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Study of European Style Options Under Itô-McKean Brownian Motion with Azzalini Skew-Normal Distribution

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Abstract. In this article, we deal European style option, with arbitrary payoff which includes both put and call options, on an asset whose price evolves as Itô-McKean skew Brownian motion with Azzalini skew-normal distribution. Initially, we investigate a condition which leads the Itô-McKean skew Brownian motion to be a martingale. Next, we price the option and show that if the payoff function is convex then so is the price function. After this, we show if the payoff is finite then the price function satisfies a partial differential equation with respect to time. Further, we provide a necessary and sufficient condition for the price function to satisfy Feymann-Kac type equation. Next, we study Black-Scholes type equation and give expressions for the delta hedge. Finally, we study the particular case of an European call option in order to compare some of our results with the existing literature. Our results can be used to investigate the optimal exercise boundary, discrete time hedging strategies etc. of the option.

1. Introduction

In 1973, Black and Scholes [2] introduced a closed form formula for an European call option in the case when the log-returns of the underlying asset price are normally distributed. Their model does not consider the skewness, time dependence, jumps, etc. of the log-return which occurs in the real market data. In this regard, Peiro [15] studied skewness and symmetry of returns in stock markets and showed that it cannot be rejected for most of the markets. Hussain and Shashiashvili [10], Hussain et al. [8, 9] studied several American style options and the corresponding discrete time hedging strategies. They found that uniform approximations of the value function of the American style options can be used to obtain uniform approximations of the corresponding delta hedging strategies. To be more precise for asset markets, researchers studied option theory under skew Brownian motion. It is a diffusion process which is characterized by a parameter $p \in [0, 1]$, with excursion from zero; positive has probability p and negative has probability $1 - p$. Through this motion, one can split the risks associated with the underlying asset into endogenous and exogenous parts. The theory of skew Brownian motion was initially introduced by Itô

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and McKean [11] and the density of which is investigated by Azzalini [1]. Lejay [12] discussed in detail the construction of Skew Brownian Motion. Several authors, Ouknine [14], studied properties of skew Brownian motion. Eling et al. [4] showed that skew Brownian motion is better able to catch the characteristics of hedge fund returns than the usual Brownian motion. They also show that skewness parameter has several advantages as compared to common measures of skewness. Moreover, Rossello [17] studied arbitrage under the skew Brownian motion models. Itô and McKean [11] also gave the concept of Itô-McKean skew Brownian motion while Harrison and Shepp [7] studied a stochastic differential equation of a random variable having properties of Itô-McKean skew Brownian motion (i.e., considering both endogenous and exogenous risks). In their investigation, they found that the part studying exogenous risk is in fact the absolute of a Brownian motion while the part studying endogenous risk is Brownian motion, that is, Itô-McKean skew Brownian motion is the sum of standard Brownian motion and an independent reflected Bronian motion. Azzalini [1] investigated the distribution of Itô-McKean skew Brownian motion. Corns and Satchell [3], and Zhu and He [20] worked with Itô-McKean skew Brownian motion and priced European call option and studied the Greeks of the option only. For more detail on Itô-McKean skew Brownian motion, its properties and distributions, we refer readers to Gairat and Sheherbakov [5], Mukherjee and Dey [13], Raqab, Shafiqah Al-Awadhi and Debasis Kundu [16] etc.

In this paper, we extend the work of Corns and Satchell [3] and Zhu and He [20]. To do this, we consider the Itô-McKean skew Brownian motion given in Corns and Satchell [3] and Zhu and He [20]

$$X_t^\delta = \sqrt{1-\delta^2}\, W_t^1 + \delta|W_t^2|, \delta \in (-1,1), 0 \le t \le T,$$

where W_t^1 and W_t^2 being two independent standard Brownian motions and $|W_t^2|$ is the absolute of W_t^2, and study European style options (which include both call and put options) with arbitrary payoff on an asset evolves under X_t^δ. Initially, we investigate a condition under which X_t^δ is sub-martingale, martingale and super-martingale. Next, we price the option and show that if the payoff function is convex then so is the price function. After that, we show that if the payoff is finite then the price function satisfies a partial differential equation with respect to time. Next, we show that the price function satisfies a Feymann-Kac type equation if and only if X_t^δ is martingale. We also study the Black-Scholes type equation, give expressions for the delta hedge and study the Greeks of the option. Finally, we study the particular case of European call option in order to compare some of our results i.e., the price function and the Greeks given in the literature. Results can be used to investigate the optimal exercise boundary, discrete time hedging strategies etc. of the corresponding option.

2. Basic Notations and Some Preliminary Results

In this section, we give basic notions and some preliminary results, similar to Corns and Satchell [3] and Zhu and He [20], which are frequently used in the investigation of our results.

Consider a probability space $(\Omega, \mathcal{F}, P)$ on which we consider the motion X_t^δ (investigated and studied in Corns and Satchell [3], Mukherjee and Dey [13], Zhu and He [20] etc.) defined as

$$X_t^\delta = \sqrt{1-\delta^2}\, W_t^1 + \delta|W_t^2|, \delta \in (-1,1), 0 \le t \le T, \tag{1}$$

where W_t^1 and W_t^2 are two independent standard Brownian motions.

Let $T > 0$ be a finite time horizon and the $\sigma-$algebras generated respectively by W_t^1 and W_t^2 are independent. Let us denote by $(\mathcal{F}_t)_{0\le t\le T}$, the $P-$completion of the natural filtration of W_t^1 and W_t^2, $0 \le t \le T$. On the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)_{0\le t\le T}$, we consider a financial market on a risky asset $S_t; 0 \le t \le T$; evolves as geometric Brownian motion (investigated and studied in Corns and Satchell [3], Zhu and He [20] etc.) in the following form

$$S_T = S_t e^{\int_t^T \mu(v)dv + \sigma(X_T^\delta - X_t^\delta)}, 0 \le t \le T, \tag{2}$$

where S_t is the value of the share of a stock at time t, while σ is the stock volatility and $(\mu(t), \mathcal{F}_t)_{0\le t\le T}$ is certain bounded progressive measurable process, and a money market account paying constant interest rate r.

Densities of the random variables $\sqrt{1-\delta^2}\, W_t^1$ and $\delta|W_t^2|$ are given as

$$f_{\sqrt{1-\delta^2}W_t^1}(x) \quad = \quad \frac{1}{\sqrt{2\pi t(1-\delta^2)}} e^{-\frac{x^2}{2t(1-\delta^2)}}, \tag{3}$$

$0 \le t \le T,\ \delta \in (-1,1)$; while

$$\begin{aligned} f_{\delta|W_t^2|}(x) &= \frac{d}{dx} P\left(\delta|W_t^2| \le x\right) \\ &= \begin{cases} \frac{d}{dx} P\left(|W_t^2| \le \frac{x}{\delta}\right), & \text{if } \delta > 0; \\ \frac{d}{dx} P\left(|W_t^2| \ge \frac{x}{\delta}\right), & \text{if } \delta < 0, \end{cases} \\ &= \begin{cases} \frac{\sqrt{2}}{\delta\sqrt{\pi t}} e^{-\frac{x^2}{2\delta^2 t}}, & \text{if } \delta > 0; \\ -\frac{\sqrt{2}}{\delta\sqrt{\pi t}} e^{-\frac{x^2}{2\delta^2 t}}, & \text{if } \delta < 0. \end{cases} \end{aligned} \tag{4}$$

