Filomat 36:17 (2022), 5885–5893
https://doi.org/10.2298/FIL2217885D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Application of Measure of Noncompactness On Integral Equations Involving Generalized Proportional Fractional and Caputo-Fabrizio Fractional Integrals

Anupam Das[a], Bipan Hazarika[b], Vahid Parvanah[c], Nihar Kumar Mahato[d]

[a]*Department of Mathematics, Cotton University, Panbazar, Guwahati-781001, Assam, India*
[b]*Department of Mathematics, Gauhati University, Guwahati-781014, Assam, India*
[c]*Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran*
[d]*Discipline of Natural Science, Indian Institute of Information Technology, Design, and Manufacturing Jabalpur, India*

Abstract. Using Petryshyn's fixed point theorem, we show the existence of solution to fractional integral equations, including generalized proportional and Caputo–Fabrizio fractional integrals. We also use appropriate examples to support our findings.

1. Introduction

Fractional calculus is a branch of mathematical analysis which really examines many possible interpretations for representing the real number powers or complex number powers of the differentiation operator $\mathcal{U}$. Here, the term $\mathcal{U}\theta(z)$ is equal to $\frac{d}{dz}\theta(z)$, as well as the integration operator $\mathcal{O}\theta(z)$ should be $\int_0^z \theta(p)dp$ and for operators such as this, and even the implementation of a calculus that extrapolates the conventional calculus.

There are some types of fractional operators such as Grünwald–Letnikov, Osler, Liouville, Caputo, Hadamard, Marchaud, Riesz, Cossar, Weyl, Coimbra, Jumarie, Hilfer, Davidson-Essex, Chen, Caputo–Fabrizio and Atangana–Baleanu [8], [31] and [35], but, the most applicable fractional operators are the Riemann-Liouville and Caputo integro-differential operators.

In applied sciences and applied analysis, a fractional derivative is a derivative of any real or complex non-integer order. The initial breakthrough was made in a letter written by Leibniz, and it was completed by Antoine de l'Hopital [23] in the 16th century. Fractional calculus was incorporated in one of Abel's earliest works [3], where these aspects can be perceived: the concept of fractional order integration and differentiation, the strictly inverse relation between them, the fact that fractional order differentiation and integration can be interpreted as the same unified operation, and thereby the coherent form for ambiguous real order differentiation and integration. Over the nineteenth and early twentieth centuries, the methods and interpretations of fractional calculus evolved dramatically, and numerous researchers contributed to

2020 *Mathematics Subject Classification*. 45G05, 47H08, 47H09, 47H10.
Keywords. Functional integral equations; Measure of non-compactness; Fixed point theorem
Received: 08 November 2021; Revised: 05 December 2021; Accepted: 10 April 2022
Communicated by Erdal Karapınar
Corresponding author: Vahid Parvanah
Email addresses: math.anupam@gmail.com; anupam.das@cottonuniversity.ac.in (Anupam Das), bh_rgu@yahoo.co.in; bh_gu@gauhati.ac.in (Bipan Hazarika), zam.dalahoo@gmail.com (Vahid Parvanah), nihar@iiitdmj.ac.in (Nihar Kumar Mahato)

the concepts of fractional derivatives and integrals. The functions $\theta(\mathcal{U})$, which are even more familiar than powers, are described in the functional calculus in spectral theory in terms of functional analysis. The pseudo-differential operators principle also allows for the consideration of the powers of $\mathcal{U}$.

The pseudo-differential operators principle also allows the powers of $\mathcal{U}$ to be taken into account. The operators that emerge are expressions of singular integral operators. As a result, there are a number of well-known theories in which fractional calculus can be explored.

The functional integral equations (FIEs) are making notable results to many real-life issues. Through using sorts of integral equations and fractional differentials, many problems can be represented in technology, engineering, astronomy, and other fields (we refer the reader to [1, 2, 14, 34] and many references therein).

Kazemi and Ezzati [25] used Petryshyn's fixed point theorem to investigate the existence of the solution of nonlinear functional integral equations (see, also [19, 24] for some works related to this fixed point theorem). Many researchers have solved many forms of integral equations to overcome various real-life situations with the help of fixed point theorems and measure of noncompactness. (e.g, one can see [4]–[10], [11–13, 17, 18, 27] and references among them).

The following list of abbreviations should be considered:

(a) generalized proportional fractional (GPF),
(b) Caputo-Fabrizio fractional integral (CFFI),
(c) fixed point theorems (FPTs),
(d) measure of noncompactness (MNC).

In this article, we have the following assumptions:

• E: A Banach space with the norm $\|\,.\,\|_E$;
• $B[\theta,\kappa]$: A closed ball with center θ and radius κ in E;
• B_ρ a closed ball around the origin in a general Banach space E.
• $\bar{A}$: the closure of A;
• Conv$A$: the convex closure of A;
• NB_E: the family of all nonempty and bounded subsets of E;
• RC_E: the subfamily consisting of all relatively compact sets;
• $\mathbb{R}$ the set of real numbers;
• $\mathbb{R}^+ = [0,\infty)$.

