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Approximation by Stancu Variant of α-Schurer-Kantorovich Operators

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Abstract. We introduce the sequence of Stancu variant of α-Schurer-Kantorovich operators and systematically investigate some basic estimates. We also obtain the uniform convergence theorem and the order of approximation in terms of suitable modulus of continuity for our newly defined operators. Moreover, we investigate rate of convergence by means of Peetre's K-functional and local direct estimate via Lipschitz-type functions. Finally, A-statistical approximation is presented.

1. Introduction and preliminaries

Operators theory is a fascinating field of research for the last two decades due to the advent of computer. It contributes important role in applied and pure mathematics, viz, fixed point theory, numerical analysis etc. In computational aspects of mathematics and shape of geometric objects, CAGD (Computer-aided Geometric design) plays an interesting role with the mathematical description. It focuses on mathematics which is compatible with computers in shape designing. To investigate the behaviour of parametric surfaces and curves, control nets and control points has a significant role respectively. CAGD is widely used as an application in applied mathematics and industries. It has several applications in other branches of sciences, e.g., approximation theory, computer graphics, data structures, numerical analysis, computer algebra etc. In 1912, Bernstein [5] was the first who introduced a sequence of polynomials to present a smallest and easiest proof of celebrated theorem named as Weierstrass approximation theorem with the aid of binomial distribution as follows:

$$B_n(f;x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right)\begin{pmatrix} n \\ i \end{pmatrix} x^i(1-x)^{n-i}, \quad x \in [0,1] \tag{1}$$

for any $f \in C[0,1]$ (the set of continuous function on $[0,1]$) and $n \in \mathbb{N}$. The basis $\begin{pmatrix} n \\ i \end{pmatrix} x^i(1-x)^{n-i}$ of Bernstein polynomials (1) has significant role in preserving the shape of the surfaces or curves (see [25]-[27]). Graphic

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design programs, viz, photoshop inkspaces and Adobe's illustrator deals with Bernstein polynomials in the form of Bèzier curves. To preserve the shape of the parametric surface or curve, it depends on basis $\binom{n}{i}x^i(1-x)^{n-i}$ which is used to design the curves.

Let s be a non-negative integer. In 1962, Schurer [47] presented the following modification of Bernstein operators (1) by introducing linear positive operators $B_{n,s} : C[0, 1 + s] \to C[0, 1]$ which are defined by

$$B_{n,s}(f;x) = \sum_{i=0}^{n+s} f\left(\frac{i}{n}\right)\binom{n+s}{i}x^i(1-x)^{n+s-i}, \ x \in [0,1] \tag{2}$$

for any $f \in C[0, 1+s]$. Barbosu [2, 3] presented some interesting modifications and results of these operators.

In the recent past, Chen et al. [10] presented a family of modified Bernstein operators which is termed as α-Bernstein operator based on parameter $\alpha \in [0, 1]$ as

$$T_{n,\alpha}(f;x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right)T_{n,i}^{(\alpha)}(x), \tag{3}$$

where $T_{n,i}^{(\alpha)}(x)$ is the α-Bernstein polynomial of degree n and are given by $T_{1,0}^{(\alpha)}(x) = 1 - x$, $T_{1,1}^{(\alpha)}(x) = x$ and

$$T_{n,i}^{(\alpha)}(x) \quad = \quad \left[\binom{n-2}{i}(1-\alpha)x + \binom{n-2}{i-2}(1-\alpha)(1-x) + \binom{n}{i}\alpha x(1-x)\right]x^{i-1}(1-x)^{n-i-1},$$

with $n \geq 2$ and $x \in [0, 1]$. Mohiuddine et al. [32] studied the Kantorovich modification of (3) and further modified α-Bernstein-Kantorovich operator in Stancu sense by Mohiuddine and Özger [37] while for the classical Bernstein-Stancu operator (see [49]). Durrmeyer modification of α-Bernstein operators were presented in [18–20, 36]. Cai et al. [7] introduced a generalization of classical Bernstein operators based on Bézier bases with the shape parameter $-1 \leq \lambda \leq 1$, and their Shape-preserving properties and Kantorovich type λ-Bernstein operators in [8] and [6], respectively.

