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On a High-Order Iteration Technique for a Wave Equation with Nonlinear Viscoelastic Term

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Abstract. In this paper, a high-order iterative technique for solving a wave equation with strong damping and nonlinear viscoelastic term is constructed. For this purpose, we adapt the high-order iterative method used in the earlier works and establish the existence theorem of a recurrent sequence associated with the proposed problem. Thereafter, we prove a N-order convergence of the obtained sequence to a unique solution of the problem.

1. Introduction

In this paper, we are interested in studying the following viscoelastic problem for a wave equation

$$\begin{cases} u_{tt} - \lambda u_{txx} - \dfrac{\partial^2}{\partial x^2}\left(\mu(x,t,u(x,t))\right) + \int_0^t g(t-s)\dfrac{\partial^2}{\partial x^2}\left(\bar{\mu}(x,s,u(x,s))\right) ds \\ \qquad\qquad\qquad\qquad = f(x,t,u), 0 < x < 1, 0 < t < T, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1.1}$$

where f, g, μ, $\bar{\mu}$, $\tilde{u}_0$, $\tilde{u}_1$ are given functions and $\lambda > 0$, is given constant.

This type of equations usually arises in the theory of viscoelasticity. It is well known that viscoelastic materials are responsible for natural damping, which is due to the special property of these materials to keep memory of their past history and possess a capacity of storage and dissipation of mechanical energy. The dynamic properties of viscoelastic materials are great importance and interest as these materials have a wide application in the natural sciences, we refer to [4]-[6], [12] and the references therein for the physical motivation.

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In the problem (1.1), as $\mu = \bar{\mu} = u$, the nonlinear quantities are Laplace operators and then the corresponding problem becomes a type of viscoelastic problem with strong damping. In this case, many related mathematical models have been studied for past years. Indeed, there have been a lot of investigations devoted to the following viscoelastic wave equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds - \lambda \Delta u_t + \gamma h(u_t) = \mathcal{F}(x,t,u). \tag{1.2}$$

As mentioned above, many results of the equation (1.2) have been discussed, however, to our best knowledge, there seems to have been few works devoted to studying of (1.1) and partial differential equations with nonlinear viscoelastic term. At first, we introduce the paper published in 1985 by Hrusa [7], in which the author considered the following one-dimensional nonlinear viscoelastic equation

$$u_{tt}(x,t) - cu_{xx}(x,t) + \int_0^t g(t-s)\left(\Psi(u_x(x,s))\right)_x ds = f(x,t),$$

established the global existence results for large data and, the exponential decay results for strong solutions when $g(s) = e^{-s}$ and Ψ satisfies some conditions.

In the above papers, the authors used the different methods such as fixed point, linear approximation, monotony, upper and lower solution for studying the solvability of the corresponding problems. During last decades, many iterative methods have been constructed for finding the solution $x^* \in D \subset \mathbb{R}^n$ of nonlinear equation $F(x) = 0$. One of the well-known iteration methods mentioned in literature for finding the solution x^* is Newton's classical scheme

$$x^{(k+1)} = x^{(k)} - \left[F'(x^{(k)})\right]^{-1} F(x^{(k)}),\ k = 0,1,2,\cdots,$$

where $F'(x^{(k)})$ is the Jacobian matrix of function F evaluated in the k th iteration. However, it is a fact that the classical Newton iteration method cannot be applied to all cases. Therefore, its abundant variants has been developed in the literature, for example, see [1], [3], [13], [14], [19] and the references therein. In [3], by adding a new step to Newton's method, Cordero et al. constructed the following two-step scheme with fifth-order of convergence:

$$\begin{cases} y^{(k)} = x^{(k)} - \left[F'(x^{(k)})\right]^{-1} F(x^{(k)}), \\ x^{(k+1)} = y^{(k)} - \left[\alpha_1 I + \alpha_2 \left[F'(y^{(k)})\right]^{-1} F(x^{(k)}) + \alpha_3 \left(\left[F'(y^{(k)})\right]^{-1} F(x^{(k)})\right)^2\right] \left[F'(y^{(k)})\right]^{-1} F(y^{(k)}), \end{cases}$$

where α_1, α_2 and α_3 are arbitrary parameters and I is the identity matrix of size $n \times n$. Also, in the paper, they used the p-order convergence of the sequence $\{x^{(k)}\}$ by the fact that there exists $M > 0$ ($0 < M < 1$ if $p = 1$) and k_0 such that

$$\left\|x^{(k+1)} - x^*\right\| \leq M \left\|x^{(k)} - x^*\right\|^p,\ \forall k \geq k_0. \tag{1.3}$$

For other results of high-order scheme, we refer to [1], [2], [14] and [19] for the schemes respect with convergence of fourth-order, sixth-order and p-order. In the case that models in hand are linear integro-differential equations, Turkyilmazoglu [17] constructed an effective and accurate algorithm based on the power series representation via ordinary polynomials to find numerical solutions of high-order linear Fredholm integro-differential equations having a weak or strong kernel. However, in case of nonlinear integro-differential equations or nonlinear viscoelastic ones, the method given in [17] and some of aforementioned methods are hardly applicable. Therefore, in [18], the author proposed a modified method to obtain exact and analytic approximate solutions of high-order nonlinear Volterra-Fredholm-Hammerstein integro-differential equations.

Based on the ideas about recurrent relations of the above methods, a high-order iterative method in sense of (1.3) is developed for solving some nonlinear wave equations, for example, see [9], [11] and [15]. In [15], Truong et al. studied the solvability of a wave equation of Kirchhoff-Carrier type

$$\begin{cases} u_{tt} - \mu(t, \|u(t)\|^2, \|u_x(t)\|^2)u_{xx} = f(x,t,u), 0 < x < 1, 0 < t < T, \\ u_x(0,t) - h_0 u(0,t) = u_x(1,t) + h_1 u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x), u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1.4}$$

where μ, f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions and $h_0 > 0$, $h_1 \geq 0$ is given constant and $\mu(t, \|u(t)\|^2, \|u_x(t)\|^2)$ depends on the integrals $\|u(t)\|^2 = \int_0^1 u^2(x,t)\,dx$, $\|u_x(t)\|^2 = \int_0^1 u_x^2(x,t)\,dx$. In this paper, the authors constructed a high-order iterative scheme by proving the existence and convergence at N-order rate of the recurrent sequence $\{u_m\}$ obtained by

$$u_m'' - \mu(t, \|u_m(t)\|^2, \|u_{mx}(t)\|^2)u_{mxx} = \sum_{i=0}^{N-1} \frac{1}{i!} \frac{\partial^i f}{\partial u^i}(x,t,u_{m-1})(u_m - u_{m-1})^i,$$

$0 < x < 1$, $0 < t < T$, where u_m satisfying $(1.4)_{2,3}$. By using the same high-order iterative method given in [15], Nhan et al. [11] considered the following problem for a wave equation with a nonlinear integral term

$$\begin{cases} u_{tt} - \frac{\partial}{\partial x}(\mu(x,t)u_x) + \lambda u_t = f(x,t,u) + \int_0^t g(x,t,s,u(x,s))ds, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1.5}$$

and constructed a recurrent sequence $\{u_m\}$ defined by

$$\begin{aligned} & u_m'' - \frac{\partial}{\partial x}(\mu(x,t)u_{mx}) + \lambda u_m' \\ = & \sum_{k=0}^{N-1} \frac{1}{k!} \frac{\partial^k f}{\partial u^k}(x,t,u_{m-1})(u_m - u_{m-1})^k + \sum_{k=0}^{N-1} \frac{1}{k!} \int_0^t \left[\frac{\partial^k g}{\partial u^k}(x,t,s,u_{m-1}(x,s))\right](u_m(x,s) - u_{m-1}(x,s))^k ds, \end{aligned}$$

where u_m satisfying $(1.5)_{2,3}$. Moreover, the convergence at N-order rate of this sequence to a weak unique solution of the problem (1.5) was also proved. In the present paper, we adapt the high-order iterative method used in [9], [11], [13] and [15], in order to prove the existence and convergence at N-order rate of a recurrent sequence associated with the problem (1.1). However, the first obstacle in this paper is the presence of the nonlinear term in the form $\frac{\partial^2}{\partial x^2}(\varphi(x,t,u(x,t)))$ which makes some technical difficulties to obtain the results of existence and uniqueness of solutions for the problem (1.1). Then, to overcome the obstacles, we decomposed the term $\frac{\partial^2}{\partial x^2}(\mu(x,t,u(x,t)))$ in $a(t;u(t),v)$ and $\frac{\partial^2}{\partial x^2}\bar{\mu}(x,t,u(x,t))$ in $\bar{a}(t;u(t),v)$ by

$$\begin{aligned} a(t;u(t),v) &= \left\langle \frac{\partial}{\partial x}(\mu(t,u(t))), v_x \right\rangle = \langle D_1\mu(t,u(t)) + D_3\mu(t,u(t))u_x(t), v_x \rangle, \\ \bar{a}(t;u(t),v) &= \left\langle \frac{\partial}{\partial x}(\bar{\mu}(t,u(t))), v_x \right\rangle = \langle D_1\bar{\mu}(t,u(t)) + D_3\bar{\mu}(t,u(t))u_x(t), v_x \rangle, \end{aligned}$$

and used the high-order iterative scheme given as in (3.3)-(3.4). Furthermore, it note more that the assumption (A_2) for the Kirchhoff-Carrier term $\mu\left(t, \|u(t)\|^2, \|u_x(t)\|^2\right)$ in the paper [15] can not be used for the nonlinear term $\frac{\partial^2}{\partial x^2}(\mu(x,t,u(x,t)))$ and $\frac{\partial^2}{\partial x^2}\bar{\mu}(x,t,u(x,t))$ in the problem (1.1), then a modified version of Lemma 2.6 in [10], that if O is closed set of $\mathbb{R}^N$ and $f \in C^0(O,\mathbb{R})$, there is a continuous nondecreasing function $\Phi_f : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\left|f(x)\right| \leq \Phi_f(\|x\|_*), \ \forall x = (x_1, \cdots, x_N) \in O, \ \|x\|_* = \sqrt{x_1^2 + \cdots + x_N^2},$$

is used. The second obstacle in this paper is to prove boundness of the Galerkin approximation solution, we here encounter with evaluating Volterra nonlinear integral inequalities, then Lemma 3.5 is used. It can be said with much confidence that these techniques haven't been applied before. In addition, the problem under consideration contains the second-order differential operator $\frac{\partial^2}{\partial x^2}$, which affects the nonlinear quantities $\mu(x,t,u(x,t))$ and $\bar{\mu}(x,s,u(x,t))$. So, it can be considered that the terms $\frac{\partial^2}{\partial x^2}(\mu(x,t,u(x,t)))$ and $\int_0^t g(t-s)\frac{\partial^2}{\partial x^2}(\bar{\mu}(x,s,u(x,s)))\,ds$ are the extended models of the Laplacian and the viscoelastic convolution as $\mu = \bar{\mu} = u$, respectively. Usually, the presence of the nonlinear term in the form $\frac{\partial^2}{\partial x^2}(\varphi(x,t,u(x,t)))$ makes certain difficulties for proving the existence of the problem under consideration. To our knowledge, there are no publications about investigating high-order iterative schemes for the problems having such nonlinear terms.

The remaining parts of this paper are arranged as follows. In Section 2, we introduce some notations. The existence of a recurrent sequence associated with the problem (1.1) is presented in Section 3. Finally, in Section 4, a high-order convergence of the recurrent sequence obtained in Section 3 is considered.

2. Preliminaries

In this section, we present some notations and materials in order to present main results.

Let $\Omega = (0,1)$, $Q_T = (0,1)\times(0,T)$ and we define the scalar product in L^2 by

$$\langle u, v\rangle = \int_0^1 u(x)v(x)dx,$$

and the corresponding norm $\|\cdot\|$, i.e., $\|u\| = \sqrt{\langle u,u\rangle}$. Let us denote the standard function spaces by $C^m(\overline{\Omega})$, $L^P = L^p(\Omega)$ and $H^m = H^m(\Omega)$ for $1 \le p \le \infty$ and $m \in \mathbb{N}$. Also, we denote that $\|\cdot\|_X$ is a norm in a certain Banach space X, and $L^p(0,T;X)$, $1 \le p \le \infty$, is the Banach space of real functions $u : (0,T) \to X$ measurable with the corresponding norm $\|\cdot\|_{L^p(0,T;X)}$ defined by

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p\,dt\right)^{1/p} < \infty \text{ for } 1 \le p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = ess\sup_{0<t<T} \|u(t)\|_X \text{ for } p = \infty.$$

On H^1, we use the following norm

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{1/2}.$$

Let $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \nabla u(t)$, $u_{xx}(t) = \Delta u(t)$, denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

With $f \in C^k([0,1]\times[0,T^*]\times\mathbb{R})$, $f = f(x,t,y)$, we define $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_3 f = \frac{\partial f}{\partial y}$, and $D^\alpha f = D_1^{\alpha_1}\cdots D_3^{\alpha_3} f$; $\alpha = (\alpha_1,\cdots,\alpha_3) \in \mathbb{Z}_+^3$, $|\alpha| = \alpha_1+\cdots+\alpha_3 \le k$, $D^{(0,\cdots,0)} f = D^{(0)} f = f$.

Similarly, with $\mu \in C^k([0,1]\times[0,T^*]\times\mathbb{R})$, $\mu = \mu(x,t,y)$, we define $D_1\mu = \frac{\partial \mu}{\partial x}$, $D_2\mu = \frac{\partial \mu}{\partial t}$, $D_3\mu = \frac{\partial \mu}{\partial y}$ and $D^\beta\mu = D_1^{\beta_1}\cdots D_3^{\beta_3}\mu$, $\beta = (\beta_1,\cdots,\beta_3) \in \mathbb{Z}_+^3$, $|\beta| = \beta_1+\cdots+\beta_3 \le k$; $D^{(0,\cdots,0)}\mu = \mu$.

3. Existence of a N-order iterative scheme

In this section, we investigate a N-order iterative scheme for the problem (1.1). For this purpose, we make the following assumptions:

(H_1) $\tilde{u}_0, \tilde{u}_1 \in H_0^1 \cap H^2$;
(H_2) $g \in H^1(0,T^*)$;
(H_3) $\mu, \bar{\mu} \in C^{N+1}([0,1]\times[0,T^*]\times\mathbb{R})$ and
$D_3\mu(x,t,y) \geq \mu_* > 0,\ \forall (x,t,y) \in [0,1]\times[0,T^*]\times\mathbb{R}$;
(H_4) $f \in C^N([0,1]\times[0,T^*]\times\mathbb{R})$, such that $f(0,t,0) = f(1,t,0) = 0,\ \forall t \in [0,T^*]$.

We state a weak solution of (1.1) by the fact that

$$\begin{aligned} u \quad \in \quad W_T = \{&u \in L^\infty(0,T;H^2\cap H_0^1) : u' \in L^\infty(0,T;H^2\cap H_0^1), \\ &u'' \in L^2(0,T;H_0^1)\cap L^\infty(0,T;L^2)\}, \end{aligned}$$

and satisfies the following variational equation

$$\langle u''(t), v\rangle + \lambda\langle u_x'(t), v_x\rangle + a(t;u(t),v) = \int_0^t g(t-s)\bar{a}(s;u(s),v)ds + \langle f[u](t), v\rangle, \tag{3.1}$$

for all $v \in H_0^1$, a.e., $t \in (0,T)$, together with the initial conditions

$$u(0) = \tilde{u}_0,\ u'(0) = \tilde{u}_1, \tag{3.2}$$

where

$$\begin{aligned} f[u](x,t) &= f(x,t,u(x,t)), \\ a(t;u(t),v) &= \left\langle \frac{\partial}{\partial x}(\mu(t,u(t))), v_x\right\rangle = \langle D_1\mu(t,u(t)) + D_3\mu(t,u(t))u_x(t), v_x\rangle, \\ \bar{a}(t;u(t),v) &= \left\langle \frac{\partial}{\partial x}(\bar{\mu}(t,u(t))), v_x\right\rangle = \langle D_1\bar{\mu}(t,u(t)) + D_3\bar{\mu}(t,u(t))u_x(t), v_x\rangle. \end{aligned}$$

For a fixed constant $T^* > 0$ and an arbitrary constant $M > 0$, we put

$$\begin{cases} \tilde{K}_M(f) = \sum_{|\alpha|\leq N} \left\|D_3^\alpha \mu\right\|_{C^0(\Omega_M)},\ K_M(\mu) = \sum_{|\alpha|\leq N+1} \left\|D_3^\alpha \mu\right\|_{C^0(\Omega_M)}, \\ K_M(\bar{\mu}) = \sum_{|\alpha|\leq N+1} \left\|D_3^\alpha \bar{\mu}\right\|_{C^0(\Omega_M)},\ \left\|f\right\|_{C^0(\Omega_M)} = \sup_{(x,t,u)\in\Omega_M} \left|f(x,t,u)\right|, \end{cases}$$

where $\Omega_M = [0,1]\times[0,T^*]\times[-M,M]$.