Now, we present the concept of measure of noncompactness. Foe more details, see the references [15], [20], [28], [29] and [32].

Definition 1.1. *[26] A mapping $\xi : NB_E \to \mathbb{R}^+$ is said to be an MNC in E if:*

(i) $Y \in NB_E$ *and* $\xi(Y) = 0$ *imply* Y *is precompact.*
(ii) $\ker \xi = \{Y \in NB_E : \xi(Y) = 0\}$ *is nonempty and* $\ker \xi \subset RC_E$.
(iii) $Y \subseteq Y_1 \implies \xi(Y) \le \xi(Y_1)$.
(iv) $\xi(\bar{Y}) = \xi(Y)$.
(v) $\xi(ConvY) = \xi(Y)$.
(vi) $\xi(\sigma Y + (1-\sigma)Y_1) \le \sigma\xi(Y) + (1-\sigma)\xi(Y_1)$ *for all* $\sigma \in [0,1]$.
(vii) *if* $Y_l \in NB_E$, $Y_l = \bar{Y}_l$, $Y_{l+1} \subset Y_l$ *for all* $l \in \mathbb{N}$ *and* $\lim_{l\to\infty} \xi(Y_l) = 0$ *then* $Y_\infty = \bigcap_{l=1}^{\infty} Y_l \neq \emptyset$.

Definition 1.2. *[30] Let $F : E \to E$ be a continuous function so that $F(B)$ is bounded for all bounded subsets $B \subset E$, and $\xi(FB) \le k\xi(B)$ for some $k \in (0,1)$. Then F is called a k-set contraction. If $\xi(FB) < \xi(B)$, where $\xi(B) > 0$, then F is called a densifying (condensing) map. A k-set contraction is condensing, but the converse is not true.*

Theorem 1.3. *[33] Let $F : B_\rho \to E$ be a condensing function so that if $F(z) = kz$ for some $z \in \partial B_\rho$, then $k \in (0,1)$. Then $fix(F)$ in B_ρ is nonempty, where $fix(F)$ is the set of fixed points of F.*

1.1. Measure of noncompactness on $C([0,1])$

Consider the space $E = C(I)$ which consists of the set of real continuous functions on I, where $I = [0,1]$. Then E is a Banach space with the norm

$$\| w \| = \sup \{ |w(s)| : s \in I \}, \ w \in E.$$

Let $A(\neq \emptyset) \subseteq E$ be fixed and bounded. For an arbitrary $w \in A$ and arbitrary $\epsilon > 0$, denote by $Q(w,\epsilon)$ the modulus of the continuity of w, i.e.,

$$Q(w,\epsilon) = \sup \{ |w(s_1) - w(s_2)| : s_1, s_2 \in I, |s_1 - s_2| \leq \epsilon \}.$$

Further, we define

$$Q(A,\epsilon) = \sup \{ Q(w,\epsilon) : w \in A \} \text{ and } Q_0(A) = \lim_{\epsilon \to 0} Q(A,\epsilon).$$

It is well-known that the function Q_0 is a MNC in E such that the Hausdorff MNC χ is given by $\chi(A) = \frac{1}{2} Q_0(A)$ (see[16]).

2. Main Result

For $\rho \in (0,1]$ and $\alpha \in \mathbb{C}$ with $Re(\alpha) > 0$, we define the left GPF integral of Θ by [22]

$$\left({}_aI^{\alpha,\rho}\Theta \right)(t) = \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t e^{\frac{(\rho-1)(t-\tau)}{\rho}} (t-\tau)^{\alpha-1} \Theta(\tau) d\tau.$$

In [36], the Caputo-Fabrizio fractional integral of function Θ is defined by

$${}_0^{CF}I_s^\sigma \left[\Theta(s) \right] = \frac{1-\sigma}{M(\sigma)} \Theta(s) + \frac{\sigma}{M(\sigma)} \int_0^s \Theta(\eta) d\eta, \ s \geq 0,$$

where $0 < \sigma < 1, \ \Theta \in H^1[0,\tau], \ \eta > 0$ and M be any smooth function with $M(0) = M(1) = 1$ and $\sigma(0) > 0$ for all $0 < \sigma < 1$.

In this part, we study the following fractional integral equation

$$\Theta(s) = D\left(s, l(s,\Theta(s)), \left({}_0I^{\alpha,\rho}\Theta \right)(s), \ {}_0^{CF}I_s^\sigma \left[\Theta(s) \right] \right), \tag{1}$$

where $0 \leq \sigma \leq 1$, $\alpha > 1$ and $s \in I = [0,1]$.

Let $B_r = \{ \Theta \in E : \| \Theta \| \leq r \}$.