Using convolution theory and densities (3) and (4), one can obtain the Azzalini [1] skew-normal density

$$\begin{aligned} f_{X_t^\delta}(x) &= \begin{cases} \int_{-\infty}^{x} f_{\sqrt{1-\delta^2}W_t^1}(z) f_{\delta|W_t^2|}(x-z)dz, & \text{if } \delta > 0; \\ f_{W_t^1}(x); & \text{if } \delta = 0; \\ \int_{x}^{\infty} f_{\sqrt{1-\delta^2}W_t^1}(z) f_{\delta|W_t^2|}(x-z)dz, & \text{if } \delta < 0, \end{cases} \\ &= \begin{cases} \frac{2}{\sqrt{t}}\phi\left(\frac{x}{\sqrt{t}}\right)\Phi\left(\frac{x\delta}{\sqrt{t(1-\delta^2)}}\right), & \text{if } \delta \ne 0, \\ \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}, & \text{if } \delta = 0, \end{cases} \end{aligned} \tag{5}$$

where $\phi\left(\frac{x}{\sqrt{t}}\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2t}}$ and $\Phi\left(\frac{x\delta}{\sqrt{t(1-\delta^2)}}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\delta x}{\sqrt{t(1-\delta^2)}}} e^{-\frac{u^2}{2}} du.$

Denoting

$$W_t = \sqrt{1-\delta^2}\, W_t^1, \text{ and } R_t = \delta|W_t^2|,$$

then using (3) and (4), one gets the conditional densities as

$$f_{\sigma W_T|\sigma W_t}(x_1|y) \quad = \quad \frac{1}{\sigma\sqrt{(1-\delta^2)(T-t)}} \phi\left(\frac{x_1 - y}{\sigma\sqrt{(1-\delta^2)(T-t)}}\right), \tag{6}$$

where $\delta \in (-1,1)$ while

$$f_{\sigma R_T|\sigma R_t}(x_2|z) = \begin{cases} \frac{1}{\sigma\delta\sqrt{T-t}}\left(\phi\left(\frac{x_2 - z}{\sigma\delta\sqrt{T-t}}\right) + \phi\left(\frac{x_2+z}{\sigma\delta\sqrt{T-t}}\right)\right), & \text{for } \delta > 0; \\ \frac{-1}{\sigma\delta\sqrt{T-t}}\left(\phi\left(\frac{x_2 - z}{\sigma\delta\sqrt{T-t}}\right) + \phi\left(\frac{x_2+z}{\sigma\delta\sqrt{T-t}}\right)\right), & \text{for } \delta < 0. \end{cases} \tag{7}$$

Next we express

$$E\left[S_T \middle| F_t\right] = S_t e^{\int_t^T \mu(v)dv - \sigma(W_t + R_t)} E\left[e^{\sigma(W_T+R_T)} \middle| F_t\right], 0 \le t \le T, \tag{8}$$

where

$$E\left(e^{\sigma(W_T+R_T)} \middle| F_t\right) \quad = \quad \begin{cases} \int_0^\infty \int_{-\infty}^\infty e^{x_1+x_2} f_{\sigma W_T, \sigma R_T|\sigma W_t, \sigma R_t}(x_1, x_2|y,z) dx_1 dx_2, & \text{if } \delta > 0; \\ \int_{-\infty}^0 \int_{-\infty}^\infty e^{x_1+x_2} f_{\sigma W_T, \sigma R_T|\sigma W_t, \sigma R_t}(x_1, x_2|y,z) dx_1 dx_2, & \text{if } \delta < 0. \end{cases}$$

To calculate the latter conditional expectation we split the density of two independent random variables W_t and R_t as

$$f_{\sigma W_T, \sigma R_T|\sigma W_t, \sigma R_t}(x_1, x_2|y,z) \quad = \quad f_{\sigma W_T|\sigma W_t}(x_1|y) f_{\sigma R_T|\sigma R_t}(x_2|z),$$

therefore

$$E\left(e^{\sigma(W_T+R_T)}\Big|F_t\right) = \begin{cases} \int_0^\infty \int_{-\infty}^\infty e^{x_1+x_2} f_{\sigma W_T|\sigma W_t}(x_1|y) f_{\sigma R_T|\sigma R_t}(x_2|z)dx_1dx_2, & \text{if } \delta > 0; \\ \int_{-\infty}^0 \int_{-\infty}^\infty e^{x_1+x_2} f_{\sigma W_T|\sigma W_t}(x_1|y) f_{\sigma R_T|\sigma R_t}(x_2|z)dx_1dx_2, & \text{if } \delta < 0. \end{cases}$$

Use of (6) and (7) further gives

$$= e^{y+z+\frac{(T-t)\sigma^2}{2}}\left[\Phi\left(\frac{z+(T-t)\sigma^2\delta^2}{\sigma\delta\sqrt{T-t}}\right)+e^{-2z}\Phi\left(\frac{-z+(T-t)\sigma^2\delta^2}{\sigma\delta\sqrt{T-t}}\right)\right], \tag{9}$$

for both δ negative or positive.

Let us denote

$$I^z(T-t) = \ln\left[\Phi\left(\frac{z+(T-t)\sigma^2\delta^2}{\sigma\delta\sqrt{T-t}}\right)+e^{-2z}\Phi\left(\frac{-z+(T-t)\sigma^2\delta^2}{\sigma\delta\sqrt{T-t}}\right)\right], \tag{10}$$

where $z = \sigma R_t$. With the above expresion, (9) becomes

$$E\left(e^{\sigma(W_T+R_T)}\Big|F_t\right) = e^{y+z+\frac{(T-t)\sigma^2}{2}+I^z(T-t)}, \tag{11}$$

where $y = \sigma W_t$ and $z = \sigma R_t$.

Moreover, for $y = \sigma W_t$, $z = \sigma R_t$, $\delta \in (-1, 1)$, density $f(u|y,z) = f_{\sigma W_T+\sigma R_T|\sigma W_t,\sigma R_t}(u|y,z)$ can be expressed as

$$\begin{aligned} & f_{\sigma W_T+\sigma R_T|\sigma W_t,\sigma R_t}(u|y,z) \\ = & \begin{cases} \int_{-\infty}^u f_{\sigma W_T|\sigma W_t}(v|y) f_{\sigma R_T|\sigma R_t}(u-v|z)dv, & \text{if } \delta > 0; \\ \int_u^\infty f_{\sigma W_T|\sigma W_t}(v|y) f_{\sigma R_T|\sigma R_t}(u-v|z)dv, & \text{if } \delta < 0, \end{cases} \\ = & \frac{e^{-\frac{(u-y-z)^2}{2(T-t)\sigma^2}}}{\sigma\sqrt{2\pi(T-t)}}\Phi\left(\frac{\delta^2(u-y-z)+z}{\sigma\delta\sqrt{(T-t)(1-\delta^2)}}\right)+\frac{e^{-\frac{(u-y+z)^2}{2(T-t)\sigma^2}}}{\sigma\sqrt{2\pi(T-t)}}\Phi\left(\frac{\delta^2(u-y+z)-z}{\sigma\delta\sqrt{(T-t)(1-\delta^2)}}\right), \end{aligned} \tag{12}$$

for $\delta \in (-1, 1)$.