Recently, Özger et al. [46] constructed the α-Bernstein-Schurer operators $T_{n,\alpha,s} : C[0, 1 + s] \to C[0, 1]$ defined for any $f \in C[0, 1 + s]$ by

$$T_{n,s,\alpha}(f;x) = \sum_{i=0}^{n+s} f\left(\frac{i}{n}\right)T_{n,s,i}^{(\alpha)}(x), \tag{4}$$

where the α-Bernstein-Schurer polynomials $T_{n,s,i}^{(\alpha)}(x)$ are defined by $T_{1,s,0}^{(\alpha)}(x) = 1 - x$, $T_{1,s,1}^{(\alpha)}(x) = x$ and

$$T_{n,s,i}^{(\alpha)}(x) \quad = \quad \left\{\binom{n+s-2}{i}(1-\alpha)x + \binom{n+s-2}{i-2}(1-\alpha)(1-x) + \binom{n+s}{i}\alpha x(1-x)\right\}x^{i-1}(1-x)^{n+s-i-1}$$

for $n \geq 2$. The bivariate form of α-Bernstein-Schurer operators and their associated GBS operators were presented by Mohiuddine [31]. Most recently, the Kantorovich form of (4) have been studied in [42], defined as

$$\mathcal{K}_{n,s,\alpha}(f;x) = (n+1)\sum_{i=0}^{n+s} T_{n,s,i}^{(\alpha)}(x)\int_{\frac{i}{n+s}}^{\frac{i+1}{n+s}} f(t)dt. \tag{5}$$

In the same paper, authors defined the bivariate form of (5) and studied several approximation properties of both the operators. Some recent work on positive linear operators, we refer to [1, 9, 23, 33, 38–41, 45].

2. Stancu-type α-Schurer-Kantorovich operators and auxiliary results

Let δ and γ be two non-negative integers such that $0 \leq \delta \leq \gamma$. Motivated by the discussion of previous section, in this section, we construct the Stancu-type α-Schurer-Kantorovich operators by

$$\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(f;x) = (n+\gamma+1)\sum_{i=0}^{n+s} T_{n,s,i}^{(\alpha)}(x)\int_{\frac{i+\delta}{n+\gamma+1}}^{\frac{i+\delta+1}{n+\gamma+1}} f(t)dt, \qquad x \in [0,1], \tag{6}$$

for any $f \in C[0,1+s]$ and $\alpha \in [0,1]$.

Let $e_j(x) = x^j$, $j \in \{0,1,2\}$. The following lemma is given in [46].

Lemma 2.1. *For the operators (4), we have*

$$\begin{aligned}
T_{n,\alpha,s}(e_0(t);x) &= 1,\\
T_{n,\alpha,s}(e_1(t);x) &= x + \frac{s}{n}x,\\
T_{n,\alpha,s}(e_2(t);x) &= x^2 + \frac{(n+s+2(1-\alpha))}{n^2}x(1-x) + \frac{s(s+2n)}{n^2}x^2,
\end{aligned}$$

Lemma 2.2. *Let $e_j(t) = t^j$, $j \in \{0,1,2\}$. For the operators (6), we have*

$$\begin{aligned}
\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(e_0(t);x) &= 1,\\
\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(e_1(t);x) &= \left(\frac{n+s}{n+\gamma+1}\right)x + \frac{2\delta+1}{2(n+\gamma+1)};\\
\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(e_2(t);x) &= \left(\frac{n+s}{n+\gamma+1}\right)^2 x^2 + \left(\frac{n+s+2(1-\alpha)}{(n+\gamma+1)^2}\right)x(1-x) + \frac{(2\delta+1)(n+s)}{(n+\gamma+1)^2}x + \frac{(3\delta^2+3\delta+1)}{3(n+\gamma+1)^2}.
\end{aligned}$$

Proof. We prove the Lemma 2.2 with the Lemma 2.1 and easy to obtain

$$\int_{\frac{i+\delta}{n+\gamma+1}}^{\frac{i+\delta+1}{n+\gamma+1}} f(t)dt = \begin{cases} \dfrac{1}{n+\gamma+1} & \text{if} \quad f(t)=1,\\ \dfrac{i+\delta}{(n+\gamma+1)^2} + \dfrac{1}{2(n+\gamma+1)^2} & \text{if} \quad f(t)=t,\\ \dfrac{i^2}{(n+\gamma+1)^3} + \dfrac{(2\delta+1)i}{(n+\gamma+1)^3} + \dfrac{(3\delta^2+3\delta+1)}{3(n+\gamma+1)^3} & \text{if} \quad f(t)=t^2. \end{cases} \tag{7}$$