For every $T \in (0,T^*]$, we consider

$$V_T = \{v \in L^\infty(0,T;H^2\cap H_0^1) : v' \in L^\infty(0,T;H^2\cap H_0^1),\ v'' \in L^2(0,T;H_0^1)\},$$

then V_T is a Banach space with respect to the norm (see Lions [8])

$$\|v\|_{V_T} = \max\{\|v\|_{L^\infty(0,T;H^2\cap H_0^1)},\ \|v'\|_{L^\infty(0,T;H^2\cap H_0^1)},\ \|v''\|_{L^2(0,T;H_0^1)}\}.$$

We also put

$$\begin{cases} W(M,T) = \{v \in V_T : \|v\|_{V_T} \leq M\}, \\ W_1(M,T) = \{v \in W(M,T) : v'' \in L^\infty(0,T;L^2)\}. \end{cases}$$

At the moment, we construct the high-order recurrent sequence $\{u_m\}$ by choosing $u_0 \equiv 0$, and suppose that

$$u_{m-1} \in W_1(M,T). \tag{3.3}$$

Then, we find $u_m \in W_1(M,T)$ $(m \geq 1)$ satisfying the following nonlinear variational problem

$$\begin{cases} \langle u_m''(t), v\rangle + \lambda\langle u_{mx}'(t), v_x\rangle + \langle D_3\mu(t, u_m(t))u_{mx}(t), v_x\rangle \\ \qquad = \int_0^t g(t-s)\langle D_3\bar{\mu}(s, u_m(s))u_{mx}(s), v_x\rangle ds \\ \qquad + \int_0^t g(t-s)\langle P_m[D_1\bar{\mu}, u_{m-1}, u_m](s), v_x\rangle ds \\ \qquad -\langle P_m[D_1\mu, u_{m-1}, u_m](t), v_x\rangle + \langle P_m[f, u_{m-1}, u_m](t), v\rangle, \ \forall v \in H_0^1, \\ u_m(0) = \tilde{u}_0, \ u_m'(0) = \tilde{u}_1, \end{cases} \tag{3.4}$$

where

$$\begin{aligned} P_m[f, u_{m-1}, u_m](x,t) &= \sum\nolimits_{i=0}^{N-1} \frac{1}{i!} D_3^i f(x,t,u_{m-1}(x,t))\,(u_m(x,t) - u_{m-1}(x,t))^i, \\ P_m[D_1\mu, u_{m-1}, u_m](x,t) &= \sum\nolimits_{i=0}^{N-1} \frac{1}{i!} D_3^i D_1\mu(x,t,u_{m-1}(x,t))\,(u_m(x,t) - u_{m-1}(x,t))^i, \\ P_m[D_1\bar{\mu}, u_{m-1}, u_m](x,t) &= \sum\nolimits_{i=0}^{N-1} \frac{1}{i!} D_3^i D_1\bar{\mu}(x,t,u_{m-1}(x,t))\,(u_m(x,t) - u_{m-1}(x,t))^i. \end{aligned}$$

Using the Faedo-Galerkin approximation method, the arguments of compactness and the lemma of estimating nonlinear integral inequalities, we confirm the existence of the sequence $\{u_m\}$ as in the following theorem.

Theorem 3.1. *Let $(H_1) - (H_4)$ hold. Then there are positive constants M, T chosen appropriately such that there exists a recurrent sequence $\{u_m\} \subset W_1(M,T)$ defined by* (3.3)-(3.4).

Proof. First, we find the Faedo-Galerkin approximation solution in form of

$$u_m^{(k)}(t) = \sum\nolimits_{j=1}^{k} c_{mj}^{(k)}(t) w_j,$$

where $w_j = \sqrt{2}\sin(j\pi x), \ j = 1, \cdots, k$ are the eigenfunctions, respect with the eigenvalues $\lambda_j = (j\pi)^2, \ j = 1, \cdots, k$, of the operator $-\frac{\partial^2}{\partial x^2}$ satisfying $-\frac{\partial^2 w_j}{\partial x^2} = \lambda_j w_j$, $w_j(0) = w_j(1) = 0$, and $c_{mj}^{(k)}, \ j = 1, \cdots, k$ satisfy the following system of nonlinear integrodifferential equations

$$\begin{cases} \langle \ddot{u}_m^{(k)}(t), w_j\rangle + \lambda\langle \dot{u}_{mx}^{(k)}(t), w_{jx}\rangle + \langle A_m^{(k)}(t) u_{mx}^{(k)}(t), w_{jx}\rangle \\ \qquad = \int_0^t g(t-s)\langle B_m^{(k)}(s) u_{mx}^{(k)}(s), w_{jx}\rangle ds + \int_0^t g(t-s)\langle Q_m^{(k)}(s), w_{jx}\rangle ds \\ \qquad -\langle P_m^{(k)}(t), w_{jx}\rangle + \langle F_m^{(k)}(t), w_j\rangle, \ 1 \leq j \leq k, \\ u_m^{(k)}(0) = \tilde{u}_{0k}, \ \dot{u}_m^{(k)}(0) = \tilde{u}_{1k}, \end{cases} \tag{3.5}$$

where

$$\begin{cases} \tilde{u}_{0k} = \sum_{j=1}^k \alpha_j^{(k)} w_j \to \tilde{u}_0 \text{ strongly in } H^2 \cap H_0^1, \\ \tilde{u}_{1k} = \sum_{j=1}^k \beta_j^{(k)} w_j \to \tilde{u}_1 \text{ strongly in } H^2 \cap H_0^1, \end{cases} \tag{3.6}$$

and

$$\begin{cases} A_m^{(k)}(x,t) = D_3\mu(x,t,u_m^{(k)}(x,t)), \\ B_m^{(k)}(x,t) = D_3\bar{\mu}(x,t,u_m^{(k)}(x,t)), \\ F_m^{(k)}(x,t) = P_m[f, u_{m-1}, u_m^{(k)}](x,t) \\ \qquad = \sum\limits_{i=0}^{N-1} \frac{1}{i!} D_3^i f(x,t,u_{m-1}(x,t)) \left(u_m^{(k)}(x,t) - u_{m-1}(x,t)\right)^i, \\ P_m^{(k)}(x,t) = P_m[D_1\mu, u_{m-1}, u_m^{(k)}](x,t) \\ \qquad = \sum\limits_{i=0}^{N-1} \frac{1}{i!} D_3^i D_1\mu(x,t,u_{m-1}(x,t)) \left(u_m^{(k)}(x,t) - u_{m-1}(x,t)\right)^i, \\ Q_m^{(k)}(x,t) = P_m[D_1\bar{\mu}, u_{m-1}, u_m^{(k)}](x,t) \\ \qquad = \sum\limits_{i=0}^{N-1} \frac{1}{i!} D_3^i D_1\bar{\mu}(x,t,u_{m-1}(x,t)) \left(u_m^{(k)}(x,t) - u_{m-1}(x,t)\right)^i. \end{cases}$$

Clearly, if u_{m-1} satisfies (3.3), then, for some $0 \leq T_m^{(k)} \leq T$, the system (3.5) admits a solution $u_m^{(k)}$ on an interval $[0,\ T_m^{(k)}]$. Additionally, if the following estimations hold, then one can take $T_m^{(k)} = T$ for all m and k. Setting

$$
\begin{aligned}
S_m^{(k)}(t) = & \left\|\dot{u}_m^{(k)}(t)\right\|^2 + \left\|\dot{u}_{mx}^{(k)}(t)\right\|^2 + \left\|\sqrt{A_m^{(k)}(t)}u_{mx}^{(k)}(t)\right\|^2 + \left\|\sqrt{A_m^{(k)}(t)}\Delta u_m^{(k)}(t)\right\|^2 \\
& + \lambda\left\|\Delta\dot{u}_m^{(k)}(t)\right\|^2 + 2\lambda\int_0^t\left(\left\|\dot{u}_{mx}^{(k)}(s)\right\|^2 + \left\|\Delta\dot{u}_m^{(k)}(s)\right\|^2\right)ds + 2\int_0^t\left\|\ddot{u}_{mx}^{(k)}(s)\right\|^2 ds.
\end{aligned}
\tag{3.7}
$$

By (3.5) and some computations, it implies that

$$
\begin{aligned}
S_m^{(k)}(t) = & S_m^{(k)}(0) + \int_0^t ds\int_0^1 \dot{A}_m^{(k)}(x,s)\left[\left|u_{mx}^{(k)}(x,s)\right|^2 + \left|\Delta u_m^{(k)}(x,s)\right|^2\right]dx \\
& - 2\int_0^t\left\langle A_{mx}^{(k)}(s)u_{mx}^{(k)}(s),\Delta\dot{u}_m^{(k)}(s)\right\rangle ds - 2\int_0^t\left\langle\frac{\partial}{\partial x}(A_m^{(k)}(s)u_{mx}^{(k)}(s)),\Delta\ddot{u}_m^{(k)}(s)\right\rangle ds \\
& - 2g(0)\int_0^t\left[\left\langle B_m^{(k)}(s)u_{mx}^{(k)}(s),u_{mx}^{(k)}(s)\right\rangle + \left\langle\frac{\partial}{\partial x}\left(B_m^{(k)}(s)u_{mx}^{(k)}(s)\right),\Delta u_m^{(k)}(s) + \Delta\dot{u}_m^{(k)}(s)\right\rangle\right]ds \\
& + 2\int_0^t g(t-s)\left[\left\langle B_m^{(k)}(s)u_{mx}^{(k)}(s),u_{mx}^{(k)}(t)\right\rangle + \left\langle\frac{\partial}{\partial x}(B_m^{(k)}(s)u_{mx}^{(k)}(s)),\Delta u_m^{(k)}(t) + \Delta\dot{u}_m^{(k)}(t)\right\rangle\right]ds \\
& - 2\int_0^t d\tau\int_0^\tau g'(\tau-s)\left[\left\langle B_m^{(k)}(s)u_{mx}^{(k)}(s),u_{mx}^{(k)}(\tau)\right\rangle + \left\langle\frac{\partial}{\partial x}(B_m^{(k)}(s)u_{mx}^{(k)}(s)),\Delta u_m^{(k)}(\tau) + \Delta\dot{u}_m^{(k)}(\tau)\right\rangle\right]ds \\
& + 2\int_0^t g(t-s)\left[\left\langle Q_m^{(k)}(s),u_{mx}^{(k)}(t)\right\rangle + \left\langle Q_{mx}^{(k)}(s),\Delta u_m^{(k)}(t) + \Delta\dot{u}_m^{(k)}(t)\right\rangle\right]ds \\
& - 2\int_0^t d\tau\int_0^\tau g'(\tau-s)\left[\left\langle Q_m^{(k)}(s),u_{mx}^{(k)}(\tau)\right\rangle + \left\langle Q_{mx}^{(k)}(s),\Delta u_m^{(k)}(\tau) + \Delta\dot{u}_m^{(k)}(\tau)\right\rangle\right]ds \\
& - 2g(0)\int_0^t\left[\left\langle Q_m^{(k)}(s),u_{mx}^{(k)}(s)\right\rangle + \left\langle Q_{mx}^{(k)}(s),\Delta u_m^{(k)}(s) + \Delta\dot{u}_m^{(k)}(s)\right\rangle\right]ds \\
& + 2\int_0^t\left[\left\langle F_m^{(k)}(s),\dot{u}_m^{(k)}(s)\right\rangle + \left\langle F_{mx}^{(k)}(s),\dot{u}_{mx}^{(k)}(s) + \ddot{u}_{mx}^{(k)}(s)\right\rangle\right]ds \\
& - 2\int_0^t\left[\left\langle P_m^{(k)}(s),\dot{u}_{mx}^{(k)}(s)\right\rangle + \left\langle P_{mx}^{(k)}(s),\Delta\dot{u}_m^{(k)}(s) + \Delta\ddot{u}_m^{(k)}(s)\right\rangle\right]ds \\
= & S_m^{(k)}(0) + \sum\nolimits_{j=1}^{11} I_j.
\end{aligned}
\tag{3.8}
$$

Next, we shall sequentially estimate the terms on the right-hand side of (3.8) as follows.
Note that, by (3.7), we have

$$
S_m^{(k)}(t) \geq \bar{\mu}_*\bar{S}_m^{(k)}(t),
$$

in which $\bar{\mu}_* = \min\{1,\mu_*,2\lambda\}$ and

$$
\begin{aligned}
\bar{S}_m^{(k)}(t) = & \left\|\dot{u}_m^{(k)}(t)\right\|^2 + \left\|\dot{u}_{mx}^{(k)}(t)\right\|^2 + \left\|u_{mx}^{(k)}(t)\right\|^2 + \left\|\Delta u_m^{(k)}(t)\right\|^2 + \left\|\Delta\dot{u}_m^{(k)}(t)\right\|^2 \\
& + \int_0^t\left(\left\|\dot{u}_{mx}^{(k)}(s)\right\|^2 + \left\|\Delta\dot{u}_m^{(k)}(s)\right\|^2 + \left\|\ddot{u}_{mx}^{(k)}(s)\right\|^2\right)ds.
\end{aligned}
\tag{3.9}
$$

Then, we have the following lemmas.

Lemma 3.2 *Let $T^* > 0$ and $h \in C^0([0,1] \times [0,T^*] \times \mathbb{R}; \mathbb{R})$. Then there is a continuous nondecreasing function $\Phi_h : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\left|\varphi(x,t,y)\right| \leq \Phi_h\left(\left|y\right|\right), \ \forall (x,t,y) \in [0,1] \times [0,T^*] \times \mathbb{R}.$$

Futhermore, if $h \in C^3([0,1] \times [0,T^*] \times \mathbb{R}; \mathbb{R})$, then there is a continuous nondecreasing function $\Phi_h^{[3]} : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\max_{|\alpha| \leq 3} \left|D^\alpha h(x,t,y)\right| \leq \Phi_h^{[3]}\left(\left|y\right|\right), \ \forall (x,t,y) \in [0,1] \times [0,T^*] \times \mathbb{R}.$$

The proofs of Lemma 3.2 can be deduced from Lemma 2.6 in [10] by choosing $N = 3, O = [0,1] \times [0,T^*] \times \mathbb{R}$, $\|x\|_* = |x_1| + |x_2| + |x_3|$, $\Phi_h\left(\left|y\right|\right) = \bar{\varphi}_h\left(1 + T^* + \left|y\right|\right)$ and $\Phi_h^{[3]}(z) = \max_{|\alpha| \leq 3} \Phi_{D^\alpha h}(z)$, respectively.

Lemma 3.3. *There is a continuous nondecreasing function $\psi_1, \bar{\psi}_1 : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that*

$$\begin{array}{llll}
\text{(i)} & \left\|A_m^{(k)}(t)\right\|_{L^\infty} \leq \psi_1\left(\sqrt{\bar{S}_m^{(k)}(t)}\right), & \text{(v)} & \left\|B_m^{(k)}(t)\right\|_{L^\infty} \leq \bar{\psi}_1\left(\sqrt{\bar{S}_m^{(k)}(t)}\right), \\
\text{(ii)} & \left\|\dot{A}_m^{(k)}(t)\right\|_{L^\infty} \leq \psi_1\left(\sqrt{\bar{S}_m^{(k)}(t)}\right), & \text{(vi)} & \left\|\dot{B}_m^{(k)}(t)\right\|_{L^\infty} \leq \bar{\psi}_1\left(\sqrt{\bar{S}_m^{(k)}(t)}\right), \\
\text{(iii)} & \left\|A_{mx}^{(k)}(t)\right\|_{L^\infty} \leq \psi_1\left(\sqrt{\bar{S}_m^{(k)}(t)}\right), & \text{(vii)} & \left\|B_{mx}^{(k)}(t)\right\|_{L^\infty} \leq \bar{\psi}_1\left(\sqrt{\bar{S}_m^{(k)}(t)}\right), \\
\text{(iv)} & \left\|\dot{A}_{mx}^{(k)}(t)\right\|_{L^\infty} \leq \psi_1\left(\sqrt{\bar{S}_m^{(k)}(t)}\right), & \text{(viii)} & \left\|\dot{B}_{mx}^{(k)}(t)\right\|_{L^\infty} \leq \bar{\psi}_1\left(\sqrt{\bar{S}_m^{(k)}(t)}\right).
\end{array}$$

Proof. We prove (i)-(iv), and, similarly, then the estimations (v)-(viii) are proved by replacing μ by $\bar{\mu}$ and taking $\bar{\psi}_1(z) = \left(1 + (1 + 2\sqrt{2})z + \sqrt{2}z^2\right)\Phi_{\bar{\mu}}^{[3]}(z)$.