3. Main Results

In this section, we extend the work of Corns and Satchell [3] and Zhu and He [20] and study new results under the setup of these authors. First, we study mean and variance of the random variable X_t^δ defined in (1) and discuss its martingale property. These results lead to the investigation of the Feymann-Kac formula, Black-Scholes type equation and the continuous time delta hedging strategy. Next, we price European style option, with arbitrary payoff, on the asset S_T formulated in (2) and show that if the payoff function is convex then so is the price function. After this, we show that if the payoff is finite, then the price function is continuous and satisfies a partial differential equation with respect to time. Next, we show that the price function satisfies a Feymann-Kac type equation if and only if the process X_T^δ is martingale. After these, we study the Black-Scholes type equation, expressions for the delta hedge and Greeks of the option. Last, we come to the particular case of European call option in order to compare some of our results (the price function and the Greeks) with the existing literature. Results can be used to investigate the optimal exercise boundary, discrete time hedging strategies etc of the corresponding option.

By notion of the absolute

$$|W_t^2| = \begin{cases} W_t^2, & \text{if } W_t^2 \geq 0; \\ -W_t^2, & \text{if } W_t^2 < 0, \end{cases}$$

we can express

$$|W_t^2| = W_t^2 \mathcal{I}_{(W_t^2\geq 0)} - W_t^2 \mathcal{I}_{(W_t^2<0)}, \tag{13}$$

where $\mathcal{I}_A$ is the indicator function on event A.

Using the latter expression and definition of the Brownian motion, we express and calculate the expected value $|W_t^2|$ as

$$\begin{aligned} E[|W_t^2|] &= E[W_t^2\mathcal{I}_{(W_t^2\geq 0)} - W_t^2\mathcal{I}_{(W_t^2<0)}] \\ &= E[2W_t^2\mathcal{I}_{(W_t^2\geq 0)} - W_t^2] \\ &= E[2W_t^2\mathcal{I}_{(W_t^2\geq 0)}], \end{aligned}$$

where we have used $\mathcal{I}_{(W_t^2\geq 0)} + \mathcal{I}_{(W_t^2<0)} = 1$. This further gives

$$\begin{aligned} E[|W_t^2|] &= E\left[2\sqrt{t}\frac{W_t^2}{\sqrt{t}}\mathcal{I}_{\left(\frac{W_t^2}{\sqrt{t}}\geq 0\right)}\right] \\ &= \frac{2\sqrt{t}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{\frac{-x^2}{2}}\mathcal{I}_{(x\geq 0)}dx \\ &= \sqrt{\frac{2t}{\pi}}, \end{aligned}$$

while variance as

$$\begin{aligned} V[|W_t^2|] &= E\left[|W_t^2|^2\right] - \left(E\left[|W_t^2|\right]\right)^2 \\ &= \frac{(\pi-2)}{\pi}t. \end{aligned}$$

Using these results we find that the expected value of X_t^δ is calculated as

$$E[X_t^\delta] = \frac{\delta(\pi-2)}{\pi}t, \quad \delta\in(-1,1),$$

while the variance as

$$\begin{aligned} V[X_t^\delta] &= V\left[\sqrt{1-\delta^2}\,W_t^1 + \delta|W_t^2|\right] \\ &= (1-\delta^2)t + \frac{\delta^2(\pi-2)}{\pi}t \\ &= \frac{\pi-2\delta^2}{\pi}t. \end{aligned}$$

In the following result, we study the martingale property of the Itô McKean skew Brownian motion X_t^δ.

Theorem 3.1. *The random variable X_t^δ, $0\leq t\leq T$, is sub-martingale if*

$$\phi\left(\frac{W_t^2}{\sqrt{T-t}}\right) > \frac{W_t^2}{\sqrt{2\pi(T-t)}}\left(1-\mathcal{I}_{(W_t^2<0)}\sqrt{2\pi}\right),$$

martingale if

$$\phi\left(\frac{W_t^2}{\sqrt{T-t}}\right) = \frac{W_t^2}{\sqrt{2\pi(T-t)}}\left(1-\mathcal{I}_{(W_t^2<0)}\sqrt{2\pi}\right), \tag{14}$$

while super-martingale if

$$\phi\left(\frac{W_t^2}{\sqrt{T-t}}\right) < \frac{W_t^2}{\sqrt{2\pi(T-t)}}\left(1-\mathcal{I}_{(W_t^2<0)}\sqrt{2\pi}\right).$$

Proof. Consider the expected value

$$\begin{aligned} E\left[X_T^\delta|F_t\right] &= E\left[X_T^\delta - X_t^\delta + X_t^\delta|F_t\right] \\ &= X_t^\delta + E\left[X_T^\delta - X_t^\delta|F_t\right] \\ &= X_t^\delta + \delta E\left[|W_T^2| - |W_t^2||F_t\right] \\ &= X_t^\delta + \delta \int_{-|W_t^2|}^{\infty} y f_{|W_T^2|-|W_t^2|}(y)dy, \end{aligned} \tag{15}$$

where the density function can be calculated as

$$\begin{aligned} f_{|W_T^2|-|W_t^2|}(y) &= \frac{d}{dy}P\left(|W_T^2| - |W_t^2| \leq y\right) \\ &= \frac{d}{dy}P\left(-y - |W_t^2| - W_t^2 \leq W_T^2 - W_t^2 \leq y + |W_t^2| - W_t^2\right). \end{aligned}$$

Using relation (13) and $W_t^2 = W_t^2\mathcal{I}_{(W_t^2\geq 0)} + W_t^2\mathcal{I}_{(W_t^2<0)}$, we further express

$$\begin{aligned} f_{|W_T^2|-|W_t^2|}(y) &= \frac{d}{dy}P\left(-\frac{y + 2W_t^2\mathcal{I}_{(W_t^2\geq 0)}}{\sqrt{T-t}} \leq \frac{W_T^2 - W_t^2}{\sqrt{T-t}} \leq \frac{y - 2W_t^2\mathcal{I}_{(W_t^2<0)}}{\sqrt{T-t}}\right) \\ &= \frac{1}{\sqrt{2\pi}}\frac{d}{dy}\int_{-\frac{y+2W_t^2\mathcal{I}_{(W_t^2\geq 0)}}{\sqrt{T-t}}}^{\frac{y-2W_t^2\mathcal{I}_{(W_t^2<0)}}{\sqrt{T-t}}} e^{\frac{-w^2}{2}}dw \\ &= \frac{1}{\sqrt{2\pi(T-t)}}\left(e^{\frac{-\left(y-2W_t^2\mathcal{I}_{(W_t^2<0)}\right)^2}{2(T-t)}} + e^{\frac{-\left(y+2W_t^2\mathcal{I}_{(W_t^2\geq 0)}\right)^2}{2(T-t)}}\right). \end{aligned}$$