Thus,

$$\begin{aligned}
\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(e_0(t);x) &= (n+\gamma+1)\sum_{i=0}^{n+s}\tilde{s}_{n,i}^{(\alpha)}(x)\int_{\frac{i+\delta}{n+\gamma+1}}^{\frac{i+\delta+1}{n+\gamma+1}} dt\\
&= \sum_{i=0}^{n+s}\tilde{s}_{n,i}^{(\alpha)}(x)\\
&= 1;\\
\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(e_1(t);x) &= (n+\gamma+1)\sum_{i=0}^{n+s}\tilde{s}_{n,i}^{(\alpha)}(x)\int_{\frac{i+\delta}{n+\gamma+1}}^{\frac{i+\delta+1}{n+\gamma+1}} tdt\\
&= \frac{s}{n+\gamma+1}\sum_{i=0}^{n+s}\tilde{s}_{n,i}^{(\alpha)}(x)f\left(\frac{i}{s}\right) + \frac{2\delta+1}{2(n+\gamma+1)}\sum_{i=0}^{n+s}\tilde{s}_{n,i}^{(\alpha)}(x)\\
&= \frac{s}{n+\gamma+1}T_{n,\alpha,s}(e_1(t);x) + \frac{2\delta+1}{2(n+\gamma+1)}T_{n,\alpha,s}(e_0(t);x)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{s}{n+\gamma+1}\right)\left(x+\frac{s}{n}x\right)+\frac{2\delta+1}{2(n+\gamma+1)};\\
\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(e_2(t);x) &= (n+\gamma+1)\sum_{i=0}^{n+s}\tilde{s}_{n,i}^{(\alpha)}(x)\int_{\frac{i+\delta}{n+\gamma+1}}^{\frac{i+\delta+1}{n+\gamma+1}} t^2dt\\
&= \frac{n^2}{(n+\gamma+1)^2}\sum_{i=0}^{n+s}\tilde{s}_{n,i}^{(\alpha)}(x)f\left(\frac{i}{s}\right)^2+\frac{(2\delta+1)n}{(n+\gamma+1)^2}\sum_{i=0}^{n+s}\tilde{s}_{n,i}^{(\alpha)}(x)f\left(\frac{i}{s}\right)+\frac{(3\delta^2+3\delta+1)}{3(n+\gamma+1)^2}\sum_{i=0}^{n+s}\tilde{s}_{n,i}^{(\alpha)}(x)\\
&= \frac{n^2}{(n+\gamma+1)^2}\sum_{i=0}^{n+s}T_{n,\alpha,s}(e_2(t);x)+\frac{(2\delta+1)n}{(n+\gamma+1)^2}T_{n,\alpha,s}(e_1(t);x)+\frac{(3\delta^2+3\delta+1)}{3(n+\gamma+1)^2}T_{n,\alpha,s}(e_0(t);x)\\
&= \left(\frac{n}{n+\gamma+1}\right)^2\left(x^2+\frac{(n+s+2(1-\alpha))(x-x^2)}{n^2}+\frac{s(s+2n)x^2}{n^2}\right)\\
&\quad +\frac{(2\delta+1)n}{(n+\gamma+1)^2}\left(x+\frac{s}{n}x\right)+\frac{(3\delta^2+3\delta+1)}{3(n+\gamma+1)^2}
\end{aligned}
$$

which completes the desired proof. □

Corollary 2.3. *Let $\tau_j(t)=(t-x)^j, =0,1,2$. Then, we get*

$$
\begin{aligned}
\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(\tau_0(t);x) &= 1,\\
\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(\tau_1(t);x) &= \left(\frac{s-\gamma-1}{n+\gamma+1}\right)x+\frac{2\delta+1}{2(n+\gamma+1)};\\
\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(\tau_2(t);x) &= \left(\frac{s-\gamma-1}{n+\gamma+1}\right)^2x^2+\left(\frac{n+s+2(1-\alpha)}{(n+\gamma+1)^2}\right)x(1-x)+\frac{(2\delta+1)(s-\gamma-1)}{(n+\gamma+1)^2}x+\frac{(3\delta^2+3\delta+1)}{3(n+\gamma+1)^2}.
\end{aligned}
$$

Proof. With the aid of Lemma 2.2 and linearity property

$$\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(\tau_2(t);x) = \mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(e_2(t);x)-2x\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(e_1(t);x)+x^2\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(e_0(t);x),$$

we can easily completes the desired proof. □

We first give the uniform convergence property of our Stancu variant of α-Schurer-Kantorovich operators (6).

