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Compactness Criteria and Spectra of Some Infinite Lower Triangular Matrices

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Abstract. In this paper, a study is made on two well-known operator matrices; the Rhaly operator R_a and the generalized difference operator Δ_{ab}. Firstly, some compactness results for the operators R_a and Δ_{ab}, whose purpose is to help in describing their spectra, are derived. Next, general results on investigating the spectra of such operators on a large class of Banach sequence spaces are established. These results give a complete description of the spectra. The obtained results unify, extend and improve many comparable results in the existing literature.

1. Introduction

We denote by ℓ^∞, c and c_0 the classical Banach spaces of all bounded, convergent and null sequences, respectively. Further, let ℓ^p $(1 \leq p < \infty)$ denote the Banach space of absolutely $p-$summable sequences with the ℓ^p-norm. By bs we denote the Banach space of all sequences $x = (x_k) = (x_k)_{k=0}^\infty$ for which $\left(\sum_{k=0}^n x_k\right)$ is bounded with the usual norm

$$\|x\|_{\mathrm{bs}} = \sup_n \left|\sum_{k=0}^n x_k\right|.$$

The space $\mathrm{cs} = \left\{x = (x_k) = (x_k)_{k=0}^\infty : \sum_{k=0}^\infty x_k \text{ converges}\right\}$ is a Banach space with the bs$-$norm. Also, we consider the Banach space bv of all sequences $x = (x_k) = (x_k)_{k=0}^\infty$ of bounded variation with the norm

$$\|x\|_{\mathrm{bv}} = |\lim_{k\to\infty} x_k| + |x_0| + \sum_{k=1}^\infty |x_k - x_{k-1}|.$$

The bv_0 denotes $\mathrm{bv}_0 = \mathrm{bv} \cap c_0$; a Banach space with the bv$-$norm. The Banach space h of all null sequences $x = (x_k) = (x_k)_{k=0}^\infty$, for which the following norm

$$\|x\|_{\mathrm{h}} = \sum_{k=0}^\infty (k+1)\,|x_{k+1} - x_k|$$

is finite, is called Hahn sequence space; cf. [32].

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For a fixed sequence (a_k) of real numbers, the Rhaly operator R_a is defined on a Banach sequence space μ by

$$R_a x := \left(a_k \sum_{n=0}^{k} x_n\right)_{k=0}^{\infty}, \qquad x = (x_k) = (x_k)_{k=0}^{\infty} \in \mu. \tag{1}$$

The operator R_a is represented by an infinite lower triangular matrix with constant row-segments. For the particular choice of $(a_k) = (1/(k+1))_{k=0}^{\infty}$ we obtain the well-known classical Cesàro operator C_1. A question that has been of recent interest (Brown, Halmos and Shields [12]) is: can one obtain the spectrum and its subdivision of the Cesàro matrix as operator on certain sequence space? Brown, Halmos and Shields started the investigation of such problems in their paper [12], where they investigated and solved the problem in the case of the Cesàro matrix C_1 as an operator on the Hilbert space ℓ^2. More papers by different authors were devoted to the spectral problem of C_1 on the Banach spaces c [25], c_0 [25, 33], ℓ^p $(1 < p < \infty)$ [15, 25], ℓ^∞ [25, 31], bv_0 [28], bv [29], the Bachelis space N^p $(1 < p < \infty)$ [16] and the weighted ℓ^p $(1 \le p < \infty)$ spaces [5, 6]. Motivated by the paper [12], in [35], Rhoades started to consider the spectral problems associated with certain classes of Hausdorff matrices.

The Rhaly operator R_a, as a generalization of the Cesàro operator C_1, and its boundedness and compactness on classical sequence spaces have been investigated deeply in [26]. Further results on the boundedness, the compactness and the spectra of R_a acting on the Banach spaces c_0 [46, 48], c [46, 52], ℓ^p $(1 < p < \infty)$ [47], bv_0 [50, 51] and bv [49, 51] have been investigated in both compact and noncompact cases of the operator R_a. In [27], a generalization of the Rhaly matrix as operator on H^2 Hardy spaces has been given, where its spectrum was calculated.

The spectra of the Rhaly operator R_a are quite similar to those of the generalized difference operator Δ_{ab}, which is defined on a Banach sequence space μ by

$$\Delta_{ab} x := (a_k x_k + b_{k-1} x_{k-1})_{k=0}^{\infty}, \quad x = (x_k) = (x_k)_{k=0}^{\infty} \in \mu, \quad b_{-1} = x_{-1} = 0, \tag{2}$$

for fixed sequences (a_k) and (b_k) of real numbers [1]. The spectra of the generalized difference operator Δ_{ab}, in various Banach sequence spaces, have attracted a lot of attention. For example, we mention the works in ℓ^1 [3, 21, 41, 42], ℓ^p $(1 \le p < \infty)$ [4, 11], c [1, 2, 7], c_0 [7, 19], bv_0 [20], h [20] and cs [18, 37].

This paper appeals for a more in-depth investigation of the boundedness and compactness of the operators R_a and Δ_{ab} in various Banach sequence spaces, where the main purpose is to investigate the spectra of such operators (in their compactness case) in a large class of Banach sequence spaces including the spaces c_0, c, ℓ^p $(1 \le p \le \infty)$, bv_0, bv, cs and h. It is noted that we are led to similar results for the different spectral problems. This is due to the common properties of the considered spaces that mainly control the spectral problem. So, we seek studying the problem in general in order to avoid repeating the same results by changing the considered space.

In fact, the natural technique for investigating spectral problems of infinite matrices involves standard operator theory and summability theory. However, for a general infinite matrix, there is no known method for obtaining its spectrum. In fact, such problems have in common that the methods of proof are closely adapted to the matrix operator and the sequence spaces under consideration. That is, the methods of proof are *ad hoc*.

It is worth mentioning that infinite matrices, in general, and their associated spectral problems play an important role in many branches of mathematics such as integral equations, difference equations, infinite systems of linear algebraic or differential equations and the theory of summability of sequences and series. For example, Hilbert studied the eigenvalues of integral operators by viewing the operators as infinite matrices [24, *p*. 1063]. Further, it is known that infinite system of linear equations can be represented alternatively by infinite "coefficient" matrix. In [38] Shivakumar and Wong discussed infinite systems for algebraic equations, while Chew, Shivakumar and Williams [13] discussed systems of differential equations. In [39], Shivakumar, Williams and Rudraiah discussed eigenvalues of infinite matrices as operators acting on ℓ^1 and ℓ^∞. A detailed study about infinite matrices is given by Bemkopf [10], while for concepts and a history of infinite matrices we refer to Cooke [14], and we refer to Shivakumar and Sivakumar [40] for a brief review.

We structure the remaining part of this paper as follows: preliminary facts and results, which are needed for our study, are included in Section 2. The boundedness and compactness of both the Rhaly operator R_a and the generalized difference operator Δ_{ab} are derived in Section 3. Solvability of the spectral problem associated with the compact Rhaly operator R_a in various sequence spaces is presented in Subsection 4.1. The spectral problem associated with the compact generalized difference operator Δ_{ab} is investigated in Subsection 4.2. The conclusion of this paper is summarized in Section 5.

2. Preliminaries

To fix terminology and notation, we will throughout the paper denote by $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{C}$ the sets of natural numbers, nonnegative integers and complex numbers. By convention, any term with negative index is equal to zero and $\sum_{k=n}^{m} c_k = 0$, for any $n, m \in \mathbb{N}_0$ with $n > m$. The zero vector is denoted by $\mathbf{0} = (0, 0, 0, \ldots)$. An operator $T : X \longrightarrow X$ is a bounded linear operator on an infinite dimensional complex Banach space X, and the set of all such is $\mathcal{B}(X)$. The symbol $\mathcal{R}(T)$ denotes the *range* of T. Also, $\mathcal{N}(T)$ denotes the *kernel* of T. Write $T_\lambda = T - \lambda \mathrm{I}$, where $\lambda \in \mathbb{C}$ and I is the identity operator.

2.1. Spectra of bounded linear operators

The *resolvent set* of an operator T is the set $\rho(T, X)$ of all $\lambda \in \mathbb{C}$ such that T_λ has a bounded inverse in $\mathcal{B}(X)$. For $\lambda \in \rho(T, X)$, the operator T_λ^{-1} is called the *resolvent operator*. The *spectrum* of T is the set $\sigma(T, X)$ of all complex numbers not in $\rho(T, X)$. The spectrum $\sigma(T, X)$ is nonempty and compact.

The spectrum $\sigma(T, X)$ can be divided into subsets in many different ways, depending on the possible behaviors of $\mathcal{R}(T_\lambda)$ and ${T_\lambda}^{-1}$ as follows (cf. [45]):

(1) The *point spectrum* $\sigma_{\mathrm{p}}(T, X)$ is the set of all scalars $\lambda \in \sigma(T, X)$ such that $\mathcal{N}(T_\lambda) \neq \{\mathbf{0}\}$. In this case, λ is called an eigenvalue of T and any $x \in \mathcal{N}(T_\lambda)$, where $x \neq \mathbf{0}$, is an eigenvector of T for λ and satisfies $Tx = \lambda x$.

