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Characterizations of EP Elements in Rings with Involution

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Abstract. In this paper, we present a number of new characterizations of EP elements in rings with involution. Some well-known results are extended to more general settings. In addition, we specially investigate the EP elements by certain equations. Proofs of relevant conclusions are also given.

1. Introduction

Let R be a ring and $a \in R$. Then a is said to be group invertible if there exists $a^{\#} \in R$ such that

$$aa^{\#}a = a, \quad a^{\#}aa^{\#} = a^{\#}, \quad aa^{\#} = a^{\#}a.$$

The element $a^{\#}$ is called a group inverse of a and it is uniquely determined by these equations [2]. We use $R^{\#}$ to denote the set of all group invertible elements of R.

An element $a \in R$ is regular if there exists some $b \in R$ satisfying $aba = a$. The set of all regular elements of R is denoted by R^{reg}.

An involution $* : a \longmapsto a^*$ in a ring R is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^*a^*.$$

We say the element $b = a^{\dagger} \in R$ the Moore-Penrose inverse (or MP-inverse) of $a \in R$, if the following conditions hold (see [12]):

$$aba = a, \quad bab = b, \quad (ab)^* = ab, \quad (ba)^* = ba.$$

There is at most one element $b(= a^{\dagger})$ satisfying the above conditions (see [4, 5, 7]). The set of all MP-invertible elements of R will be denoted by $R^{\dagger}$. We say $a \in R$ is EP if a belongs to $R^{\#} \cap R^{\dagger}$ and satisfies $a^{\#} = a^{\dagger}$ [3]. The set of all EP elements of R is denoted by R^{EP}.

According to [8], an element $a \in R$ is called Hermitian if $a^* = a$. Clearly, Hermitian elements are *EP*. We denote the set of all Hermitian elements of R by R^{Her}.

Let S be a semigroup, Drazin[17] introduced a class of outer inverses. For any $a, b, c \in S$, a is said to be (b, c)-invertible if there exists $y \in S$ such that

$$y \in bSy \cap ySc, \quad yab = b, \quad cay = c.$$

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If such a y exists, it is unique and called a (b, c)-inverse of a, denoted by $a^{\|(b,c)}$.

For any element $a \in R$, we define ${}^0a = \{x \in R | xa = 0\}$. Clearly, ${}^0a$ is a left ideal of R.

EP elements have been investigated by many authors. In [8], many new characterizations of EP elements in rings with involution are given by the equivalent relationship between invertible elements. In 2019, Xu, Chen and Benítez gave that the EP elements in R can be characterized by three equations in [13]. Some sufficient and necessary conditions for an element in a ring to be an EP element are investigated in [15]. More results on EP elements can also be found in [6, 9–11, 15, 16].

Motivated by these results above, this paper mainly consider the new characterizations of EP elements. A number of equivalent conditions are given to characterize these generalized inverses. Moreover, we specially consider the characterizations of EP elements from the perspective of the solutions of equations.

Throughout this paper, R is a $*-$ring with 1. As usual, denote by $J(R)$ the Jacobson radical of R, $U(R)$ the set of all invertible elements of R, $E(R)$ the set of all idempotents of R and $C(R)$ the centre of R. For any element $a \in R$, we define the commutant of a by $comm(a) = \{x \in R | ax = xa\}$.

2. Some extended studies on EP elements

We give the following lemma at first.

Lemma 2.1. *[8, Theorem 1.1.3] Let $a \in R$. Then $a \in R^{EP}$ if and only if $a \in R^{\#}$ and $(a^{\#})^* = aa^{\#}(a^{\#})^*$.*

The following result is a generalization of Lemma 2.1.

Theorem 2.2. *Let R be a ring and $a \in R$. Then $a \in R^{EP}$ if and only if $a \in R^{\#}$ and $(a^{\#})^* - aa^{\#}(a^{\#})^* \in C(R) \cap J(R)$.*

Proof. "$\Longrightarrow$" It is obvious by Lemma 2.1.

