\chi_D\|_\varphi\cdot\|A(x)\cdot\chi_D\|_{\varphi_2}\\
&\leq \|g\cdot\chi_D\|_\varphi + k_1\cdot b_0\cdot\|x\cdot\chi_D\|_\varphi\cdot\|A(x)\|_{\varphi_2}\\
&\leq \|g\cdot\chi_D\|_\varphi + k_1\cdot b_0\cdot\|x\cdot\chi_D\|_\varphi\cdot\frac{2C}{\Gamma(\alpha)}\|k\|_{\varphi_2}\|F_f(x)\|_N\\
&\leq \|g\cdot\chi_D\|_\varphi + \frac{2Ck_1\cdot b_0\|k\|_{\varphi_2}}{\Gamma(\alpha)}\|x\cdot\chi_D\|_\varphi\|b+R(|x(s)|))\|_N\\
&\leq \|g\cdot\chi_D\|_\varphi + \frac{2Ck_1\cdot b_0\|k\|_{\varphi_2}}{\Gamma(\alpha)}\|x\cdot\chi_D\|_\varphi\left(\|b\|_N+R(r_0)\right),
\end{aligned}$$

where

$$r_0 = \frac{1}{2\eta_0 \vartheta}\left(\frac{\Gamma(\alpha)}{2Ck_1 \cdot b_0 \|k\|_{\varphi_2})} - \|b\|_N\right).$$

As in Theorem 3.3, we obtain

$$\beta_H(B(X)) \leq \frac{2Ck_1 \cdot b_0 \|k\|_{\varphi_2}}{\Gamma(\alpha)} \|x \cdot \chi_D\|_{\varphi}\Big(\|b\|_N + R(r_0)\Big) \cdot \beta_H(X).$$

Since $\frac{2Ck_1 \cdot b_0 \|k\|_{\varphi_2}}{\Gamma(\alpha)} \|x \cdot \chi_D\|_{\varphi}\Big(\|b\|_N + R(r_0)\Big) < 1$, then by Theorem 2.11, we have wrapped up. □

3.3. *The case of Δ_2-condition.*

Now, we will discuss the case when the N-function fulfills the Δ_2-condition. Write the next assumptions:

(C6) Assume that φ be N-functions and the function N fulfills the Δ_2-condition:

1. There exist $\gamma \geq 0$ s.t.
$$R(u) \leq \gamma N^{-1}(\varphi(u)) \quad \text{for} \quad u \geq 0.$$
2. Assume that there exists $r^* > 0$ on the interval $I = [0, a] \subset [0, d]$ s.t.
$$\int_I \varphi\Big(|g(t)| + \frac{2k_1 b_0 |k(t)|}{\Gamma(\alpha)} \cdot r^* \Big(\|b\|_N + \gamma \cdot r^*\Big)\Big) dt \leq r^*.$$

Remark 3.6. *By using assumption (C6)1 and ([21, Theorem 10.5 with k = 1]), then for any $x \in E_\varphi, \gamma > 0$, we have*

$$\|R(|x \cdot \chi_{[0,t]}|)\|_N \leq \gamma \left\| N^{-1}\left(\varphi\left(|x \cdot \chi_{[0,t]}|\right)\right) \right\|_N \leq \gamma + \gamma \int_0^t \varphi\left(|x(s)|\right) ds \tag{3}$$

and then by the Hölder inequality and our assumptions we get

$$|A(x)(t)| \leq |k(t)| \Big(\|b\|_N + \|R(|x \cdot \chi_{[0,t]}|)\|_N \Big).$$

Theorem 3.7. *Let the assumptions (G1), (G2), (C1)-(C3), (C6) and (K1) be fulfilled. If*

$$\left(\frac{2k_1 b_0 \|k\|_{\varphi_2}}{\Gamma(\alpha)} \Big(\|b\|_N + \gamma \cdot r^*\Big)\right) < 1,$$

then there exists an a.e. nondecreasing solution $x \in E_\varphi(I)$ of (1) *on $I = [0, a] \subset \mathbb{I}$.*

Proof. **Step I″** is equivalent to **Step I** i.e. $B : B_1(E_\varphi(I)) \to E_\varphi(I)$ is continuous.