Setting $\psi_1(z) = \left(1 + (1 + 2\sqrt{2})z + \sqrt{2}z^2\right)\Phi_\mu^{[3]}(z)$, and using Lemma 3.2 with $h = D_3\mu$, we get

$$\begin{aligned}
\left|A_m^{(k)}(x,t)\right| &= \left|D_3\mu(x,t,u_m^{(k)}(x,t))\right| \\
&\leq \Phi_\mu^{[3]}\left(\left|u_m^{(k)}(x,t)\right|\right) \leq \Phi_\mu^{[3]}\left(\left\|u_{mx}^{(k)}(t)\right\|\right) \\
&\leq \Phi_\mu^{[3]}\left(\sqrt{\bar{S}_m^{(k)}(t)}\right) \leq \psi_1\left(\sqrt{\bar{S}_m^{(k)}(t)}\right), \text{ for all } (x,t) \in [0,1] \times [0,T^*].
\end{aligned}$$

By taking the derivative $\dot{A}_m^{(k)}(x,t) = D_2D_3\mu(x,t,u_m^{(k)}(x,t)) + D_3^2\mu(x,t,u_m^{(k)}(x,t))\dot{u}_m^{(k)}(x,t)$, and using Lemma 3.2 with replacing h by $D_2D_3\mu$ and $D_3^2\mu$, we have

$$\begin{aligned}
\left|\dot{A}_m^{(k)}(x,t)\right| &\leq \left|D_2D_3\mu(x,t,u_m^{(k)}(x,t))\right| + \left|D_3^2\mu(x,t,u_m^{(k)}(x,t))\right| \left\|\dot{u}_{mx}^{(k)}(t)\right\| \\
&\leq \Phi_\mu^{[3]}\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)\left(1 + \sqrt{\bar{S}_m^{(k)}(t)}\right) \leq \psi_1\left(\sqrt{\bar{S}_m^{(k)}(t)}\right).
\end{aligned}$$

By taking the derivative $A_{mx}^{(k)}(x,t) = D_1D_3\mu(x,t,u_m^{(k)}(x,t)) + D_3^2\mu(x,t,u_m^{(k)}(x,t))u_{mx}^{(k)}(x,t)$, note that

$$\begin{aligned}
\left|u_{mx}^{(k)}(x,t)\right| &\leq \sqrt{2}\left\|u_{mx}^{(k)}(t)\right\|_{H^1} \\
&= \sqrt{2}\sqrt{\left\|u_{mx}^{(k)}(t)\right\|^2 + \left\|\Delta u_m^{(k)}(t)\right\|^2} \leq \sqrt{2}\sqrt{\bar{S}_m^{(k)}(t)},
\end{aligned}$$

and using Lemma 3.2 with replacing h by $D_1D_3\mu$ và $D_3^2\mu$, we obtain

$$\begin{aligned}
\left|A_{mx}^{(k)}(x,t)\right| &\leq \left|D_1D_3\mu(x,t,u_m^{(k)}(x,t))\right| + \left|D_3^2\mu(x,t,u_m^{(k)}(x,t))\right| \left|u_{mx}^{(k)}(x,t)\right| \\
&\leq \Phi_\mu^{[3]}\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)\left(1 + \sqrt{2}\sqrt{\bar{S}_m^{(k)}(t)}\right) \leq \psi_1\left(\sqrt{\bar{S}_m^{(k)}(t)}\right).
\end{aligned}$$

Similarly

$$\begin{aligned}\dot{A}^{(k)}_{mx}(x,t) &= D_1D_2D_3\mu(x,t,u^{(k)}_m(x,t)) + D_1D_3^2\mu(x,t,u^{(k)}_m(x,t))\dot{u}^{(k)}_m(x,t)\\ &+D_3^2\mu(x,t,u^{(k)}_m(x,t))\dot{u}^{(k)}_{mx}(x,t)\\ &+\left[D_2D_3^2\mu(x,t,u^{(k)}_m(x,t)) + D_3^3\mu(x,t,u^{(k)}_m(x,t))\dot{u}^{(k)}_m(x,t)\right]u^{(k)}_{mx}(x,t),\end{aligned}$$

then we have

$$\begin{aligned}\left|\dot{A}^{(k)}_{mx}(x,t)\right| &\le \Phi^{[3]}_\mu\left(\sqrt{\bar{S}^{(k)}_m(t)}\right)\left(1+\sqrt{\bar{S}^{(k)}_m(t)}\right)+\Phi^{[3]}_\mu\left(\sqrt{\bar{S}^{(k)}_m(t)}\right)\sqrt{2}\sqrt{\bar{S}^{(k)}_m(t)}\\ &+\Phi^{[3]}_\mu\left(\sqrt{\bar{S}^{(k)}_m(t)}\right)\left(1+\sqrt{\bar{S}^{(k)}_m(t)}\right)\sqrt{2}\sqrt{\bar{S}^{(k)}_m(t)}\\ &= \left[1+(1+2\sqrt{2})\sqrt{\bar{S}^{(k)}_m(t)}+\sqrt{2}\bar{S}^{(k)}_m(t)\right]\Phi^{[3]}_\mu\left(\sqrt{\bar{S}^{(k)}_m(t)}\right)\\ &\equiv \psi_1\left(\sqrt{\bar{S}^{(k)}_m(t)}\right).\end{aligned}$$

Lemma 3.3 is proved completely. □

For estimating $I_1 - I_{11}$ of (3.8) below, we always choose $\beta = \frac{1}{10}\bar{\mu}_*$.

By $\bar{S}^{(k)}_m(t) \ge \left\|u^{(k)}_{mx}(t)\right\|^2 + \left\|\Delta u^{(k)}_m(t)\right\|^2$ and Lemma 3.3-(ii), we get

$$\begin{aligned}I_1 = &\int_0^t ds\int_0^1 \dot{A}^{(k)}_m(x,s)\left[\left|u^{(k)}_{mx}(x,s)\right|^2+\left|\Delta u^{(k)}_m(x,s)\right|^2\right]dx\\ \le &\int_0^t \psi_1\left(\sqrt{\bar{S}^{(k)}_m(s)}\right)\left(\left\|u^{(k)}_{mx}(s)\right\|^2+\left\|\Delta u^{(k)}_m(s)\right\|^2\right)ds\\ \le &\int_0^t \psi_1\left(\sqrt{\bar{S}^{(k)}_m(s)}\right)\bar{S}^{(k)}_m(s)ds \equiv \int_0^t \chi_1\left(\bar{S}^{(k)}_m(s)\right)ds,\end{aligned} \tag{3.10}$$

where $\chi_1(z) = z\psi_1\left(\sqrt{z}\right)$.

By $\bar{S}^{(k)}_m(t) \ge \left\|u^{(k)}_{mx}(t)\right\|^2 + \left\|\Delta \dot{u}^{(k)}_m(t)\right\|^2$ and Lemma 3.3-(iii), we have

$$\begin{aligned}I_2 &= -2\int_0^t \langle A^{(k)}_{mx}(s)u^{(k)}_{mx}(s),\Delta\dot{u}^{(k)}_m(s)\rangle ds\\ &\le 2\int_0^t \psi_1\left(\sqrt{\bar{S}^{(k)}_m(s)}\right)\left\|u^{(k)}_{mx}(s)\right\|\left\|\Delta\dot{u}^{(k)}_m(s)\right\|ds\\ &\le 2\int_0^t \psi_1\left(\sqrt{\bar{S}^{(k)}_m(s)}\right)\bar{S}^{(k)}_m(s)ds \equiv \int_0^t \chi_2\left(\bar{S}^{(k)}_m(s)\right)ds,\end{aligned} \tag{3.11}$$

where $\chi_2(z) = 2z\psi_1\left(\sqrt{z}\right)$.

Using the integral by part formula, we obtain

$$\begin{aligned}I_3 &= -2\int_0^t \langle \frac{\partial}{\partial x}(A^{(k)}_m(s)u^{(k)}_{mx}(s)),\Delta\ddot{u}^{(k)}_m(s)\rangle ds\\ &= 2\langle \frac{\partial}{\partial x}\left(D_3\mu(0,\tilde{u}_{0k})\tilde{u}_{0kx}\right),\Delta\tilde{u}_{1k}\rangle - 2\langle \frac{\partial}{\partial x}(A^{(k)}_m(t)u^{(k)}_{mx}(t)),\Delta\dot{u}^{(k)}_m(t)\rangle + 2\int_0^t \langle \frac{\partial^2}{\partial x\partial s}(A^{(k)}_m(s)u^{(k)}_{mx}(s)),\Delta\dot{u}^{(k)}_m(s)\rangle ds\\ &= 2\langle \frac{\partial}{\partial x}\left[D_3\mu(0,\tilde{u}_{0k})\tilde{u}_{0kx}\right],\Delta\tilde{u}_{1k}\rangle + I_3^{(1)} + I_3^{(2)}.\end{aligned}$$

We shall estimate $I_3^{(1)}$and $I_3^{(2)}$ as follows.

Put $H_m^{(k)}(x,t) = \frac{\partial}{\partial x}(A_m^{(k)}(t)u_{mx}^{(k)}(t))$ and $\dot{H}_m^{(k)}(x,t) = \frac{\partial^2}{\partial x \partial t}(A_m^{(k)}(x,t)u_{mx}^{(k)}(x,t))$, we have

$$\begin{aligned} H_m^{(k)}(x,t) &= \frac{\partial}{\partial x}(A_m^{(k)}(t)u_{mx}^{(k)}(t)) = A_{mx}^{(k)}(t)u_{mx}^{(k)}(t) + A_m^{(k)}(t)\Delta u_m^{(k)}(t), \\ \dot{H}_m^{(k)}(x,t) &= \dot{A}_{mx}^{(k)}(t)u_{mx}^{(k)}(t) + A_{mx}^{(k)}(t)\dot{u}_{mx}^{(k)}(t) + \dot{A}_m^{(k)}(t)\Delta u_m^{(k)}(t) + A_m^{(k)}(t)\Delta \dot{u}_m^{(k)}(t), \end{aligned}$$

so

$$\begin{aligned} \left\|\dot{H}_m^{(k)}(t)\right\| &\leq \psi_1\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)\left\|u_{mx}^{(k)}(t)\right\| + \psi_1\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)\left\|\dot{u}_{mx}^{(k)}(t)\right\| \\ &\quad + \psi_1\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)\left\|\Delta u_m^{(k)}(t)\right\| + \psi_1\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)\left\|\Delta \dot{u}_m^{(k)}(t)\right\| \\ &= \psi_1\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)\left(\left\|u_{mx}^{(k)}(t)\right\| + \left\|\dot{u}_{mx}^{(k)}(t)\right\| + \left\|\Delta u_m^{(k)}(t)\right\| + \left\|\Delta \dot{u}_m^{(k)}(t)\right\|\right) \\ &\leq 2\psi_1\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)\left(\left\|u_{mx}^{(k)}(t)\right\|^2 + \left\|\dot{u}_{mx}^{(k)}(t)\right\|^2 + \left\|\Delta u_m^{(k)}(t)\right\|^2 + \left\|\Delta \dot{u}_m^{(k)}(t)\right\|^2\right)^{1/2} \\ &\leq 2\psi_1\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)\sqrt{\bar{S}_m^{(k)}(t)}, \end{aligned}$$

and then

$$\begin{aligned} \left\|\frac{\partial}{\partial x}(A_m^{(k)}(t)u_{mx}^{(k)}(t))\right\| &= \left\|H_m^{(k)}(t)\right\| \\ &\leq \left\|H_m^{(k)}(0)\right\| + \int_0^t \left\|\dot{H}_m^{(k)}(s)\right\| ds \\ &\leq \left\|\frac{\partial}{\partial x}\left[D_3\mu(0,\tilde{u}_{0k})\tilde{u}_{0kx}\right]\right\| + 2\int_0^t \sqrt{\bar{S}_m^{(k)}(s)}\psi_1\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)ds. \end{aligned} \tag{3.12}$$

Therefore, $I_3^{(1)}$ and $I_3^{(2)}$ are estimated by

$$\begin{aligned} I_3^{(1)} &= -2\langle\frac{\partial}{\partial x}(A_m^{(k)}(t)u_{mx}^{(k)}(t)), \Delta \dot{u}_m^{(k)}(t)\rangle \\ &\leq \beta\left\|\Delta \dot{u}_m^{(k)}(t)\right\|^2 + \frac{1}{\beta}\left\|\frac{\partial}{\partial x}(A_m^{(k)}(t)u_{mx}^{(k)}(t))\right\|^2 \\ &\leq \beta\bar{S}_m^{(k)}(t) + \frac{2}{\beta}\left[\left\|\frac{\partial}{\partial x}\left(D_3\mu(0,\tilde{u}_{0k})\tilde{u}_{0kx}\right)\right\|^2 + 4T^*\int_0^t \bar{S}_m^{(k)}(s)\psi_1^2\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)ds\right]; \end{aligned}$$

$$\begin{aligned} I_3^{(2)} &= 2\int_0^t \langle\frac{\partial^2}{\partial x \partial s}(A_m^{(k)}(s)u_{mx}^{(k)}(s)), \Delta \dot{u}_m^{(k)}(s)\rangle ds \\ &= 2\int_0^t \langle\dot{H}_m^{(k)}(s), \Delta \dot{u}_m^{(k)}(s)\rangle ds \\ &\leq 2\int_0^t \left\|\dot{H}_m^{(k)}(s)\right\|\left\|\Delta \dot{u}_m^{(k)}(s)\right\| ds \leq 4\int_0^t \psi_1\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)\bar{S}_m^{(k)}(s)ds. \end{aligned}$$

Then

$$\begin{aligned}
I_3 &= -2\int_0^t \langle \frac{\partial}{\partial x}(A_m^{(k)}(s)u_{mx}^{(k)}(s)), \Delta \ddot{u}_m^{(k)}(s)\rangle ds \\
&\leq 2\langle \frac{\partial}{\partial x}\left[D_3\mu(0,\tilde{u}_{0k})\tilde{u}_{0kx}\right], \Delta\tilde{u}_{1k}\rangle + \beta \bar{S}_m^{(k)}(t) \\
&+ \frac{2}{\beta}\left[\left\|\frac{\partial}{\partial x}\left(D_3\mu(0,\tilde{u}_{0k})\tilde{u}_{0kx}\right)\right\|^2 + 4T^*\int_0^t \bar{S}_m^{(k)}(s)\psi_1^2\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)ds\right] + 4\int_0^t \psi_1\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)\bar{S}_m^{(k)}(s)ds \\
&= 2\langle \frac{\partial}{\partial x}\left(D_3\mu(0,\tilde{u}_{0k})\tilde{u}_{0kx}\right), \Delta\tilde{u}_{1k}\rangle + \frac{2}{\beta}\left\|\frac{\partial}{\partial x}\left(D_3\mu(0,\tilde{u}_{0k})\tilde{u}_{0kx}\right)\right\|^2 \\
&+ \beta\bar{S}_m^{(k)}(t) + 4\int_0^t \left[1 + \frac{2}{\beta}T^*\psi_1\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)\right]\psi_1\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)\bar{S}_m^{(k)}(s)ds \\
&\equiv \beta\bar{S}_m^{(k)}(t) + 2\langle \frac{\partial}{\partial x}\left(D_3\mu(0,\tilde{u}_{0k})\tilde{u}_{0kx}\right), \Delta\tilde{u}_{1k}\rangle + \frac{2}{\beta}\left\|\frac{\partial}{\partial x}\left(D_3\mu(0,\tilde{u}_{0k})\tilde{u}_{0kx}\right)\right\|^2 + \int_0^t \chi_3\left(\bar{S}_m^{(k)}(s)\right)ds,
\end{aligned} \tag{3.13}$$

where $\chi_3(z) = 4\left(1 + \frac{2}{\beta}T^*\psi_1\left(\sqrt{z}\right)\right)z\psi_1\left(\sqrt{z}\right)$.

Similarly to (3.12), after replacing μ by $\bar{\mu}$, we get that

$$\left\|\frac{\partial}{\partial x}\left(B_m^{(k)}(t)u_{mx}^{(k)}(t)\right)\right\| \leq \left\|\frac{\partial}{\partial x}\left[D_3\bar{\mu}(0,\tilde{u}_{0k})\tilde{u}_{0kx}\right]\right\| + 2\int_0^t \sqrt{\bar{S}_m^{(k)}(s)}\bar{\psi}_1\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)ds.$$

Therefore

$$\begin{aligned}
I_4 &= -2g(0)\int_0^t \left\langle B_m^{(k)}(s)u_{mx}^{(k)}(s), u_{mx}^{(k)}(s)\right\rangle ds - 2g(0)\int_0^t \left\langle \frac{\partial}{\partial x}(B_m^{(k)}(s)u_{mx}^{(k)}(s)), \Delta u_m^{(k)}(s) + \Delta\dot{u}_m^{(k)}(s)\right\rangle ds \\
&\leq 2\left|g(0)\right|\int_0^t \bar{\psi}_1\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)\bar{S}_m^{(k)}(s)ds + 2\sqrt{2}\left|g(0)\right|\int_0^t \left\|\frac{\partial}{\partial x}(B_m^{(k)}(s)u_{mx}^{(k)}(s))\right\|\sqrt{\bar{S}_m^{(k)}(s)}ds,
\end{aligned}$$

hence

$$\begin{aligned}
I_4 &\leq 2\left|g(0)\right|\int_0^t \bar{\psi}_1\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)\bar{S}_m^{(k)}(s)ds + 2\left|g(0)\right|\int_0^t \bar{S}_m^{(k)}(s)ds + \left|g(0)\right|\int_0^t \left\|\frac{\partial}{\partial x}(B_m^{(k)}(s)u_{mx}^{(k)}(s))\right\|^2 ds \\
&\leq 2\left|g(0)\right|\int_0^t \bar{\psi}_1\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)\bar{S}_m^{(k)}(s)ds + 2\left|g(0)\right|\int_0^t \bar{S}_m^{(k)}(s)ds \\
&+ 2\left|g(0)\right|\int_0^t \left[\left\|\frac{\partial}{\partial x}\left[D_3\bar{\mu}(0,\tilde{u}_{0k})\tilde{u}_{0kx}\right]\right\|^2 + 4T^*\int_0^\tau \bar{S}_m^{(k)}(s)\bar{\psi}_1^2\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)ds\right]d\tau \\
&\leq 2\left|g(0)\right|\int_0^t \bar{\psi}_1\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)\bar{S}_m^{(k)}(s)ds + 2\left|g(0)\right|\int_0^t \bar{S}_m^{(k)}(s)ds \\
&+ 2T^*\left|g(0)\right|\left\|\frac{\partial}{\partial x}\left[D_3\bar{\mu}(0,\tilde{u}_{0k})\tilde{u}_{0kx}\right]\right\|^2 + 8\left(T^*\right)^2\left|g(0)\right|\int_0^t \bar{S}_m^{(k)}(s)\bar{\psi}_1^2\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)ds \\
&= 2T^*\left|g(0)\right|\left\|\frac{\partial}{\partial x}\left(D_3\bar{\mu}(0,\tilde{u}_{0k})\tilde{u}_{0kx}\right)\right\|^2 + 2\left|g(0)\right|\int_0^t \left[1 + \bar{\psi}_1\left(\sqrt{\bar{S}_m^{(k)}(s)}\right) + 4\left(T^*\right)^2\bar{\psi}_1^2\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)\right]\bar{S}_m^{(k)}(s)ds \\
&\equiv 2T^*\left|g(0)\right|\left\|\frac{\partial}{\partial x}\left(D_3\bar{\mu}(0,\tilde{u}_{0k})\tilde{u}_{0kx}\right)\right\|^2 + \int_0^t \chi_4\left(\bar{S}_m^{(k)}(s)\right)ds,
\end{aligned} \tag{3.14}$$

where $\chi_4(z) = 2\left|g(0)\right|\left(1 + \bar{\psi}_1\left(\sqrt{z}\right) + 4\left(T^*\right)^2\bar{\psi}_1^2\left(\sqrt{z}\right)\right)z$.