"$\Longleftarrow$" Set $(a^{\#})^* - aa^{\#}(a^{\#})^* = x \in C(R) \cap J(R)$. Then $xa = ax = a((a^{\#})^* - aa^{\#}(a^{\#})^*) = 0$.
Moreover, $x(aa^{\#})^* = x = (a^{\#})^* - aa^{\#}(a^{\#})^*$, this gives $(1 - xa^*)(a^{\#})^* = aa^{\#}(a^{\#})^*$.
Noting that $aa^{\#}(1 - xa^*) = aa^{\#} = (1 - a^*x)aa^{\#} = (1 - xa^*)aa^{\#}$.
Then $aa^{\#}(1 - xa^*)^{-1} = (1 - xa^*)^{-1}aa^{\#}$, so $(a^{\#})^* = (1 - xa^*)^{-1}aa^{\#}(a^{\#})^* = aa^{\#}(1 - xa^*)^{-1}(a^{\#})^*$.
Hence we have $a(a^{\#})^* = a^2a^{\#}(1-xa^*)^{-1}(a^{\#})^* = a(1-xa^*)^{-1}(a^{\#})^*$ and $(a^{\#})^* = aa^{\#}(1-xa^*)^{-1}(a^{\#})^* = aa^{\#}(a^{\#})^*$, showing that $a \in R^{EP}$ by Lemma 2.1. □

Notice that $J(R)^* = J(R)$. Therefore, Theorem 2.2 implies the following corollary.

Corollary 2.3. *$a \in R^{EP}$ if and only if $a \in R^{\#}$ and $a^{\#} - a^{\#}(aa^{\#})^* \in C(R) \cap J(R)$.*

Similar to the proof of Theorem 2.2, the following conclusion can be obtained.

Corollary 2.4. *$a \in R^{EP}$ if and only if $a \in R^{\#}$ and $a^* - a^*aa^{\#} \in C(R) \cap J(R)$.*

It is known that an element a is *EP* if and only if $a \in R^{\#} \cap R^{\dagger}$ and $a^* = a^*aa^{\#}$ [8, Theorem 1.2.1]. It is obvious that Corollary 2.4 is a generalization of this result.

Lemma 2.5. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a^*aa^{\#} \in R^{EP}$ and $(a^*aa^{\#})^{\#} = (a^*aa^{\#})^{\dagger} = a^{\dagger}a(a^{\dagger})^*$.*

Proof. It is routine. □

Consequently, we obtain the following corollary by Lemma 2.5 and [8, Theorem 1.2.1].

Corollary 2.6. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{EP}$ if and only if $(a^{\dagger})^* = a^{\dagger}a(a^{\dagger})^*$.*

Since $a \in R^{\dagger}$ is *EP* if and only if $a^{\dagger}a = aa^{\dagger}$, we have $a^{\dagger} \in (a^{\dagger})^*R \cap Ra^{\dagger}$. Hence Corollary 2.6 implies the following corollary.

Corollary 2.7. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{EP}$ if and only if $a^{\|((a^{\dagger})^*, a^{\dagger})} = a^{\dagger}$.*

By observing Corollary 2.6, we can draw the following theorem.

Theorem 2.8. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{EP}$ if and only if $(a^{\#})^* = a^{\dagger}a(a^{\dagger})^*$.*

Proof. $'' \Longrightarrow ''$ If $a \in R^{EP}$, then $a^{\#} = a^{\dagger}$ and $(a^{\dagger})^* = a^{\dagger}a(a^{\dagger})^*$ by Corollary 2.6. Hence $(a^{\#})^* = a^{\dagger}a(a^{\dagger})^*$.