Using the latter density, we write

$$\begin{aligned} &E\left[X_T^\delta|F_t\right] \\ &= X_t^\delta + \frac{\delta}{\sqrt{2\pi(T-t)}}\int_{-|W_t^2|}^{\infty} y\left(e^{\frac{-\left(y-2W_t^2\mathcal{I}_{(W_t^2<0)}\right)^2}{2(T-t)}} + e^{\frac{-\left(y+2W_t^2\mathcal{I}_{(W_t^2\geq 0)}\right)^2}{2(T-t)}}\right)dy \\ &= X_t^\delta - \frac{\delta(T-t)}{\sqrt{2\pi(T-t)}}\int_{-|W_t^2|}^{\infty}\left[\frac{-y + 2W_t^2\mathcal{I}_{(W_t^2<0)} - 2W_t^2\mathcal{I}_{(W_t^2<0)}}{T-t}e^{\frac{-\left(y-2W_t^2\mathcal{I}_{(W_t^2<0)}\right)^2}{2(T-t)}}\right. \\ &\quad\left. + \frac{-y - 2W_t^2\mathcal{I}_{(W_t^2\geq 0)} + 2W_t^2\mathcal{I}_{(W_t^2\geq 0)}}{T-t}e^{\frac{-\left(y+2W_t^2\mathcal{I}_{(W_t^2\geq 0)}\right)^2}{2(T-t)}}\right]dy \\ &= X_t^\delta + \frac{\delta\sqrt{T-t}}{\sqrt{2\pi}}\left[e^{-\frac{\left(|W_t^2|+2W_t^2\mathcal{I}_{(W_t^2<0)}\right)^2}{2(T-t)}} + e^{-\frac{\left(|W_t^2|-2W_t^2\mathcal{I}_{(W_t^2\geq 0)}\right)^2}{2(T-t)}}\right] \\ &\quad + \frac{2\delta W_{2,t}\mathcal{I}_{(W_t^2<0)}}{\sqrt{2\pi(T-t)}}\int_{-|W_t^2|}^{\infty} e^{\frac{-\left(y-2W_t^2\mathcal{I}(W_{2,t}<0)\right)^2}{2(T-t)}}dy - \frac{2\delta W_t^2\mathcal{I}_{(W_t^2\geq 0)}}{\sqrt{2\pi(T-t)}}\int_{-|W_t^2|}^{\infty} e^{\frac{-\left(y+2W_t^2\mathcal{I}_{(W_t^2\geq 0)}\right)^2}{2(T-t)}}dy \end{aligned}$$

$$
\begin{aligned}
&= X_t^\delta + \delta\sqrt{T-t}\left[\phi\left(\frac{|W_t^2| + 2W_t^2\mathcal{I}_{(W_t^2<0)}}{\sqrt{T-t}}\right) + \phi\left(\frac{|W_t^2| - 2W_t^2\mathcal{I}_{(W_t^2\geq 0)}}{T-t}\right)\right] \\
&\quad + \frac{2\delta W_t^2\mathcal{I}_{(W_t^2<0)}}{\sqrt{2\pi}}\int_{\frac{-W_t^2}{\sqrt{T-t}}}^{\infty} e^{\frac{-y^2}{2}}\,dy - \frac{2\delta W_t^2\mathcal{I}_{(W_t^2\geq 0)}}{\sqrt{2\pi}}\int_{\frac{W_t^2}{\sqrt{T-t}}}^{\infty} e^{\frac{-y^2}{2}}\,dy \\
&= X_t^\delta + \delta\sqrt{T-t}\left[\phi\left(\frac{W_t^2}{\sqrt{T-t}}\right) + \phi\left(\frac{-W_t^2}{\sqrt{T-t}}\right)\right] \\
&\quad + \frac{2\delta W_t^2}{\sqrt{2\pi}}\left(\mathcal{I}_{(W_t^2<0)}\int_{-\infty}^{\frac{W_t^2}{\sqrt{T-t}}} e^{\frac{-y^2}{2}}\,dy - \mathcal{I}_{(W_t^2\geq 0)}\int_{\frac{W_t^2}{\sqrt{T-t}}}^{\infty} e^{\frac{-y^2}{2}}\,dy\right).
\end{aligned}
$$

Using relation $\mathcal{I}_{(W_t^2\geq 0)} + \mathcal{I}_{(W_t^2<0)} = 1$, we get

$$
\begin{aligned}
E\left[X_T^\delta|F_t\right] &= X_t^\delta + 2\delta\sqrt{T-t}\,\phi\left(\frac{W_t^2}{\sqrt{T-t}}\right) + \frac{2\delta W_t^2}{\sqrt{2\pi}}\left(\mathcal{I}_{(W_t^2<0)}\sqrt{2\pi} - 1\right) \\
&= X_t^\delta + 2\delta\sqrt{T-t}\left[\phi\left(\frac{W_t^2}{\sqrt{T-t}}\right) + \frac{W_t^2}{\sqrt{2\pi(T-t)}}\left(\mathcal{I}_{(W_t^2<0)}\sqrt{2\pi} - 1\right)\right].
\end{aligned} \tag{16}
$$

Results follow from the latter expression. □

Now we are ready to valuate European style options on the asset evolves as (2) with arbitrary payoff that include both European put and call type of options.

Theorem 3.2. *Value function with arbitrary finite payoff $H(x)$ of the European style option on the asset S_T, evolves as (2), can be expressed as*

$$
c(\tau,x) = e^{-r\tau}\int_0^\infty \frac{H(w)}{w} f\left(h^{x,\tau}(w) + y + z|y,z\right)dw, \tag{17}
$$

with

$$
\begin{aligned}
& f\left(h^{x,\tau}(w) + y + z|y,z\right) \\
&= \frac{e^{\frac{-\left(h^{x,\tau}(w)\right)^2}{2\tau\sigma^2}}}{\sigma\sqrt{2\pi\tau}}\Phi\left(\frac{\delta^2 h^{x,\tau}(w) + z}{\sigma\delta\sqrt{\tau(1-\delta^2)}}\right) + \frac{e^{\frac{-\left(h^{x,\tau}(w)+2z\right)^2}{2\tau\sigma^2}}}{\sigma\sqrt{2\pi\tau}}\Phi\left(\frac{\delta^2\left(h^{x,\tau}(w)+2z\right) - z}{\sigma\delta\sqrt{\tau(1-\delta^2)}}\right) \\
&= \frac{1}{\sigma\sqrt{\tau}}\left[\phi\left(\frac{h^{x,\tau}(w)}{\sigma\sqrt{\tau}}\right)\Phi\left(\frac{\delta^2 h^{x,\tau}(w) + z}{\sigma\delta\sqrt{\tau(1-\delta^2)}}\right)\right. \\
&+ \left.\phi\left(\frac{h^{x,\tau}(w) + 2z}{\sigma\sqrt{\tau}}\right)\Phi\left(\frac{\delta^2\left(h^{x,\tau}(w)+2z\right) - z}{\sigma\delta\sqrt{\tau(1-\delta^2)}}\right)\right],
\end{aligned} \tag{18}
$$

where $h^{x,\tau}(w) = \ln\frac{w}{x} - \left(r - \frac{\sigma^2}{2}\right)\tau + I^z(\tau)$, $\tau = T - t$, $x = S_t$, $y = \sigma W_t$ and $z = \sigma R_t$.

Proof. Using expressions (8) and (11), we can express the mathematical expectation of the discounted stock price $\tilde{S}_t = e^{-rt}S_t$ as

$$
\begin{aligned}
E\left(\tilde{S_T}|F_t\right) &= E\left(e^{-rT}S_T|F_t\right) \\
&= \tilde{S}_t e^{\int_t^T \mu(v)dv - \left(r-\frac{\sigma^2}{2}\right)(T-t) + I^z(T-t)}.
\end{aligned}
$$

Hence, the discounted stock $\tilde{S}_t$ is a martingale under risk neutral measure $\tilde{P}$, i.e.,

$$\tilde{E}\left(e^{-rT}S_T \middle| F_t\right) = e^{-rt}S_t, 0 \le t \le T,$$

if and only if

$$\int_t^T \mu(v)dv - \left(r - \frac{\sigma^2}{2}\right)(T-t) + I^z(T-t) = 0,\ 0 \le t \le T, \tag{19}$$

where $\tilde{E}$ denotes the mathematical expectation under risk neutral measure $\tilde{P}$.