The terms I_5, I_6 are estimated by

$$\begin{aligned}
I_5 &= 2\int_0^t g(t-s)\left[\left\langle B_m^{(k)}(s)u_{mx}^{(k)}(s), u_{mx}^{(k)}(t)\right\rangle + \left\langle \frac{\partial}{\partial x}(B_m^{(k)}(s)u_{mx}^{(k)}(s)), \Delta u_m^{(k)}(t) + \Delta \dot{u}_m^{(k)}(t)\right\rangle\right] ds \\
&\le 2\int_0^t \left|g(t-s)\right|\left[\left\|B_m^{(k)}(s)\right\|_{L^\infty}\left\|u_{mx}^{(k)}(s)\right\|\left\|u_{mx}^{(k)}(t)\right\| + \left\|\frac{\partial}{\partial x}(B_m^{(k)}(s)u_{mx}^{(k)}(s))\right\|\left(\left\|\Delta u_m^{(k)}(t)\right\| + \left\|\Delta \dot{u}_m^{(k)}(t)\right\|\right)\right] ds \\
&\le 2\int_0^t \left|g(t-s)\right|\left[\bar{\psi}_1\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)\sqrt{\bar{S}_m^{(k)}(s)}\sqrt{\bar{S}_m^{(k)}(t)} + \sqrt{2}\left\|\frac{\partial}{\partial x}(B_m^{(k)}(s)u_{mx}^{(k)}(s))\right\|\sqrt{\bar{S}_m^{(k)}(t)}\right] ds \\
&\le \beta \bar{S}_m^{(k)}(t) + \frac{1}{\beta}\left(\int_0^t \left|g(t-s)\right|\left[\bar{\psi}_1\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)\sqrt{\bar{S}_m^{(k)}(s)} + \sqrt{2}\left\|\frac{\partial}{\partial x}(B_m^{(k)}(s)u_{mx}^{(k)}(s))\right\|\right] ds\right)^2,
\end{aligned}$$

so

$$\begin{aligned}
I_5 &\le \beta \bar{S}_m^{(k)}(t) + \frac{2}{\beta}\left\|g\right\|_{L^2(0,T^*)}^2 \int_0^t \left[\bar{\psi}_1^2\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)\bar{S}_m^{(k)}(s) + 2\left\|\frac{\partial}{\partial x}(B_m^{(k)}(s)u_{mx}^{(k)}(s))\right\|^2\right] ds \\
&= \beta \bar{S}_m^{(k)}(t) + \frac{2}{\beta}\left\|g\right\|_{L^2(0,T^*)}^2 \int_0^t \bar{\psi}_1^2\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)\bar{S}_m^{(k)}(s) ds \\
&\quad + \frac{4}{\beta}\left\|g\right\|_{L^2(0,T^*)}^2 \int_0^t \left\|\frac{\partial}{\partial x}\left[B_m^{(k)}(r)u_{mx}^{(k)}(r)\right]\right\|^2 dr \\
&\le \beta \bar{S}_m^{(k)}(t) + \frac{2}{\beta}\left\|g\right\|_{L^2(0,T^*)}^2 \int_0^t \bar{\psi}_1^2\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)\bar{S}_m^{(k)}(s) ds \\
&\quad + \frac{8}{\beta}\left\|g\right\|_{L^2(0,T^*)}^2 \int_0^t \left[\left\|\frac{\partial}{\partial x}\left[D_3\bar{\mu}(0,\tilde{u}_{0k})\tilde{u}_{0kx}\right]\right\|^2 + 4T^*\int_0^r \bar{S}_m^{(k)}(s)\bar{\psi}_1^2\left(\sqrt{\bar{S}_m^{(k)}(s)}\right) ds\right] dr \\
&\le \beta \bar{S}_m^{(k)}(t) + \frac{8}{\beta}T^*\left\|g\right\|_{L^2(0,T^*)}^2 \left\|\frac{\partial}{\partial x}\left[D_3\bar{\mu}(0,\tilde{u}_{0k})\tilde{u}_{0kx}\right]\right\|^2 \\
&\quad + \frac{2}{\beta}\left\|g\right\|_{L^2(0,T^*)}^2\left(1+16\left(T^*\right)^2\right)\int_0^t \bar{\psi}_1^2\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)\bar{S}_m^{(k)}(s) ds \\
&\equiv \beta \bar{S}_m^{(k)}(t) + \frac{8}{\beta}T^*\left\|g\right\|_{L^2(0,T^*)}^2 \left\|\frac{\partial}{\partial x}\left[D_3\bar{\mu}(0,\tilde{u}_{0k})\tilde{u}_{0kx}\right]\right\|^2 + \int_0^t \chi_5\left(\bar{S}_m^{(k)}(s)\right) ds,
\end{aligned} \tag{3.15}$$

where $\chi_5(z) = \frac{2}{\beta}\left\|g\right\|_{L^2(0,T^*)}^2\left(1+16\left(T^*\right)^2\right) z\bar{\psi}_1^2\left(\sqrt{z}\right)$;

$$\begin{aligned}
I_6 &= -2\int_0^t d\tau \int_0^\tau g'(\tau-s)\left[\langle B_m^{(k)}(s)u_{mx}^{(k)}(s), u_{mx}^{(k)}(\tau)\rangle + \langle \frac{\partial}{\partial x}(B_m^{(k)}(s)u_{mx}^{(k)}(s)), \Delta u_m^{(k)}(\tau) + \Delta \dot{u}_m^{(k)}(\tau)\rangle\right] ds \\
&\le 2\int_0^t d\tau \int_0^\tau \left|g'(\tau-s)\right|\left[\left\|B_m^{(k)}(s)\right\|_{L^\infty}\left\|u_{mx}^{(k)}(s)\right\|\left\|u_{mx}^{(k)}(\tau)\right\|\right. \\
&\qquad\qquad \left. + \left\|\frac{\partial}{\partial x}(B_m^{(k)}(s)u_{mx}^{(k)}(s))\right\|\left(\left\|\Delta u_m^{(k)}(\tau)\right\| + \left\|\Delta \dot{u}_m^{(k)}(\tau)\right\|\right)\right] ds \\
&\le 2\int_0^t d\tau \int_0^\tau \left|g'(\tau-s)\right|\left[\bar{\psi}_1\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)\sqrt{\bar{S}_m^{(k)}(s)}\sqrt{\bar{S}_m^{(k)}(\tau)} + \sqrt{2}\left\|\frac{\partial}{\partial x}(B_m^{(k)}(s)u_{mx}^{(k)}(s))\right\|\sqrt{\bar{S}_m^{(k)}(\tau)}\right] ds \\
&\le 4\sqrt{T^*}\left\|g'\right\|_{L^2(0,T^*)}\left[\int_0^t \bar{S}_m^{(k)}(\tau) d\tau\right]^{1/2}\left[\int_0^t \left[\bar{\psi}_1^2\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)\bar{S}_m^{(k)}(s) + 2\left\|\frac{\partial}{\partial x}(B_m^{(k)}(s)u_{mx}^{(k)}(s))\right\|^2\right] ds\right]^{1/2}
\end{aligned}$$

thus

$$
\begin{aligned}
I_6 &\le 4T^* \left\|g'\right\|^2_{L^2(0,T^*)} \int_0^t \bar{S}_m^{(k)}(\tau)d\tau \\
&+ \int_0^t \left[\bar{\psi}_1^2\left(\sqrt{\bar{S}_m^{(k)}(s)}\right) \bar{S}_m^{(k)}(s) + 2\left\| \frac{\partial}{\partial x}(B_m^{(k)}(s)u_{mx}^{(k)}(s))\right\|^2 \right] ds \\
&\le 4T^* \left\|g'\right\|^2_{L^2(0,T^*)} \int_0^t \bar{S}_m^{(k)}(s)ds + \int_0^t \bar{\psi}_1^2\left(\sqrt{\bar{S}_m^{(k)}(s)}\right) \bar{S}_m^{(k)}(s)ds \\
&+ 4\int_0^t \left[\left\| \frac{\partial}{\partial x}\left[D_3\bar{\mu}(0,\tilde{u}_{0k})\tilde{u}_{0kx}\right]\right\|^2 + 4T^* \int_0^r \bar{S}_m^{(k)}(s)\bar{\psi}_1^2\left(\sqrt{\bar{S}_m^{(k)}(s)}\right) ds \right] dr \\
&\le 4T^* \left\|g'\right\|^2_{L^2(0,T^*)} \int_0^t \bar{S}_m^{(k)}(s)ds + \int_0^t \bar{\psi}_1^2\left(\sqrt{\bar{S}_m^{(k)}(s)}\right) \bar{S}_m^{(k)}(s)ds \\
&+ 4T^* \left\| \frac{\partial}{\partial x}\left[D_3\bar{\mu}(0,\tilde{u}_{0k})\tilde{u}_{0kx}\right]\right\|^2 + 16\,(T^*)^2 \int_0^t \bar{S}_m^{(k)}(s)\bar{\psi}_1^2\left(\sqrt{\bar{S}_m^{(k)}(s)}\right) ds \\
&= 4T^* \left\| \frac{\partial}{\partial x}\left[D_3\bar{\mu}(0,\tilde{u}_{0k})\tilde{u}_{0kx}\right]\right\|^2 \\
&+ \int_0^t \left[4T^* \left\|g'\right\|^2_{L^2(0,T^*)} + (1+6\,(T^*)^2)\bar{\psi}_1^2\left(\sqrt{\bar{S}_m^{(k)}(s)}\right) \right] \bar{S}_m^{(k)}(s)ds \\
&\equiv 4T^* \left\| \frac{\partial}{\partial x}\left[D_3\bar{\mu}(0,\tilde{u}_{0k})\tilde{u}_{0kx}\right]\right\|^2 + \int_0^t \chi_6\left(\bar{S}_m^{(k)}(s)\right) ds,
\end{aligned}
\tag{3.16}
$$

where $\chi_6(z) = \left[4T^* \left\|g'\right\|^2_{L^2(0,T^*)} + (1+6\,(T^*)^2)\bar{\psi}_1^2\left(\sqrt{z}\right)\right] z.$

For $I_7 - I_{11}$, first, we have to estimate $F_m^{(k)}(x,t)$, $F_{mx}^{(k)}(x,t)$, $P_m^{(k)}(x,t)$, $P_{mx}^{(k)}(x,t)$, $\dot{P}_{mx}^{(k)}(x,t)$, $Q_m^{(k)}(x,t)$, $Q_{mx}^{(k)}(x,t)$. Then, we need the following lemma.

Lemma 3.4. *There is a positive constant $\bar{C}_*(M)$ that*

(i) $\left\|F_m^{(k)}(t)\right\|_{L^\infty} \le \bar{C}_*(M)\tilde{K}_M(f)\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right]$,

(ii) $\left\|\dot{F}_m^{(k)}(t)\right\|_{L^\infty} \le \bar{C}_*(M)\tilde{K}_M(f)\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right]$,

(iii) $\left\|F_{mx}^{(k)}(t)\right\| \le \bar{C}_*(M)\tilde{K}_M(f)\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right]$,

(iv) $\left\|P_m^{(k)}(t)\right\|_{L^\infty} \le \bar{C}_*(M)K_M(\mu)\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right]$,

(v) $\left\|P_{mx}^{(k)}(t)\right\| \le \bar{C}_*(M)K_M(\mu)\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right]$,

(vi) $\left\|\dot{P}_{mx}^{(k)}(t)\right\| \le \bar{C}_*(M)K_M(\mu)\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right]$,

(vii) $\left\|Q_m^{(k)}(t)\right\|_{L^\infty} \le \bar{C}_*(M)K_M(\bar{\mu})\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right]$,

(viii) $\left\|Q_{mx}^{(k)}(t)\right\| \le \bar{C}_*(M)K_M(\bar{\mu})\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right]$.

Moreover, $\bar{C}_*(M)$ is presented by

$$\begin{aligned}
\bar{C}_*(M) &= \left(\bar{a}_* M^2 + \bar{b}_* M + \bar{c}_*\right)\left(2 + M^{N-1}\right), \\
\bar{a}_* &= \sum_{i=0}^{N-1} \frac{2^{i-1}}{i!}, \\
\bar{b}_* &= 3\bar{a}_* + 2\sum_{i=0}^{N-2} \frac{2^i}{i!}, \\
\bar{c}_* &= \bar{a}_* + 3\sum_{i=0}^{N-2} \frac{2^i}{i!} + 2\sum_{i=0}^{N-3} \frac{2^i}{i!}.
\end{aligned}$$

Proof. (i) Evaluating $\left\|F_m^{(k)}(t)\right\|_{L^\infty}$. By

$$\begin{aligned}
\bar{C}_*(M) &= \left(\bar{a}_* M^2 + \bar{b}_* M + \bar{c}_*\right)\left(2 + M^{N-1}\right) \\
&\geq \bar{c}_*\left(2 + M^{N-1}\right) \geq \sum_{i=0}^{N-1} \frac{2^{i-1}}{i!}\left(2 + M^{N-1}\right),
\end{aligned}$$

then we have

$$\begin{aligned}
\left|F_m^{(k)}(x,t)\right| &\leq \sum_{i=0}^{N-1} \frac{1}{i!}\left|D_3^i f(x,t,u_{m-1}(x,t))\right|\left(\left|u_m^{(k)}(x,t)\right| + |u_{m-1}(x,t)|\right)^i \\
&\leq \sum_{i=0}^{N-1} \frac{2^{i-1}}{i!}\tilde{K}_M(f)\left(\left|u_m^{(k)}(x,t)\right|^i + |u_{m-1}(x,t)|^i\right) \\
&\leq \sum_{i=0}^{N-1} \frac{2^{i-1}}{i!}\tilde{K}_M(f)\left[\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^i + M^i\right] \\
&\leq \sum_{i=0}^{N-1} \frac{2^{i-1}}{i!}\left(2 + M^{N-1}\right)\tilde{K}_M(f)\left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right] \\
&\leq \bar{C}_*(M)\tilde{K}_M(f)\left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right].
\end{aligned}$$

(ii) Evaluating $\left\|\dot{F}_m^{(k)}(t)\right\|_{L^\infty}$. Taking the derivative of $F_m^{(k)}(x,t)$ with respect to time variable, we get that

$$\begin{aligned}
&\dot{F}_m^{(k)}(x,t) \\
= &\ D_2 f(x,t,u_{m-1}(x,t)) + D_3 f(x,t,u_{m-1}(x,t))u'_{m-1}(x,t) \\
&+ \sum_{i=1}^{N-1} \frac{1}{i!}\left[D_2 D_3^i f(x,t,u_{m-1}(x,t)) + D_3^{i+1} f(x,t,u_{m-1}(x,t))u'_{m-1}(x,t)\right]\left(u_m^{(k)}(x,t) - u_{m-1}(x,t)\right)^i \\
&+ \sum_{i=1}^{N-1} \frac{i}{i!} D_3^i f(x,t,u_{m-1}(x,t))\left(u_m^{(k)}(x,t) - u_{m-1}(x,t)\right)^{i-1}\left(\dot{u}_m^{(k)}(x,t) - u'_{m-1}(x,t)\right).
\end{aligned}$$