We consider the following assumptions to solve the Eq. (1):

(A) $D : I \times \mathbb{R}^3 \to \mathbb{R}, \ l : I \times \mathbb{R} \to \mathbb{R}$ are continuous and there exist constants $\beta_1, \beta_2, \beta_3, \beta_4 \geq 0$ satisfying

$$\left| D(s,l,I_1,I_2) - D(s,\bar{l},\bar{I}_1,\bar{I}_2) \right| \leq \beta_1 \left| l - \bar{l} \right| + \beta_2 \left| I_1 - \bar{I}_1 \right| + \beta_3 \left| I_2 - \bar{I}_2 \right|, \ s \in I, \ l, I_1, I_2, \bar{l}, \bar{I}_1, \bar{I}_2 \in \mathbb{R}$$

and

$$|l(s,J_1) - l(s,J_2)| \leq \beta_4 |J_1 - J_2|, J_1, J_2 \in \mathbb{R}.$$

Also, $|D(s,0,0,0)| = 0$ and $l(s,0) = 0$ for all $s \in I$.

(B) There exists $r > 0$ such that

$$\bar{D} = \sup \{ |D(s,l,I_1,I_2)| : s \in I, l \in [-L,L], I_1 \in [-K_{GPF}, K_{GPF}], I_2 \in [-K_{CF}, K_{CF}] \} \leq r,$$

where

$$L = \sup \{ |l(s,\Theta)| : s \in I, \ \Theta \in [-r,r] \},$$

$$K_{GPF} = \sup\left\{\left|\left({}_0I^{\alpha,\rho}\Theta\right)(s)\right| : s \in I, \Theta \in [-r,r]\right\},$$

$$K_{CF} = \sup\left\{\left|{}_0^{CF}I_s^{\sigma}\left[\Theta(s)\right]\right| : s \in I, \Theta \in [-r,r]\right\}$$

and

$$\mathcal{M} = \sup_{0<\sigma<1} \frac{1}{|M(\sigma)|}.$$

(C) There exists a positive solution r of the inequality

$$\beta_1\beta_4 r + \frac{\beta_2 r}{\rho^{\alpha}\Gamma(\alpha+1)}.e^{\frac{(\rho-1)}{\rho}} + 2\mathcal{M}\beta_3 r \le r,$$

and $\beta_3\mathcal{M} < 1$.

Theorem 2.1. *If conditions from (A)-(C) hold, then the Eq. (1) has a solution in $E = C(I)$.*

Proof. Let the operator $A : E \to E$ be defined by

$$(A\Theta)(s) = D\left(s, l(s,\Theta(s)), \left({}_0I^{\alpha,\rho}\Theta\right)(s),\ {}_0^{CF}I_s^{\sigma}\left[\Theta(s)\right]\right).$$

Step 1: We prove that the function A maps B_r into B_r. Let $\Theta \in B_r$. We have

$$\begin{aligned}
&|(A\Theta)(s)| \\
&\le \left|D\left(s, l(s,\Theta(s)), \left({}_0I^{\alpha,\rho}\Theta\right)(s),\ {}_0^{CF}I_s^{\sigma}\left[\Theta(s)\right]\right) - D(s,0,0,0)\right| + |\Delta(s,0,0,0)| \\
&\le \beta_1 \left|l(s,\Theta(s)) - 0\right| + \beta_2 \left|\left({}_0I^{\alpha,\rho}\Theta\right)(s) - 0\right| + \beta_3 \left|{}_0^{CF}I_s^{\sigma}\left[\Theta(s)\right] - 0\right| \\
&\le \beta_1\beta_4 |\Theta(s)| + \beta_2 \left|\left({}_0I^{\alpha,\rho}\Theta\right)(s)\right| + \beta_3 \left|{}_0^{CF}I_s^{\sigma}\left[\Theta(s)\right]\right|.
\end{aligned}$$

Also,

$$\begin{aligned}
&\left|\left({}_0I^{\alpha,\rho}\Theta\right)(s)\right| \\
&= \left|\frac{1}{\rho^{\alpha}\Gamma(\alpha)} \int_0^s e^{\frac{(\rho-1)(s-\tau)}{\rho}} (s-\tau)^{\alpha-1}\Theta(\tau)d\tau\right| \\
&\le \frac{1}{\rho^{\alpha}\Gamma(\alpha)} \int_0^s e^{\frac{(\rho-1)(s-\tau)}{\rho}} (s-\tau)^{\alpha-1} |\Theta(\tau)|\, d\tau \\
&\le \frac{re^{\frac{(\rho-1)}{\rho}}}{\rho^{\alpha}\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} d\tau \\
&\le \frac{re^{\frac{(\rho-1)}{\rho}}}{\rho^{\alpha}\Gamma(\alpha+1)}
\end{aligned}$$

and

$$\begin{aligned}
&\left|{}_0^{CF}I_s^{\sigma}\left[\Theta(s)\right]\right| \\
&\le \left|\frac{1-\sigma}{M(\sigma)}\Theta(s) + \frac{\sigma}{M(\sigma)} \int_0^s \Theta(\eta)d\eta\right| \\
&\le \frac{|1-\sigma|}{|M(\sigma)|} |\Theta(s)| + \frac{|\sigma|}{|M(\sigma)|} \int_0^s \left|\Theta(\eta)\right| d\eta \\
&\le 2\mathcal{M}r.
\end{aligned}$$

Hence, $\| \Theta \| < r$ gives

$$\| A \| \leq \beta_1\beta_4 r + \frac{\beta_2 r}{\rho^{\alpha}\Gamma(\alpha+1)}.e^{\frac{(\rho-1)}{\rho}} + 2\beta_3\mathcal{M}r \leq r.$$

Due to the assumption (C), A maps B_r into B_r.