Theorem 2.4. *Let f be any function in $C[0,1+s]$. Then, for any $0\leq\alpha\leq 1$, it follows that*

$$\lim_{n\to\infty}\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(f;x)=f(x)$$

uniformly on $[0,1]$.

Proof. We can see from Lemma 2.2 that

$$\lim_{n\to\infty}\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}\left(e_j(t);x\right)=x^j \quad (j=0,1,2).$$

Therefore, by the well-known Bohman-Korovkin-Popoviciu theorem, we obtain that the sequence of operators $\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(f;x)$ are converge to f uniformly on $[0,1]$. □

3. Order of convergence

In this section, we study the order of convergence of the sequence of operators $\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}$ by means suitable the modulus of continuity. For any $\delta^* > 0$, let $\omega(f;\delta^*)$ (the modulus of continuity of f of order one) be define to measure the maximum oscillation of f such that $\lim_{\delta^*\to 0+}\omega(f;\delta^*) = 0$. Then, for $f \in C[0,1]$, one has

$$\omega(f;\delta^*) = \sup_{|t_1-t_2|\leq\delta^*} | f(t_1) - f(t_2) |; \quad t_1, t_2 \in [0,1] \tag{8}$$

and

$$| f(t_1) - f(t_2) | \leq \left(1 + \frac{| t_1 - t_2 |}{\delta^*}\right)\omega(f;\delta^*). \tag{9}$$

Theorem 3.1. *[48] Let $[u,v] \subseteq [x,y]$. Then, for the sequences of positive linear operators $\{L\}_{s\geq 1}$ which acting from $C[x,y]$ to $C[u,v]$, we immediately see*

1. *if $f \in C[x_1,y_1]$ and $x \in [u,v]$, then we have*

$$|L_s(f;x) - f(x)| \leq |f(x)||L_s(1;x) - 1| + \left\{L_s(1;x) + \frac{1}{\delta^*}\sqrt{L_s((t-x)^2;x)}\sqrt{L_s(1;x)}\right\}\omega(f;\delta^*),$$

2. *for any $f' \in C[x_1,y_1]$ and $x \in [u,v]$, we have*

$$\begin{aligned}|L_s(f;x) - f(x)| &\leq |f(x)||L_s(1;x) - 1| + |f'(x)||L_s(t-x;x)| \\ &\quad + L_s((t-x)^2;x)\left\{\sqrt{L_s(1;x)} + \frac{1}{\delta^*}\sqrt{L_s((t-x)^2;x)}\right\}\omega(f';\delta^*).\end{aligned}$$

Theorem 3.2. *For all $f \in C[0,1+s]$, the operators $\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}$ satisfying the inequality*

$$|\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(f;x) - f(x)| \leq 2\omega\left(f;\sqrt{\mu_{n,s,\alpha}^{\delta,\gamma}(x)}\right),$$

where $\theta_{n,\gamma} = \sqrt{\mu_{n,s,\alpha}^{\delta,\gamma}(x)} = \sqrt{\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(\tau_2(t);x)}$ and $\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(\tau_2(t);x)$ is given in Corollary 2.3

Proof. If we consider the Condition 1 of Theorem 3.1 and Lemma 2.2, then we can write

$$\begin{aligned}|\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(f;x) - f(x)| &\leq |f(x)||\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(e_0(t);x) - 1| + \left\{\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(e_0(t);x)\right. \\ &\quad \left. + \frac{1}{\theta_{n,\gamma}}\sqrt{\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(\tau_2(t);x)}\sqrt{\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(e_0(t);x)}\right\}\omega(f;\delta^*)\end{aligned}$$

If we choose $\delta^* = \theta_{n,\gamma} = \sqrt{\mu_{n,s,\alpha}^{\delta,\gamma}(x)} = \sqrt{\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(\tau_2(t);x)}$, then we get

$$|\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(f;x) - f(x)| \leq 2\omega\left(f;\sqrt{\mu_{n,s,\alpha}^{\delta,\gamma}(x)}\right).$$

□

Theorem 3.3. *If $f' \in C[0,1+s]$ then, for every $x \in [0,1]$, we have*

$$\left|\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(f;x) - f(x)\right| \leq \left|\left(\frac{s-\gamma-1}{n+\gamma+1}\right)x + \frac{2\delta+1}{2(n+\gamma+1)}\right|\left|f'(x)\right| + 2\mu_{n,s,\alpha}^{\delta,\gamma}(x)\omega\left(f';\sqrt{\mu_{n,s,\alpha}^{\delta,\gamma}(x)}\right),$$

where $\theta_{n,\gamma} = \sqrt{\mu_{n,s,\alpha}^{\delta,\gamma}(x)} = \sqrt{\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(\tau_2(t);x)}$ and $\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(\tau_2(t);x)$ is given in Corollary 2.3.