(2) The *residual spectrum* $\sigma_{\mathrm{r}}(T, X)$ is the set of all scalars $\lambda \in \sigma(T, X)$ such that λ is not an eigenvalue but $\mathcal{R}(T_\lambda)$ is not dense.

(3) The *continuous spectrum* $\sigma_{\mathrm{c}}(T, X)$ is the set of all scalars $\lambda \in \sigma(T, X)$ such that λ is not an eigenvalue and $\mathcal{R}(T_\lambda)$ is dense but T_λ^{-1} is unbounded.

(4) The *defect spectrum* $\sigma_\delta(T, X)$ (or *surjectivity spectrum*) is the set of all scalars $\lambda \in \mathbb{C}$ such that $\mathcal{R}(T_\lambda) \neq X$.

(5) The *compression spectrum* $\sigma_{\mathrm{co}}(T, X)$ is the set of all scalars $\lambda \in \mathbb{C}$ such that $\mathcal{R}(T_\lambda)$ is not dense in X.

(6) The *approximate point spectrum* $\sigma_{\mathrm{ap}}(T, X)$ is defined to be the set of all scalars $\lambda \in \mathbb{C}$ such that $\lim_{k\to\infty} \|T_\lambda x_k\| = 0$ for some sequence (x_k) in X such that $\|x_k\| = 1$ for all $k \in \mathbb{N}_0$.

Another classification of the spectrum is also considered. Following Taylor and Halberg [44], T_λ is classified I, II or III, according as its range, $R(T_\lambda)$, is all of X; is not all of X, but is dense in X; or is not dense in X. In addition T_λ is classified 1, 2 or 3 according as T_λ is boundedly invertible; T_λ is invertible but not boundedly; or T_λ is not invertible. The state of an operator is the combination of its Roman and Arabic numerical classifications and is denoted by the Roman numeral with the Arabic numeral as a subscript. Then, the operator $T_\lambda \in \mathrm{I}_2$ if $\mathcal{R}(T_\lambda)$ is all of X and T_λ is invertible but not boundedly, and so on.

Clearly, $\lambda \in \rho(T, X)$ (the *resolvent set*) if and only if $T_\lambda \in \mathrm{I}_1$; otherwise $\lambda \in \sigma(T, X)$. So, the spectrum is subdivided into I_3, II_2, II_3, III_1, III_2 and III_3. We usually use the notation $\mathrm{I}_3\sigma(T, X)$, $\mathrm{II}_2\sigma(T, X)$, $\mathrm{II}_3\sigma(T, X)$, $\mathrm{III}_1\sigma(T, X)$, $\mathrm{III}_2\sigma(T, X)$ and $\mathrm{III}_3\sigma(T, X)$. It is clear from the definition that

$$\begin{aligned}\sigma_{\mathrm{p}}(T, X) &= \mathrm{I}_3\sigma(T, X) \cup \mathrm{II}_3\sigma(T, X) \cup \mathrm{III}_3\sigma(T, X),\\ \sigma_{\mathrm{r}}(T, X) &= \mathrm{III}_1\sigma(T, X) \cup \mathrm{III}_2\sigma(T, X)\end{aligned}$$

and

$$\sigma_{\mathrm{c}}(T, X) = \mathrm{II}_2\sigma(T, X). \tag{3}$$

Further, from the definition, we learn

$$\sigma_\delta(T, X) = \sigma(T, X) \setminus \mathrm{I}_3\sigma(T, X). \tag{4}$$

The following relation holds

$$\sigma_{\mathrm{ap}}(T, X) = \sigma(T, X) \setminus \mathrm{III}_1\sigma(T, X); \tag{5}$$

cf. [45, *p*. 282]. Observe also:

$$\sigma_{\mathrm{r}}(T, X) = \sigma_{\mathrm{p}}(T^*, X^*) \setminus \sigma_{\mathrm{p}}(T, X); \tag{6}$$

cf. [9, Relation 1.56 and Proposition 1.3(e)]. A non-disjoint spectral decomposition of an infinite matrix (approximate point spectrum, defect spectrum and compression spectrum) has been discussed for the first time in [8] and [17]. Next, many authors have made similar studies of different classes of infinite matrices.

We give the following Lemma, which is needed in the sequel:

Lemma 2.1. *[22, Theorems 3.3 and 4.2], [23, Corollaries 2.2 and 2.3] Let T be a bounded linear operator on a complex Banach space X. Then* $\mathrm{III}_1\sigma(T, X)$ *and* $\mathrm{I}_3\sigma(T, X)$ *are open sets.*

2.2. *Matrix transformations between sequence spaces*

For an infinite matrix $A = (a_{n,k})$ of complex entries and for a sequence $x = (x_k)$, we put

$$Ax = ((Ax)_n) = \left(\sum_{k=0}^{\infty} a_{n,k}x_k\right)$$

if this expression exists. Now, if μ_1 and μ_2 are sequence spaces, then the matrix $A = (a_{n,k})$ is identified with the linear operator $A : \mu_1 \longrightarrow \mu_2$ if $Ax \in \mu_2$ for every $x \in \mu_1$. The class $(\mu_1 : \mu_2)$ is defined as

$$(\mu_1 : \mu_2) = \{A = (a_{n,k}) : Ax \in \mu_2 \text{ for every } x \in \mu_1\}.$$

If $A \in (\mu_1 : \mu_2)$, then A is called a *matrix transformation* (or *summability method*) from μ_1 into μ_2.

It is worthwhile to mention that the study of bounded linear operators between sequence spaces is so related to matrix transformations. Precisely, in many cases, the most general linear operator transforming one sequence space into another determines and is determined by an infinite matrix. So, we sometimes are interested in infinite matrices instead of general bounded linear operators.

Next, we invoke some results from summability theory which are needed for our study.

Lemma 2.2. *[43, Formula (45)] Let $A = (a_{n,k})$ be an infinite matrix. Then $A \in (\mathrm{cs} : \mathrm{cs})$ if and only if:*

(1) $\sum_{n=0}^{\infty} a_{n,k}$ *converges for all* $k \in \mathbb{N}_0$.

(2) $\sup_N \sum_{n=0}^{\infty} \left|\sum_{k=0}^{N}(a_{k,n} - a_{k,n-1})\right| < \infty$.

Lemma 2.3. *[32, Proposition 10] Let $A = (a_{n,k})$ be an infinite matrix. Then $A \in (h : h)$ if and only if:*

(1) $\lim_{n\to\infty} a_{n,k} = 0$, *for all* $k \in \mathbb{N}_0$.

(2) $\sum_{n=0}^{\infty} (n+1) \left|a_{n,k} - a_{n+1,k}\right| < \infty$, *for all* $k \in \mathbb{N}_0$.

(3) $\sup_N \frac{1}{N+1} \sum_{n=0}^{\infty} (n+1) \left|\sum_{k=0}^{N}(a_{n,k} - a_{n+1,k})\right| < \infty$.

Lemma 2.4. *[43, Formula (77)] Let $A = (a_{n,k})$ be an infinite matrix. Then $A \in \left(\ell^1 : \ell^1\right)$ if and only if $\|A\| = \sup\limits_k \sum_{n=0}^{\infty} \left|a_{n,k}\right| < \infty$.*

Lemma 2.5. *[43, Formula (111)] Let $A = (a_{n,k})$ be an infinite matrix. Then $A \in (\mathrm{bv}_0 : \mathrm{bv}_0)$ if and only if:*

(1) $\lim_{n\to\infty} a_{n,k} = 0$, *for all* $k \in \mathbb{N}_0$.

(2) $\sup_N \sum_{n=0}^{\infty} \left|\sum_{k=0}^{N}(a_{n,k} - a_{n-1,k})\right| < \infty$.

3. Compactness criteria for R_a and Δ_{ab}

3.1. Compactness of R_a

In this subsection we deal with the following main question: for what conditions on (a_k) is R_a a compact operator?. Partial answers for this question have been settled in the Banach spaces c_0 [46], c [46], ℓ^p $(1 < p < \infty)$ [26], bv_0 [51] and bv [51]. Here, we give new compactness criteria for the operator R_a on the Banach spaces cs, h and ℓ^1, in which it is shown that boundedness and compactness of the operator R_a are equivalent. For completeness, we give some modifications to the recent results for the compactness of R_a in c_0, c, ℓ^p $(1 < p \le \infty)$, bv_0 and bv.

Let $\mu \in \left\{\mathrm{cs}, \mathrm{h}, \ell^1\right\}$. For the next proofs, define the operator $R_a^m : \mu \longrightarrow \mu$, where $m \in \mathbb{N}_0$, by

$$R_a^m x = R_a\left(\left(\chi_{[0,m]}(k)\, x_k\right)_{k=0}^{\infty}\right), \qquad x = (x_k) \in \mu.$$

Then R_a^m is finite rank since its range is spanned by $\{R_a(e_k) : k = 0, 1, 2, ..., m\}$, where e_k's are the standard unit vectors. So R_a^m is compact for every $m \in \mathbb{N}_0$. Let R_a be represented by the matrix $(a_{n,k})$, where $a_{n,k} = 0$ for all $n < k$ and $a_{n,k} = a_n$ for all $n \ge k$. Then the operator R_a^m is represented by the matrix $\left(\chi_{[0,m]}(k)\, a_{n,k}\right)$, and so $R_a - R_a^m$ is represented by $\left(\chi_{[m+1,\infty]}(k)\, a_{n,k}\right)$.