Then, by the fact that

$$\begin{aligned}\bar{C}_*(M) &= \left(\bar{a}_*M^2+\bar{b}_*M+\bar{c}_*\right)\left(2+M^{N-1}\right)\\ &\geq \left(\bar{b}_*M+\bar{c}_*\right)\left(2+M^{N-1}\right)\\ &= \left[\left(3\sum_{i=0}^{N-1}\frac{2^{i-1}}{i!}+2\sum_{i=0}^{N-2}\frac{2^i}{i!}\right)M+\sum_{i=0}^{N-1}\frac{2^{i-1}}{i!}+3\sum_{i=0}^{N-2}\frac{2^i}{i!}+2\sum_{i=0}^{N-3}\frac{2^i}{i!}\right]\left(2+M^{N-1}\right)\\ &\geq \left[\sum_{i=0}^{N-1}\frac{2^{i-1}}{i!}(1+M)+\sum_{i=0}^{N-2}\frac{2^i}{i!}\right]\left(2+M^{N-1}\right),\end{aligned}$$

we have

$$\begin{aligned}\left|\dot{F}_m^{(k)}(x,t)\right| &\leq \tilde{K}_M(f)(1+M)+\sum_{i=1}^{N-1}\frac{1}{i!}\tilde{K}_M(f)(1+M)\left(\sqrt{\bar{S}_m^{(k)}(t)}+M\right)^i+\sum_{i=1}^{N-1}\frac{i}{i!}\tilde{K}_M(f)\left(\sqrt{\bar{S}_m^{(k)}(t)}+M\right)^i\\ &= \sum_{i=0}^{N-1}\frac{1}{i!}\tilde{K}_M(f)(1+M)\left(\sqrt{\bar{S}_m^{(k)}(t)}+M\right)^i+\sum_{i=1}^{N-1}\frac{i}{i!}\tilde{K}_M(f)\left(\sqrt{\bar{S}_m^{(k)}(t)}+M\right)^i\\ &\leq \sum_{i=0}^{N-1}\frac{1}{i!}2^{i-1}\tilde{K}_M(f)(1+M)\left[\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^i+M^i\right]+\sum_{i=1}^{N-1}\frac{i}{i!}2^{i-1}\tilde{K}_M(f)\left[\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^i+M^i\right]\\ &\leq \sum_{i=0}^{N-1}\frac{2^{i-1}}{i!}(1+M)\left(2+M^{N-1}\right)\tilde{K}_M(f)\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right]\\ &\quad+\sum_{i=1}^{N-1}\frac{i}{i!}2^{i-1}\left(2+M^{N-1}\right)\tilde{K}_M(f)\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right]\\ &= \left[\sum_{i=0}^{N-1}\frac{2^{i-1}}{i!}(1+M)+\sum_{i=0}^{N-2}\frac{2^i}{i!}\right]\left(2+M^{N-1}\right)\tilde{K}_M(f)\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right]\\ &\leq \bar{C}_*(M)\tilde{K}_M(f)\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right].\end{aligned}$$

(iii) Evaluating $\left\|F_{mx}^{(k)}(t)\right\|$. Taking the derivative of $F_m^{(k)}(x,t)$ with respect to spatial variable, we get that

$$\begin{aligned}F_{mx}^{(k)}(x,t) &= D_1f(x,t,u_{m-1}(x,t))+D_3f(x,t,u_{m-1}(x,t))\nabla u_{m-1}(x,t)\\ &\quad+\sum_{i=1}^{N-1}\frac{1}{i!}\left[D_1D_3^if(x,t,u_{m-1}(x,t))+D_3^{i+1}f(x,t,u_{m-1}(x,t))\nabla u_{m-1}(x,t)\right]\left(u_m^{(k)}(x,t)-u_{m-1}(x,t)\right)^i\\ &\quad+\sum_{i=1}^{N-1}\frac{i}{i!}D_3^if(x,t,u_{m-1}(x,t))\left(u_m^{(k)}(x,t)-u_{m-1}(x,t)\right)^{i-1}\left(u_{mx}^{(k)}(x,t)-\nabla u_{m-1}(x,t)\right),\end{aligned}$$

then

$$\begin{aligned}\left\|F_{mx}^{(k)}(t)\right\| &\leq \tilde{K}_M(f)(1+M)+\sum_{i=1}^{N-1}\frac{1}{i!}\tilde{K}_M(f)(1+M)\left(\sqrt{\bar{S}_m^{(k)}(t)}+M\right)^i+\sum_{i=1}^{N-1}\frac{i}{i!}\tilde{K}_M(f)\left(\sqrt{\bar{S}_m^{(k)}(t)}+M\right)^i\\ &= \sum_{i=0}^{N-1}\frac{1}{i!}\tilde{K}_M(f)(1+M)\left(\sqrt{\bar{S}_m^{(k)}(t)}+M\right)^i+\sum_{i=1}^{N-1}\frac{i}{i!}\tilde{K}_M(f)\left(\sqrt{\bar{S}_m^{(k)}(t)}+M\right)^i\\ &\leq \sum_{i=0}^{N-1}\frac{2^{i-1}}{i!}\tilde{K}_M(f)(1+M)\left[\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^i+M^i\right]+\sum_{i=1}^{N-1}\frac{i}{i!}2^{i-1}\tilde{K}_M(f)\left[\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^i+M^i\right]\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=0}^{N-1} \frac{2^{i-1}}{i!}(1+M)\left(2+M^{N-1}\right)\tilde{K}_M(f)\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right] \\
&\quad +\sum_{i=1}^{N-1} \frac{i}{i!}2^{i-1}\left(2+M^{N-1}\right)\tilde{K}_M(f)\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right] \\
&= \left[\sum_{i=0}^{N-1} \frac{2^{i-1}}{i!}(1+M)+\sum_{i=1}^{N-1} \frac{i}{i!}2^{i-1}\right]\left(2+M^{N-1}\right)\tilde{K}_M(f)\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right] \\
&\leq \bar{C}_*(M)\tilde{K}_M(f)\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right].
\end{aligned}
$$

(iv), (v) Evaluating $\left\|P_m^{(k)}(t)\right\|_{L^\infty}$ and $\left\|P_{mx}^{(k)}(t)\right\|$.

Replacing $\tilde{K}_M(f)$ by $K_M(\mu)$ in the estimations of $\left\|F_m^{(k)}(t)\right\|_{L^\infty}$ and $\left\|F_{mx}^{(k)}(t)\right\|$, we obtain

$$
\begin{aligned}
\left\|P_m^{(k)}(t)\right\|_{L^\infty} &\leq \bar{C}_*(M)K_M(\mu)\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right], \\
\left\|P_{mx}^{(k)}(t)\right\| &\leq \bar{C}_*(M)K_M(\mu)\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right].
\end{aligned}
$$

(vi) Evaluating $\left\|\dot{P}_{mx}^{(k)}(t)\right\|$. Setting $h = D_1\mu$, we have

$$
\begin{aligned}
P_m^{(k)}(x,t) &= P_m[h,u_{m-1},u_m^{(k)}](x,t) \equiv \sum_{i=0}^{N-1} \frac{1}{i!}D_3^i h(x,t,u_{m-1})\left(u_m^{(k)}-u_{m-1}\right)^i, \\
P_{mx}^{(k)}(x,t) &= D_1 h(x,t,u_{m-1}) + D_3 h(x,t,u_{m-1})\nabla u_{m-1} \\
&\quad +\sum_{i=1}^{N-1} \frac{1}{i!}\left[D_1 D_3^i h(x,t,u_{m-1}) + D_3^{i+1}h(x,t,u_{m-1})\nabla u_{m-1}\right]\left(u_m^{(k)}-u_{m-1}\right)^i \\
&\quad +\sum_{i=1}^{N-1} \frac{i}{i!}D_3^i h(x,t,u_{m-1})\left(u_m^{(k)}-u_{m-1}\right)^{i-1}\left(u_{mx}^{(k)}-\nabla u_{m-1}\right).
\end{aligned}
$$

Taking the derivative of $P_{mx}^{(k)}(x,t)$ with respect to time variable, we get that

$$
\begin{aligned}
\dot{P}_{mx}^{(k)}(x,t) &= D_2 D_1 h(x,t,u_{m-1}) + D_3 D_1 h(x,t,u_{m-1})u'_{m-1} \\
&\quad +\left[D_2 D_3 h(x,t,u_{m-1}) + D_3^2 h(x,t,u_{m-1})u'_{m-1}\right]\nabla u_{m-1} + D_3 h(x,t,u_{m-1})\nabla u'_{m-1} \\
&\quad +\sum_{i=1}^{N-1} \frac{i}{i!}\left[D_1 D_3^i h(x,t,u_{m-1}) + D_3^{i+1}h(x,t,u_{m-1})\nabla u_{m-1}\right]\left(u_m^{(k)}-u_{m-1}\right)^{i-1}\left(\dot{u}_m^{(k)}-u'_{m-1}\right) \\
&\quad +\sum_{i=1}^{N-1} \frac{1}{i!}\Big[D_2 D_1 D_3^i h(x,t,u_{m-1}) + D_1 D_3^{i+1}h(x,t,u_{m-1})u'_{m-1} + D_3^{i+1}h(x,t,u_{m-1})\nabla u'_{m-1} \\
&\quad +\left(D_2 D_3^{i+1}h(x,t,u_{m-1}) + D_3^{i+2}h(x,t,u_{m-1})u'_{m-1}\right)\nabla u_{m-1}\Big]\left(u_m^{(k)}-u_{m-1}\right)^i \\
&\quad +\left[D_2 D_3 h(x,t,u_{m-1}) + D_3^2 h(x,t,u_{m-1})u'_{m-1}\right]\left(u_{mx}^{(k)}-\nabla u_{m-1}\right) \\
&\quad +D_3 h(x,t,u_{m-1})\left(\dot{u}_{mx}^{(k)}-\nabla u'_{m-1}\right)
\end{aligned}
$$

$$+\sum_{i=2}^{N-1}\frac{i}{i!}(i-1)D_3^i h(x,t,u_{m-1})\left(u_m^{(k)}-u_{m-1}\right)^{i-2}\left(\dot{u}_m^{(k)}-u'_{m-1}\right)\left(u_{mx}^{(k)}-\nabla u_{m-1}\right)$$
$$+\sum_{i=2}^{N-1}\frac{i}{i!}D_3^i h(x,t,u_{m-1})\left(u_m^{(k)}-u_{m-1}\right)^{i-1}\left(\dot{u}_{mx}^{(k)}-\nabla u'_{m-1}\right)$$
$$+\sum_{i=2}^{N-1}\frac{i}{i!}\left[D_2D_3^i h(x,t,u_{m-1})+D_3^{i+1}h(x,t,u_{m-1})u'_{m-1}\right]\left(u_m^{(k)}-u_{m-1}\right)^{i-1}\left(u_{mx}^{(k)}-\nabla u_{m-1}\right),$$

then

$$\begin{aligned}
\left\|\dot{P}_{mx}^{(k)}(t)\right\| \leq\ & K_M(\mu)(1+M)+K_M(\mu)(1+M)M+K_M(\mu)M \\
&+\sum_{i=1}^{N-1}\frac{i}{i!}K_M(\mu)(1+M)\left(\sqrt{\bar{S}_m^{(k)}(t)}+M\right)^i \\
&+\sum_{i=1}^{N-1}\frac{1}{i!}\left[K_M(\mu)(1+M)+K_M(\mu)M+K_M(\mu)(1+M)M\right]\left(\sqrt{\bar{S}_m^{(k)}(t)}+M\right)^i \\
&+K_M(\mu)(1+M)\left(\sqrt{\bar{S}_m^{(k)}(t)}+M\right)+K_M(\mu)\left(\sqrt{\bar{S}_m^{(k)}(t)}+M\right) \\
&+\sum_{i=2}^{N-1}\frac{i}{i!}(i-1)K_M(\mu)\left(\sqrt{\bar{S}_m^{(k)}(t)}+M\right)^i+\sum_{i=2}^{N-1}\frac{i}{i!}K_M(\mu)\left(\sqrt{\bar{S}_m^{(k)}(t)}+M\right)^i \\
&+\sum_{i=2}^{N-1}\frac{i}{i!}K_M(\mu)(1+M)\left(\sqrt{\bar{S}_m^{(k)}(t)}+M\right)^i \\
=\ & \sum_{i=1}^{N-1}\frac{i}{i!}(1+M)K_M(\mu)\left(\sqrt{\bar{S}_m^{(k)}(t)}+M\right)^i+\sum_{i=0}^{N-1}\frac{1}{i!}\left(1+3M+M^2\right)K_M(\mu)\left(\sqrt{\bar{S}_m^{(k)}(t)}+M\right)^i \\
&+\sum_{i=2}^{N-1}\frac{i}{i!}(i-1)K_M(\mu)\left(\sqrt{\bar{S}_m^{(k)}(t)}+M\right)^i+\sum_{i=1}^{N-1}\frac{i}{i!}K_M(\mu)\left(\sqrt{\bar{S}_m^{(k)}(t)}+M\right)^i \\
&+\sum_{i=1}^{N-1}\frac{i}{i!}(1+M)K_M(\mu)\left(\sqrt{\bar{S}_m^{(k)}(t)}+M\right)^i \\
\leq\ & \sum_{i=1}^{N-1}\frac{i}{i!}2^{i-1}(1+M)K_M(\mu)\left[\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^i+M^i\right] \\
&+\sum_{i=0}^{N-1}\frac{1}{i!}2^{i-1}\left(1+3M+M^2\right)K_M(\mu)\left[\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^i+M^i\right] \\
&+\sum_{i=2}^{N-1}\frac{i}{i!}(i-1)2^{i-1}K_M(\mu)\left[\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^i+M^i\right]+\sum_{i=1}^{N-1}\frac{i}{i!}2^{i-1}K_M(\mu)\left[\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^i+M^i\right] \\
&+\sum_{i=1}^{N-1}\frac{i}{i!}2^{i-1}(1+M)K_M(\mu)\left[\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^i+M^i\right] \\
\leq\ & \sum_{i=1}^{N-1}\frac{i}{i!}2^{i-1}(1+M)K_M(\mu)\left(2+M^{N-1}\right)\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=0}^{N-1}\frac{1}{i!}2^{i-1}\left(1+3M+M^2\right)K_M(\mu)\left(2+M^{N-1}\right)\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right]\\
&+\sum_{i=2}^{N-1}\frac{i}{i!}(i-1)2^{i-1}K_M(\mu)\left(2+M^{N-1}\right)\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right]\\
&+\sum_{i=1}^{N-1}\frac{i}{i!}2^{i-1}K_M(\mu)\left(2+M^{N-1}\right)\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right]\\
&+\sum_{i=1}^{N-1}\frac{i}{i!}2^{i-1}(1+M)K_M(\mu)\left(2+M^{N-1}\right)\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right]\\
=\;&\left[\sum_{i=0}^{N-1}\frac{2^{i-1}}{i!}\left(1+3M+M^2\right)+\sum_{i=0}^{N-2}\frac{2^i}{i!}(3+2M)+2\sum_{i=0}^{N-3}\frac{2^i}{i!}\right]\left(2+M^{N-1}\right)K_M(\mu)\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right]\\
=\;&\left(\bar{a}_*M^2+\bar{b}_*M+\bar{c}_*\right)\left(2+M^{N-1}\right)K_M(\mu)\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right]\\
=\;&\bar{C}_*(M)K_M(\mu)\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right],
\end{aligned}
$$

(vii), (viii) Evaluating $\left\|Q_m^{(k)}(t)\right\|_{L^\infty}$ and $\left\|Q_{mx}^{(k)}(t)\right\|$. Replacing $K_M(\mu)$ by $K_M(\bar{\mu})$ in the above estimations of $\left\|P_m^{(k)}(t)\right\|_{L^\infty}$ and $\left\|P_{mx}^{(k)}(t)\right\|$, we get that

$$
\begin{aligned}
\left\|Q_m^{(k)}(t)\right\|_{L^\infty} &\le \bar{C}_*(M)K_M(\bar{\mu})\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right],\\
\left\|Q_{mx}^{(k)}(t)\right\| &\le \bar{C}_*(M)K_M(\bar{\mu})\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right].
\end{aligned}
$$

Lemma 3.4 is proved completely. □

Now, we continue to estimate $I_7 - I_{11}$ of (3.8).