Proof. We consider Condition 2 of Theorem 3.1 and Lemma 2.2. Then, we get

$$\begin{aligned}|\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(f;x)-f(x)| &\leq |f(x)||\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(e_0(t);x)-1|+|f'(x)||\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(\tau_1(t);x)| \\ &\quad+\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(\tau_2(t);x)\Big\{\sqrt{\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(e_0(t);x)}+\frac{1}{\theta_{n,\gamma}}\sqrt{\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(\tau_2(t);x)}\Big\}\omega(f';\delta^*).\end{aligned}$$

If we put $\delta^*=\theta_{n,\gamma}=\sqrt{\mu^{\delta,\gamma}_{n,s,\alpha}(x)}=\sqrt{\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(\tau_2(t);x)}$ then we can get our result. □

Theorem 3.4. *For $f\in C[0,1+s]$ and $\omega(f;\theta_{n,\gamma})$ denotes the modulus of smoothness. Then*

$$\left|\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(f;x)-f(x)\right|\leq\left\{1+\sqrt{\Gamma^{\delta,\gamma}_{s}(x)}\right\}\omega\left(f;\theta_{n,\gamma}\right),$$

where $\theta_{n,\gamma}=(n+\gamma+1)^{-\frac{1}{2}}$ and

$$\Gamma^{\delta,\gamma}_{s}(x)=\frac{(s-\gamma-1)^2x^2+(n+s+2(1-\alpha))\,x(1-x)+(2\delta+1)(s-\gamma-1)x+\left(\delta^2+\delta+3^{-1}\right)}{(n+\gamma+1)}.$$

Proof. For any $f\in C[0,1+s]$, $x\in[0,1]$ and in view of monotonicity and linearity of the operators (6), we can easily find

$$\begin{aligned}&\left|\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(f;x)-f(x)\right| \\ &\quad\leq\left\{1+\theta^{-1}_{n,\gamma}\sqrt{\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}\left(\tau_2(t);x\right)}\right\}\omega\left(f;\theta_{n,\gamma}\right) \\ &\quad\leq\left\{1+\sqrt{\frac{(s-\gamma-1)^2x^2+(n+s+2(1-\alpha))\,x(1-x)+(2\delta+1)(s-\gamma-1)x+(\delta^2+\delta+3^{-1})}{(n+\gamma+1)}}\right\}\omega\left(f;\theta_{n,\gamma}\right),\end{aligned}$$

where $\theta_{n,\gamma}>0$ and $\theta_{n,\gamma}=(n+\gamma+1)^{-\frac{1}{2}}$. Thus, we arrive to the assertion. □

Now, we give the order of convergence for the sequence of operators defined by (6) using modulus of smoothness which has first order continuous derivatives , i.e., $\omega\left(f';\theta_{n,\gamma}\right)=\omega_1\left(f;\theta_{n,\gamma}\right)$.

Theorem 3.5. *For the operators defined in (6) and $0\leq\theta_{n,\gamma}\leq1$, we have*

$$\left|\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(f;x)-f(x)\right|\leq\omega_1\left((n+\gamma)^{-1}\right)\sqrt{\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}\left(\tau_2(t);x\right)}\left\{1+\sqrt{(n+\gamma)}\sqrt{\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}\left(\tau_2(t);x\right)}\right\}.$$

Proof. For any $a\leq x_1,x_2\leq b$, we know that

$$\begin{aligned}f(x_1)-f(x_2) &= (x_1-x_2)f'(\xi) \\ &= (x_1-x_2)f'(x_1)+(x_1-x_2)\left[f'(\xi)-f'(x_1)\right],\end{aligned}\tag{10}$$

where $\xi\in(x_1,x_2)$. Further, we have

$$\left|(x_1-x_2)\left[f'(\xi)-f'(x_1)\right]\right|\leq|x_1-x_2|(\lambda+1)\omega_1(\theta_{n,\gamma}),\quad \lambda=\lambda\left(x_1,x_2;\theta_{n,\gamma}\right).\tag{11}$$

Next, we obtain

$$\left|\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(f;x)-f(x)\right|=\left|(n+\gamma+1)\sum_{i=0}^{n+s}\tilde{s}^{(\alpha)}_{n,i}(x)\int_{\frac{i+\delta}{n+\gamma+1}}^{\frac{i+\delta+1}{n+\gamma+1}}\Big(f(t)-f(x)\Big)dt\right|.\tag{12}$$