Step 2: We prove that A is continuous on B_r. Let $\epsilon > 0$ and $\Theta, \bar{\Theta} \in B_r$ such that $\| \Theta - \bar{\Theta} \| < \epsilon$. We have

$$\begin{aligned}
&\left|(A\Theta)(s) - (A\bar{\Theta})(s)\right| \\
&\leq \left|D\left(s, l(s,\Theta(s)), ({}_0I^{\alpha,\rho}\Theta)(s), {}_0^{CF}I_s^{\sigma}[\Theta(s)]\right) - D\left(s, l(s,\bar{\Theta}(s)), ({}_0I^{\alpha,\rho}\bar{\Theta})(s), {}_0^{CF}I_s^{\sigma}[\bar{\Theta}(s)]\right)\right| \\
&\leq \beta_1\left|l(s,\Theta(s)) - l(s,\bar{\Theta}(s))\right| + \beta_2\left|({}_0I^{\alpha,\rho}\Theta)(s) - ({}_0I^{\alpha,\rho}\bar{\Theta})(s)\right| + \beta_3\left|{}_0^{CF}I_s^{\sigma}[\Theta(s)] - {}_0^{CF}I_s^{\sigma}[\bar{\Theta}(s)]\right|.
\end{aligned}$$

Also,

$$\begin{aligned}
&\left|({}_0I^{\alpha,\rho}\Theta)(s) - ({}_0I^{\alpha,\rho}\bar{\Theta})(s)\right| \\
&= \left|\frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_0^s e^{\frac{(\rho-1)(s-\tau)}{\rho}}(s-\tau)^{\alpha-1}\left\{\Theta(\tau) - \bar{\Theta}(\tau)\right\}d\tau\right| \\
&\leq \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_0^s e^{\frac{(\rho-1)(s-\tau)}{\rho}}(s-\tau)^{\alpha-1}\left|\Theta(\tau) - \bar{\Theta}(\tau)\right|d\tau \\
&< \frac{\epsilon e^{\frac{(\rho-1)}{\rho}}}{\rho^{\alpha}\Gamma(\alpha+1)},
\end{aligned}$$

and

$$\begin{aligned}
&\left|{}_0^{CF}I_s^{\sigma}[\Theta(s)] - {}_0^{CF}I_s^{\sigma}[\bar{\Theta}(s)]\right| \\
&= \left|\frac{1-\sigma}{M(\sigma)}(\Theta(s) - \bar{\Theta}(s)) + \frac{\sigma}{M(\sigma)}\int_0^s(\Theta(\eta) - \bar{\Theta}(\eta))\,d\eta\right| \\
&\leq \frac{|1-\sigma|}{|M(\sigma)|}\left|\Theta(s) - \bar{\Theta}(s)\right| + \frac{\sigma}{|M(\sigma)|}\int_0^s\left|\Theta(\eta) - \bar{\Theta}(\eta)\right|d\eta \\
&\leq 2\mathcal{M}\epsilon.
\end{aligned}$$

Hence, $\| \Theta - \bar{\Theta} \| < \epsilon$ gives

$$\left|(A\Theta)(s) - (A\bar{\Theta})(s)\right| < \beta_1\beta_4\epsilon + \frac{\beta_2\epsilon e^{\frac{(\rho-1)}{\rho}}}{\rho^{\alpha}\Gamma(\alpha+1)} + 2\beta_3\mathcal{M}\epsilon.$$

As $\epsilon \to 0$, we get $\left|(A\Theta)(s) - (A\bar{\Theta})(s)\right| \to 0$. This shows that A is continuous on B_r.

Step 3: An estimate of χ with respect to Q_0. Assume that $\emptyset \neq \Omega \subseteq B_{r_0}$. Choose $\Theta \in \Omega$. Let $\epsilon > 0$ be arbitrary and let $s_1, s_2 \in I$ such that $|s_2 - s_1| \leq \epsilon$ and $s_2 \geq s_1$.