$'' \Longleftarrow ''$ Suppose that $(a^{\#})^* = a^{\dagger}a(a^{\dagger})^*$. Post-multiplying the equality by a^*, one has $(aa^{\#})^* = a^{\dagger}a^2a^{\dagger}$. Noting that $(a^{\dagger})^* = aa^{\dagger}(a^{\dagger})^*$. Then $(a^{\#})^* = a^{\dagger}a(a^{\dagger})^* = a^{\dagger}a^2a^{\dagger}(a^{\dagger})^* = (aa^{\#})^*(a^{\dagger})^*$, it follows that $a^{\#} = a^{\dagger}a^{\#}a$. Since $a^{\#} = aa^{\dagger}a^{\#}$, $aa^{\dagger}a^{\#} = a^{\dagger}a^{\#}a$. Therefore $a \in R^{EP}$ by [8, Theorem 1.2.1]. □

Corollary 2.9. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{EP}$ if and only if $(a^{\#})^* - a^{\dagger}a(a^{\dagger})^*$ is Hermitian.*

Proof. $'' \Longrightarrow ''$ It is an immediate result of Theorem 2.8.

$'' \Longleftarrow ''$ Assume that $(a^{\#})^* - a^{\dagger}a(a^{\dagger})^*$ is Hermitian. Then

$$(a^{\#})^* - a^{\dagger}a(a^{\dagger})^* = a^{\#} - a^{\dagger}a^{\dagger}a.$$

Post-multiplying the equality by $a^{\dagger}a$, one yields

$$(a^{\#})^* = (a^{\#})^*a^{\dagger}a.$$

Pre-multiplying the last equality by a^*, one has

$$(aa^{\#})^* = a^{\dagger}a.$$

Hence $a \in R^{EP}$. □

Corollary 2.10. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{EP}$ if and only if $(a^{\#})^* - a^{\dagger}a(a^{\dagger})^* \in comm(a)$.*

Proof. $'' \Longrightarrow ''$ It follows from Theorem 2.8.

$'' \Longleftarrow ''$ Assume that $(a^{\#})^* - a^{\dagger}a(a^{\dagger})^* \in comm(a)$. Then

$$((a^{\#})^* - a^{\dagger}a(a^{\dagger})^*)a = a((a^{\#})^* - a^{\dagger}a(a^{\dagger})^*) = a(a^{\#})^* - a(a^{\dagger})^*.$$

Post-multiplying the equality by $a^{\dagger}a$, one yields

$$a(a^{\#})^* = a(a^{\#})^*a^{\dagger}a.$$

Pre-multiplying the last equality by $a^{\dagger}$, one has

$$(a^{\#})^* = (a^{\#})^*a^{\dagger}a.$$

By the proof of Corollary 2.9, we have $a \in R^{EP}$. □

Noting that $(a^{\#})^* = a^{\dagger}a(a^{\#})^*$. Then Theorem 2.8 leads to the following corollary.

Corollary 2.11. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{EP}$ if and only if $Ra \subseteq^0 ((a^{\#})^* - (a^{\dagger})^*)$.*

Proof. $'' \Longrightarrow ''$ Since $a \in R^{EP}$, $a^{\dagger}a(a^{\#})^* = a^{\dagger}a(a^{\dagger})^*$ by Theorem 2.8. Hence $Ra = Ra^{\dagger}a \subseteq^0 ((a^{\#})^* - (a^{\dagger})^*)$.

$'' \Longleftarrow ''$ Assume that $Ra \subseteq^0 ((a^{\#})^* - (a^{\dagger})^*)$. Then $a(a^{\#})^* = a(a^{\dagger})^*$.
Pre-multiplying the equality by $a^{\dagger}$, then we have $(a^{\#})^* = a^{\dagger}a(a^{\#})^*$. Hence $a \in R^{EP}$ by Theorem 2.8. □

3. Invertible construction of EP elements

Theorem 3.1. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{EP}$ if and only if $a^*aa^{\#}+1-a^{\dagger}a \in U(R)$ and $(a^*aa^{\#}+1-a^{\dagger}a)^{-1} = (a^{\#})^*+1-a^{\dagger}a$.*

Proof. $'' \Longrightarrow ''$ Certainly, by Lemma 2.5, we have

$$a^*aa^{\#} + 1 - a^{\dagger}a \in U(R)$$

and

$$(a^*aa^{\#} + 1 - a^{\dagger}a)^{-1} = a^{\dagger}a(a^{\dagger})^* + 1 - a^{\dagger}a.$$

If $a \in R^{EP}$, then $a^{\dagger}a(a^{\dagger})^* = (a^{\#})^*$ by Theorem 2.8.
Hence $(a^*aa^{\#} + 1 - a^{\dagger}a)^{-1} = (a^{\#})^* + 1 - a^{\dagger}a$.