In view of (10), (11) and (12), we obtain

$$\begin{aligned}
\left|\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(f;x)-f(x)\right| &\leq \left|(n+\gamma+1)\sum_{i=0}^{n+s}\tilde{s}^{(\alpha)}_{n,i}(x)\int_{\frac{i+\delta}{n+\gamma+1}}^{\frac{i+\delta+1}{n+\gamma+1}}(t-x)\,f'(x)dt\right| \\
&\leq \omega_1\left(\theta_{n,\gamma}\right)(n+\gamma+1)(\lambda+1)\sum_{i=0}^{n+s}\tilde{s}^{(\alpha)}_{n,i}(x)\int_{\frac{i+\delta}{n+\gamma+1}}^{\frac{i+\delta+1}{n+\gamma+1}}|t-x|\,dt \\
&\leq \omega_1\left(\theta_{n,\gamma}\right)\Bigg\{(n+\gamma+1)\sum_{i=0}^{n+s}\tilde{s}^{(\alpha)}_{n,i}(x)\int_{\frac{i+\delta}{n+\gamma+1}}^{\frac{i+\delta+1}{n+\gamma+1}}|t-x|\,dt \\
&\quad+\sum_{\lambda\geq n}(n+\gamma+1)\lambda\left(x_1,t;\theta_{n,\gamma}\right)\sum_{i=0}^{n+s}\tilde{s}^{(\alpha)}_{n,i}(x)\int_{\frac{i+\delta}{n+\gamma+1}}^{\frac{i+\delta+1}{n+\gamma+1}}|t-x|\,dt\Bigg\} \\
&\leq \omega_1\left(\theta_{n,\gamma}\right)\Bigg\{(n+\gamma+1)\sum_{i=0}^{n+s}\tilde{s}^{(\alpha)}_{n,i}(x)\int_{\frac{i+\delta}{n+\gamma+1}}^{\frac{i+\delta+1}{n+\gamma+1}}|t-x|\,dt \\
&\quad+\theta^{-1}_{n,\gamma}(n+\gamma+1)\sum_{i=0}^{n+s}\tilde{s}^{(\alpha)}_{n,i}(x)\int_{\frac{i+\delta}{n+\gamma+1}}^{\frac{i+\delta+1}{n+\gamma+1}}(t-x)^2\,dt\Bigg\} \\
&\leq \omega_1\left(\theta_{n,\gamma}\right)\sqrt{\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(\tau_2(t);x)}\left\{1+\theta^{-1}_{n,\gamma}\sqrt{\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(\tau_2(t);x)}\right\}.
\end{aligned}$$

Choosing $\theta_{n,\gamma}=(n+\gamma+1)^{-1}$, we get

$$\left|\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(f;x)-f(x)\right| \leq \omega_1\left((n+\gamma+1)^{-1}\right)\sqrt{\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(\tau_2(t);x)}\left\{1+\sqrt{(n+\gamma+1)}\sqrt{\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(\tau_2(t);x)}\right\},$$

which completes the proof of Theorem 3.5. □

4. Direct approximation

For and $g\in C[0,1]$ and any $\delta^*>0$, the Peetre's K-functional is defined as

$$K_2(g;\delta^*)=\inf\left\{\delta^*\|f''\|_{C[0,1+s]}+\|g-f\|_{C[0,1+s]}:f\in C^2[0,1+s]\right\},$$

where

$$C^2[0,1+s]=\{f\in C[0,1+s]:f',f''\in C[0,1+s]\}.$$

The second-order modulus of smoothness $\omega_2(g;\sqrt{\delta^*})$ is given by

$$\omega_2(g;\sqrt{\delta^*})=\sup_{0<h<\sqrt{\delta^*}}\ \sup_{x,x+2h\in[0,1]}|g(x+2h)-2g(x+h)+g(x)|.$$

From [12], for any $g\in C[0,1+s]$, there is an absolute constant $C>0$, we have

$$K_2(g;\delta^*)\leq C\omega_2(g;\sqrt{\delta^*}).$$

Note that the usual modulus of continuity is

$$\omega(g;\delta^*)=\sup_{0<h\leq\delta^*}\ \sup_{x,x+h\in[0,1]}|g(x+h)-g(x)|.$$