Step II″. We will construct an invariant set $V \subset B_1(E_\varphi(I))$ for the operator B is bounded in $L_\varphi(I)$.
Denote by Q the set of all numbers $r^* > 0$ for which

$$\int_I \varphi\Big(|g(t)| + \frac{2k_1 b_0 |k(t)|}{\Gamma(\alpha)} \cdot r^* \Big(\|b\|_N + \gamma \cdot r^*\Big)\Big) dt \leq r^*.$$

Let V refers to the closure of the set $\{x \in E_\varphi(I) : \int_0^a \varphi(|x(s)|)\, ds \leq r^* - 1\}$. Clearly V is not a ball in $E_\varphi(I)$, but $V \subset B_{r^*}(E_\varphi(I))$ (cf. [21, p. 222]). Notice that $\overline{V}$ is a closed, bounded and convex subset of $E_\varphi(I)$.

For arbitrary $x \in V$ and $t \in I$, we have

$$\begin{aligned}
|B(x)(t)| &\le |g(t)| + k_1 |G(x)| \cdot |A(x)(t)| \\
&\le |g(t)| + k_1 b_0 \|x\|_{\varphi} \cdot \frac{2|k(t)|}{\Gamma(\alpha)} \Big(\|b\|_N \; + \; \|R(|x \cdot \chi_{[0,t]}|)\|_N \Big) \\
&\le |g(t)| + \frac{2k_1 b_0 |k(t)|}{\Gamma(\alpha)} \left(1 + \int_0^a \varphi(|x(t)|)\, dt \right) \left(\|b\|_N \; + \; \gamma + \gamma \int_0^a \varphi\,(|x(s)|)\; ds \right) \\
&\le |g(t)| + \frac{2k_1 b_0 |k(t)|}{\Gamma(\alpha)} \cdot r^* \Big(\|b\|_N \; + \; \gamma + \gamma (r^* - 1) \Big).
\end{aligned}$$

Therefor,

$$\int_I \varphi(B(x)(t))\; dt \le \int_I \varphi \Big(|g(t)| + \frac{2k_1 b_0 |k(t)|}{\Gamma(\alpha)} \cdot r^* \Big(\|b\|_N \; + \; \gamma \cdot r^* \Big) \Big) dt.$$

By the definition of r^* we get $\int_I \varphi(B(x)(t))\; dt \le r^*$ and then $B(V) \subset V$. Consequently $B(\overline{V}) \subset \overline{B(V)} \subset \overline{V} = V$. Then $B : V \to V$ is continuous on $V \subset B_{r^*}(E_{\varphi}(I))$.

Step III″ and **Step IV″** are equivalent to **Step III** and **Step IV** for $Q_{r^*} \subset B_{r^*}(E_{\varphi}(I))$.

Step V″. Assume that $X \subset Q_{r^*}$ is nonempty set and let $\epsilon > 0$ be arbitrary. Then for $x \in X$ and a set $D \subset I$, meas $D \le \epsilon$, we obtain

$$\begin{aligned}
\|B(x) \cdot \chi_D\|_{\varphi} &\le \|g \cdot \chi_D\|_{\varphi} \; + \; \frac{2k_1 b_0 \|k\|_{\varphi_2}}{\Gamma(\alpha)} \cdot \|x \cdot \chi_D\|_{\varphi} \Big\| b + R(|x(\cdot)|) \Big\|_N \\
&\le \|g \cdot \chi_D\|_{\varphi} \; + \; \frac{2k_1 b_0 \|k\|_{\varphi_2}}{\Gamma(\alpha)} \|x \cdot \chi_D\|_{\varphi} \left(\|b\|_N + \gamma + \gamma \int_0^a \varphi\,(|x(s)|)\; ds \right) \\
&\le \|g \cdot \chi_D\|_{\varphi} \; + \; \frac{2k_1 b_0 \|k\|_{\varphi_2}}{\Gamma(\alpha)} \|x \cdot \chi_D\|_{\varphi} \Big(\|b\|_N + \gamma \cdot r^* \Big).
\end{aligned}$$