$'' \Longleftarrow ''$ Assume that $(a^*aa^{\#} + 1 - a^{\dagger}a)^{-1} = (a^{\#})^* + 1 - a^{\dagger}a$. Then

$$a^{\dagger}a(a^{\dagger})^* + 1 - a^{\dagger}a = (a^{\#})^* + 1 - a^{\dagger}a,$$

this gives

$$a(a^{\#})^* = a(a^{\dagger})^*.$$

Hence $a \in R^{EP}$ by Theorem 2.8. □

Theorem 3.2. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{EP}$ if and only if $aa^*a^{\#} + 1 - aa^{\dagger} \in U(R)$ and $(aa^*a^{\#} + 1 - aa^{\dagger})^{-1} = a(a^{\#})^*a^{\dagger} + 1 - aa^{\dagger}$.*

Proof. $'' \Longrightarrow ''$ By Theorem 3.1, we have

$$(1 - (a^{\dagger} - a^*a^{\#})a)^{-1} = (a^{\#})^* + 1 - a^{\dagger}a,$$

this implies $(1 - a(a^{\dagger} - a^*a^{\#}))^{-1} = 1 + a((a^{\#})^* + 1 - a^{\dagger}a)(a^{\dagger} - a^*a^{\#})$, that is

$$(aa^*a^{\#} + 1 - aa^{\dagger})^{-1} = a(a^{\#})^*a^{\dagger} + 1 - a(aa^{\#})^*a^{\#}.$$

Since $a \in R^{EP}$, thus $a(aa^{\#})^*a^{\#} = a(aa^{\#})^*a^{\dagger} = aa^{\dagger}$.
Hence $(aa^*a^{\#} + 1 - aa^{\dagger})^{-1} = a(a^{\#})^*a^{\dagger} + 1 - aa^{\dagger}$.

$'' \Longleftarrow ''$ Assume that $(aa^*a^{\#} + 1 - aa^{\dagger})^{-1} = a(a^{\#})^*a^{\dagger} + 1 - aa^{\dagger}$. Then

$$1 = (aa^*a^{\#} + 1 - aa^{\dagger})(a(a^{\#})^*a^{\dagger} + 1 - aa^{\dagger}) = 1 - aa^{\dagger} + aa^*a^{\#} - aa^*a^{\#}aa^{\dagger} + aa^*a^{\#}a(a^{\#})^*a^{\dagger}.$$

This gives

$$-aa^{\dagger} + aa^*a^{\#} - aa^*a^{\#}aa^{\dagger} + aa^*a^{\#}a(a^{\#})^*a^{\dagger} = 0.$$

Post-multiplying the equality by $aa^{\dagger}$, one has

$$aa^*a^{\#} = aa^*a^{\#}aa^{\dagger}.$$

Pre-multiplying the last equality by $(a^{\dagger})^*a^{\dagger}$, one obtains $a^{\#} = a^{\#}aa^{\dagger}$. Hence $a \in R^{EP}$ by [8, Theorem 1.2.1]. □

Corollary 3.3. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{EP}$ if and only if $aa^*a^{\#}aa^{\dagger} + 1 - aa^{\dagger} \in U(R)$ and $(aa^*a^{\#}aa^{\dagger} + 1 - aa^{\dagger})^{-1} = a(a^{\#})^*a^{\#} + 1 - aa^{\dagger}$.*