Theorem 4.1. *Let $g \in C[0,1+s]$. Then, the inequality*

$$\left|\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(g;x)-g(x)\right| \leq 4K_2\left(g;\frac{1}{4}\left(\mu_{n,s,\alpha}^{\delta,\gamma}(x)+\nu_{n,s}^{\delta,\gamma}(x)\right)\right)+\omega\left(g;\sqrt{\nu_{n,s}^{\delta,\gamma}(x)}\right)$$

holds for any $x \in [0,1]$, where $\mu_{n,s,\alpha}^{\delta,\gamma}(x)=\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(\tau_2(t);x)$,

$$\nu_{n,s}^{\delta,\gamma}(x)=\left(\left(\frac{n+s}{n+\gamma+1}-1\right)x+\frac{2\delta+1}{2(n+\gamma+1)}\right)^2$$

and $\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(\tau_2(t);x)$ is given in Corollary 2.3.

Proof. For any $g \in C[0,1+s]$ and $x \in [0,1]$, we define the auxiliary operators by

$$\mathcal{T}_{n,s,\alpha}^{\delta,\gamma}(g;x)=\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(g;x)+g(x)-g\left(\left(\frac{n+s}{n+\gamma+1}\right)x+\frac{2\delta+1}{2(n+\gamma+1)}\right). \tag{13}$$

We can easily see that

$$\mathcal{T}_{n,s,\alpha}^{\delta,\gamma}(e_0(t);x)=1$$

and

$$\mathcal{T}_{n,s,\alpha}^{\delta,\gamma}(e_1(t);x)=\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(e_1(t);x)+x-\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(e_1(t);x)=x.$$

Let $\Theta \in C^2[0,1+s]$. We know by Taylor series expression that

$$\Theta(t)=\Theta(x)+(t-x)\Theta'(x)+\int_x^t (t-\chi)\Theta''(\chi)\mathrm{d}\chi.$$

By applying $\mathcal{T}_{n,s,\alpha}^{\delta,\gamma}$, we get

$$\begin{aligned}
\mathcal{T}_{n,s,\alpha}^{\delta,\gamma}(\Theta;x)-\Theta(x) &= \Theta'(x)\mathcal{T}_{n,s,\alpha}^{\delta,\gamma}(t-x;x)+\mathcal{T}_{n,s,\alpha}^{\delta,\gamma}\Big(\int_x^t (t-\chi)\Theta''(\chi)\mathrm{d}\chi;x\Big)\\
&= \mathcal{T}_{n,s,\alpha}^{\delta,\gamma}\Big(\int_x^t (t-\chi)\Theta''(\chi)\mathrm{d}\chi;x\Big)\\
&= \mathcal{D}_{n,s,\alpha}^{\delta,\gamma}\Big(\int_x^t (t-\chi)\Theta''(\chi)\mathrm{d}\chi;x\Big)+\int_x^x (x-\chi)\Theta''(\chi)\mathrm{d}\chi;x\\
&\quad -\int_x^{\left(\frac{n+s}{n+\gamma+1}\right)x+\frac{2\delta+1}{2(n+\gamma+1)}}\left(\left(\frac{n+s}{n+\gamma+1}\right)x+\frac{2\delta+1}{2(n+\gamma+1)}-\chi\right)\Theta''(\chi)\mathrm{d}\chi
\end{aligned}$$

which yields

$$\begin{aligned}
\mid \mathcal{T}_{n,s,\alpha}^{\delta,\gamma}(\Theta;x)-\Theta(x)\mid &\leq \left|\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}\Big(\int_x^t (t-\chi)\Theta''(\chi)\mathrm{d}\chi;x\Big)\right|\\
&\quad +\left|\int_x^{\left(\frac{n+s}{n+\gamma+1}\right)x+\frac{2\delta+1}{2(n+\gamma+1)}}\left(\left(\frac{n+s}{n+\gamma+1}\right)x+\frac{2\delta+1}{2(n+\gamma+1)}-\chi\right)\Theta''(\chi)\mathrm{d}\chi\right|.
\end{aligned}$$