Now,

$$
\begin{aligned}
&\left|(A\Theta)(s_2)-(A\Theta)(s_1)\right| \\
&\leq \left|D\left(s_2, l(s_2,\Theta(s_2)), ({}_0I^{\alpha,\rho}\Theta)(s_2),\ {}_0^{CF}I_{s_2}^{\sigma}[\Theta(s_2)]\right)-D\left(s_1, l(s_1,\Theta(s_1)), ({}_0I^{\alpha,\rho}\Theta)(s_1),\ {}_0^{CF}I_{s_1}^{\sigma}[\Theta(s_1)]\right)\right| \\
&\leq \left|D\left(s_2, l(s_2,\Theta(s_2)), ({}_0I^{\alpha,\rho}\Theta)(s_2),\ {}_0^{CF}I_{s_2}^{\sigma}[\Theta(s_2)]\right)-D\left(s_2, l(s_2,\Theta(s_2)), ({}_0I^{\alpha,\rho}\Theta)(s_2),\ {}_0^{CF}I_{s_1}^{\sigma}[\Theta(s_1)]\right)\right| \\
&+\left|D\left(s_2, l(s_2,\Theta(s_2)), ({}_0I^{\alpha,\rho}\Theta)(s_2),\ {}_0^{CF}I_{s_1}^{\sigma}[\Theta(s_1)]\right)-D\left(s_2, l(s_2,\Theta(s_2)), ({}_0I^{\alpha,\rho}\Theta)(s_1),\ {}_0^{CF}I_{s_1}^{\sigma}[\Theta(s_1)]\right)\right| \\
&+\left|D\left(s_2, l(s_2,\Theta(s_2)), ({}_0I^{\alpha,\rho}\Theta)(s_1),\ {}_0^{CF}I_{s_1}^{\sigma}[\Theta(s_1)]\right)-D\left(s_2, l(s_1,\Theta(s_1)), ({}_0I^{\alpha,\rho}\Theta)(s_1),\ {}_0^{CF}I_{s_1}^{\sigma}[\Theta(s_1)]\right)\right| \\
&+\left|D\left(s_2, l(s_1,\Theta(s_1)), ({}_0I^{\alpha,\rho}\Theta)(s_1),\ {}_0^{CF}I_{s_1}^{\sigma}[\Theta(s_1)]\right)-D\left(s_1, l(s_1,\Theta(s_1)), ({}_0I^{\alpha,\rho}\Theta)(s_1),\ {}_0^{CF}I_{s_1}^{\sigma}[\Theta(s_1)]\right)\right| \\
&\leq \beta_3\left|{}_0^{CF}I_{s_2}^{\sigma}[\Theta(s_2)]-{}_0^{CF}I_{s_1}^{\sigma}[\Theta(s_1)]\right|+\beta_2\left|({}_0I^{\alpha,\rho}\Theta)(s_2)-({}_0I^{\alpha,\rho}\Theta)(s_1)\right| \\
&+\beta_1\left|l(s_2,\Theta(s_2))-l(s_1,\Theta(s_1))\right|+Q_D(I,\epsilon),
\end{aligned}
$$

where

$$
Q_D(I,\epsilon)=\sup\left\{\begin{array}{c}|D(s_2,l,H_1,H_2)-D(s_1,l,H_1,H_2)|:|s_2-s_1|\leq\epsilon; s_1,s_2\in I;\\ l\in[-L,L]; H_1\in[-K_{GPF},K_{GPF}]; H_2\in[-K_{CF},K_{CF}]\end{array}\right\}.
$$

Also,

$$
Q_l(I,\epsilon)=\sup\left\{|l(s_2,\Theta(s_2))-l(s_1,\Theta(s_1))|: s_1,s_2\in I;\Theta(s_1),\Theta(s_2)\in[-r,r]\right\}
$$

and

$$
\begin{aligned}
\left|{}_0^{CF}I_{s_2}^{\sigma}[\Theta(s_2)]-{}_0^{CF}I_{s_1}^{\sigma}[\Theta(s_1)]\right| &\leq \mathcal{M}\left|\Theta(s_2)-\Theta(s_1)\right|+\mathcal{M}\parallel\Theta\parallel(s_2-s_1)\\
&\leq \mathcal{M}Q(\Theta,\epsilon)+\mathcal{M}r\epsilon,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|({}_0I^{\alpha,\rho}\Theta)(s_2)-({}_0I^{\alpha,\rho}\Theta)(s_1)\right| \\
&=\left|\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^{s_2}e^{\frac{(\rho-1)(s_2-\tau)}{\rho}}(s_2-\tau)^{\alpha-1}\Theta(\tau)d\tau-\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^{s_1}e^{\frac{(\rho-1)(s_1-\tau)}{\rho}}(s_1-\tau)^{\alpha-1}\Theta(\tau)d\tau\right| \\
&\leq\frac{1}{\rho^\alpha\Gamma(\alpha)}\left|\int_0^{s_2}e^{\frac{(\rho-1)(s_2-\tau)}{\rho}}(s_2-\tau)^{\alpha-1}\Theta(\tau)d\tau-\int_0^{s_1}e^{\frac{(\rho-1)(s_1-\tau)}{\rho}}(s_1-\tau)^{\alpha-1}\Theta(\tau)d\tau\right| \\
&\leq\frac{1}{\rho^\alpha\Gamma(\alpha)}\left|\int_0^{s_2}e^{\frac{(\rho-1)(s_2-\tau)}{\rho}}(s_2-\tau)^{\alpha-1}\Theta(\tau)d\tau-\int_0^{s_1}e^{\frac{(\rho-1)(s_2-\tau)}{\rho}}(s_2-\tau)^{\alpha-1}\Theta(\tau)d\tau\right| \\
&+\frac{1}{\rho^\alpha\Gamma(\alpha)}\left|\int_0^{s_1}e^{\frac{(\rho-1)(s_2-\tau)}{\rho}}(s_2-\tau)^{\alpha-1}\Theta(\tau)d\tau-\int_0^{s_1}e^{\frac{(\rho-1)(s_1-\tau)}{\rho}}(s_1-\tau)^{\alpha-1}\Theta(\tau)d\tau\right| \\
&\leq\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_{s_1}^{s_2}e^{\frac{(\rho-1)(s_2-\tau)}{\rho}}(s_2-\tau)^{\alpha-1}\left|\Theta(\tau)\right|d\tau \\
&+\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^{s_1}\left|\left(e^{\frac{(\rho-1)(s_2-\tau)}{\rho}}(s_2-\tau)^{\alpha-1}-e^{\frac{(\rho-1)(s_1-\tau)}{\rho}}(s_1-\tau)^{\alpha-1}\right)\Theta(\tau)\right|d\tau \\
&\leq\frac{-e^{\frac{(\rho-1)}{\rho}}}{\rho^\alpha\Gamma(\alpha+1)}\parallel\Theta\parallel(s_2-s_1)^\alpha \\
&+\frac{\parallel\Theta\parallel}{\rho^\alpha\Gamma(\alpha)}\int_0^{s_1}\left|e^{\frac{(\rho-1)(s_2-\tau)}{\rho}}(s_2-\tau)^{\alpha-1}-e^{\frac{(\rho-1)(s_1-\tau)}{\rho}}(s_1-\tau)^{\alpha-1}\right|d\tau.
\end{aligned}
$$