Proof. $'' \Longrightarrow ''$ By Theorem 3.2, we have

$$(1 - aa^{\dagger}(1 - aa^*a^{\#}))^{-1} = a(a^{\#})^*a^{\dagger} + 1 - aa^{\dagger},$$

so

$$(1 - (1 - aa^*a^{\#})aa^{\dagger})^{-1} = 1 + (1 - aa^*a^{\#})(a(a^{\#})^*a^{\dagger} + 1 - aa^{\dagger})aa^{\dagger}.$$

It follows that

$$(1-aa^{\dagger}+aa^{*}a^{\#}aa^{\dagger})^{-1}=1-aa^{*}a^{\#}a(a^{\#})^{*}a^{\dagger}+a(a^{\#})^{*}a^{\dagger}.$$

Noting that $a\in R^{EP}$. Then

$$aa^{*}a^{\#}a(a^{\#})^{*}a^{\dagger}=aa^{*}a^{\#}a(a^{\dagger})^{*}a^{\dagger}=aa^{*}(a^{\dagger})^{*}a^{\dagger}=aa^{\dagger}$$

and

$$a(a^{\#})^{*}a^{\dagger}=a(a^{\#})^{*}a^{\#}.$$

Hence $(aa^{*}a^{\#}aa^{\dagger}+1-aa^{\dagger})^{-1}=a(a^{\#})^{*}a^{\#}+1-aa^{\dagger}$.

$''\Longleftarrow''$ Assume that $(aa^{*}a^{\#}aa^{\dagger}+1-aa^{\dagger})^{-1}=a(a^{\#})^{*}a^{\#}+1-aa^{\dagger}$. Then

$$1=(aa^{*}a^{\#}aa^{\dagger}+1-aa^{\dagger})(a(a^{\#})^{*}a^{\#}+1-aa^{\dagger})=1-aa^{\dagger}+aa^{*}a^{\#}aa^{\dagger}a(a^{\#})^{*}a^{\#}.$$

This gives

$$aa^{\dagger}=aa^{*}a^{\#}a(a^{\#})^{*}a^{\#}.$$

Post-multiplying the equality by $a^{\dagger}a$, one has

$$aa^{\dagger}=aa^{\dagger}a^{\dagger}a.$$

Hence $a\in R^{EP}$. $\square$

Noting that $aa^{*}a^{\dagger}\in R^{EP}$ with $(aa^{*}a^{\dagger})^{\dagger}=a(a^{\#})^{*}a^{\dagger}$. Hence Theorem 3.2 infers the following corollary.

Corollary 3.4. *Let $a\in R^{\#}\cap R^{\dagger}$. Then $a\in R^{EP}$ if and only if $aa^{*}a^{\#}+1-aa^{\dagger}\in U(R)$ and $(aa^{*}a^{\#}+1-aa^{\dagger})^{-1}=(aa^{*}a^{\dagger})^{\dagger}+1-aa^{\dagger}$.*

4. Characterizing *EP* elements by idempotents

Lemma 4.1. *[14, Theorem 2.11] An element $a\in R$ is EP if and only if there exists $g\in E(R)$ such that $Ra=Rg, aR=gR, gg^{*}=g^{*}g, 1+(g^{*}-g)^{*}(g^{*}-g)\in U(R)$.*

Lemma 4.2. *$a\in R^{EP}$ if and only if there exists $e\in E(R)$ such that*

$$Ra=Re, aR=eR, ee^{*}=e^{*}e, e^{*}-e\in J(R). \tag{1}$$

Proof. $''\Longrightarrow''$ If $a\in R^{EP}$, then, by taking $e=a^{\dagger}a$, we get (1).

$''\Longleftarrow''$ If $e^{*}-e\in J(R)$, then $(e^{*}-e)^{*}(e^{*}-e)\in J(R)$, it follows that $1+(e^{*}-e)^{*}(e^{*}-e)\in U(R)$. Hence $a\in R^{EP}$ by Lemma 4.1. $\square$

Theorem 4.3. *$a\in R^{EP}$ if and only if there exists $e\in E(R)$ such that*

$$Ra=Re, aR=e^{*}R, ee^{*}-e^{*}e\in C(R)\cap J(R), 1+(e^{*}-e)^{*}(e^{*}-e)\in U(R). \tag{2}$$

Proof. $''\Longrightarrow''$ Suppose that $a\in R^{EP}$, then, by taking $e=a^{\dagger}a$, we get (2).