Using Lemma 3.4-(vii), (viii) and the inequality $\|u_{mx}^{(k)}(t)\| + \|\Delta u_m^{(k)}(t)\| + \|\Delta \dot{u}_m^{(k)}(t)\| \le \sqrt{3}\sqrt{\bar{S}_m^{(k)}(t)}$, we have

$$
\begin{aligned}
I_7 &= 2\int_0^t g(t-s)\left[\left\langle Q_m^{(k)}(s), u_{mx}^{(k)}(t)\right\rangle + \left\langle Q_{mx}^{(k)}(s), \Delta u_m^{(k)}(t)+\Delta\dot{u}_m^{(k)}(t)\right\rangle\right]ds\\
&\le 2\sqrt{3}\bar{C}_*(M)K_M(\bar{\mu})\int_0^t \left|g(t-s)\right|\left[1+\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)^{N-1}\right]\sqrt{\bar{S}_m^{(k)}(t)}ds\\
&\le \beta\bar{S}_m^{(k)}(t)+\frac{3}{\beta}\bar{C}_*^2(M)K_M^2(\bar{\mu})\left(\int_0^t\left|g(t-s)\right|\left[1+\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)^{N-1}\right]ds\right)^2\\
&\le \beta\bar{S}_m^{(k)}(t)+\frac{6}{\beta}\bar{C}_*^2(M)K_M^2(\bar{\mu})\left\|g\right\|_{L^2(0,T^*)}^2\int_0^t\left[1+\left(\bar{S}_m^{(k)}(s)\right)^{N-1}\right]ds\\
&= \beta\bar{S}_m^{(k)}(t)+\int_0^t \chi_7\left(\bar{S}_m^{(k)}(s)\right)ds,
\end{aligned}
\tag{3.17}
$$

where $\chi_7(z) = \frac{6}{\beta}\bar{C}_*^2(M)K_M^2(\bar{\mu})\left\|g\right\|_{L^2(0,T^*)}^2\left(1+z^{N-1}\right)$.

$$
\begin{aligned}
I_8 &= -2\int_0^t d\tau\int_0^\tau g'(\tau-s)\left[\left\langle Q_m^{(k)}(s),u_{mx}^{(k)}(\tau)\right\rangle + \left\langle Q_{mx}^{(k)}(s),\Delta u_m^{(k)}(\tau)+\Delta\dot{u}_m^{(k)}(\tau)\right\rangle\right]ds \\
&\leq 2\sqrt{3}\bar{C}_*(M)K_M(\bar{\mu})\int_0^t d\tau\int_0^\tau\left|g'(\tau-s)\right|\left[1+\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)^{N-1}\right]\sqrt{\bar{S}_m^{(k)}(\tau)}ds \\
&\leq \sqrt{3}\bar{C}_*(M)K_M(\bar{\mu})\int_0^t\left[\bar{S}_m^{(k)}(\tau)+\left(\int_0^\tau\left|g'(\tau-s)\right|\left(1+\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)^{N-1}\right)ds\right)^2\right]d\tau \\
&\leq \sqrt{3}\bar{C}_*(M)K_M(\bar{\mu})\int_0^t\left[\bar{S}_m^{(k)}(\tau)+2\left\|g'\right\|_{L^2(0,T^*)}^2\int_0^\tau\left(1+\left(\bar{S}_m^{(k)}(s)\right)^{N-1}\right)ds\right]d\tau \\
&\leq \sqrt{3}\bar{C}_*(M)K_M(\bar{\mu})\int_0^t\left[\bar{S}_m^{(k)}(s)+2T^*\left\|g'\right\|_{L^2(0,T^*)}^2\left(1+\left(\bar{S}_m^{(k)}(s)\right)^{N-1}\right)\right]ds \\
&= \int_0^t\chi_8\left(\bar{S}_m^{(k)}(s)\right)ds,
\end{aligned}
\tag{3.18}
$$

where $\chi_8(z) = \sqrt{3}\bar{C}_*(M)K_M(\bar{\mu})\left[z+2T^*\left\|g'\right\|_{L^2(0,T^*)}^2\left(1+z^{N-1}\right)\right]$.

$$
\begin{aligned}
I_9 &= -2g(0)\int_0^t\left[\left\langle Q_m^{(k)}(s),u_{mx}^{(k)}(s)\right\rangle+\left\langle Q_{mx}^{(k)}(s),\Delta u_m^{(k)}(s)+\Delta\dot{u}_m^{(k)}(s)\right\rangle\right]ds \\
&\leq 2\sqrt{3}\left|g(0)\right|\bar{C}_*(M)K_M(\bar{\mu})\int_0^t\left[1+\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)^{N-1}\right]\sqrt{\bar{S}_m^{(k)}(s)}ds \\
&= \int_0^t\chi_9\left(\bar{S}_m^{(k)}(s)\right)ds,
\end{aligned}
\tag{3.19}
$$

where $\chi_9(z)\,ds = 2\sqrt{3}\left|g(0)\right|\bar{C}_*(M)K_M(\bar{\mu})\left(\sqrt{z}+\sqrt{z^N}\right)$.

Using Lemma 3.4-(vii), (viii) and the inequality $\left\|\dot{u}_m^{(k)}(t)\right\|+\left\|\dot{u}_{mx}^{(k)}(t)\right\| \leq \sqrt{2}\sqrt{\bar{S}_m^{(k)}(t)}$, we get that

$$
\begin{aligned}
I_{10} &= 2\int_0^t\left[\left\langle F_m^{(k)}(s),\dot{u}_m^{(k)}(s)\right\rangle+\left\langle F_{mx}^{(k)}(s),\dot{u}_{mx}^{(k)}(s)+\ddot{u}_{mx}^{(k)}(s)\right\rangle\right]ds \\
&\leq 2\bar{C}_*(M)\tilde{K}_M(f)\int_0^t\left[1+\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)^{N-1}\right]\left[\left\|\dot{u}_m^{(k)}(s)\right\|+\left\|\dot{u}_{mx}^{(k)}(s)\right\|+\left\|\ddot{u}_{mx}^{(k)}(s)\right\|\right]ds \\
&= 2\bar{C}_*(M)\tilde{K}_M(f)\int_0^t\left[1+\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)^{N-1}\right]\left[\left\|\dot{u}_m^{(k)}(s)\right\|+\left\|\dot{u}_{mx}^{(k)}(s)\right\|\right]ds \\
&+ 2\bar{C}_*(M)\tilde{K}_M(f)\int_0^t\left[1+\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)^{N-1}\right]\left\|\ddot{u}_{mx}^{(k)}(s)\right\|ds
\end{aligned}
$$

$$\begin{aligned}
&\leq 2\sqrt{2}\bar{C}_*(M)\tilde{K}_M(f)\int_0^t \left[1+\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)^{N-1}\right]\sqrt{\bar{S}_m^{(k)}(s)}ds \\
&+\frac{2}{\beta}\bar{C}_*^2(M)\tilde{K}_M^2(f)\int_0^t \left[1+\left(\bar{S}_m^{(k)}(s)\right)^{N-1}\right]ds+\beta\int_0^t \left\|\ddot{u}_{mx}^{(k)}(s)\right\|^2 ds \\
&\leq \beta\bar{S}_m^{(k)}(t)+2\sqrt{2}\bar{C}_*(M)\tilde{K}_M(f)\int_0^t \left[\sqrt{\bar{S}_m^{(k)}(s)}+\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)^{N}\right]ds \\
&+\frac{2}{\beta}\bar{C}_*^2(M)\tilde{K}_M^2(f)\int_0^t \left[1+\left(\bar{S}_m^{(k)}(s)\right)^{N-1}\right]ds \\
&\equiv \beta\bar{S}_m^{(k)}(t)+\int_0^t \chi_{10}\left(\bar{S}_m^{(k)}(s)\right)ds,
\end{aligned} \tag{3.20}$$

where $\chi_{10}(z) = 2\sqrt{2}\bar{C}_*(M)\tilde{K}_M(f)\left(\sqrt{z}+\sqrt{z^N}\right)+\frac{2}{\beta}\bar{C}_*^2(M)\tilde{K}_M^2(f)\left(1+z^{N-1}\right)$.

Rewriting I_{11} by

$$\begin{aligned}
I_{11} &= -2\int_0^t \left[\left\langle P_m^{(k)}(s),\dot{u}_{mx}^{(k)}(s)\right\rangle+\left\langle P_{mx}^{(k)}(s),\Delta\dot{u}_m^{(k)}(s)+\Delta\ddot{u}_m^{(k)}(s)\right\rangle\right]ds \\
&= -2\int_0^t \left[\left\langle P_m^{(k)}(s),\dot{u}_{mx}^{(k)}(s)\right\rangle+\left\langle P_{mx}^{(k)}(s),\Delta\dot{u}_m^{(k)}(s)\right\rangle\right]ds-2\int_0^t \left\langle P_{mx}^{(k)}(s),\Delta\ddot{u}_m^{(k)}(s)\right\rangle ds \\
&= I_{11}^{(1)}+I_{11}^{(2)}.
\end{aligned}$$

Using Lemma 3.4-(vii), (viii) and the inequality $\|\dot{u}_{mx}^{(k)}(t)\|+\|\Delta\dot{u}_m^{(k)}(t)\|\leq\sqrt{2}\sqrt{\bar{S}_m^{(k)}(t)}$, we have

$$\begin{aligned}
I_{11}^{(1)} &= -2\int_0^t \left[\left\langle P_m^{(k)}(s),\dot{u}_{mx}^{(k)}(s)\right\rangle+\left\langle P_{mx}^{(k)}(s),\Delta\dot{u}_m^{(k)}(s)\right\rangle\right]ds \\
&\leq 2\sqrt{2}\bar{C}_*(M)K_M(\mu)\int_0^t \left[1+\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)^{N-1}\right]\sqrt{\bar{S}_m^{(k)}(s)}ds.
\end{aligned}$$

Integrating $I_{11}^{(2)}$ by part, we obtain

$$\begin{aligned}
I_{11}^{(2)} &= -2\int_0^t \left[\left\langle P_{mx}^{(k)}(s),\Delta\ddot{u}_m^{(k)}(s)\right\rangle\right]ds \\
&= 2\left\langle P_{mx}^{(k)}(0),\Delta\tilde{u}_{1k}\right\rangle-2\left\langle P_{mx}^{(k)}(t),\Delta\dot{u}_m^{(k)}(t)\right\rangle+2\int_0^t \left\langle \dot{P}_{mx}^{(k)}(s),\Delta\dot{u}_m^{(k)}(s)\right\rangle ds.
\end{aligned}$$

Using Lemma 3.4-(vi), we get

$$\begin{aligned}
\left\|P_{mx}^{(k)}(t)\right\| &\leq \left\|P_{mx}^{(k)}(0)\right\|+\int_0^t \left\|\dot{P}_{mx}^{(k)}(s)\right\|ds \\
&\leq \left\|P_{mx}^{(k)}(0)\right\|+\bar{C}_*(M)K_M(\mu)\int_0^t \left[1+\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)^{N-1}\right]ds,
\end{aligned}$$

then

$$\begin{aligned}
-2\left\langle P_{mx}^{(k)}(t),\Delta\dot{u}_m^{(k)}(t)\right\rangle &\leq \beta\left\|\Delta\dot{u}_m^{(k)}(t)\right\|^2+\frac{1}{\beta}\left\|P_{mx}^{(k)}(t)\right\|^2 \\
&\leq \beta\bar{S}_m^{(k)}(t)+\frac{2}{\beta}\left[\left\|P_{mx}^{(k)}(0)\right\|^2+T^*\bar{C}_*^2(M)K_M^2(\mu)\int_0^t \left[1+\left(\bar{S}_m^{(k)}(s)\right)^{N-1}\right]ds\right],
\end{aligned}$$

and

$$2\int_0^t \left\langle \dot{P}_{mx}^{(k)}(s), \Delta \dot{u}_m^{(k)}(s)\right\rangle ds \le 2\bar{C}_*(M)K_M(\mu)\int_0^t \left[1+\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)^{N-1}\right]\sqrt{\bar{S}_m^{(k)}(s)}ds.$$

Therefore

$$\begin{aligned} I_{11}^{(2)} &\le 2\left\langle P_{mx}^{(k)}(0), \Delta\tilde{u}_{1k}\right\rangle + \beta\bar{S}_m^{(k)}(t) \\ &+\frac{2}{\beta}\left[\left\|P_{mx}^{(k)}(0)\right\|^2 + T^*\bar{C}_*^2(M)K_M^2(\mu)\int_0^t\left[1+\left(\bar{S}_m^{(k)}(s)\right)^{N-1}\right]ds\right] \\ &+2\bar{C}_*(M)K_M(\mu)\int_0^t\left[1+\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)^{N-1}\right]\sqrt{\bar{S}_m^{(k)}(s)}ds. \end{aligned}$$

By the above estimations of $I_{11}^{(1)}$ and $I_{11}^{(2)}$, we deduce that

$$\begin{aligned} I_{11} &\le 2\left\langle P_{mx}^{(k)}(0), \Delta\tilde{u}_{1k}\right\rangle + \beta\bar{S}_m^{(k)}(t) + \frac{2}{\beta}\left\|P_{mx}^{(k)}(0)\right\|^2 \\ &+\frac{2}{\beta}T^*\bar{C}_*^2(M)K_M^2(\mu)\int_0^t\left[1+\left(\bar{S}_m^{(k)}(s)\right)^{N-1}\right]ds \\ &+2(1+\sqrt{2})\bar{C}_*(M)K_M(\mu)\int_0^t\left[\sqrt{\bar{S}_m^{(k)}(s)}+\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)^{N}\right]ds \\ &\le 2\left\langle P_{mx}^{(k)}(0), \Delta\tilde{u}_{1k}\right\rangle + \beta\bar{S}_m^{(k)}(t) + \frac{2}{\beta}\left\|P_{mx}^{(k)}(0)\right\|^2 + \int_0^t \chi_{11}\left(\bar{S}_m^{(k)}(s)\right)ds, \end{aligned} \tag{3.21}$$

where $\chi_{11}(z) = \frac{2}{\beta}T^*\bar{C}_*^2(M)K_M^2(\mu)\left(1+z^{N-1}\right) + 2(1+\sqrt{2})\bar{C}_*(M)K_M(\mu)\left(\sqrt{z}+\sqrt{z^N}\right)$.

By the estimations (3.10)-(3.21), we get from (3.8) and (3.9) that

$$\bar{S}_m^{(k)}(t) \le \bar{S}_{0mk} + \int_0^t \chi\left(\bar{S}_m^{(k)}(s)\right)ds, \tag{3.22}$$

where

$$\begin{aligned} \chi(z) &= \frac{2}{\bar{\mu}_*}\sum_{i=1}^{11}\chi_i(z), \\ \bar{S}_{0mk} &= \frac{2}{\bar{\mu}_*}S_m^{(k)}(0) + \frac{4}{\bar{\mu}_*}\langle\frac{\partial}{\partial x}\left[D_3\mu(0,\tilde{u}_{0k})\tilde{u}_{0kx}\right], \Delta\tilde{u}_{1k}\rangle + \frac{4}{\bar{\mu}_*}\left\langle P_{mx}^{(k)}(0), \Delta\tilde{u}_{1k}\right\rangle + \frac{40}{\bar{\mu}_*^2}\left\|P_{mx}^{(k)}(0)\right\|^2 \\ &+\frac{4}{\bar{\mu}_*}\left[T^*\left(2+\left|g(0)\right|\right)+\frac{10}{\bar{\mu}_*}\left(1+4T^*\left\|g\right\|_{L^2(0,T^*)}^2\right)\right]\left\|\frac{\partial}{\partial x}\left[D_3\bar{\mu}(0,\tilde{u}_{0k})\tilde{u}_{0kx}\right]\right\|^2. \end{aligned}$$

We shall check that $\bar{S}_{0mk}$ is bounded by a constant independent of m and k.

Note that, due to $A_m^{(k)}(0) = D_3\mu(x,0,\tilde{u}_{0k}(x))$ is independent of m and

$$\begin{aligned} S_m^{(k)}(0) &= \|\tilde{u}_{1k}\|^2 + \|\tilde{u}_{1kx}\|^2 + \left\|\sqrt{A_m^{(k)}(0)}\tilde{u}_{0kx}\right\|^2 + \left\|\sqrt{A_m^{(k)}(0)}\Delta\tilde{u}_{0k}\right\|^2 + \lambda\|\Delta\tilde{u}_{1k}\|^2, \\ P_{mx}^{(k)}(x,0) &= D_1D_1\mu(x,0,\tilde{u}_0(x)) + D_3D_1\mu(x,0,\tilde{u}_0(x))\tilde{u}_{0x}(x) \\ &+\sum_{i=1}^{N-1}\frac{1}{i!}\left[D_1D_3^iD_1\mu(x,0,\tilde{u}_0(x)) + D_3^{i+1}D_1\mu(x,0,\tilde{u}_0(x))\tilde{u}_{0x}(x)\right](\tilde{u}_{0k}(x)-\tilde{u}_0(x))^i \\ &+\sum_{i=1}^{N-1}\frac{i}{i!}D_3^iD_1\mu(x,0,\tilde{u}_0(x))(\tilde{u}_{0k}(x)-\tilde{u}_0(x))^{i-1}(\tilde{u}_{0kx}(x)-\tilde{u}_{0x}(x)), \end{aligned}$$

so $S_m^{(k)}(0)$ and $P_{mx}^{(k)}(0)$, too.

This shows that there exists a constant $M > 0$ independent of k and m such that

$$\bar{S}_{0mk} \leq \frac{M^2}{2}, \text{ for all } m,\ k \in \mathbb{N}. \tag{3.23}$$

Lemma 3.5. *There is a constant $T > 0$ such that*

$$\bar{S}_m^{(k)}(t) \leq M^2, \textit{ for all } t \in [0, T],\ \forall m,\ k \in \mathbb{N}. \tag{3.24}$$

Proof. From (3.22) and (3.23), we have

$$\bar{S}_m^{(k)}(t) \leq \frac{M^2}{2} + \int_0^t \chi\left(\bar{S}_m^{(k)}(s)\right) ds.$$

By setting $y(t) = \frac{M^2}{2} + \int_0^t \chi\left(\bar{S}_m^{(k)}(s)\right) ds$, and χ is a continuous and strictly increasing function, we get that

$$\begin{aligned} 0 &\leq \bar{S}_m^{(k)}(t) \leq y(t), y(0) = \frac{M^2}{2}, \\ y'(t) &= \chi\left(\bar{S}_m^{(k)}(t)\right) \leq \chi\left(y(t)\right). \end{aligned}$$

From the above inequality, we obtain

$$H(y(t)) - H(\frac{M^2}{2}) = \int_{\frac{M^2}{2}}^{y(t)} \frac{dz}{\chi(z)} = \int_0^t \frac{y'(s)ds}{\chi(y(s))} \leq t,$$

where $H(y) = \int_0^y \dfrac{dz}{\chi(z)}$ is a continuous and strictly increasing function on $\mathbb{R}_+$.