As done in Theorem 3.3, we have

$$\beta_H(B(X)) \le \left(\frac{2k_1 b_0 \|k\|_{\varphi_2}}{\Gamma(\alpha)} \Big(\|b\|_N + \gamma \cdot r^* \Big) \right) \beta_H(X).$$

Since $\left(\frac{2k_1 b_0 \|k\|_{\varphi_2}}{\Gamma(\alpha)} \Big(\|b\|_N + \gamma \cdot r^* \Big) \right) < 1$, then by Theorem 2.11, we have wrapped up. □

4. Remarks and Example

Allow us to introduce a few remarks and examples, that outline the significance and validity of our outcomes.

Remark 4.1. *The quadratic integral equations are frequently applicable in radiative transfer theory, neutron transport, the kinetic theory of gases, and astrophysics [8, 9].*

Remark 4.2. *The functions* $M_1(u) = \exp|u| - |u| - 1$, $M_2(u) = (1+|u|) \cdot \ln(1+|u|) - |u|$ *are examples of complementary N-functions, s.t* M_1 *fulfills the* Δ_3*-condition and* M_2 *fulfills the* Δ'*-condition. The N-functions* $M_3(u) = \frac{u^p}{p}$, $p > 1$ *and* $M_4(u) = |u|^{\alpha}(|\ln|u| + 1)$ *for* $\alpha \ge \frac{3+\sqrt{5}}{2}$ *fulfill the* Δ_2*-condition. Moreover, the complement functions to* $M_5(u) = \exp u^2 - 1$ *and* $M_6(u) = \exp|u| - |u| - 1$ *fulfill the* Δ_2*-condition while the original functions* M_5 *and* M_6 *do not.*

Example 4.3. *Choose the N-functions* $M(u) = N(u) = u^2$ *and* $\varphi_2(u) = \exp|u| - |u| - 1$. *We need to show that, the operator* $J^\alpha : L_N(I) \to L_{\varphi_2}(I)$ *is continuous and Lemma 3.1 is fulfilled.*

Indeed: For $t \in [0, d]$ *and any* $\alpha \in (0, 1)$, *we have*

$$k(t) = \int_0^t M\Big(s^{\alpha-1}\Big)\, ds = \int_0^t s^{2\alpha-2}\, ds = \frac{t^{2\alpha-1}}{2\alpha-1}.$$

This implies that Proposition 2.1 is fulfilled. Moreover,

$$\int_0^d \varphi_2\Big(k(t)\Big)\, ds = \int_0^d \Big(e^{\frac{t^{2\alpha-1}}{2\alpha-1}} - \frac{t^{2\alpha-1}}{2\alpha-1} - 1\Big) dt$$

which is finite. Then for $x \in L_N(I)$, *we have* $J^\alpha : L_N(I) \to L_{\varphi_2}(I)$ *is continuous.*

For more details and different examples of the N-functions M, N and φ_2 fulfill Lemma 3.1 (see [21, Theorem 15.4]).

Remark 4.4. *The acting and continuity conditions of the operator* $G(x) = b_0 \cdot x(t)$, $b_0 \geq 0$ *in Orlicz spaces are discussed in [21, Theorem 18.2] (cf. assumption (G2)).*

Example 4.5. *Let* $G(x) = b_0 \cdot x(t)$, *we have*

$$x(t) = g(t) + b_0 \cdot x(t) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big(b(s) + \log\Big(1 + \sqrt{x(s)}\Big)\Big)\, ds,\ t \in [0, 1], \tag{4}$$

which represent a particular case of equation (1) with $R(x) = \log\Big(1 + \sqrt{x(s)}\Big)$.

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