As $\epsilon \to 0$, then

$$s_2 \to s_1,$$

$$\left|\left({}_0I^{\alpha,\rho}\Theta\right)(s_2) - \left({}_0I^{\alpha,\rho}\Theta\right)(s_1)\right| \to 0$$

and

$$\left|{}_0^{CF}I_{s_2}^{\sigma}\left[\Theta(s_2)\right] - {}_0^{CF}I_{s_1}^{\sigma}\left[\Theta(s_1)\right]\right| \to \mathcal{M}Q(\Theta,\epsilon).$$

Hence,

$$\begin{aligned}&|(A\Theta)(s_2) - (A\Theta)(s_1)|\\ &\leq \beta_3\left|{}_0^{CF}I_{s_2}^{\sigma}\left[\Theta(s_2)\right] - {}_0^{CF}I_{s_1}^{\sigma}\left[\Theta(s_1)\right]\right| + \beta_2\left|\left({}_0I^{\alpha,\rho}\Theta\right)(s_2) - \left({}_0I^{\alpha,\rho}\Theta\right)(s_1)\right| + \beta_1 Q_l(I,\epsilon) + Q_D(I,\epsilon),\end{aligned}$$

i.e.,

$$Q(A\Theta,\epsilon) \leq \beta_3\left|{}_0^{CF}I_{s_2}^{\sigma}\left[\Theta(s_2)\right] - {}_0^{CF}I_{s_1}^{\sigma}\left[\Theta(s_1)\right]\right| + \beta_2\left|\left({}_0I^{\alpha,\rho}\Theta\right)(s_2) - \left({}_0I^{\alpha,\rho}\Theta\right)(s_1)\right| + \beta_1 Q_l(I,\epsilon) + Q_D(I,\epsilon).$$

By the uniform continuity of l, D on $I \times [-r, r]$ and $I \times [-L, L] \times [-K_{GPF}, K_{GPF}] \times [-K_{CF}, K_{CF}]$, respectively, we have $Q_l(I,\epsilon) \to 0$ and $Q_D(I,\epsilon) \to 0$ as $\epsilon \to 0$.
Taking $\sup\limits_{\Theta\in\Omega}$ as $\epsilon \to 0$, we have

$$Q_0(A\Omega) \leq [\beta_3\mathcal{M}]\, Q_0(\Omega),$$

which shows that A is a condensing map by assumption (B).
Now, if $\Theta \in \partial B_r$ and $A\Theta = k\Theta$ then $\| A\Theta \| = k \| \Theta \| = kr$ and by assumption (B),

$$|(A\Theta)(s)| = \left|D\left(s, l(s,\Theta(s)), \left({}_0I^{\alpha,\rho}\Theta\right)(s),\ {}_0^{CF}I_s^{\sigma}\left[\Theta(s)\right]\right)\right| \leq r,\ s \in I.$$

Hence, $\| A\Theta \| \leq r$ which implies $k \leq 1$.

Thus, by Petryshyn's fixed point theorem, A has a fixed point in $\Omega \subseteq B_r$, i.e., equation (1) has a solution in E. □

Example 2.2. Consider the following equation:

$$\Theta(s) = \frac{\Theta(s)}{5+s^3} + \frac{\left({}_0I^{2,\frac{2}{3}}\Theta\right)(s)}{10} + \frac{{}_0^{CF}I_s^{\frac{1}{3}}\left[\Theta(s)\right]}{10}, \tag{2}$$

where $s \in [0,1] = I$.