We shall prove that $H(\infty) = \int_0^\infty \dfrac{dz}{\chi(z)}$ is a convergent integral.

Indeed, by the definitions of $\psi_1(z),\ \chi_3(z),\ \chi_{11}(z)$ as in Lemma 3.3, (3.13) and (3.21) respectively, we have

$$\begin{aligned} \psi_1(z) &= \left(1 + (1 + 2\sqrt{2})z + \sqrt{2}z^2\right) \Phi_\mu^{[3]}(z) \\ &\geq \Phi_\mu^{[3]}(z) z^2 \geq \mu_* z^2; \chi_3(z) = 4\left(1 + \frac{20}{\bar{\mu}_*} T^* \psi_1\left(\sqrt{z}\right)\right) z \psi_1\left(\sqrt{z}\right) \\ &\geq \frac{80}{\bar{\mu}_*} T^* z \psi_1^2\left(\sqrt{z}\right) \geq \frac{80}{\bar{\mu}_*} T^* z \left(\mu_* z\right)^2 \geq 80 T^* \mu_* z^3; \end{aligned}$$

$$\begin{aligned} \chi_{11}(z) &= \frac{20}{\bar{\mu}_*} T^* \bar{C}_*^2(M) K_M^2(\mu) \left(1 + z^{N-1}\right) + 2(1 + \sqrt{2}) \bar{C}_*(M) K_M(\mu) \left(\sqrt{z} + \sqrt{z^N}\right) \\ &\geq \frac{20}{\bar{\mu}_*} T^* \bar{C}_*^2(M) K_M^2(\mu) \geq \frac{20}{\bar{\mu}_*} T^* \left(2\bar{c}_*\right)^2 \mu_*^2 \geq 80 T^* \bar{c}_*^2 > 0. \end{aligned}$$

So

$$\begin{aligned} \chi(z) &\geq \frac{2}{\bar{\mu}_*} \left(\chi_3(z) + \chi_{11}(z)\right) \geq \frac{2}{\bar{\mu}_*} \left(80 T^* \mu_* z^3 + 80 T^* \bar{c}_*^2\right) \\ &= 160 T^* \left(\frac{\mu_*}{\bar{\mu}_*} z^3 + \frac{\bar{c}_*^2}{\bar{\mu}_*}\right) \geq 160 T^* \left(z^3 + \bar{c}_*^2\right), \end{aligned}$$

with $\bar{c}_* > 0$.

On the other hand, because $\int_0^\infty \frac{dz}{z^3+\bar{c}_*^2}$ is convergent, then $H(\infty) = \int_0^\infty \frac{dz}{\chi(z)}$ is, too.

By this, we have that $H : \mathbb{R}_+ \to [0, H(\infty))$ is a continuous bijection, then $H^{-1} : [0, H(\infty)) \to \mathbb{R}_+$ is also continuous and strictly increasing.

Due to the fact that $H(M^2) - H(\frac{M^2}{2}) > 0$, we can choose T such that

$$0 < T < H(M^2) - H(\frac{M^2}{2}), \tag{3.25}$$

and

$$k_T = M\hat{\mu}_T^{\frac{-1}{N-1}} < 1, \tag{3.26}$$

where

$$\begin{aligned}
\hat{\mu}_T &= 3D_1^*(M)\sqrt{T}\exp\left(TD_2^*(M)\right),\\
D_1^*(M) &= \frac{1}{\bar{\mu}_*}\bar{d}_M(f) + \frac{40}{\bar{\mu}_*^2}\left(\bar{d}_M^2(\mu) + \bar{d}_M^2(\bar{\mu})T^*\left\|g\right\|^2_{L^2(0,T^*)}\right),\\
D_2^*(M) &= \frac{1}{\bar{\mu}_*}K_M(\mu)(1+M) + \frac{10}{\bar{\mu}_*^2}\left(M^2K_M^2(\mu) + \left(1+M^2\right)K_M^2(\bar{\mu})T^*\left\|g\right\|^2_{L^2(0,T^*)}\right)\\
&\quad + \frac{4}{\bar{\mu}_*}\bar{d}_M(f) + \frac{20}{\bar{\mu}_*^2}\left(\bar{d}_M^2(\mu) + \bar{d}_M^2(\bar{\mu})T^*\left\|g\right\|^2_{L^2(0,T^*)}\right).
\end{aligned}$$

By (3.25), for all $t \in [0, T]$, we have

$$0 \le H(y(t)) \le t + H(\frac{M^2}{2}) \le T + H(\frac{M^2}{2}) < H(M^2) < H(\infty).$$

Due to $H^{-1} : [0, H(\infty)) \to \mathbb{R}_+$ is strictly increasing, we get

$$\bar{S}_m^{(k)}(t) \le y(t) = H^{-1}(H(y(t))) \le H^{-1}\left(H(M^2)\right) = M^2.$$

Lemma 3.5 is proved. □

Finally, by using some argument of compactness, we shall prove the convergence of the Faedo-Galerkin approximation solution $u_m^{(k)}$. This is presented as follows.

Because of the boundness of $\bar{S}_m^{(k)}(t)$ given by (3.24), there exists a subsequence of $\{u_m^{(k)}\}$ still denoted by the same symbol such that

$$\begin{cases}
u_m^{(k)} \to u_m & \text{in} \quad L^\infty(0,T;H^2\cap H_0^1) \text{ weak*},\\
\dot{u}_m^{(k)} \to u_m' & \text{in} \quad L^\infty(0,T;H^2\cap H_0^1) \text{ weak*},\\
\ddot{u}_m^{(k)} \to u_m'' & \text{in} \quad L^2(0,T;H_0^1) \text{ weak},\\
u_m \in W(M,T).
\end{cases} \tag{3.27}$$

Applying the compactness lemma of Aubin-Lions, we can deduce from $(3.27)_{1,2,3}$ that there exists a subsequence still denoted by $\{u_m^{(k)}\}$ such that

$$\begin{cases}
u_m^{(k)} \to u_m & \text{strongly in} \quad C^0\left(0,T;H_0^1\right),\\
\dot{u}_m^{(k)} \to u_m' & \text{strongly in} \quad C^0\left(0,T;H_0^1\right).
\end{cases} \tag{3.28}$$

On the other hand, due to the fact that

$$\begin{aligned}
\left|A_m^{(k)}(x,t) - D_3\mu(x,t,u_m(x,t))\right| &= \left|D_3\mu(x,t,u_m^{(k)}(x,t)) - D_3\mu(x,t,u_m(x,t))\right|\\
&\le K_M(\mu)\left|u_m^{(k)}(x,t) - u_m(x,t)\right| \le K_M(\mu)\left\|u_{mx}^{(k)}(t) - u_{mx}(t)\right\|\\
&\le K_M(\mu)\left\|u_m^{(k)} - u_m\right\|_{C^0\left(0,T;H_0^1\right)},
\end{aligned}$$

and $(3.28)_1$, we get

$$A_m^{(k)} \to D_3\mu(\cdot,\cdot,u_m) \text{ strongly in } L^\infty(Q_T). \tag{3.29}$$

Similarly

$$B_m^{(k)} \to D_3\bar{\mu}(\cdot,\cdot,u_m) \text{ strongly in } L^\infty(Q_T). \tag{3.30}$$

We also have

$$\begin{aligned}
&\left|F_m^{(k)}(x,t) - P_m[f,u_{m-1},u_m](x,t)\right| \\
=\ & \left|P_m[f,u_{m-1},u_m^{(k)}](x,t) - P_m[f,u_{m-1},u_m](x,t)\right| \\
\le\ & \sum_{i=1}^{N-1} \frac{1}{i!} D_3^i f(x,t,u_{m-1}(x,t)) \left|\left(u_m^{(k)}(x,t) - u_{m-1}(x,t)\right)^i - (u_m(x,t) - u_{m-1}(x,t))^i\right| \\
\le\ & \tilde{K}_M(f) \sum_{i=1}^{N-1} \frac{1}{i!} \left|u_m^{(k)}(x,t) - u_m(x,t)\right| \sum_{j=0}^{i-1} \left|\left(u_m^{(k)}(x,t) - u_{m-1}(x,t)\right)^{i-1-j} (u_m(x,t) - u_{m-1}(x,t))^j\right| \\
\le\ & \tilde{K}_M(f) \sum_{i=0}^{N-2} \frac{(2M)^i}{i!} \left\|u_m^{(k)} - u_m\right\|_{C^0(0,T;H_0^1)}.
\end{aligned}$$

Hence

$$F_m^{(k)} \to P_m[f,u_{m-1},u_m] \text{ strongly in } L^\infty(Q_T). \tag{3.31}$$

Similarly

$$\begin{aligned}
P_m^{(k)} \to\ & P_m[D_1\mu,u_{m-1},u_m] \text{ strongly in } L^\infty(Q_T), \\
Q_m^{(k)} \to\ & P_m[D_1\bar{\mu},u_{m-1},u_m] \text{ strongly in } L^\infty(Q_T).
\end{aligned} \tag{3.32}$$

Because of the convergences given in (3.27), (3.29)-(3.32), and passing the limit as $k \to \infty$ in (3.5), we have that u_m satisfies

$$\begin{cases}
\langle u_m''(t),v\rangle + \lambda\langle u_{mx}'(t),v_x\rangle + \langle D_3\mu(t,u_m(t))u_{mx}(t),v_x\rangle \\
\qquad = \int_0^t g(t-s)\langle D_3\bar{\mu}(s,u_m(s))u_{mx}(s),v_x\rangle ds \\
\qquad\quad + \int_0^t g(t-s)\langle P_m[D_1\bar{\mu},u_{m-1},u_m](s),v_x\rangle ds \\
\qquad\quad - \langle P_m[D_1\mu,u_{m-1},u_m](t),v_x\rangle + \langle P_m[f,u_{m-1},u_m](t),v\rangle,\ \forall v \in H_0^1, \\
u_m(0) = \tilde{u}_0,\ u_m'(0) = \tilde{u}_1.
\end{cases} \tag{3.33}$$

On the other hand, we deduce from $(3.27)_4$ and $(3.33)_1$ that

$$\begin{aligned}
u_m'' =\ & \lambda u_{mxx}' + \frac{\partial}{\partial x}\left[D_3\mu(t,u_m(t))u_{mx}(t) + P_m[D_1\mu,u_{m-1},u_m](t)\right] \\
& - \int_0^t g(t-s)\frac{\partial}{\partial x}\left[D_3\bar{\mu}(s,u_m(s))u_{mx}(s) + P_m[D_1\bar{\mu},u_{m-1},u_m](s)\right]ds + P_m[f,u_{m-1},u_m](t) \\
\equiv\ & \tilde{F}_m \in L^\infty(0,T;L^2).
\end{aligned}$$

Thus $u_m \in W_1(M,T)$. The existence result follows. Theorem 3.1 is proved. □

4. Convergence of a N-order iterative scheme

In this section, by using Theorem 3.1 and the compact imbedding theorems, we prove the existence and uniqueness of weak local solution to the problem (1.1). First, we introduce the Banach space (see Lions [8])

$$W_1(T) = \left\{u \in C^0([0,T];H_0^1) \cap C^1([0,T];L^2) : u' \in L^2(0,T;H_0^1)\right\},$$

with respect to the norm $\|u\|_{W_1(T)} = \|u\|_{C^0([0,T];H_0^1)} + \|u'\|_{C^0(0,T;L^2)} + \|u'\|_{L^2(0,T;H_0^1)}$.

Then, we have the following theorem.

Theorem 4.1. *Suppose $(H_1)-(H_4)$ hold. Then, there exist constants $M>0$ and $T>0$ such that the problem (1.1) admits a unique weak solution $u \in W_1(M,T)$ and the recurrent sequence $\{u_m\}$ obtained by Theorem 3.1 strongly converges at $N-$order rate to u in $W_1(T)$ in sense of*

$$\|u_m - u\|_{W_1(T)} \leq C\,\|u_{m-1} - u\|_{W_1(T)}^N, \tag{4.1}$$

for all $m \geq 1$, where C is a suitable constant. Moreover, we have the following estimate

$$\|u_m - u\|_{W_1(T)} \leq C_T\,(k_T)^{N^m},\ \text{for all } m \in \mathbb{N}, \tag{4.2}$$

where C_T and $0<k_T<1$ are the constants depending only on T, f, g, μ, $\bar{\mu}$, $\tilde{u}_0$ and $\tilde{u}_1$.

Proof. First, we prove the local existence of (1.1). It is necessary to show that $\{u_m\}$ (in Theorem 3.1) is a Cauchy sequence in $W_1(T)$. Let $w_m = u_{m+1} - u_m$, then w_m satisfies the variational problem

$$\begin{cases} \langle w_m''(t), v\rangle + \lambda\langle w_{mx}'(t), v_x\rangle + \langle D_3\mu(t,u_{m+1}(t))w_{mx}(t), v_x\rangle \\ \qquad = -\langle [D_3\mu(t,u_{m+1}(t)) - D_3\mu(t,u_m(t))]\,u_{mx}(t), v_x\rangle \\ \qquad\quad + \int_0^t g(t-s)\langle D_3\bar{\mu}(s,u_{m+1}(s))w_{mx}(s), v_x\rangle ds \\ \qquad\quad + \int_0^t g(t-s)\langle [D_3\bar{\mu}(s,u_{m+1}(s)) - D_3\bar{\mu}(s,u_m(s))]\,u_{mx}(s), v_x\rangle ds \\ \qquad\quad + \langle P_{m+1}[f,u_m,u_{m+1}](t) - P_m[f,u_{m-1},u_m](t), v\rangle \\ \qquad\quad + \int_0^t g(t-s)\langle P_{m+1}[D_1\bar{\mu},u_m,u_{m+1}](s) - P_m[D_1\bar{\mu},u_{m-1},u_m](s), v_x\rangle ds \\ \qquad\quad - \langle P_{m+1}[D_1\mu,u_m,u_{m+1}](t) - P_m[D_1\mu,u_{m-1},u_m](t), v_x\rangle,\ \forall v \in H_0^1, \\ w_m(0) = w_m'(0) = 0, \end{cases} \tag{4.3}$$

where $P_m[f,u_{m-1},u_m](x,t)$, $P_m[D_1\mu,u_{m-1},u_m](x,t)$, $P_m[D_1\bar{\mu},u_{m-1},u_m](x,t)$ are defined by (3.4).

Taking $v = w_m'(t)$ in $(4.3)_1$ and then integrating in t, we get

$$\begin{aligned} \bar{\mu}_*\bar{S}_m(t) &\leq \int_0^t ds\int_0^1 \frac{\partial}{\partial s}[D_3\mu(x,s,u_{m+1}(x,s))]\,w_{mx}^2(x,s)dx \\ &\quad + 2\int_0^t d\tau\int_0^\tau g(\tau-s)\langle D_3\bar{\mu}(s,u_{m+1}(s))w_{mx}(s), w_{mx}'(\tau)\rangle ds \\ &\quad + 2\int_0^t d\tau\int_0^\tau g(\tau-s)\langle [D_3\bar{\mu}(s,u_{m+1}(s)) - D_3\bar{\mu}(s,u_m(s))]\,u_{mx}(s), w_{mx}'(\tau)\rangle ds \\ &- 2\int_0^t \langle [D_3\mu(s,u_{m+1}(s)) - D_3\mu(s,u_m(s))]\,u_{mx}(s), w_{mx}'(s)\rangle ds \\ &+ 2\int_0^t \langle P_{m+1}[f,u_m,u_{m+1}](s) - P_m[f,u_{m-1},u_m](s), w_m'(s)\rangle ds \\ &+ 2\int_0^t d\tau\int_0^\tau g(\tau-s)\langle P_{m+1}[D_1\bar{\mu},u_m,u_{m+1}](s) - P_m[D_1\bar{\mu},u_{m-1},u_m](s), w_{mx}'(\tau)\rangle ds \\ &- 2\int_0^t \langle P_{m+1}[D_1\mu,u_m,u_{m+1}](s) - P_m[D_1\mu,u_{m-1},u_m](s), w_{mx}'(s)\rangle ds \\ &= \sum_{i=1}^7 J_i, \end{aligned} \tag{4.4}$$

where $\bar{\mu}_* = \min\{1, \mu_*, 2\lambda\}$ and

$$\bar{S}_m(t) = \left\|w_m'(t)\right\|^2 + \|w_{mx}(t)\|^2 + \int_0^t \left\|w_{mx}'(s)\right\|^2 ds. \tag{4.5}$$

Next, we need to estimate the integrals on the right-hand side of (4.4).

For estimating $J_1 - J_7$ of (4.4) below, we always choose $\beta = \frac{1}{10}\bar{\mu}_*$.