Let us give the first main result:

Theorem 3.1. *The following are equivalent:*

(1) *The operator R_a is bounded on* cs.

(2) *The assumption that $(a_k) \in \ell^1$ holds.*

(3) *The operator R_a is compact on* cs.

Proof. **(1)⟶(2)** We need to applying the result in Lemma 2.2. Indeed, the boundedness of R_a implies that $\sup_N A_N < \infty$, where

$$A_N = \sum_{n=1}^{\infty} \left| \sum_{k=0}^{N} (a_{k,n} - a_{k,n-1}) \right|, \qquad N \in \mathbb{N}_0.$$

An arithmetic shows

$$\sup_N A_N = \sum_{n=0}^{\infty} |a_n|,$$

which is finite thanks to Lemma 2.2. Thus $(a_k) \in \ell^1$.

(2)⟶(3) Let m be a positive integer. Then, for every $x = (x_k) \in \mathrm{cs}$, we have

$$\begin{aligned} \left\| (R_a - R_a^m)\, x \right\|_{\mathrm{cs}} &= \sup_{N \geq 0} \left| \sum_{n=0}^{N} \left(\sum_{k=0}^{\infty} \chi_{[m+1,\infty]}(k)\, a_{n,k} x_k \right) \right| \\ &\leq \sup_{N \geq 0} \sum_{n=m+1}^{N} |a_n| \left| \sum_{k=m+1}^{n} x_k \right| \\ &\leq 2\, \|x\|_{\mathrm{cs}} \sum_{n=m+1}^{\infty} |a_n|. \end{aligned}$$

That is

$$\left\| R_a^m - R_a \right\| \leq 2 \sum_{n=m+1}^{\infty} |a_n| \longrightarrow 0$$

as $m \to \infty$. Then R_a is compact, as it is the norm limit of a sequence of compact operators.

(3)⟶(1) Follows immediately.
□

Next, we prove that the boundedness and compactness of the operator R_a are also equivalent in the Banach space h.

Theorem 3.2. *The following are equivalent:*

(1) *The operator R_a is bounded on* h.

(2) *The assumption that $(a_k) \in$* h *holds.*

(3) *The operator R_a is compact on* h.

Proof. **(1)⟶(2)** We apply the result in Lemma 2.3. Indeed, the boundedness of R_a implies

$$\lim_{n\to\infty} a_{n,k} = \lim_{n\to\infty} a_n = 0, \quad \text{for all } k \in \mathbb{N}_0.$$

Then $a = (a_k) \in c_0$. Further

$$A_k = \sum_{n=0}^{\infty} (n+1) \left| a_{n+1,k} - a_{n,k} \right|$$

converges, for all $k \in \mathbb{N}_0$. Thus, $A_0 = \sum_{n=0}^{\infty} (n+1) |a_{n+1} - a_n|$ is finite, and so, $a = (a_k) \in$ h. Alternatively, the result follows immediately since $R_a(1, 0, 0, ...) = (a_0, a_1, a_2, ...) \in$ h.

(2)⟶(3) For $m \in \mathbb{N}$, we have

$$\begin{aligned}
\left\|\left(R_a - R_a^m\right)x\right\|_{\mathrm{h}} &= \sum\nolimits_{n=0}^{\infty}(n+1)\left|\sum\nolimits_{k=0}^{\infty}\chi_{[m+1,\infty]}(k)\,a_{n+1,k}x_k - \sum\nolimits_{k=0}^{\infty}\chi_{[m+1,\infty]}(k)\,a_{n,k}x_k\right| \\
&= \sum\nolimits_{n=0}^{\infty}(n+1)\left|a_{n+1}\sum\nolimits_{k=m+1}^{n+1}x_k - a_n\sum\nolimits_{k=m+1}^{n}x_k\right| \\
&\leq (m+1)\left|a_{m+1}\right|\left|x_{m+1}\right| + \sum\nolimits_{n=m+1}^{\infty}(n+1)\left|\sum\nolimits_{k=m+1}^{n}\left(a_{n+1}-a_n\right)x_k\right| + \\
&\quad + \sum\nolimits_{n=m+1}^{\infty}(n+1)\left|a_{n+1}\right|\left|x_{n+1}\right| \\
&\leq (m+1)\left|a_{m+1}\right|\left|x_{m+1}\right| + \\
&\quad + \sup\nolimits_{n\geq m+1}\left|\sum\nolimits_{k=m+1}^{n}x_k\right|\sum\nolimits_{n=m+1}^{\infty}(n+1)\left|a_{n+1}-a_n\right| + \\
&\quad + \sup\nolimits_{n\geq m+1}(n+1)\left|x_{n+1}\right|\sum\nolimits_{n=m+1}^{\infty}\left|a_{n+1}\right| \\
&\leq \|x\|_{\mathrm{h}}\left(\left|a_{m+1}\right| + 2\sum\nolimits_{n=m+1}^{\infty}(n+1)\left|a_{n+1}-a_n\right| + \sum\nolimits_{n=m+1}^{\infty}\left|a_{n+1}\right|\right),
\end{aligned}$$

for all $x = (x_k) \in \mathrm{h}$. Therefore

$$\begin{aligned}
\left\|R_a - R_a^m\right\| &\leq \left|a_{m+1}\right| + 2\sum\nolimits_{n=m+1}^{\infty}(n+1)\left|a_{n+1}-a_n\right| + \sum\nolimits_{n=m+1}^{\infty}\left|a_{n+1}\right| \\
&\longrightarrow 0
\end{aligned}$$

as $m \to \infty$, where we used the fact that, for $(a_k) \in \mathrm{h}$,

$$\left|a_{m+1}\right| \leq (m+1)\left|a_{m+1}\right| \leq \sum\nolimits_{n=m+1}^{\infty}(n+1)\left|a_{n+1}-a_n\right| \longrightarrow 0$$

as $m \to \infty$. Thus R_a is compact.

(3)⟶(1) Follows immediately.
□

Now, we give the following result, which improves the result in [26, Proposition 3.4]:

Theorem 3.3. *The following are equivalent:*

(1) *The operator R_a is bounded on ℓ^1.*

(2) *The assumption that $(a_k) \in \ell^1$ holds.*

(3) *The operator R_a is compact on ℓ^1.*

Proof. **(1)⟶(2)** Follows immediately since $R_a(1,0,0,...) = (a_0,a_1,a_2,...) \in \ell^1$.

(2)⟶(3) Again, it can be shown that the operator R_a is the limit in $\mathcal{B}\left(\ell^1\right)$ of the sequence $\left(R_a^m\right)$ of operators of finite rank. Thus R_a is compact.

(3)⟶(1) Follows immediately.
□

Remark 3.4. *Theorems 3.1, 3.2 and 3.3 assert the fact that the well known Cesàro operator C_1 is not well defined on* cs, h *or ℓ^1. The same assertion was declared in [36].*

Now, let $\eta \in \{c_0, c, \ell^p, \mathrm{bv}_0, \mathrm{bv}\}$, where $1 < p \leq \infty$. To the end of this section, we comment on some recent results related to the compactness of the operator R_a on η. It is well known that the Rhaly matrix has the factorization

$$R_a = D \circ C_1,$$

where C_1 is the Cesàro matrix and D is the diagonal matrix $\operatorname{diag}((k+1)\,a_k)$. Since C_1 is bounded on every sequence space in η [15, 25, 28, 29, 31, 33], then the compactness of the operator R_a follows from that of D. Depending on this fact, many authors tried to find sufficient conditions for the compactness of D, and so for R_a. Consider the following result:

Theorem 3.5. *[26, Proposition 3.1(b)] The operator R_a is compact on ℓ^p $(1 < p < \infty)$ if $((k+1)\,a_k) \in c_0$.*

However, in [26, Example 2], it is shown that the condition $((k+1)\,a_k) \in \ell^\infty$ is not a necessary condition for the boundedness of the operator R_a on ℓ^2. Furthermore, in [34, Corollary 2.2], a result on the compactness of the operator R_a on ℓ^2 has been given by Rhaly. A similar result can be obtained as follows:

Theorem 3.6. *Let (a_k) be a strictly decreasing sequence of positive real numbers and $((k+1)\,a_k) \in c$. Then, the operator R_a is compact on ℓ^p $(1 < p < \infty)$ if and only if $((k+1)\,a_k) \in c_0$.*

Proof. It suffices to prove the necessity of the condition. Indeed, if $((k+1)\,a_k) \notin c_0$, then, the spectrum $\sigma(R_a, \ell^p)$ contains the set $\left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{qL}{2}\right| \leq \frac{qL}{2}\right\}$ [47, Theorem 3.3], where q is the dual of p and $L = \lim_{k\to\infty} (k+1)\,a_k$. That is, R_a is not compact since it has uncountable spectrum. □

The following is a weak, but important, result about the compactness of the operator R_a on the ℓ^p spaces; it is nothing but the result in [26, Corollary 3.5].