Here,

$$\left({}_0I^{2,\frac{2}{3}}\Theta\right)(s) = -\frac{9}{4\Gamma(2)}\int_0^s e^{\frac{s-\tau}{2}}(s-\tau)\,\Theta(\tau)d\tau,$$

and

$${}_0^{CF}I_s^{\frac{1}{3}}\left[\Theta(s)\right] = \frac{2}{3M(\frac{1}{3})}\Theta(s) + \frac{1}{3M(\frac{1}{3})}\int_0^s \Theta(\eta)d\eta,$$

where $M(\sigma) = 1$ and $\mathcal{M} = 1$.
Also, $D(s, l, H_1, H_2) = l + \frac{H_1}{10} + \frac{H_2}{10}$ and $l(s,\Theta) = \frac{\Theta}{5+s^3}$.
It is trivial that both D and l are continuous and

$$|l(s, J_1) - l(s, J_2)| \leq \frac{|J_1 - J_2|}{5}$$

and

$$\left|D(s,l,H_1,H_2) - D(s,\bar{l},\bar{H}_1,\bar{H}_2)\right| \le \left|l-\bar{l}\right| + \frac{1}{10}\left|H_1-\bar{H}_1\right| + \frac{1}{10}\left|H_2-\bar{H}_2\right|.$$

Therefore, $\beta_1 = 1,\ \beta_2 = \frac{1}{10},\ \beta_3 = \frac{1}{10}$ and $\beta_4 = \frac{1}{5}$.
It is obvious that

$$|D(s,0,0,0)| = 0 \text{ and } l(s,0) = 0.$$

If $\| \Theta \| \le r$, then

$$K_{GPF} = r,\ K_{CF} = 2r, \text{ and } L = \frac{r}{5}.$$

Further,

$$|D(s,l,H_1,H_2)| \le \frac{r}{5} + \frac{1}{10}\{r\} + \frac{2r}{10} = (0.5)\, r \le r.$$

If we choose $r = 2$, then

$$L = \frac{2}{5},\ K_{CF} = 4 \text{ and } K_{GPF} = 2$$

which gives

$$\bar{D} \le 2.$$

We observe that all the assumptions from $(A)-(C)$ of Theorem 2.1 are satisfied. By Theorem 2.1 we conclude that the equation (2) has a solution in E.

References

[1] S. Abbas, M. Benchohra and G.M. N'Guérékata, Advances Fractional Differential and Integral Equations, Nova Science Publishers, New York, (2015).

[2] A. Abdeljawad, R.P. Agarwal, E. Karapınar and P.S. Kumari, Solutions of the Nonlinear Integral Equation and Fractional Differential Equation Using the Technique of a Fixed Point with a Numerical Experiment in Extended b-Metric Space, Symmetry, 2019, 11, 686.

[3] N.H. Abel, Oplosning af et par opgaver ved hjelp af bestemte integraler, Magazin for Naturvidenskaberne. Kristiania, (Oslo) 2(55) 1823:2.

[4] R.S. Adiguzel, U. Aksoy, E. Karapınar and I.M. Erhan, On the solution of a boundary value problem associated with a fractional differential equation, Math. Methods Appl. Sci., (2020). https://doi.org/10.1002/mma.6652.

[5] R.S. Adiguzel, U. Aksoy, E. Karapınar and I.M. Erhan, On The Solutions Of Fractional Differential Equations Via Geraghty Type Hybrid Contractions, Appl. Comput. Math., 20(2), 2021, 313–333.

[6] R.S. Adiguzel, U. Aksoy, E. Karapınar and I.M. Erhan, Uniqueness of solution for higher-order nonlinear fractional differential equations with multi-point and integral boundary conditions, RACSAM (2021) 115:155; https://doi.org/10.1007/s13398-021-01095-3.

[7] H. Afshari, H. Hosseinpour, H.R. Marasi, Application of some new contractions for existence and uniqueness of differential equations involving Caputo-Fabrizio derivative, Adv. Differ. Equ., 2021, 321 (2021).

[8] H. Afshari, S. Kalantari and E. Karapınar, Solution of fractional differential equations via coupled fixed point, Electron. J. Differ. Equ., Vol. 2015 (2015), No. 286, 1-12.

[9] H. Afshari and E. Karapınar, A discussion on the existence of positive solutions of the boundary value problems via ψ-Hilfer fractional derivative on b-metric spaces, Adv. Differ. Equ., 2020, 616 (2020). https://doi.org/10.1186/s13662-020-03076-z

[10] H. Afshari, H. Shojaat, and M. Siahkali Moradi, Existence of the positive solutions for a tripled system of fractional differential equations via integral boundary conditions, Results in Nonlinear Analysis, 4(3) (2021). 186-199.

[11] J.A. Alamo and J. Rodríguez, Operational calculus for modified Eedélyi-Kober operators, Serdica Bulgaricae Math. Publ., 20 (1994) 351–363.

[12] B. Alqahtani, H. Aydi, E. Karapınar, V. Rakočević, A Solution for Volterra Fractional Integral Equations by Hybrid Contractions, Mathematics, 2019, 7, 694.https://doi.org/10.3390/math7080694.