With this, the stochastic differential equation of the stock price (2) becomes

$$dS_t = S_t\left(rdt - dI^z(t) + \sigma dX_t^\delta\right), S_0 \ge 0, \tag{20}$$

where

$$dI^z(t) = \frac{-e^{-I^z(t)}}{2\sigma\delta t\sqrt{t}}\Big[(\sigma^2\delta^2 t - z)\phi\left(\frac{\sigma^2\delta^2 t + z}{\sigma\delta\sqrt{t}}\right) + e^{-2z}(\sigma^2\delta^2 t + z)\phi\left(\frac{\sigma^2\delta^2 t - z}{\sigma\delta\sqrt{t}}\right)\Big]dt, \tag{21}$$

where $y = \sigma W_t$ while $z = \sigma R_t$.

Let T be the option expiry time and $H(x)$ be the payoff of an European style option on an asset S_t evolves as (20), then the discount of the expected value of $H(S_T)$ up to current time t is denoted and expressed as

$$\begin{aligned} c(T-t,x) &= e^{-r(T-t)}\tilde{E}\left[H(S_T)\middle|F_t\right] \\ &= e^{-r(T-t)}\tilde{E}\Big[H\Big(xe^{\sigma(W_T-W_t)+\left(r-\frac{\sigma^2}{2}\right)(T-t)-I^z(T-t)+\sigma(R_T-R_t)}\Big)\Big|F_t\Big] \\ &= e^{-r(T-t)}\int_{-\infty}^{\infty} H\Big(xe^{\left(r-\frac{\sigma^2}{2}\right)(T-t)-I^z(T-t)-y-z+u}\Big)f(u|y,z)du. \end{aligned} \tag{22}$$

Using change of variable

$$v = \left(r - \frac{\sigma^2}{2}\right)(T-t) - I^z(T-t) - y - z + u \tag{23}$$

in the density (12) and denote $\tau = T - t$, we get

$$c(\tau,x) = e^{-r\tau}\int_{-\infty}^{\infty} H(xe^v)\, f\Big(v - \left(r - \frac{\sigma^2}{2}\right)\tau + I^z(\tau) + y + z|y,z\Big)dv, \tag{24}$$

with

$$\begin{aligned} & f\left(v - \left(r - \frac{\sigma^2}{2}\right)\tau + I^z(\tau) + y + z|y,z\right) \\ = {} & \frac{e^{\frac{-\left(v-\left(r-\frac{\sigma^2}{2}\right)\tau+I^z(\tau)\right)^2}{2\tau\sigma^2}}}{\sigma\sqrt{2\pi\tau}}\Phi\left(\frac{\delta^2\left(v - \left(r - \frac{\sigma^2}{2}\right)\tau + I^z(\tau)\right) + z}{\sigma\delta\sqrt{\tau(1-\delta^2)}}\right) \\ + {} & \frac{e^{\frac{-\left(v-\left(r-\frac{\sigma^2}{2}\right)\tau+I^z(\tau)+2z\right)^2}{2\tau\sigma^2}}}{\sigma\sqrt{2\pi\tau}}\Phi\left(\frac{\delta^2\left(v - \left(r - \frac{\sigma^2}{2}\right)\tau + I^z(\tau) + 2z\right) - z}{\sigma\delta\sqrt{\tau(1-\delta^2)}}\right). \end{aligned} \tag{25}$$

Making substitution $w = xe^v$, we get the required result. □

Now, we are ready to study some properties of the value function $c(\tau, x), 0 \leq t < T, x > 0$, through the payoff function $H(x)$. Particular cases are European call and put options with payoffs $(S_T - L)^+$ and $(L - S_T)^+$, respectively. These properties include continuity and convexity of the value function and also satisfies the well known Feyman-Kac type equation. And under certain conditions, it satisfies Black Scholes equation. Our results agree with Corns and Satchell [3] and Zhu and He [20] for the special cases of European Call Options where the Greeks can also be calculated. Hence our approach is a generalization of [3, 20] and opens a new direction which needs to be tested in future studies.

Theorem 3.3. *If the payoff $H(x)$ is convex, then so is the value function $c(\tau, x)$.*

Proof. Since $H(x)$ is convex, thus the first order left/right derivative $H'(x\mp)$ exists and is increasing (see Royden [18]). Differentiating (24) with respect to x we get left/right derivative as

$$c_{x\mp}(\tau, x) = e^{-r\tau} \int_{-\infty}^{\infty} e^v H'\left(xe^v \mp\right) f\left(v - \left(r - \frac{\sigma^2}{2}\right)\tau + I^z(\tau) + y + z | y, z\right) dv, \tag{26}$$

where $x = S_t$, $y = \sigma W_t$, $z = \sigma R_t$ and $\delta \in (-1, 1)$.

As $H'(x\mp)$ exists and increases with respect to x, thus $c_{x\mp}(\tau, x)$ also exists and increases with respect to x. Thus $c(\tau, x)$ is convex with respect to x. □

Theorem 3.4. *The value function $c(\tau, x)$, with finite payoff function $H(\cdot)$, is continuous on $[0, T)$ and satisfies*

$$c_\tau(\tau, x) = -\left(r + \frac{1}{2\tau}\right)c + \frac{e^{-r\tau}}{\sigma\sqrt{\tau}} \int_0^{\infty} \frac{H(w)}{w} g_\tau\left(h^{x,\tau}(w) + y + z | y, z\right) dw, \tag{27}$$

where

$$\begin{aligned}
& g_\tau\left(h^{x,\tau}(w) + y + z | y, z\right) \\
= \; & \frac{h^{x,\tau}(w)}{\sigma^2 \tau} \phi\left(\frac{h^{x,\tau}(w)}{\sigma\sqrt{\tau}}\right) \Phi\left(\frac{\delta^2 h^{x,\tau}(w) + z}{\sigma\delta\sqrt{\tau(1-\delta^2)}}\right)\left(\frac{h^{x,\tau}(w)}{2\tau} - \frac{dI^z(\tau)}{d\tau} + r - \frac{\sigma^2}{2}\right) \\
- \; & \frac{1}{\sigma\delta\sqrt{\tau(1-\delta^2)}} \phi\left(\frac{h^{x,\tau}(w)}{\sigma\sqrt{\tau}}\right) \phi\left(\frac{\delta^2 h^{x,\tau}(w) + z}{\sigma\delta\sqrt{\tau(1-\delta^2)}}\right) \\
\times \; & \left(\frac{\delta^2 h^{x,\tau}(w) + z}{2\tau} - \delta^2\left(\frac{dI^z(\tau)}{d\tau} + r - \frac{\sigma^2}{2}\right)\right) + \frac{h^{x,\tau}(w) + 2z}{\sigma^2\tau} \phi\left(\frac{h^{x,\tau}(w) + 2z}{\sigma\sqrt{\tau}}\right) \\
\times \; & \Phi\left(\frac{\delta^2\left(h^{x,\tau}(w) + 2z\right) - z}{\sigma\delta\sqrt{\tau(1-\delta^2)}}\right)\left(\frac{h^{x,\tau}(w) + 2z}{2\tau} - \frac{dI^z(\tau)}{d\tau} + r - \frac{\sigma^2}{2}\right) \\
- \; & \frac{1}{\sigma\delta\sqrt{\tau(1-\delta^2)}} \phi\left(\frac{h^{x,\tau}(w) + 2z}{\sigma\sqrt{\tau}}\right) \phi\left(\frac{\delta^2\left(h^{x,\tau}(w) + 2z\right) - z}{\sigma\delta\sqrt{\tau(1-\delta^2)}}\right) \\
\times \; & \left(\frac{\delta^2\left(h^{x,\tau}(w) + 2z\right) - z}{2\tau} - \delta^2\left(\frac{dI^z(\tau)}{d\tau} + r - \frac{\sigma^2}{2}\right)\right),
\end{aligned} \tag{28}$$

where $h^{x,\tau}(w) = \ln\frac{w}{x} - \left(r - \frac{\sigma^2}{2}\right)\tau + I^z(\tau)$, $x = S_t$, $y = \sigma W_t$, $z = \sigma R_t$, $\delta \in (-1, 1)$, while $I^z(\tau)$ and $\frac{d}{d\tau}I^z(\tau)$ are given by (10) and (21), respectively.