Theorem 3.7. *Let (a_k) be a decreasing sequence of positive real numbers. Then, the operator R_a is compact on ℓ^p $(1 < p < \infty)$ if $(a_k) \in \ell^1$.*

Proof. The result follows immediately by applying the well known classical Olivier's result about the speed of convergence to zero of the terms of a convergent series with positive and decreasing terms. So, it remains to apply Theorem 3.5. □

It should be observed that, if (a_k) is a decreasing sequence of positive real numbers, then the condition $(a_k) \in \ell^1$ is not necessary for the compactness of R_a in ℓ^2. For example, let (a_k) be such that

$$a_0 = 2, \quad a_k = \frac{1}{(k+1)\log(k+1)}, \quad \text{for all } k \in \mathbb{N}.$$

Then, using the result in [26, Proposition 4.2], the operator R_a will be compact on ℓ^2. Indeed,

$$\sum\nolimits_{k=0}^{\infty} (k+1)\,|a_k|^2 = 4 + \sum\nolimits_{k=1}^{\infty} \frac{1}{(k+1)\,(\log(k+1))^2},$$

which is convergent. Alternatively, we can use Theorem 3.6. On the other hand, $(a_k) \notin \ell^1$.

Remark 3.8. *It can be shown that, in general, the condition $(a_k) \in c_0$ is necessary but not sufficient for the compactness of R_a in ℓ^p $(1 < p < \infty)$. So far, to the author's knowledge, there are no conditions on the matrix R_a have been given which are necessary as well as sufficient for the compactness of the corresponding operator in ℓ^p $(1 < p < \infty)$. However, while this may be of concern for those interested in summability theory, it will definitely not affect our study of the spectra of the operator R_a.*

Next, we recall the following result.

Theorem 3.9. *[46, Theorems 2 and 9] Let (a_k) be a strictly decreasing sequence of positive real numbers such that $((k+1)\,a_k) \in c_0$. Then the operator R_a is compact on both c_0 and c.*

We observed that, for every $x = (x_k) \in c_0$,

$$\left\|(R_a - R_a^m)\,x\right\|_{c_0} \leq \|x\|_{c_0} \sup_{n\geq m+1} (n+1)\,|a_n|\,.$$

So, it appears that the result in Theorem 3.9 remains true if we omit the conditions that (a_k) be a strictly decreasing sequence of positive real numbers. Much more surprisingly, in the following theorem, it can be proved that the condition $((k+1)\,a_k) \in c_0$ is actually necessary and sufficient for the compactness of the operator R_a on both c_0 and c, so that we have the following important theorem.

Theorem 3.10. *The following hold:*

(1) *The operator R_a is compact on c_0 if and only if $((k+1)\,a_k) \in c_0$.*

(2) *The operator R_a is compact on c if and only if $((k+1)\,a_k) \in c_0$.*

Proof. **(1)** The sufficiency of the condition $((k+1)\,a_k) \in c_0$ is clear from the above argument, and the necessity can be proved as follows: suppose that R_a is a compact operator and consider the bounded sequence $\left(y_j\right)$, where $y_j = \left(\chi_{[0,j]}\,(k)\right)_{k=0}^{\infty}$. Suppose, to the contrary, that $\lim_{j\to\infty} (j+1)\,a_j \neq 0$. Then, there exists $\kappa > 0$ so that $(j+1)\,|a_j| > 2\kappa$ for infinitely many j. Let us write $\mathcal{J}$ for the set of all such j. Then if $j, j' \in \mathcal{J}$ satisfies $j' > j$, we have

$$\|R_a\left(y_j\right) - R_a\left(y_{j'}\right)\|_{c_0} > (j'-j)\left|a_{j'}\right|.$$

Then, for $j' = 2j+1$, we obtain

$$\|R_a\left(y_j\right) - R_a\left(y_{j'}\right)\|_{c_0} > \frac{1}{2}(j'+1)\left|a_{j'}\right| > \kappa.$$

This proves that the sequence $\left(R_a\left(y_j\right)\right)$ has no Cauchy subsequence, and so, it has no convergent subsequence. This contradicts the compactness of R_a, where $\left(y_j\right)$ is a bounded sequence.

(2) Similar to the proof of statement **(1)**.
□

As the following result shows, the condition $((k+1)\,a_k) \in c_0$ is also necessary and sufficient for the compactness of the operator R_a on ℓ^∞.

Theorem 3.11. *The operator R_a is compact on ℓ^∞ if and only if $((k+1)\,a_k) \in c_0$.*

Proof. The result can be proved by adopting the method presented in the proof of Theorem 3.10 to the space ℓ^∞. □

Finally, we recall the following result:

Theorem 3.12. *[51, Theorems 2.3 and 3.2] If $\frac{a_n}{a_{n-1}} < \frac{n}{n+1}$ and $((k+1)\,a_k) \in c_0$, then R_a is compact on both* bv$_0$ *and* bv.

The conditions in Theorem 3.12 means that the sequence $((k+1)\,a_k)$ is a strictly decreasing sequence of positive real numbers with the zero limit. Accordingly, we observe that, by applying the result in [36, Lemma 3.3], the operator D will be compact on both bv_0 and bv. Thus, the boundedness of C_1 together with the compactness of D imply the compactness of R_a. This gives alternative proof of Theorem 3.12. On the contrary, this proof can not be adapted in the spaces cs, h or ℓ^1. In fact, the Cesàro operator C_1 is not well defined on cs, h or ℓ^1.

Under the assumptions that $((k+1)\,a_k)$ is a strictly decreasing sequence of positive real numbers, it can be shown that the condition $((k+1)\,a_k) \in c_0$ is necessary and sufficient for the compactness of the operator R_a on both bv_0 and bv. Thus we can conclude the following result, where the proof is omitted as it is easy.

Theorem 3.13. *Let $((k+1)\,a_k)$ be a strictly decreasing sequence of positive real numbers. Then, the following hold:*

(1) *The operator R_a is compact on* bv_0 *if and only if* $((k+1)\,a_k) \in c_0$.

(2) *The operator R_a is compact on* bv *if and only if* $((k+1)\,a_k) \in c_0$.

3.2. Compactness of Δ_{ab}

In view of the main purpose of the current study, necessary and sufficient conditions for the boundedness and compactness of the operator Δ_{ab} in the sequence spaces bv_0, cs, h, c_0, c and ℓ^1 appeared to be important results.

Define the operator Δ_{ab}^m, where $m \in \mathbb{N}_0$, by

$$\Delta_{ab}^m x = \Delta_{ab}\left((\chi_{[0,m]}(k)\, x_k)_{k=0}^{\infty}\right), \qquad x = (x_k) \in \mu,$$

where $\mu \in \{\mathrm{bv}_0, \mathrm{cs}, \mathrm{h}, c_0, c\}$. Then Δ_{ab}^m is finite rank, and so is compact. Further, let Δ_{ab} be represented by the matrix $(b_{n,k})$, where $b_{n,n} = a_n$, $b_{n+1,n} = b_n$, for all $n \in \mathbb{N}_0$ and $b_{n,k} = 0$, otherwise. Then Δ_{ab}^m is represented by the matrix $(\chi_{[0,m]}(k)\, b_{n,k})$, and so $\Delta_{ab} - \Delta_{ab}^m$ is represented by $(\chi_{[m+1,\infty]}(k)\, b_{n,k})$.