$''\Longleftarrow''$ Write $s=1+(e^{*}-e)^{*}(e^{*}-e)\in U(R)$. Then

$$see^{*}=ee^{*}ee^{*}=ee^{*}s, se=ee^{*}e=es, se^{*}=e^{*}ee^{*}=e^{*}s, se^{*}e=e^{*}ee^{*}e=e^{*}es, s=s^{*}.$$

Take $p=ee^{*}s^{-1}=s^{-1}ee^{*}$. Then we have $p=p^{*}=p^{2}, pe=e, p=ep$.
Set $ee^{*}-e^{*}e=t\in C(R)\cap J(R)$, it follows that

$$e^{*}=(pe)^{*}=e^{*}p=e^{*}ee^{*}s^{-1}=(ee^{*}-t)e^{*}s^{-1}=ee^{*}s^{-1}-te^{*}s^{-1}=p-ts^{-1}e^{*},$$

Since $t\in C(R)$ and $se=es$, then $et=te, s^{-1}e=es^{-1}$, we have $ets^{-1}e^{*}=tes^{-1}e^{*}=ts^{-1}ee^{*}$. Hence

$$p=ep=e(e^{*}+ts^{-1}e^{*})=ee^{*}+ets^{-1}e^{*}=ee^{*}+ts^{-1}ee^{*}=(1+ts^{-1})ee^{*},$$

$$ee^{*}=(1+ts^{-1})^{-1}p.$$

Moreover, we have $p=e^{*}+ts^{-1}e^{*}=(1+ts^{-1})e^{*}$, then $e^{*}=(1+ts^{-1})^{-1}p$, It is obvious that $e^{*}=ee^{*}$. Consequently, $e^{*}=e$. Hence $a\in R^{EP}$ by Lemma 4.1. $\square$

Observing the proof of Theorem 4.3, we have the following corollary by Lemma 4.2.

Corollary 4.4. *$a \in R^{EP}$ if and only if there exists $e \in E(R)$ such that*

$$Ra = Re, aR = e^*R, ee^* - e^*e \in C(R) \cap J(R), e^* - e \in J(R). \tag{3}$$

5. Characterizing *EP* elements by the general solution of certain equation

Let $a \in R^\# \cap R^\dagger$. Then we can construct the following equation.

$$aa^\dagger x - ya = c. \tag{4}$$

Lemma 5.1. *[1] Let $a \in R^\# \cap R^\dagger$. Then Eq.(4) has solutions if and only if $(1 - aa^\dagger)c(1 - a^\dagger a) = 0$. Moreover, the general solution of Eq.(4) is given by*

$$\begin{cases} x = aa^\dagger c + aa^\dagger za + (1 - aa^\dagger)w \\ y = (aa^\dagger - 1)ca^\dagger + z - (1 - aa^\dagger)zaa^\dagger \end{cases}, where\ \ z, w \in R. \tag{5}$$

Theorem 5.2. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{EP}$ if and only if the general solution of Eq.(4) is given by*

$$\begin{cases} x = aa^\dagger c + aa^\dagger za + (1 - aa^\dagger)w \\ y = (aa^\dagger - 1)ca^\dagger + z - (1 - a^\dagger a)zaa^\dagger \end{cases}, where\ \ z, w \in R. \tag{6}$$

Proof. "$\Longrightarrow$" Assume that $a \in R^{EP}$, then $a^\dagger a = aa^\dagger$. Hence the formula (5) is the same as the formula (6).