Proof. The partial derivative of $c(\tau, x)$, given in Theorem 3.2, with respect to τ can be calculated as

$$c_\tau(\tau, x) \;=\; -rc(\tau, x) + e^{-r\tau} \int_0^{\infty} \frac{H(w)}{w} f_\tau\left(h^{x,\tau}(w) + y + z | y, z\right) dw\Big], \tag{29}$$

where $f_\tau\left(h^{x,\tau}(w)+y+z|y,z\right)$ is calculated from (18) as

$$\begin{aligned}
& f_\tau\left(h^{x,\tau}(w)+y+z|y,z\right) \\
= & -\frac{1}{2\tau}f\left(h^{x,\tau}(w)+y+z|y,z\right)+\frac{1}{\sigma\sqrt{\tau}}\Bigg[\frac{h^{x,\tau}(w)}{\sigma^2\tau}\phi\left(\frac{h^{x,\tau}(w)}{\sigma\sqrt{\tau}}\right)\Phi\left(\frac{\delta^2h^{x,\tau}(w)+z}{\sigma\delta\sqrt{\tau(1-\delta^2)}}\right) \\
\times & \left(\frac{h^{x,\tau}(w)}{2\tau}-\frac{dI^z(\tau)}{d\tau}+r-\frac{\sigma^2}{2}\right)-\frac{1}{\sigma\delta\sqrt{\tau(1-\delta^2)}}\phi\left(\frac{h^{x,\tau}(w)}{\sigma\sqrt{\tau}}\right) \\
\times & \phi\left(\frac{\delta^2h^{x,\tau}(w)+z}{\sigma\delta\sqrt{\tau(1-\delta^2)}}\right)\left(\frac{\delta^2h^{x,\tau}(w)+z}{2\tau}-\delta^2\left(\frac{dI^z(\tau)}{d\tau}+r-\frac{\sigma^2}{2}\right)\right) \\
+ & \frac{h^{x,\tau}(w)+2z}{\sigma^2\tau}\phi\left(\frac{h^{x,\tau}(w)+2z}{\sigma\sqrt{\tau}}\right)\Phi\left(\frac{\delta^2\left(h^{x,\tau}(w)+2z\right)-z}{\sigma\delta\sqrt{\tau(1-\delta^2)}}\right)\left(\frac{h^{x,\tau}(w)+2z}{2\tau}-\frac{dI^z(\tau)}{d\tau}+r-\frac{\sigma^2}{2}\right) \\
- & \frac{1}{\sigma\delta\sqrt{\tau(1-\delta^2)}}\phi\left(\frac{h^{x,\tau}(w)+2z}{\sigma\sqrt{\tau}}\right)\phi\left(\frac{\delta^2\left(h^{x,\tau}(w)+2z\right)-z}{\sigma\delta\sqrt{\tau(1-\delta^2)}}\right) \\
\times & \left(\frac{\delta^2\left(h^{x,\tau}(w)+2z\right)-z}{2\tau}-\delta^2\left(\frac{dI^z(\tau)}{d\tau}+r-\frac{\sigma^2}{2}\right)\right)\Bigg] \\
= & -\frac{1}{2\tau}f\left(h^{x,\tau}(w)+y+z|y,z\right)+\frac{1}{\sigma\sqrt{\tau}}g\left(h^{x,\tau}(w)+y+z|y,z\right). \qquad (30)
\end{aligned}$$

Inserting (30) in (29) we get the result. □

In the following result, we study the Feymann-Kac type differential equation for the corresponding option. Solution of this equation gives fair price of the corresponding option.

Theorem 3.5. *If the left/right derivative $H'(x\pm)$ of the payoff function exists then the value function $c(\tau,x)$ is differentiable with respect to τ and the left/right second order derivative $c_{xx\mp}(\tau,x)$ exists. Moreover, this function satisfies the Feymann-Kac type equation of the form*

$$\begin{aligned}
c_t & + x\left[r+\frac{e^{-I^z(t)}}{2\sigma\delta t\sqrt{2\pi t}}\left((\sigma^2\delta^2t-z)e^{-\frac{1}{2}\left(\frac{\sigma^2\delta^2t+z}{\sigma\delta\sqrt{t}}\right)^2}+e^{-2z}(\sigma^2\delta^2t+z)e^{-\frac{1}{2}\left(\frac{\sigma^2\delta^2t-z}{\sigma\delta\sqrt{t}}\right)^2}\right)\right]c_x \\
& + \frac{x^2\sigma^2}{2}c_{xx}=rc, \qquad (31)
\end{aligned}$$

with condition

$$\phi\left(\frac{W_t^2}{\sqrt{T-t}}\right)=\frac{W_t^2}{\sqrt{2\pi(T-t)}}\left(1-\mathcal{I}_{(W_t^2<0)}\sqrt{2\pi}\right),$$

where $z=\sigma\delta|W_t^2|$, $\delta\in(-1,1)$ while $I^z(\tau)$ is given in (10).

Proof. Using Theorem 3.4, we found that $c(\tau,x)$ is differentiable with respect to time τ. To show the existence of the second order derivative $c_{xx\mp}(\tau,x)$, first we use the change of variable $u=xe^v$ in (26) and obtain the alternate form of $c_{x\mp}(\tau,x)$ as

$$c_{x\mp}(\tau,x) = \frac{e^{-r\tau}}{x}\int_0^\infty H'(u\mp)\,f\left(\ln\frac{u}{x}-\left(r-\frac{\sigma^2}{2}\right)\tau+I^z(\tau)+y+z|y,z\right)du. \qquad (32)$$

Using product rule, we calculate the left/right second order partial derivative as

$$c_{xx\mp}(\tau,x) = -\frac{1}{x}c_{x\mp}(\tau,x)+\frac{e^{-r\tau}}{x}\int_0^\infty H'(u\mp)\,f_x\left(\ln\frac{u}{x}-\left(r-\frac{\sigma^2}{2}\right)\tau+I^z(\tau)+y+z|y,z\right)du, \qquad (33)$$

where the partial derivative $f_x\Big(\ln\frac{u}{x}-\Big(r-\frac{\sigma^2}{2}\Big)\tau+I^z(\tau)+y+z|y,z\Big)$ is given as