One can check that

$$\begin{aligned} \mathcal{M}_1 &= \sup_{j\geq 0} \sum_{n=1}^{\infty} \left| \sum_{k=0}^{j} (b_{n,k} - b_{n-1,k}) \right| \\ &= \sup_{j\geq 0} \left(\left|b_j\right| + \left|a_j - b_j + b_{j-1}\right| + \sum_{k=0}^{j-1} |a_{k+1} - a_k + b_k - b_{k-1}| \right); \end{aligned}$$

see [20]. If $\mathcal{M}_1$ is finite, then the operator $\Delta_{ab} : \mathrm{bv}_0 \longrightarrow \mathrm{bv}_0$ is well defined, and so is bounded. This follows immediately by applying Lemma 2.5. Indeed, for every $x = (x_k) \in \mathrm{bv}_0$, it can be shown that

$$\begin{aligned} \|\Delta_{ab} x\|_{\mathrm{bv}_0} &= \left| \sum_{k=0}^{\infty} b_{0,k} x_k \right| + \sum_{n=1}^{\infty} \left| \sum_{k=0}^{\infty} b_{n,k} x_k - \sum_{k=0}^{\infty} b_{n-1,k} x_k \right| \\ &\leq |a_0 x_0| + \left(\sup_{j\geq 0} \sum_{n=1}^{\infty} \left| \sum_{k=0}^{j} (b_{n,k} - b_{n-1,k}) \right| \right) \sum_{j=0}^{\infty} \left| x_j - x_{j+1} \right| \\ &= |a_0 x_0| + \mathcal{M}_1 \sum_{j=1}^{\infty} \left| x_j - x_{j-1} \right| \\ &\leq \mathcal{M}_1 \left(|x_0| + \sum_{j=1}^{\infty} \left| x_j - x_{j-1} \right| \right) \\ &\leq \mathcal{M}_1 \|x\|_{\mathrm{bv}_0}. \end{aligned}$$

Consider now the following result:

Theorem 3.14. *The following hold:*

(1) *The Δ_{ab} is bounded on* bv_0 *if and only if*

$$\sup_j\left(\left|b_j\right| + \left|a_j - b_j + b_{j-1}\right| + \sum_{k=0}^{j-1} |a_{k+1} - a_k + b_k - b_{k-1}|\right) < \infty.$$

(2) *The Δ_{ab} is compact on* bv_0 *if and only if*

$$\sum_{k=0}^{\infty} |a_{k+1} - a_k + b_k - b_{k-1}| < \infty \quad \text{and} \quad (a_k), (b_k) \in c_0.$$

Proof. **(1)** We can argue as follows:

Consider the isomorphism

$$U : (x_k) \in \ell^1 \mapsto \left(\sum_{j=k}^{\infty} x_j\right) \in \mathrm{bv}_0$$

and its inverse

$$U^{-1} : (x_k) \in \mathrm{bv}_0 \mapsto (x_k - x_{k+1}) \in \ell^1.$$

Define the operator $K := U^{-1} \circ \Delta_{ab} \circ U$ on ℓ^1. Then, for every $x = (x_k) \in \ell^1$, we have

$$\begin{aligned} Kx &= U^{-1} \circ \Delta_{ab} \circ Ux \\ &= \left(a_k \sum_{j=k}^{\infty} x_j + b_{k-1} \sum_{j=k-1}^{\infty} x_j - a_{k+1} \sum_{j=k+1}^{\infty} x_j - b_k \sum_{j=k}^{\infty} x_j\right). \end{aligned}$$

Then K is a linear operator given by

$$Kx = \left(b_{k-1}x_{k-1} + (a_k - a_{k+1} + b_{k-1} - b_k) \sum_{j=k+1}^{\infty} x_j + (a_k - b_k + b_{k-1})x_k\right).$$

Clearly Δ_{ab} is bounded on bv_0 if and only if K is bounded on ℓ^1. Then, applying Lemma 2.4 to the operator K, we obtain the desired result.

(2) Let the conditions in statement **(2)** be satisfied and $m \in \mathbb{N}_0$. Then

$$\begin{aligned} \|\Delta_{ab} - \Delta_{ab}^m\| \quad &\leq \quad \sup_{j\geq m+1}\left|b_j\right| + \sup_{j\geq m+1}\left|a_j - b_j + b_{j-1}\right| + |b_m| + |a_{m+1}| + \\ &\quad + \sup_{j\geq m+1} \sum_{k=m+1}^{j} |a_{k+1} - a_k + b_k - b_{k-1}| + |b_m| \\ &\longrightarrow \quad 0 \end{aligned}$$

as $m \to \infty$. So Δ_{ab} is a compact operator.

Conversely, assume that Δ_{ab} is a compact operator. Then the condition in statement **(1)** holds, so that we have

$$\sum_{k=0}^{\infty} |a_{k+1} - a_k + b_k - b_{k-1}| < \infty.$$

Now, consider the bounded sequence $\left(e_j\right)$, where e_j is the j-th elementary vector. Suppose, to the contrary, that $\lim_{j\to\infty} a_j \neq 0$ or $\lim_{j\to\infty} b_j \neq 0$. Then, there exists $\kappa > 0$ so that either $|a_j| > \kappa$ for infinitely many j or $|b_j| > \kappa$ for infinitely many j. Let us write $\mathcal{J}$ for the set of all such j. If $j, j' \in \mathcal{J}$ satisfies $j + 2 < j'$, then

$$\|\Delta_{ab}\left(e_j\right) - \Delta_{ab}(e_{j'})\|_{\mathrm{bv}_0} = \|a_j e_j + b_j e_{j+1} - a_{j'} e_{j'} - b_{j'} e_{j'+1}\|_{\mathrm{bv}_0} > \kappa.$$

This contradicts the assumption that Δ_{ab} is compact.

□

The result in Theorem 3.14 improves and extends the recent result in [20, Corollary 2.2].

Next, we consider the operator $\Delta_{ab} : \mathrm{cs} \longrightarrow \mathrm{cs}$. Let

$$\mathcal{M}_2 = \sup_{j\geq 1}\left(\left|a_j\right| + \left|a_j - a_{j-1} - b_{j-1}\right| + \sum_{k=0}^{j-2} |a_{k+1} - a_k + b_{k+1} - b_k|\right).$$

If $\mathcal{M}_2$ is finite, then the operator $\Delta_{ab} : \mathrm{cs} \longrightarrow \mathrm{cs}$ is well defined and is bounded; see [37, Corollary 3.1]. Indeed, for every $x = (x_k) \in \mathrm{cs}$, it has been shown that $\|\Delta_{ab}x\|_{\mathrm{cs}} \leq \mathcal{M}_2 \|x\|_{\mathrm{cs}}$.

The following theorem gives necessary and sufficient conditions for the boundedness and compactness of the operator Δ_{ab} on cs, which improves and extends the result in [37, Corollary 3.1].

Theorem 3.15. *The following hold:*

(1) *The Δ_{ab} is bounded on* cs *if and only if*

$$\sup_j\left(\left|a_j\right| + \left|a_j - a_{j-1} - b_{j-1}\right| + \sum_{k=0}^{j-2} |a_{k+1} - a_k + b_{k+1} - b_k|\right) < \infty.$$

(2) *The Δ_{ab} is compact on* cs *if and only if*

$$\sum_{k=0}^{\infty} |a_{k+1} - a_k + b_{k+1} - b_k| < \infty \quad \textit{and} \quad (a_k), (b_k) \in c_0.$$

Proof. **(1)** See [37, Corollary 3.1]. Alternatively, by applying Lemma 2.2, we obtain the desired result.

(2) Let the conditions in statement **(2)** be satisfied and $m \in \mathbb{N}_0$. Then, for all $x = (x_k) \in \mathrm{cs}$, we have

$$\begin{aligned}\|\Delta_{ab} - \Delta_{ab}^m\| \quad &\leq \quad \sup_{j\geq m+1}\left|a_j\right| + \sup_{j\geq m+1}\left|a_j - a_{j-1} - b_{j-1}\right| + |a_m + b_m| + \\ &\quad + |a_{m+1} + b_{m+1}| + \sup_{j\geq m+1}\sum_{k=m+1}^{j} |a_{k+1} - a_k + b_{k+1} - b_k| \\ &\longrightarrow \quad 0\end{aligned}$$

as $m \to \infty$. So, Δ_{ab} is a compact operator.

Conversely, assume that Δ_{ab} is a compact operator. Then the condition

$$\sum_{k=0}^{\infty} |a_{k+1} - a_k + b_{k+1} - b_k| < \infty$$

holds. Now, consider the bounded sequence $\left(e_j\right)$. Suppose, to the contrary, that $\lim_{j\to\infty} a_j \neq 0$ or $\lim_{j\to\infty}\left(a_j + b_j\right) \neq 0$. Then, there exists $\kappa > 0$ so that either $|a_j| > \kappa$ for infinitely many j or $|a_j + b_j| > \kappa$ for infinitely many j. If $\mathcal{J}$ is the set of all such j and $j, j' \in \mathcal{J}$ satisfies $j + 2 < j'$, then

$$\left\|\Delta_{ab}\left(e_j\right) - \Delta_{ab}(e_{j'})\right\|_{\mathrm{cs}} = \|a_j e_j + b_j e_{j+1} - a_{j'} e_{j'} - b_{j'} e_{j'+1}\|_{\mathrm{cs}} > \kappa.$$

The last inequality holds for infinitely many j and j'. This contradicts the assumption that Δ_{ab} is compact.

□

Finally, we consider the operator $\Delta_{ab} : \mathrm{h} \longrightarrow \mathrm{h}$. Let

$$\mathcal{M}_3 = \sup_j\left(\frac{1}{j+2}\sum_{k=0}^{j} (k+1) |a_{k+1} - a_k + b_k - b_{k-1}|\right).$$

If $\mathcal{M}_3$ is finite, then the operator $\Delta_{ab} : \mathrm{h} \longrightarrow \mathrm{h}$ is well defined and is bounded; cf., [20, Corollary 3.1].

The following theorem gives necessary and sufficient conditions for the boundedness and compactness of the operator Δ_{ab} on h.