"$\Longleftarrow$" From the hypothesis and Lemma 5.1, we have

$$aa^\dagger(aa^\dagger c + aa^\dagger za + (1 - aa^\dagger)w) - ((aa^\dagger - 1)ca^\dagger + z - (1 - a^\dagger a)zaa^\dagger)a = c$$

and

$$(1 - aa^\dagger)c(1 - a^\dagger a) = 0.$$

then

$$\begin{aligned} c &= aa^\dagger c + aa^\dagger za - (aa^\dagger - 1)ca^\dagger a - za + (1 - a^\dagger a)za \\ &= aa^\dagger c + ca^\dagger a - aa^\dagger ca^\dagger a + aa^\dagger za - a^\dagger aza \\ &= (aa^\dagger - 1)c(1 - a^\dagger a) + c + aa^\dagger za - a^\dagger aza \\ &= c + aa^\dagger za - a^\dagger aza \end{aligned}$$

It follows that $aa^\dagger za = a^\dagger aza$ holds for all $z \in R$. In particular, set $z = a^\#$, then $a^\# a = a^\dagger a$. Consequently, we have $a \in R^{EP}$. □

Similarly, we can construct the following equation.

$$aa^\dagger x - ya^\dagger = c. \tag{7}$$

And we have the following theorem.

Theorem 5.3. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{EP}$ if and only if the general solution of Eq.(4) is given by*

$$\begin{cases} x = aa^\dagger c + aa^\dagger za^\dagger + (1 - aa^\dagger)w \\ y = (aa^\dagger - 1)ca + z - (1 - a^\dagger a)za^\dagger a \end{cases}, where\ \ z, w \in R. \tag{8}$$

Also we have the following theorem.

Theorem 5.4. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{EP}$ if and only if the general solution of Eq.(4) is given by*

$$\begin{cases} x = aa^\dagger c + aa^\dagger za^\# + (1 - aa^\dagger)w \\ y = (aa^\dagger - 1)ca + z - (1 - a^\dagger a)za^\dagger a \end{cases}, where\ \ z, w \in R. \tag{9}$$

6. Characterizing *EP* elements by strongly regularity

Theorem 6.1. *$a \in R$ is EP if and only if there exists $x \in R$ such that $a = xa^2, ax^2 - x \in C(R), (xa)^* = xa$.*

Proof. $'' \Longrightarrow ''$ Choose $x = a^\dagger$. We are done.

$'' \Longleftarrow ''$ Write $ax^2 - x = t \in C(R)$. Then $a = xa^2 = x^2a^3, a^2 = ax^2a^3 = (x+t)a^3$.
It follows that $a = xa^2 = x(x+t)a^3 = x^2a^3 + xta^3 = a + xta^3$, so $xta^3 = 0$.
Notice that $a(x^2a)a = (ax^2)a^2 = (x+t)a^2 = xa^2 + ta^2 = a + ta^2$ and $ta^2 = t(xa^2)a = txa^3 = xta^3 = 0$, that is $a(x^2a)a = a$;
Since $(x^2a)a(x^2a) = x(xa^2)x^2a = xax^2a = x(x+t)a = x^2a + xta$ and $xta = xt(x^2a^3) = x^2(xta^3) = 0$, then $(x^2a)a(x^2a) = x^2a$;
We get $a(x^2a) = (x+t)a = xa + ta$ and $ta = t(x^2a^3) = x(xta^3) = 0$. Hence $a(x^2a) = xa = (xa)^* = (a(x^2a))^*$;
Moreover, we have $(x^2a)a = x(xa^2) = xa = a(x^2a)$, which implies $a \in R^\#$ and $a^\# = x^2a$.
Noting that $(aa^\#)^* = (a(x^2a))^* = a(x^2a) = aa^\#$, Then $a \in R^{EP}$ by [8, Theorem 1.1.3]. □

Theorem 6.1 implies the following corollary.