[13] R. Arab, H.K. Nashine, N.H. Can and T.T. Binh, Solvability of functional-integral equations (fractional order) using measure of noncompactness, Adv. Differ. Equ., 2020, 12 (2020). https://doi.org/10.1186/s13662-019-2487-4

[14] A. Atangana and D. Baleanu, New fractional derivative with non-local and non-singular kernel, Therm. Sci., 20(2) (2016) 763–769.
[15] Sh. Banaei, M. Mursaleen, V. Parvaneh, Some fixed point theorems via measure of noncompactness with applications to differential equations, Comput. Appl. Math., (2020) 39:139.
[16] J. Banaś and K. Goebel, Measure of Noncompactness in Banach Spaces, Lecture Notes in Pure and Applied Mathematics, Vol. 60, Marcel Dekker, New York, (1980).
[17] J. Banaś, M. Jleli, M. Mursaleen, B. Samet and C. Vetro, *Advances in Nonlinear Analysis via the Concept of Measure of Noncompactness*, Springer Nature Singapore Pvt. Ltd. (2017).
[18] J. Banaś and M. Mursaleen, *Sequence Spaces and Measures of Noncompactness with Applications to Differential and Integral Equations*, Springer India (2014).
[19] A. Deep, Deepmala and R. Ezzati, Application of Petryshyn's fixed pooint theorem to solvability for functional integral equations, Appl. Math. Comput., 395 (2021), 125878.
[20] M. Gabeleh, M. Asadi and E. Karapınar, Best proximity results on condensing operators via measure of noncompactness with application to integral equations, Thai J. Math., 18(3): 1519–1535, 2020.
[21] B. Hazarika, R. Arab and M. Mursaleen, Applications of Measure of Noncompactness and Operator Type Contraction for Existence of Solution of Functional Integral Equations, Complex Anal. Oper. Theory, 13 (2019) 3837–3851.
[22] F. Jarad, T. Abdeljawad and J. Alzabut, Generalized fractional derivatives generated by a class of local proportional derivatives, Eur. Phys. J., Special Topics, 226 (2017) 3457–3471.
[23] U.N. Katugampola, A New Approach To Generalized Fractional Derivatives, Bull. Math. Anal. Appl., 6(4) (2014) 1–15.
[24] M. Kazemi and R. Ezzati, Existence of solutions for some nonlinear Volterra integral equations via Petryshyn's fixed point theorem, Int. J. Anal. Appl., 9 (2018), 1–12.
[25] M. Kazemi and R. Ezzati, Existence of solution for some nonlinear two-dimensional Volterra integral equations via measures of noncompactness, Appl. Math. Comput., 275 (2016), 165–171.
[26] K. Kuratowski, Sur les espaces complets, Fund. Math., 15 (1930) 301–309.
[27] J.E. Lazreg, S. Abbas, M. Benchohra and E. Karapınar, Impulsive Caputo-Fabrizio fractional differential equations in b-metric spaces, Open Mathematics, 19 (2021) 363-372 https://doi.org/10.1515/math-2021-0040.
[28] F. Nikbakhtsarvestani, S.M. Vaezpour and M. Asadi. A characterization of the generalized KKM mappings via the measure of noncompactness in complete geodesic spaces, J. Nonlinear Funct. Anal., 2017 (8), 2017.
[29] F. Nikbakhtsarvestani, S.M. Vaezpour and M. Asadi, $F(\psi,\varphi)$-contraction in terms of measure of noncompactness with application for nonlinear integral equations, J. Inequal. Appl., 2017, 271 (2017). https://doi.org/10.1186/s13660-017-1545-2
[30] R D. Nussbaum, The fixed point index and fixed point theorem for k-set contractions, Proquest LLC, Ann Arbor, MI, 1969. Thesis(Ph.D)-The University of Chicago.
[31] E.C. de Oliveira and J.A.T. Machado, *A Review of Definitions for Fractional Derivatives and Integral*, Math. Probl. Eng., **2014**, Article ID 238459, 6 pages.
[32] V. Parvaneh, Sh. Banaei, J.R. Roshan and M. Mursaleen, On tripled fixed point theorems via measure of noncompactness with applications to a system of fractional integral equations, Filomat, 35(14), (2021) 4897-4915. https://doi.org/10.2298/FIL2114897P
[33] W.V. Petryshyn, Structure of the fixed points sets of k-set-contractions, Arch. Rational Mech. Anal., 40 (1970-1971) 312–328.
[34] C. Ravichandran, K. Logeswari, S.K. Panda and K.S. Nisar, On new approach of fractional derivative by Mittag-Leffler kernel to neutral integro-differential systems with impulsive conditions, Chaos Solit. Fractals, 139 (2020): 110012.
[35] A. Salim, B. Benchohra, E. Karapınar and J. E. Lazreg, Existence and Ulam stability for impulsive generalized Hilfer-type fractional differential equations, Adv. Differ. Equ., 2020, 601 (2020).
[36] K. Shah, F. Jarad and T. Abdeljawad, On a nonlinear fractional order model of dengue fever disease under Caputo-Fabrizio derivative, Alexandria Eng. J., 59 (2020) 2305–2313.