Theorem 3.16. *The following hold:*

(1) *The Δ_{ab} is bounded on* h *if and only if*

$$\sup_j \left(\frac{1}{j+2} \sum_{k=0}^{j} (k+1) \left|a_{k+1} - a_k + b_k - b_{k-1}\right| \right) < \infty.$$

(2) *The Δ_{ab} is compact on* h *if and only if*

$$((k+1) a_k), ((k+1) b_k) \in c_0.$$

Proof. **(1)** See [20, Corollary 3.1].

(2) Let the conditions in statement **(2)** be satisfied. Then

$$\lim_{j\to\infty} (j+1) \left|a_{j+1} - a_j + b_j - b_{j-1}\right| = 0.$$

Then, its corresponding sequence of arithmetic mean converges to zero;

$$\lim_{j\to\infty} \left(\frac{1}{j+1} \sum_{k=0}^{j} (k+1) \left|a_{k+1} - a_k + b_k - b_{k-1}\right| \right) = 0.$$

Therefore, we obtain

$$\begin{aligned} \|\Delta_{ab} - \Delta_{ab}^{m}\| &\leq (m+1) |a_{m+1}| \sup_{j\geq m+1} \frac{1}{j+1} + (m+2) |b_m| \sup_{j\geq m+2} \frac{1}{j+1} + \\ &\quad + \sup_{j\geq m+1} \left(\frac{1}{j+2} \sum_{k=m+1}^{j} (k+1) \left|a_{k+1} - a_k + b_k - b_{k-1}\right| \right) \\ &\leq (m+1) |a_{m+1}| + (m+2) |b_m| + \\ &\quad + \sup_{j\geq m+1} \left(\frac{1}{j+2} \sum_{k=0}^{j} (k+1) \left|a_{k+1} - a_k + b_k - b_{k-1}\right| \right) \\ &\longrightarrow 0 \end{aligned}$$

as $m \to \infty$. So, Δ_{ab} is a compact operator.

Conversely, assume that Δ_{ab} is a compact operator and consider the bounded sequence $\left(e_j\right)$. Suppose, to the contrary, that

$$\lim_{j\to\infty} (j+1) a_j \neq 0 \quad \text{or} \quad \lim_{j\to\infty} (j+1) b_{j+1} \neq 0.$$

Then, there exists $\kappa > 0$ so that either $\left|(j+1) a_j\right| > \kappa$ for infinitely many j or $|(j+1) b_j| > \kappa$ for infinitely many j. If $\mathcal{J}$ is the set of all such j and $j, j' \in \mathcal{J}$ satisfies $j + 2 < j'$, then

$$\left\|\Delta_{ab}\left(e_j\right) - \Delta_{ab}(e_{j'})\right\|_{\mathrm{h}} = \|a_j e_j + b_j e_{j+1} - a_{j'} e_{j'} - b_{j'} e_{j'+1}\|_{\mathrm{h}} > \kappa.$$

This contradicts the assumption that Δ_{ab} is compact.
□

Using similar arguments, we can derive the following theorem:

Theorem 3.17. *The following hold:*

(1) *The Δ_{ab} is bounded on c_0 if and only if* $(|a_k| + |b_{k-1}|) \in \ell^{\infty}$.

(2) *The Δ_{ab} is compact on c_0 if and only if $(a_k), (b_k) \in c_0$.*

(3) *The Δ_{ab} is bounded on c if and only if*

$$(|a_k| + |b_{k-1}|) \in \ell^\infty \quad \text{and} \quad (a_k + b_{k-1}) \in c.$$

(4) *The Δ_{ab} is compact on c if and only if $(a_k), (b_k) \in c_0$.*

(5) *The Δ_{ab} is bounded on ℓ^1 if and only if $(|a_k| + |b_k|) \in \ell^\infty$.*

(6) *The Δ_{ab} is compact on ℓ^1 if and only if $(a_k), (b_k) \in c_0$.*

Finally, we recall the following result which is a special case of [30, Theorem 3.2].

Theorem 3.18. *Let $1 < p < \infty$. The operator Δ_{ab} is compact on ℓ^p if and only if $(a_k), (b_k) \in c_0$.*

4. Spectra of the compact operators R_a and Δ_{ab}

Throughout this section, we assume that (a_k) and (b_k) are two sequences of nonzero real numbers such that the operators R_a and Δ_{ab} are compact in the sequence spaces under consideration.

We shall determine the spectrum of R_a in two stages: in the first it is shown that the eigenvalues of R_a^*, the adjoint operator of R_a, contains the set $\{a_n : n \in \mathbb{N}_0\}$, while the latter set contains the eigenvalues of R_a; in the second it is shown that, due to the compactness of the operator R_a, the spectrum is precisely the set $\{a_n : n \in \mathbb{N}_0\} \cup \{0\}$. The spectrum of Δ_{ab} can be obtained with similar argument.

The main results in this section are Theorems 4.5, 4.8, 4.13 and 4.15.

4.1. Spectra of R_a

Recall the following lemma, which is analogy to [53, Lemma 3.6].

Lemma 4.1. *[36, Lemma 2.2] Let T be a linear operator on a Banach sequence space X that has a lower triangular matrix representation $A = (a_{n,k})$. Then the point spectrum of T on X satisfies $\sigma_{\mathrm{p}}(T, X) \subseteq \{a_{n,n} : n \in \mathbb{N}_0\}$.*

Now consider the following general result for the Rhaly operator R_a.

Lemma 4.2. *Let X be a Banach sequence space that contains c_{00}; the subspace of sequences with finite support. Then, the point spectrum of R_a^T, the transpose of R_a, on X satisfies*

$$\{a_n : n \in \mathbb{N}_0\} \subseteq \sigma_{\mathrm{p}}(R_a^T, X) \quad \text{and} \quad 0 \notin \sigma_{\mathrm{p}}(R_a^T, X).$$

Proof. Suppose that $R_a^T f = \lambda f$ for $f = (f_k) \in X$. Then

$$(a_n - \lambda) f_n + \sum_{k=n+1}^{\infty} a_k f_k = 0, \qquad n \in \mathbb{N}_0. \tag{7}$$

If $\lambda = 0$, then we obtain

$$a_0 f_0 + a_1 f_1 + \sum_{k=2}^{\infty} a_k f_k = 0$$

and

$$a_1 f_1 + \sum_{k=2}^{\infty} a_k f_k = 0.$$

We deduce

$$a_0 f_0 = 0.$$

Therefore $f_0 = 0$ since $a_0 \neq 0$. Going through a similar argument, by induction, we can prove that $f_n = 0$ for all $n \in \mathbb{N}_0$. Therefore $f = \mathbf{0}$. So, $0 \notin \sigma_{\mathrm{p}}(R_a^T, X)$.

Furthermore, if $\lambda = a_{n_0}$ for some $n_0 \in \mathbb{N}$, then, from Eq. (7), we have two cases:

(i) If $a_{n_0} \neq a_k$, for all $k < n_0$, then $f = (f_0, f_1, f_2, \ldots, f_{n_0}, 0, 0, \ldots) \in c_{00} \subseteq X$ is an eigenvector associated with the eigenvalue $\lambda = a_{n_0}$, where all other f_k $(k = 1, 2, \ldots, n_0)$ can be inductively calculated in terms of $f_0 = 1$.

(ii) If $a_{n_0} = a_k$, for some $k < n_0$, then $f = (f_0 = 1, f_1, f_2, \ldots, f_{k_0}, 0, 0, \ldots)$ is an eigenvector associated with the eigenvalue $\lambda = a_{n_0}$, where $k_0 = \min\{k \leq n_0 : a_k = a_{n_0}\}$.

□

Recall the following lemma.

Lemma 4.3. *[45, Problem 7, p. 233] Let X be a Banach sequence space with the standard countable basis (e_k). Suppose that T is a bounded linear operator on X into itself that has a matrix representation $A = (a_{n,k})$. Then, the adjoint operator T^* is represented by the transpose $A^T = (a_{k,n})$.*

It will be of some interest to combine Lemmas 4.1, 4.2 and 4.3, so that we obtain the following general result for the Rhaly operator R_a, which will be the key tool to derive the spectra of R_a.

Proposition 4.4. *Let X be a Banach sequence space with the standard countable basis (e_k) and $R_a \in \mathcal{B}(X)$. If X^*, the dual space of X, contains c_{00}, then R_a^* is represented by the transpose R_a^T and*

$$\sigma_{\mathrm{p}}(R_a, X) \subseteq \{a_n : n \in \mathbb{N}_0\} \subseteq \sigma_{\mathrm{p}}(R_a^*, X^*) \quad \textit{and} \quad 0 \notin \sigma_{\mathrm{p}}(R_a^*, X^*).$$

We are now in a position to give the first main result in this section.

Theorem 4.5. *Let X be a complex infinite dimensional Banach sequence space. In addition to the conditions in Proposition 4.4, let the operator $R_a : X \longrightarrow X$ be compact. Then the following hold:*

(1) $\sigma(R_a, X) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.

(2) $\sigma_{\mathrm{p}}(R_a^*, X^*) = \{a_n : n \in \mathbb{N}_0\}$.

(3) $\sigma_{\mathrm{p}}(R_a, X) = \{a_n : n \in \mathbb{N}_0\}$.