Corollary 6.2. *$a \in R$ is EP if and only if $a \in R^\dagger$ and $a = a^\dagger a^2, aa^\dagger a^\dagger - a^\dagger \in C(R)$.*

Theorem 6.3. *$a \in R$ is EP if and only if $a \in R^\dagger$ and $a = a^\dagger a^2, aa^\dagger a^\dagger - a^\dagger \in J(R)$.*

Proof. $'' \Longrightarrow ''$ Clearly.

$'' \Longleftarrow ''$ Write $aa^\dagger a^\dagger - a^\dagger = t \in J(R)$.

Post-multiplying the equality by $aa^\dagger$, one has $t = taa^\dagger$. Then $aa^\dagger a^\dagger = a^\dagger + t = a^\dagger + taa^\dagger = (1+ta)a^\dagger$. Notice that $ta \in J(R)$, that is $1 + ta \in U(R)$. Then we have $(1+ta)^{-1}aa^\dagger a^\dagger = a^\dagger$.

Post-multiplying the equality by a^2, hence $(1+ta)^{-1}a = a$, implying $a^\dagger = (1+ta)^{-1}aa^\dagger a^\dagger = aa^\dagger a^\dagger$. Therefore, $a \in R^{EP}$ by Corollary 6.2. □

Theorem 6.4. *$a \in R$ is EP if and only if $a \in R^\dagger$ and $a = a^\dagger a^2, aa^\dagger a^\dagger a - a^\dagger a \in C(R)$.*

Proof. $'' \Longrightarrow ''$ It is evident.

$'' \Longleftarrow ''$ Write $aa^\dagger a^\dagger - a^\dagger = t$. Then $ta \in C(R)$ by hypothesis. Moreover we have $a = a^\dagger a^2$, $ta^2 = 0$. Noting that $t = taa^\dagger$. Then $t = a^\dagger(ta) = a^\dagger(aa^\dagger)(ta) = a^\dagger(ta^2)a^\dagger = 0$, this gives $aa^\dagger a^\dagger = a^\dagger$. Hence $a \in R^{EP}$ by Corollary 6.2. □

We do not know whether the following theorem has been given by anyone.

Theorem 6.5. *$a \in R$ is EP if and only if $aR = aa^*R = a^*aR$.*

Proof. $'' \Longrightarrow ''$ Assume that $a \in R^{EP}$, then $aa^\dagger = a^\dagger a$. Hence

$$aR = aa^\dagger R = aa^*R$$

and

$$aR = aa^\dagger R = a^\dagger aR = a^\dagger R = a^*R = a^*aa^\dagger R = a^*aR.$$

$'' \Longleftarrow ''$ On one hand, we have $aR = aa^*R$. Let $a = aa^*x$ for some $x \in R$, then

$$x^*a = x^*aa^*x = (x^*a)(x^*a)^*,$$

i.e., $x^*a = (x^*a)^*$; Notice that $a^* = (aa^*x)^* = x^*aa^*$, thus $a = aa^*x = a(x^*aa^*)x = ax^*a$.
On the other hand, since $a^*R = a^*aR$, let $a^* = a^*ay$ for some $y \in R$. Then

$$a = (a^*ay)^* = y^*a^*a = (ay)^*a,$$

this gives $ay = (ay)^*$. Of course $a = y^*a^*a = y^*(a^*ay)a = aya$.
Let $b = x^*ay$. Then

$$ab = a(x^*ay) = ay = (ay)^* = (ab)^*;$$

$$aba = aya = a;$$

$$ba = (x^*ay)a = x^*a = (x^*a)^* = (ba)^*;$$

$$bab = x^*ab = x^*ax^*ay = x^*ay = b.$$

It follows that $a \in R^\dagger$ and $a^\dagger = b = x^*ay$.

Since $aR = a^*aR = a^*(aa^\dagger R) = a^*R$, then $a \in R^{EP}$ by [8, Theorem 1.2.5]. □

It is well known that $a \in R^{EP}$ if and only if $a^* \in R^{EP}$, so Theorem 6.5 implies the following result.

Corollary 6.6. *$a \in R^{EP}$ if and only if $a^*R = aa^*R = a^*aR$.*

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