$$\begin{aligned}
& f_x\bigg(\ln\frac{u}{x}-\bigg(r-\frac{\sigma^2}{2}\bigg)\tau+I(\tau)+y+z|y,z\bigg) \\
= & \frac{1}{\sigma x\sqrt{\tau}}\Bigg[\phi\bigg(\frac{h^{x,\tau}(u)}{\sigma\sqrt{\tau}}\bigg)\bigg(\frac{h^{x,\tau}(u)}{\sigma^2\tau}\Phi\bigg(\frac{\delta^2 h^{x,\tau}(u)+z}{\sigma\delta\sqrt{\tau(1-\delta^2)}}\bigg)-\frac{\delta}{\sigma\sqrt{\tau(1-\delta^2)}} \\
\times & \phi\bigg(\frac{\delta^2 h^{x,\tau}(u)+z}{\sigma\delta\sqrt{\tau(1-\delta^2)}}\bigg)\bigg)+\phi\bigg(\frac{h^{x,\tau}(u)+2z}{\sigma\sqrt{\tau}}\bigg)\bigg(\frac{h^{x,\tau}(u)+2z}{\sigma^2\tau}\Phi\bigg(\frac{\delta^2\,(h^{x,\tau}(u)+2z)-z}{\sigma\delta\sqrt{\tau(1-\delta^2)}}\bigg) \\
- & \frac{\delta}{\sigma\sqrt{\tau(1-\delta^2)}}\phi\bigg(\frac{\delta^2\,(h^{x,\tau}(u)+2z)-z}{\sigma\delta\sqrt{\tau(1-\delta^2)}}\bigg)\bigg)\Bigg],
\end{aligned} \tag{34}$$

with $h^{x,\tau}(u)=\ln\frac{u}{x}-\Big(r-\frac{\sigma^2}{2}\Big)\tau+I^z(\tau)$, $x=S_t$, $y=\sigma W_t^1$ and $z=\sigma\delta W_t^2$, $\delta\in(-1,1)$.

From Theorem 3.4 and relation (33), we conclude that $c(\tau,x)$ is in $C^{1,2}((0,T)\times(0,\infty))$ in weak sense.

Next, through Itô formula we calculate the differential of the discounted value function $e^{-rt}c(T-t,S_t)$ as

$$d(e^{-rt}c(T-t,S_t))=e^{-rt}\left[(-rc+c_t)\,dt+c_x dS_t+\frac{1}{2}c_{xx}d<S>_t\right],\ x=S_t,\ 0\le t\le T.$$

Using (20), we find the differential of the quadratic variation $<S>_t$ as $\sigma^2S_t^2dt$. Using this, equations (20) and (21) further give

$$\begin{aligned}
& d(e^{-rt}c(T-t,S_t)) \\
= & e^{-rt}\left[(-rc+c_t)\,dt+c_xS_t\left(rdt-dI^z(t)+\sigma dX_t^\delta\right)+\frac{1}{2}c_{xx}\sigma^2S_t^2dt\right] \\
= & e^{-rt}\Bigg[(-rc+c_t)\,dt+S_tc_x\Bigg[rdt-\frac{-e^{-I^z(t)}}{2\sigma\delta t\sqrt{t}}\bigg((\sigma^2\delta^2t-z)\phi\bigg(\frac{\sigma^2\delta^2t+z}{\sigma\delta\sqrt{t}}\bigg) \\
+ & e^{-2z}(\sigma^2\delta^2t+z)\phi\bigg(\frac{\sigma^2\delta^2t-z}{\sigma\delta\sqrt{t}}\bigg)\bigg)dt+\sigma dX_t^\delta\Bigg]+\frac{1}{2}c_{xx}\sigma^2S_t^2dt\Bigg].
\end{aligned} \tag{35}$$

From Theorem 3.1, the random variable X_t^δ is a martingale if

$$\phi\left(\frac{W_t^2}{\sqrt{T-t}}\right)=\frac{W_t^2}{\sqrt{2\pi(T-t)}}\left(1-\mathcal{I}_{(W_t^2<0)}\sqrt{2\pi}\right),$$

and since the discounted value function $e^{-rt}c(T-t,S_t)$ is a martingale, we can put the dt term equal to zero (see for detail Shreve [19]) and get the required equation (31) with the given condition. □

In the following result, we come to the Black-Scholes type equation and the delta hedging rule:

Theorem 3.6. *For arbitrary payoff function $H(.)$, the delta hedge is given as $\Delta(t)=c_x$, and when the payoff function $H(.)$ is one time left/right differentiable then the value function $c(T-t,x)$ satisfies the Black-Scholes type equation of the form*

$$rc(T-t,x)\ =\ c_t(T-t,x)+rxc_x(T-t,x)+\frac{\sigma^2}{2}x^2c_{xx}(T-t,x),$$

for all $0 \leq t < T$, $x = S_t$, *with condition* $\Pi(0) = c(T, S_0)$, *while the delta hedging strategy has the following alternate forms*

$$\begin{aligned}
\Delta(t) &= e^{-rt}\int_0^\infty \frac{H(w)}{w} f_x\left(\ln\frac{w}{x} - \left(r - \frac{\sigma^2}{2}\right)t + I^z(t) + y + z|y,z\right)dw \\
&= e^{-rt}\int_{-\infty}^\infty H'\left(xe^v\mp\right)e^v f\left(v - \left(r - \frac{\sigma^2}{2}\right)t + I^z(t) + y + z|y,z\right)dv \\
&= \frac{e^{-r\tau}}{x}\int_0^\infty H'\left(u\mp\right) f\left(\ln\frac{u}{x} - \left(r - \frac{\sigma^2}{2}\right)\tau + I^z(\tau) + y + z|y,z\right)du
\end{aligned}$$

where $f_x\left(\ln\frac{w}{x} - \left(r - \frac{\sigma^2}{2}\right)t + I^z(t) + y + z|y,z\right)$ *is given by (34).*

Proof. Let $\Pi(t)$ denotes the value of self-financing portfolio which invests in money market account paying constant interest rate r and a stock given in (2) by holding $\Delta(t)$ shares at each time t. The differential $d\Pi(t)$ of the portfolio value at each time t can be expressed (see Glonti, Purtukhiya [6], Shreve [19] Chapter 4) as

$$\begin{aligned}
d\Pi(t) &= \Delta(t)dS_t + r\left(\Pi(t) - \Delta(t)S_t\right)dt \\
&= \Delta(t)S_t\left(rdt - dI^z(t) + \sigma dX_t^\delta\right) + r\left(\Pi(t) - \Delta(t)S_t\right)dt,
\end{aligned}$$

where we have used (20).

Using the latter expression, differential of the discounted portfolio value $d\left(e^{-rt}\Pi(t)\right)$ can be expressed as

$$\begin{aligned}
d\left(e^{-rt}\Pi(t)\right) &= -re^{-rt}\Pi(t)dt + e^{-rt}d\Pi(t) \\
&= e^{-rt}\Delta(t)S_t\left(\sigma dX_t^\delta - dI^z(t)\right).
\end{aligned} \tag{36}$$

Next, assume the short option self-financing hedging portfolio value $\Pi(t)$ agrees with the value function $c(T-t, S_t)$, at each time t, then $e^{-rt}\Pi(t) = e^{-rt}c(T-t, S_t)$. And this insures that

$$d\left(e^{-rt}\Pi(t)\right) = d\left(e^{-rt}c(T-t, S_t)\right) \text{ for all } 0 \leq t < T,$$

with initial condition $\Pi(0) = c(T, S_0)$.

Using the expressions (35) and (36), the latter expression further gives

$$\Delta(t)S_t\left(\sigma dX_t^\delta - dI^z(t)\right) = \left(-rc + c_t + rS_tc_x + \frac{1}{2}c_{xx}\sigma^2S_t^2\right)dt - S_tc_xdI^z(t) + \sigma c_xS_tdX_t^\delta.$$

Comparing the dX_t^δ terms on both sides we get the delta hedging rule as

$$\Delta(t) = c_x(T-t, S_t) \text{ for all } 0 \leq t < T,$$

while, by inserting the expression of $dI^z(t)$ from (21) and comparing the dt terms, we obtain the Black-Scholes type partial differential equation as

$$rc(T-t,x) = c_t(T-t,x) + rxc_x(T-t,x) + \frac{\sigma^2}{2}x^2c_{xx}(T-t,x),$$

for all $0 \leq t < T$, $x = S_t$, with condition $\Pi(0) = c(T, S_0)$.