(4) $\sigma_{\mathrm{r}}(R_a, X) = \emptyset$.

(5) $\sigma_{\mathrm{c}}(R_a, X) = \{0\}$.

(6) $\mathrm{II}_2\sigma(R_a, X) = \{0\}$.

(7) $\mathrm{III}_3\sigma(R_a, X) = \{a_n : n \in \mathbb{N}_0\}$.

(8) $\sigma_\delta(R_a, X) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.

(9) $\sigma_{\mathrm{co}}(R_a, X) = \{a_n : n \in \mathbb{N}_0\}$.

(10) $\sigma_{\mathrm{ap}}(R_a, X) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.

Proof. **(1)** Applying Proposition 4.4, we obtain

$$\{a_n : n \in \mathbb{N}_0\} \subseteq \sigma_{\mathrm{p}}(R_a^*, X^*) \subseteq \sigma(R_a^*, X^*) = \sigma(R_a, X).$$

Further, since the space X is infinite dimensional and R_a is compact, we learn that $0 \in \sigma(R_a, X)$. Hence

$$\{a_n : n \in \mathbb{N}_0\} \cup \{0\} \subseteq \sigma(R_a, X).$$

Again, since R_a is compact, then

$$\sigma(R_a, X) \subseteq \sigma_{\mathrm{p}}(R_a, X) \cup \{0\} \subseteq \{a_n : n \in \mathbb{N}_0\} \cup \{0\}.$$

The required result follows.

(2) The result follows by applying Proposition 4.4 and statement **(1)**. Indeed, we have

$$\{a_n : n \in \mathbb{N}_0\} \subseteq \sigma_{\mathrm{p}}(R_a^*, X^*) \subseteq \sigma(R_a, X) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$$

and

$$0 \notin \sigma_{\mathrm{p}}(R_a^*, X^*).$$

(3) Since $R_a x = \mathbf{0}$ implies $x = \mathbf{0}$, then we obtain that $0 \notin \sigma_{\mathrm{p}}(R_a, X)$. Now, using Proposition 4.4 and the fact that all non-zero spectral values are eigenvalues, we obtain

$$\{a_n : n \in \mathbb{N}_0\} \subseteq \sigma_{\mathrm{p}}(R_a, X) \subseteq \{a_n : n \in \mathbb{N}_0\}.$$

The desired result follows.

(4) Observe that $\sigma_{\mathrm{r}}(R_a, X) \cup \sigma_{\mathrm{c}}(R_a, X) = \{0\}$. Since $0 \notin \sigma_{\mathrm{p}}(R_a^*, X^*)$, then, using relation (6), we obtain that $0 \notin \sigma_{\mathrm{r}}(R_a, X)$. Thus $\sigma_{\mathrm{r}}(R_a, X) = \varnothing$.

(5) Observe that $\sigma_{\mathrm{c}}(R_a, X) = \sigma(R_a, X) \backslash \left(\sigma_{\mathrm{p}}(R_a, X) \cup \sigma_{\mathrm{r}}(R_a, X)\right)$. It remains to apply statements **(3)** and **(4)**.

(6) The result follows from relation (3) with the application of statement **(5)**; $\mathrm{II}_2\sigma(R_a, X) = \sigma_{\mathrm{c}}(R_a, X) = \{0\}$.

(7) It is known that

$$\begin{aligned} \sigma_{\mathrm{p}}(R_a, X) &= \mathrm{I}_3\sigma(R_a, X) \cup \mathrm{II}_3\sigma(R_a, X) \cup \mathrm{III}_3\sigma(R_a, X) \\ &= \{a_n : n \in \mathbb{N}_0\}. \end{aligned}$$

But, using statements **(2)** and **(3)**, for any $\lambda \in \{a_n : n \in \mathbb{N}_0\}$, we have $\lambda \in \sigma_{\mathrm{p}}(R_a, X) \cap \sigma_{\mathrm{p}}(R_a^*, X^*)$. This shows that $(R_a - \lambda \mathrm{I})^{-1}$ does not exist and $R_a^* - \lambda \mathrm{I}$ is not one to one. Using [44, Theorem 1], we obtain further that $\mathcal{R}(R_a - \lambda \mathrm{I})$ is not dense. That is $\lambda \in \mathrm{III}_3\sigma(R_a, X)$.

(8) Observe that $\mathrm{I}_3\sigma(R_a, X) \subseteq \sigma_{\mathrm{p}}(R_a, X)$. Then $\mathrm{I}_3\sigma(R_a, X) = \varnothing$ since it is open; Lemma 2.1. It follows immediately from relation (4) that

$$\sigma_\delta(R_a, X) = \sigma(R_a, X).$$

It remains to apply statement **(1)**.

(9) It is known that $\sigma_{\mathrm{co}}(R_a, X) = \sigma_{\mathrm{p}}(R_a^*, X^*)$; cf. [9, Proposition 1.3(e), *p.* 28]. Then, the result follows by using statement **(2)**.

(10) Observe that $\varnothing = \sigma_{\mathrm{r}}(R_a, X) = \mathrm{III}_1\sigma(R_a, X) \cup \mathrm{III}_2\sigma(R_a, X)$. Then $\mathrm{III}_1\sigma(R_a, X) = \varnothing$. Therefore, the desired result follows from relation (5);

$$\sigma_{\mathrm{ap}}(R_a, X) = \sigma(R_a, X)\backslash \mathrm{III}_1\sigma(R_a, X),$$

with the application of statement **(1)**.
□

By application of Theorem 4.5, we obtain the spectra of the Rhaly operator R_a on the sequence spaces $\mathrm{bv}_0, \mathrm{h}, \mathrm{cs}, c_0$ and ℓ^p, where $1 \le p < \infty$ as follows:

Corollary 4.6. *Let $\mu \in \{\mathrm{bv}_0, \mathrm{h}, \mathrm{cs}, c_0, \ell^p\}$, where $1 \le p < \infty$. If the operator $R_a : \mu \longrightarrow \mu$ is compact, then the following hold:*

(1) $\sigma(R_a, \mu) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.

(2) $\sigma_{\mathrm{p}}(R_a, \mu) = \{a_n : n \in \mathbb{N}_0\}$.

(3) $\sigma_{\mathrm{r}}(R_a, \mu) = \varnothing$.

(4) $\sigma_{\mathrm{c}}(R_a, \mu) = \{0\}$.

(5) $\mathrm{II}_2\sigma(R_a, \mu) = \{0\}$.

(6) $\mathrm{III}_3\sigma(R_a, \mu) = \{a_n : n \in \mathbb{N}_0\}$.

(7) $\sigma_\delta(R_a, \mu) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.

(8) $\sigma_{\mathrm{co}}(R_a, \mu) = \{a_n : n \in \mathbb{N}_0\}$.

(9) $\sigma_{\mathrm{ap}}(R_a, \mu) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.

Remark 4.7. *An alternative proof of Corollary 4.6(2), in the case where $\mu = \ell^1$, can be based on the case where $\mu = \mathrm{h}$. Indeed, we have*

$$\{a_n : n \in \mathbb{N}_0\} = \sigma_{\mathrm{p}}(R_a, h) \subseteq \sigma_{\mathrm{p}}\left(R_a, \ell^1\right).$$

The second inclusion follows by applying Lemma 4.1.
Furthermore, since h is a proper dense subspace of ℓ^1, then

$$\sigma_{\mathrm{r}}\left(R_a, \ell^1\right) \subseteq \sigma_{\mathrm{r}}(R_a, h) = \varnothing;$$

Corollary 4.6(3) with $\mu = \mathrm{h}$. Thus $\sigma_{\mathrm{r}}\left(R_a, \ell^1\right) = \varnothing$. This gives another proof of Corollary 4.6(3) for $\mu = \ell^1$, based on the case where $\mu = \mathrm{h}$.

We observe that the results related to the spectra of the operator R_a on c_0, bv_0 and ℓ^p $(2 \le p < \infty)$, which have been given in [46, 47, 51], are included in Corollary 4.6. However, in ℓ^1, h and cs, the results are completely new.

By similar arguments with minor changes, the spectra of R_a on ℓ^∞ can be established; see the following theorem:

Theorem 4.8. *Let the operator $R_a : \ell^\infty \longrightarrow \ell^\infty$ be compact. Then the following hold:*

(1) $\sigma(R_a, \ell^\infty) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.

(2) $\sigma_{\mathrm{p}}(R_a, \ell^\infty) = \{a_n : n \in \mathbb{N}_0\}$.

(3) $\sigma_{\mathrm{r}}(R_a, \ell^\infty) = \{0\}$.

(4) $\sigma_{\mathrm{c}}(R_a, \ell^\infty) = \varnothing$.

(5) $\mathrm{III}_2\sigma(R_a, \ell^\infty) = \{0\}$.

(6) $\mathrm{III}_3\sigma(R_a, \ell^\infty) = \{a_n : n \in \mathbb{N}_0\}$.

(7) $\sigma_\delta(R_a, \ell^\infty) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.

(8) $\sigma_{\mathrm{co}}(R_a, \ell^\infty) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.