By the following inequalities

$$\begin{aligned} \left|D_3\bar{\mu}(x,t,u_{m+1}(x,t))\right| &\leq K_M(\bar{\mu}), \\ \left|\frac{\partial}{\partial t}\left[D_3\mu(x,t,u_{m+1}(x,t))\right]\right| &\leq \left|D_2D_3\mu(x,t,u_{m+1}(x,t))\right| + \left|D_3\mu(x,t,u_{m+1}(x,t))\right|\left|u'_{m+1}(x,t)\right| \\ &\leq K_M(\mu)\left(1+\left\|\nabla u'_{m+1}(t)\right\|\right) \\ &\leq K_M(\mu)(1+M), \left|D_3\mu(x,t,u_{m+1}(x,t)) - D_3\mu(x,t,u_m(x,t))\right| \\ &\leq K_M(\mu)\left|w_m(x,t)\right| \leq K_M(\mu)\left\|w_{mx}(t)\right\|, \end{aligned}$$

$$\left|D_3\bar{\mu}(x,t,u_{m+1}(x,t)) - D_3\bar{\mu}(x,t,u_m(x,t))\right| \leq K_M(\bar{\mu})\left|w_m(x,t)\right| \leq K_M(\bar{\mu})\left\|w_{mx}(t)\right\|,$$

we estimate J_1, J_2, J_3, J_4 as follows

$$\begin{aligned} J_1 &= \int_0^t ds \int_0^1 \frac{\partial}{\partial s}\left[D_3\mu(x,s,u_{m+1}(x,s))\right] w_{mx}^2(x,s)dx \\ &\leq K_M(\mu)(1+M)\int_0^t \left\|w_{mx}(s)\right\|^2 ds \leq K_M(\mu)(1+M)\int_0^t \bar{S}_m(s)ds, \\ J_2 &= 2\int_0^t d\tau \int_0^\tau g(\tau-s)\langle D_3\bar{\mu}(s,u_{m+1}(s))w_{mx}(s), w'_{mx}(\tau)\rangle ds \\ &\leq 2K_M(\bar{\mu})\int_0^t d\tau \int_0^\tau \left|g(\tau-s)\right| \left\|w_{mx}(s)\right\| \left\|w'_{mx}(\tau)\right\| ds \\ &\leq 2K_M(\bar{\mu})\int_0^t \left\|w'_{mx}(\tau)\right\| d\tau \int_0^\tau \left|g(\tau-s)\right| \sqrt{\bar{S}_m(s)}ds \\ &\leq 2K_M(\bar{\mu})\sqrt{T^*}\left\|g\right\|_{L^2(0,T^*)} \sqrt{\bar{S}_m(t)}\left[\int_0^t \bar{S}_m(s)ds\right]^{1/2} \\ &\leq \beta\bar{S}_m(t) + \frac{1}{\beta}K_M^2(\bar{\mu})T^*\left\|g\right\|^2_{L^2(0,T^*)}\int_0^t \bar{S}_m(s)ds, \\ J_3 &= 2\int_0^t d\tau \int_0^\tau g(\tau-s)\langle\left[D_3\bar{\mu}(s,u_{m+1}(s)) - D_3\bar{\mu}(s,u_m(s))\right]u_{mx}(s), w'_{mx}(\tau)\rangle ds \\ &\leq 2K_M(\bar{\mu})\int_0^t d\tau \int_0^\tau \left|g(\tau-s)\right| \left\|w_{mx}(s)\right\| \left\|u_{mx}(s)\right\| \left\|w'_{mx}(\tau)\right\| ds \\ &\leq 2MK_M(\bar{\mu})\sqrt{T^*}\left\|g\right\|_{L^2(0,T^*)} \sqrt{\bar{S}_m(t)}\left[\int_0^t \bar{S}_m(s)ds\right]^{1/2} \\ &\leq \beta\bar{S}_m(t) + \frac{1}{\beta}M^2K_M^2(\bar{\mu})T^*\left\|g\right\|^2_{L^2(0,T^*)}\int_0^t \bar{S}_m(s)ds, \\ J_4 &= -2\int_0^t \langle\left[D_3\mu(s,u_{m+1}(s)) - D_3\mu(s,u_m(s))\right]u_{mx}(s), w'_{mx}(s)\rangle ds \\ &\leq 2K_M(\mu)\int_0^t \left\|w_{mx}(s)\right\| \left\|u_{mx}(s)\right\| \left\|w'_{mx}(s)\right\| ds \\ &\leq \beta\int_0^t \left\|w'_{mx}(s)\right\|^2 ds + \frac{1}{\beta}M^2K_M^2(\mu)\int_0^t \left\|w_{mx}(s)\right\|^2 ds \\ &\leq \beta\bar{S}_m(t) + \frac{1}{\beta}M^2K_M^2(\mu)\int_0^t \bar{S}_m(s)ds. \end{aligned} \tag{4.6}$$

For J_5, J_6, J_7, we shall need the following lemma.
Lemma 4.2. *We have*

$$\begin{aligned}
&\text{(i)} && \left\|P_{m+1}[f,u_m,u_{m+1}](t)-P_m[f,u_{m-1},u_m](t)\right\| &\le& \bar{d}_M(f)\left(\sqrt{\bar{S}_m(t)}+\|w_{m-1}\|^N_{W_1(T)}\right),\\
&\text{(ii)} && \left\|P_m[D_1\mu,u_m,u_{m+1}](t)-P_m[D_1\mu,u_{m-1},u_m](t)\right\| &\le& \bar{d}_M(\mu)\left(\sqrt{\bar{S}_m(t)}+\|w_{m-1}\|^N_{W_1(T)}\right),\\
&\text{(iii)} && \left\|P_m[D_1\bar{\mu},u_m,u_{m+1}](t)-P_m[D_1\bar{\mu},u_{m-1},u_m](t)\right\| &\le& \bar{d}_M(\bar{\mu})\left(\sqrt{\bar{S}_m(t)}+\|w_{m-1}\|^N_{W_1(T)}\right),
\end{aligned}$$

where

$$\begin{aligned}
\bar{d}_M(f) &= \tilde{K}_M(f)\rho(M),\ \bar{d}_M(\mu)=K_M(\mu)\rho(M),\ \bar{d}_M(\bar{\mu})=K_M(\bar{\mu})\rho(M),\\
\rho(M) &= \sum\nolimits_{i=1}^{N-1}\frac{M^{i-1}}{i!}+\frac{1}{N!}.
\end{aligned}$$

Proof. (i) We note that

$$\begin{aligned}
&P_{m+1}[f,u_m,u_{m+1}](x,t)-P_m[f,u_{m-1},u_m](x,t)\\
=\ &\sum\nolimits_{i=0}^{N-1}\frac{1}{i!}D_3^if(x,t,u_m(x,t))w_m^i(x,t)-\sum\nolimits_{i=0}^{N-1}\frac{1}{i!}D_3^if(x,t,u_{m-1}(x,t))w_{m-1}^i(x,t)\\
=\ &\sum\nolimits_{i=1}^{N-1}\frac{1}{i!}D_3^if(x,t,u_m(x,t))w_m^i(x,t)+f(x,t,u_m(x,t))-f(x,t,u_{m-1}(x,t))\\
&-\sum\nolimits_{i=1}^{N-1}\frac{1}{i!}D_3^if(x,t,u_{m-1}(x,t))w_{m-1}^i(x,t).
\end{aligned}$$

Using Taylor's expansion of the function $f(x,t,u_m(x,t))=f(x,t,u_{m-1}(x,t)+w_{m-1}(x,t))$ around the point $u_{m-1}(x,t)$ up to order N, we obtain

$$\begin{aligned}
f(x,t,u_m(x,t))-f(x,t,u_{m-1}(x,t)) &= \sum\nolimits_{i=1}^{N-1}\frac{1}{i!}D_3^if(x,t,u_{m-1}(x,t))w_{m-1}^i(x,t)\\
&\quad+\frac{1}{N!}D_3^Nf(x,t,\tilde{u}_m)w_{m-1}^N(x,t),
\end{aligned}$$

where $\tilde{u}_m=u_{m-1}(x,t)+\theta w_{m-1}(x,t)$, $0<\theta<1$.
This leads to

$$\begin{aligned}
&P_{m+1}[f,u_m,u_{m+1}](x,t)-P_m[f,u_{m-1},u_m](x,t)\\
=\ &\sum\nolimits_{i=1}^{N-1}\frac{1}{i!}D_3^if(x,t,u_m(x,t))w_m^i(x,t)+\frac{1}{N!}D_3^Nf(x,t,\tilde{u}_m)w_{m-1}^N(x,t),
\end{aligned}$$

therefore, we have

$$\begin{aligned}
&\left\|P_{m+1}[f,u_m,u_{m+1}](t)-P_m[f,u_{m-1},u_m](t)\right\|\\
\le\ &\tilde{K}_M(f)\left(\sum\nolimits_{i=1}^{N-1}\frac{1}{i!}\|w_{mx}(t)\|^i+\frac{1}{N!}\|\nabla w_{m-1}(t)\|^N\right)\\
\le\ &\tilde{K}_M(f)\left(\sum\nolimits_{i=1}^{N-1}\frac{M^{i-1}}{i!}\sqrt{\bar{S}_m(t)}+\frac{1}{N!}\|w_{m-1}\|^N_{W_1(T)}\right)\\
\le\ &\bar{d}_M(f)\left(\sqrt{\bar{S}_m(t)}+\|w_{m-1}\|^N_{W_1(T)}\right).
\end{aligned}$$

Similarly, we also have (ii), (iii) hold. Lemma 4.2 is proved. □

Using Lemma 4.2, the integrals J_5, J_6, J_7, can be estimated by

$$
\begin{aligned}
J_5 &= 2\int_0^t \langle P_{m+1}[f,u_m,u_{m+1}](s) - P_m[f,u_{m-1},u_m](s), w_m'(s)\rangle ds \\
&\le 2\bar{d}_M(f)\int_0^t \left[\sqrt{\bar{S}_m(s)} + \|w_{m-1}\|_{W_1(T)}^N\right]\sqrt{\bar{S}_m(s)}ds \\
&\le \frac{1}{2}T\bar{d}_M(f)\,\|w_{m-1}\|_{W_1(T)}^{2N} + 4\bar{d}_M(f)\int_0^t \bar{S}_m(s)ds; \\
J_6 &= 2\int_0^t d\tau\int_0^\tau g(\tau-s)\langle P_{m+1}[D_1\bar{\mu},u_m,u_{m+1}](s) - P_m[D_1\bar{\mu},u_{m-1},u_m](s), w_{mx}'(\tau)\rangle ds \\
&\le 2\bar{d}_M(\bar{\mu})\int_0^t \left\|w_{mx}'(\tau)\right\|d\tau\int_0^\tau \left|g(\tau-s)\right|\left[\sqrt{\bar{S}_m(s)} + \|w_{m-1}\|_{W_1(T)}^N\right]ds \\
&\le \beta\bar{S}_m(t) + \frac{2}{\beta}\bar{d}_M^2(\bar{\mu})T^*\left\|g\right\|_{L^2(0,T^*)}^2\left[T\,\|w_{m-1}\|_{W_1(T)}^{2N} + \int_0^t \bar{S}_m(s)ds\right]; \\
J_7 &= -2\int_0^t \langle P_{m+1}[D_1\mu,u_m,u_{m+1}](s) - P_m[D_1\mu,u_{m-1},u_m](s), w_{mx}'(s)\rangle ds \\
&\le \beta\int_0^t \left\|w_{mx}'(s)\right\|^2 ds + \frac{1}{\beta}\int_0^t \left\|P_{m+1}[D_1\mu,u_m,u_{m+1}](s) - P_m[D_1\mu,u_{m-1},u_m](s)\right\|^2 ds \\
&\le \beta\bar{S}_m(t) + \frac{2}{\beta}\bar{d}_M^2(\mu)\left[T\,\|w_{m-1}\|_{W_1(T)}^{2N} + \int_0^t \bar{S}_m(s)ds\right].
\end{aligned}
\tag{4.7}
$$

By (4.6), (4.7), it follows from (4.4) that

$$
\bar{S}_m(t) \le TD_1^*(M)\,\|w_{m-1}\|_{W_1(T)}^{2N} + 2D_2^*(M)\int_0^t \bar{S}_m(s)ds, \tag{4.8}
$$

where

$$
\begin{aligned}
D_1^*(M) &= \frac{1}{\bar{\mu}_*}\bar{d}_M(f) + \frac{40}{\bar{\mu}_*^2}\left(\bar{d}_M^2(\mu) + \bar{d}_M^2(\bar{\mu})T^*\left\|g\right\|_{L^2(0,T^*)}^2\right), \\
D_2^*(M) &= \frac{1}{\bar{\mu}_*}K_M(\mu)\,(1+M) + \frac{10}{\bar{\mu}_*^2}\left(M^2K_M^2(\mu) + \left(1+M^2\right)K_M^2(\bar{\mu})T^*\left\|g\right\|_{L^2(0,T^*)}^2\right) \\
&\quad + \frac{4}{\bar{\mu}_*}\bar{d}_M(f) + \frac{20}{\bar{\mu}_*^2}\left(\bar{d}_M^2(\mu) + \bar{d}_M^2(\bar{\mu})T^*\left\|g\right\|_{L^2(0,T^*)}^2\right).
\end{aligned}
$$

Using Gronwall's lemma, we deduce from (4.8) that

$$
\|w_m\|_{W_1(T)} \le \hat{\mu}_T\,\|w_{m-1}\|_{W_1(T)}^N, \tag{4.9}
$$

where $\hat{\mu}_T = 3D_1^*(M)\sqrt{T}\exp\left(TD_2^*(M)\right)$ and $k_T = M\hat{\mu}_T^{\frac{-1}{N-1}} < 1$. This confirms (4.1).
It follows from (4.11) that

$$
\left\|u_m - u_{m+p}\right\|_{W_1(T)} \le (1-\gamma_T)^{-1}(\mu_T)^{\frac{-1}{N-1}}\left(\gamma_T\right)^{N^m}, \text{ for all } m \text{ and } p \in \mathbb{N}. \tag{4.10}
$$

This shows that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Thus, there exists $u \in W_1(T)$ such that

$$
u_m \to u \text{ strongly in } W_1(T). \tag{4.11}
$$

Moreover, by letting $p \to \infty$, we get from (4.10) that (4.2) is valid.

Note that, because of $u_m \in W_1(M,T)$, then there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{cases} u_{m_j} \to u & \text{in} \quad L^\infty(0,T;H^2 \cap H_0^1) \text{ weak*}, \\ u'_{m_j} \to u' & \text{in} \quad L^\infty(0,T;H^2 \cap H_0^1) \text{ weak*}, \\ u''_{m_j} \to u'' & \text{in} \quad L^2(0,T;H_0^1) \text{ weak}, \\ u \in W(M,T). \end{cases} \tag{4.12}$$

On the other hand

$$\begin{aligned} & \left|P_m[f,u_{m-1},u_m](x,t) - f(x,t,u(x,t))\right| \\ \leq \; & \left|f(x,t,u_{m-1}(x,t)) - f(x,t,u(x,t))\right| + \sum_{i=1}^{N-1} \frac{1}{i!} \left|D_i^3 f(x,t,u_{m-1}(x,t)\right| \left|(u_m(x,t) - u_{m-1}(x,t))^i\right| \\ \leq \; & \tilde{K}_M(f)\left[\|u_{m-1} - u\|_{W_1(T)} + \sum_{i=1}^{N-1} \frac{1}{i!} \|u_m - u_{m-1}\|_{W_1(T)}^i\right], \end{aligned}$$

we have

$$P_m[f,u_{m-1},u_m] \to f(\cdot,\cdot,u) \text{ strongly in } L^\infty(Q_T). \tag{4.13}$$

Simiilarly

$$\begin{cases} P_m[D_1\mu,u_{m-1},u_m] \to D_1\mu(\cdot,\cdot,u) & \text{strongly in} \quad L^\infty(Q_T), \\ P_m[D_1\bar{\mu},u_{m-1},u_m] \to D_1\bar{\mu}(\cdot,\cdot,u) & \text{strongly in} \quad L^\infty(Q_T), \\ D_3\mu(\cdot,\cdot,u_m) \to D_3\mu(\cdot,\cdot,u) & \text{strongly in} \quad L^\infty(Q_T), \\ D_3\bar{\mu}(\cdot,\cdot,u_m) \to D_3\bar{\mu}(\cdot,\cdot,u) & \text{strongly in} \quad L^\infty(Q_T). \end{cases} \tag{4.14}$$

Due to (4.13), (4.14), and passing the limit in (3.4) as $m = m_j \to \infty$, there exists $u \in W(M,T)$ satisfying the equation

$$\begin{aligned} & \langle u''(t),v\rangle + \lambda\langle u_x'(t),v_x\rangle + \langle D_3\mu(t,u(t))u_x(t),v_x\rangle \\ & = \int_0^t g(t-s)\langle D_3\bar{\mu}(s,u(s))u_x(s),v_x\rangle ds \\ & + \int_0^t g(t-s)\langle D_1\bar{\mu}(s,u(s)),v_x\rangle ds - \langle D_1\mu(t,u(t)),v_x\rangle + \langle f(t,u(t)),v\rangle, \end{aligned} \tag{4.15}$$

for all $v \in H_0^1$ and $u(0) = \tilde{u}_0$, $u'(0) = \tilde{u}_1$.

On the other hand, it follows from $(4.12)_4$ and (4.15) that

$$u'' \;=\; \lambda u'_{xx} + \frac{\partial^2}{\partial x^2}\left[\mu(t,u(t))u_x(t)\right] - \int_0^t g(t-s)\frac{\partial^2}{\partial x^2}\left[\mu(s,u(s))u_x(s)\right]ds + f(x,t,u) \in L^\infty(0,T;L^2),$$

hence, $u \in W_1(M,T)$.

Applying the similar arguments used in the proof of Theorem 3.1, one can prove that u is a unique local weak solution of (1.1). Therefore, Theorem 4.1 is proved completely. □

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