For the first form of $\Delta(t)$, we differentiate (17) with respect to x then find that c_x exists and calculated as

$$c_x(\tau, x) = e^{-r\tau}\int_0^\infty \frac{H(w)}{w} f_x\left(h^{x,\tau}(w) + y + z|y,z\right)dw,$$

where $f_x\left(\ln\frac{w}{x} - \left(r - \frac{\sigma^2}{2}\right)t + I(t) + y + z|y,z\right)$ is calculated as in (34).

Second form in the result is calculated in (26), while the third in (32). □

Theorem 3.7. *The partial derivative of $c(\tau, x)$ defined in (17) with respect to the interest rate r is given as*

$$c_r(\tau, x) = -\tau c(\tau, x) + e^{-r\tau} \int_0^\infty \frac{H(w)}{w} f_r\left(h^{x,\tau}(w) + y + z|y, z\right) dw, \tag{37}$$

where

$$\begin{aligned}
& f_r\left(h^{x,\tau}(w) + y + z|y, z\right) \\
= \quad & \frac{h^{x,\tau}(w)}{\sigma^3\sqrt{\tau}}\phi\left(\frac{h^{x,\tau}(w)}{\sigma\sqrt{\tau}}\right)\Phi\left(\frac{\delta^2 h^{x,\tau}(w) + z}{\sigma\delta\sqrt{\tau(1-\delta^2)}}\right) + \frac{h^{x,\tau}(w) + 2z}{\sigma^3\sqrt{\tau}}\phi\left(\frac{h^{x,\tau}(w) + 2z}{\sigma\sqrt{\tau}}\right) \\
\times \quad & \Phi\left(\frac{\delta^2\left(h^{x,\tau}(w) + 2z\right) - z}{\sigma\delta\sqrt{\tau(1-\delta^2)}}\right) - \frac{\delta}{\sigma^2\sqrt{1-\delta^2}}\left[\phi\left(\frac{h^{x,\tau}(w)}{\sigma\sqrt{\tau}}\right)\Phi\left(\frac{\delta^2 h^{x,\tau}(w) + z}{\sigma\delta\sqrt{\tau(1-\delta^2)}}\right)\right. \\
+ \quad & \left.\phi\left(\frac{h^{x,\tau}(w) + 2z}{\sigma\sqrt{\tau}}\right)\Phi\left(\frac{\delta^2\left(h^{x,\tau}(w) + 2z\right) - z}{\sigma\delta\sqrt{\tau(1-\delta^2)}}\right)\right],
\end{aligned} \tag{38}$$

where $h^{x,\tau}(w) = \ln\frac{w}{x} - \left(r - \frac{\sigma^2}{2}\right)\tau + I^z(\tau)$, $\tau = T - t$, $x = S_t$, $y = \sigma W_t$ and $z = \sigma R_t$.

Greeks: Greeks of the option are:
$T(t) = c_t(T - t, S_t)$ given by Theorem 3.5.
$\Gamma(t) = c_{xx\mp}(T - t, S_t)$ investigated in the proof of Theorem 3.5.
$\Delta(t) = c_{x\mp}(T - t, S_t)$ investigated in Theorem 3.6.
$R(t) = c_r(T - t, S_t)$ is investigated in Theorem 3.7.
$L(t) = c_L(T - t, S_t)$ is investigated in Lemma 3.8.

In the next results, we study the particular forms of our results. Putting the payoff function $H(S_T) = (S_T - k)^+$, we get the same value function (20) of European call option and the same Greeks as in [20] given in the following lemma.

Lemma 3.8. *Value function of the European call option with payoff $(S_T - L)^+$ can be expressed as*

$$\begin{aligned}
c(T-t, x) \quad = \quad & x e^{-0.5\sigma^2(T-t) - I^z(T-t) - y - z} \int_{\ln\frac{L}{x} - \left(r - \frac{\sigma^2}{2}\right)(T-t) + I^z(T-t) + y + z}^{\infty} e^u f(u|y, z) du \\
- \quad & L e^{-r(T-t)} \int_{\ln\frac{L}{x} - \left(r - \frac{\sigma^2}{2}\right)(T-t) + I^z(T-t) + y + z}^{\infty} f(u|y, z) du,
\end{aligned} \tag{39}$$

where $x = S_t$ and $u = \sigma W_T + \sigma R_T$, $y = \sigma W_t$, $z = \sigma R_t$, $\delta \in (-1, 1)$ while the density $f(u|y, z) = f_{\sigma W_T + \sigma R_T|\sigma W_t, \sigma R_t}(u|y, z)$ is given (23).

The partial derivative $L(t) = c_L(T - t, S_t)$ is given as

$$c_L(T-t, x) = -e^{-r(T-t)} \int_{\ln\frac{L}{x} - \left(r - \frac{\sigma^2}{2}\right)(T-t) + I^z(T-t) + y + z}^{\infty} f(u|y, z) du, \tag{40}$$

where $f(u|y, z)$ is given in (12).

Moreover, Greeks of the European call option satisfies the conditions

$$\Delta(t) = c_{x\mp}(T - t, S_t) \geq 0,$$

$$\Gamma(t) = c_{xx\mp}(T - t, S_t) \geq 0,$$

$$R(t) = c_r(T - t, S_t) \geq 0,$$

and

$$L(t) = c_L(T - t, S_t) \leq 0,$$

and coincide with the Greeks given in Zhu and He [20].

Proof. Using the payoff $H(S_T) = (S_T - L)^+$ in (22), we express the value function $c(T-t, x)$ as

$$\begin{aligned} & c(T-t,x) \\ = \;& e^{-r(T-t)} \int_{-\infty}^{\infty} \left(x e^{\left(r-\frac{\sigma^2}{2}\right)(T-t) - I^z(T-t) - y - z + u} - L \right)^+ f(u|y,z)du \\ = \;& e^{-r(T-t)} \int_{\ln\frac{L}{x} - \left(r-\frac{\sigma^2}{2}\right)(T-t) + I^z(T-t) + y + z}^{\infty} \left(x e^{\left(r-\frac{\sigma^2}{2}\right)(T-t) - I^z(T-t) - y - z + u} - L \right) f(u|y,z)du, \end{aligned} \tag{41}$$

where $f(u|y,z)$ is given in (12). This gives the required result.

As the value function (41) is the same as the value function (20) of Zhu and He [20]. Thus the Greeks of the option calculated from (41) must be the same as in Zhu and He [20]. □

4. Conclusion

In this work, we studied European style option, with arbitrary payoff which includes both put and call options, on an asset whose price evolves as geometric Itô-McKean skew Brownian motion with Azzalini skew-normal distribution. It is found that this motion is not generally a martingale, it is a martingale under certain conditions. Next, we priced the option and find that if the payoff function is convex then so is the price function. It is also found that if the payoff is finite then the price function satisfies a partial differential equation with respect to time. Further, we found that the price function satisfies a Feymann-Kac type equation if and only if the motion is a martingale. We also show that the price function satisfies a Black-Scholes type equation, give expressions for the delta hedge and the Greeks of the option. Last, we compared some of our results with the results found in the literature.

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