(9) $\sigma_{\mathrm{ap}}(R_a, \ell^\infty) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.

Proof. We only give the proofs of statements **(2)** and **(3)**. All other statements can be proved similarly as in the proof of Theorem 4.5.

(2) Combine Corollary 4.6(2), for $\mu = c_0$, and Lemma 4.1, we obtain

$$\{a_n : n \in \mathbb{N}_0\} = \sigma_{\mathrm{p}}(R_a, c_0) \subseteq \sigma_{\mathrm{p}}(R_a, \ell^\infty) \subseteq \{a_n : n \in \mathbb{N}_0\}.$$

(3) Using statements **(1)** and **(2)**, we have $\sigma_{\mathrm{r}}(R_a, \ell^\infty) \cup \sigma_{\mathrm{c}}(R_a, \ell^\infty) = \{0\}$. Since c_0 is a closed subspace of ℓ^∞ and

$$\mathcal{R}(R_a) \subseteq c_0,$$

then $\overline{\mathcal{R}(R_a)} \subseteq c_0 \neq \ell^\infty$. Thus R_a does not have a dense range, and so, $0 \in \sigma_{\mathrm{r}}(R_a, \ell^\infty)$. Thus $\sigma_{\mathrm{r}}(R_a, \ell^\infty) = \{0\}$. □

Remark 4.9. *Theorem 4.8 still valid as well for the space c. This gives a complete description of the spectra of R_a on c, which was determined in [46]. Under suitable conditions, one can similarly derive the spectra of R_a on* bv.

4.2. Spectra of Δ_{ab}

The following lemma is an analogy to Lemma 4.2.

Lemma 4.10. *Let X be a Banach sequence space that contains c_{00}. Then, the point spectrum of Δ_{ab}^T, the transpose of Δ_{ab}, on X satisfies*

$$\{a_n : n \in \mathbb{N}_0\} \subseteq \sigma_{\mathrm{p}}(\Delta_{ab}^T, X).$$

Proof. The proof is omitted since it is similar to that of Lemma 4.2. In fact, the proof can be easily adapted to the operator Δ_{ab}. □

Combining Lemmas 4.1, 4.3 and 4.10, we obtain the following proposition.

Proposition 4.11. *Let X be a Banach sequence space with the standard countable basis (e_k) and $\Delta_{ab} \in \mathcal{B}(X)$. If X^*, the dual space of X, contains c_{00}, then Δ_{ab}^* is represented by the transpose Δ_{ab}^T and*

$$\sigma_{\mathrm{p}}(\Delta_{ab}, X) \subseteq \{a_n : n \in \mathbb{N}_0\} \subseteq \sigma_{\mathrm{p}}(\Delta_{ab}^*, X^*).$$

Remark 4.12. *Under the assumptions of Lemma 4.10, unlike for the operator R_a, the element 0 may or may not belong to $\sigma_{\mathrm{p}}(\Delta_{ab}^*, X^*)$. In fact, this will depend on the choice of the sequences (a_k) and (b_k). For example, if $a_k = 1/(2k+2)$ and $b_k = 1/(k+1)$ for all $k \in \mathbb{N}_0$, then the conditions in Theorem 3.14(2) hold, so that the operator Δ_{ab} is compact on* bv_0. *So, $\Delta_{ab}^* = \Delta_{ab}^T$ is compact on $\mathrm{bv}_0^* \simeq \mathrm{bs}$. However, $0 \in \sigma_{\mathrm{p}}(\Delta_{ab}^T, \mathrm{bs})$. Whereas, if $a_k = 1/(k+1)$ and $b_k = 1/(2k+2)$ for all $k \in \mathbb{N}_0$, then $0 \notin \sigma_{\mathrm{p}}(\Delta_{ab}^T, \mathrm{bs})$.*

Taking into account Remark 4.12 and Proposition 4.11, we may consider, without no loss of generality, the case where $0 \in \sigma_{\mathrm{p}}(\Delta_{ab}^*, X^*)$ since the opposite case, $0 \notin \sigma_{\mathrm{p}}(\Delta_{ab}^*, X^*)$, may be treated similarly. Next, we give the result about the spectra of Δ_{ab}, where the proof is a routine adaptation of the argument in the proof of Theorem 4.5.

Theorem 4.13. *Let X be a complex infinite dimensional Banach sequence space. In addition to the conditions in Proposition 4.11, let the operator $\Delta_{ab} : X \longrightarrow X$ be compact with $0 \in \sigma_{\mathrm{p}}(\Delta_{ab}^*, X^*)$. Then the following hold:*

(1) $\sigma(\Delta_{ab}, X) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.

(2) $\sigma_{\mathrm{p}}(\Delta_{ab}^*, X^*) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.

(3) $\sigma_{\mathrm{p}}(\Delta_{ab}, X) = \{a_n : n \in \mathbb{N}_0\}$.

(4) $\sigma_{\mathrm{r}}(\Delta_{ab}, X) = \{0\}$.

(5) $\sigma_{\mathrm{c}}(\Delta_{ab}, X) = \emptyset$.

(6) $\mathrm{III}_2\sigma(\Delta_{ab}, X) = \{0\}$.

(7) $\mathrm{III}_3\sigma(\Delta_{ab}, X) = \{a_n : n \in \mathbb{N}_0\}$.

(8) $\sigma_{\delta}(\Delta_{ab}, X) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.

(9) $\sigma_{\mathrm{co}}(\Delta_{ab}, X) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.

(10) $\sigma_{\mathrm{ap}}(\Delta_{ab}, X) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.

Application of Theorem 4.13 yields the following corollary:

Corollary 4.14. *Let $\mu \in \{\mathrm{bv}_0, \mathrm{h}, \mathrm{cs}, c_0, \ell^p\}$, where $1 \le p < \infty$. If the operator $\Delta_{ab} : \mu \longrightarrow \mu$ is compact with $0 \in \sigma_{\mathrm{p}}(\Delta_{ab}^*, \mu^*)$, then the following hold:*

(1) $\sigma(\Delta_{ab}, \mu) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.

(2) $\sigma_{\mathrm{p}}(\Delta_{ab}, \mu) = \{a_n : n \in \mathbb{N}_0\}$.

(3) $\sigma_{\mathrm{r}}(\Delta_{ab}, \mu) = \{0\}$.

(4) $\sigma_{\mathrm{c}}(\Delta_{ab}, \mu) = \emptyset$.

(5) $\mathrm{III}_2\sigma(\Delta_{ab}, \mu) = \{0\}$.

(6) $\mathrm{III}_3\sigma(\Delta_{ab}, \mu) = \{a_n : n \in \mathbb{N}_0\}$.

(7) $\sigma_{\delta}(\Delta_{ab}, \mu) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.

(8) $\sigma_{\mathrm{co}}(\Delta_{ab}, \mu) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.

(9) $\sigma_{\mathrm{ap}}(\Delta_{ab}, \mu) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.

For the compactness case of the operator $\Delta_{ab} : c \longrightarrow c$, we can prove that $\sigma_{\mathrm{r}}(\Delta_{ab}, c) = \{0\}$. Indeed, $\mathcal{R}(\Delta_{ab}) \subseteq c_0$. Then $\overline{\mathcal{R}(\Delta_{ab})} \subseteq c_0 \neq c$. Thus Δ_{ab} does not have a dense range. Further, in this case, we have $0 \in \sigma_{\mathrm{p}}(\Delta_{ab}^*, c^*)$, where $c^* \simeq \ell^1$. So, we have the following result:

Theorem 4.15. *Let the operator $\Delta_{ab} : c \longrightarrow c$ be compact. Then the following hold:*

(1) $\sigma(\Delta_{ab}, c) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.

(2) $\sigma_{\mathrm{p}}(\Delta_{ab}, c) = \{a_n : n \in \mathbb{N}_0\}$.

(3) $\sigma_{\mathrm{r}}(\Delta_{ab}, c) = \{0\}$.

(4) $\sigma_{\mathrm{c}}(\Delta_{ab}, c) = \emptyset$.

(5) $\mathrm{III}_2\sigma(\Delta_{ab}, c) = \{0\}$.

(6) $\mathrm{III}_3\sigma(\Delta_{ab}, c) = \{a_n : n \in \mathbb{N}_0\}$.

(7) $\sigma_{\delta}(\Delta_{ab}, c) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.

(8) $\sigma_{\mathrm{co}}(\Delta_{ab}, c) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.

(9) $\sigma_{\mathrm{ap}}(\Delta_{ab}, c) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.

5. Conclusion

This paper is a follow-up to the recent articles about the spectra of the operators R_a and Δ_{ab} on Banach sequence spaces, where the main purpose is to close the gaps to obtain comparable results for the spectra of such operators in general setting (in a large class of sequence spaces). In fact, a general technique to prove the spectral results of the operators R_a and Δ_{ab} has been given. However, in particular, we consider, among other questions, the more precise problem of determining the spectra of R_a and Δ_{ab} in the Hahn sequence space h, the space of convergent series cs and the sequence space bv_0.

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