It follows from the inequalities

$$\left|\int_x^t (t-\chi)\Theta''(\chi)\mathrm{d}\chi\right| \leq (t-x)^2\|\Theta''\|$$

and

$$\left|\int_x^{\left(\frac{n+s}{n+\gamma+1}\right)x+\frac{2\delta+1}{2(n+\gamma+1)}}\left(\left(\frac{n+s}{n+\gamma+1}\right)x+\frac{2\delta+1}{2(n+\gamma+1)}-\chi\right)\Theta''(\chi)\mathrm{d}\chi\right|$$
$$\leq\left(\left(\frac{n+s}{n+\gamma+1}\right)x+\frac{2\delta+1}{2(n+\gamma+1)}-x\right)^2\|\Theta''\|$$

that

$$|\mathcal{T}^{\delta,\gamma}_{n,s,\alpha}(\Theta;x)-\Theta(x)|\leq\left(\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(\tau_2(t);x)+\left(\left(\frac{n+s}{n+\gamma+1}\right)x+\frac{2\delta+1}{2(n+\gamma+1)}-x\right)^2\right)\|\Theta''\|. \tag{14}$$

On the other hand, we conclude that

$$\|\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(g;x)\|\leq\|g\|, \tag{15}$$

and

$$|\mathcal{T}^{\delta,\gamma}_{n,s,\alpha}(g;x)|\leq 3\|g\|. \tag{16}$$

Using the inequalities (13)-(16), we obtain

$$\begin{aligned}\left|\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(g;x)-g(x)\right| &\leq \left|\mathcal{T}^{\delta,\gamma}_{n,s,\alpha}(g-\Theta;x)-(g-\Theta)(x)\right|+\left|\mathcal{T}^{\delta,\gamma}_{n,s,\alpha}(\Theta;x)-\Theta(x)\right|\\ &\quad+\left|g(x)-g\left(\left(\frac{n+s}{n+\gamma+1}\right)x+\frac{2\delta+1}{2(n+\gamma+1)}\right)\right|\\ &\leq 4\left\|g-\Theta\right\|+\left[\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(\tau_2(t);x)+\left(\left(\frac{n+s}{n+\gamma+1}-1\right)x+\frac{2\delta+1}{2(n+\gamma+1)}\right)^2\right]\|\Theta''\|\\ &\quad+\omega\left(g;\left(\frac{n+s}{n+\gamma+1}-1\right)x+\frac{2\delta+1}{2(n+\gamma+1)}\right).\end{aligned}$$

Taking infimum over all $\Theta\in C^2[0,1+s]$, we get

$$\begin{aligned}\left|\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(g;x)-g(x)\right| &\leq 4K_2\left(g;\frac{\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(\tau_2(t);x)}{4}+\frac{1}{4}\left(\left(\frac{n+s}{n+\gamma+1}-1\right)x+\frac{2\delta+1}{2(n+\gamma+1)}\right)^2\right)\\ &\quad+\omega\left(g;\left(\frac{n+s}{n+\gamma+1}-1\right)x+\frac{2\delta+1}{2(n+\gamma+1)}\right)\end{aligned}$$

which completes the proof □

Corollary 4.2. *Let $g\in C[0,1+s]$. Then, the inequality*

$$\left|\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(g;x)-g(x)\right| \leq C\omega_2\left(g;\sqrt{\frac{1}{4}\left(\mu^{\delta,\gamma}_{n,s,\alpha}(x)+\nu^{\delta,\gamma}_{n,s}(x)\right)}\right)+\omega\left(g;\sqrt{\nu^{\delta,\gamma}_{n,s}(x)}\right)$$

holds for any $x\in[0,1]$, where $C>0$ is a constant, and $\mu^{\delta,\gamma}_{n,s,\alpha}(x)$ and $\nu^{\delta,\gamma}_{n,s}(x)$ are same as in Theorem 4.1.

Proof. The result follows from the previous Theorem 4.1 and using the inequality $K_2(g;\delta^*)\leq C\omega_2(g;\sqrt{\delta^*})$ due to [12]. □

Now we give the local direct estimate for the operators $\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}$ via the well-known Lipschitz-type maximal function involving the parameters $\beta_1,\ \beta_2 > 0$ and number $\sigma \in (0,1]$. Thus, from [43], we recall that

$$Lip_K^{(\beta_1,\beta_2)}(\sigma) := \Big\{f \in C[0,1+s] : |f(t)-f(x)| \leq K\frac{|t-x|^{\sigma}}{(\beta_1 x^2+\beta_2 x+t)^{\frac{\sigma}{2}}};\ x,\ t \in [0,1]\Big\},$$

where K is a positive constant .

Theorem 4.3. *For any $f \in Lip_K^{(\beta_1,\beta_2)}(\sigma)$ and $\sigma \in (0,1]$, there exits constant $K > 0$ such that*

$$|\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(f;x)-f(x)| \leq K.(\beta_1 x^2+\beta_2 x)^{-\sigma/2}\left(\mu_{n,s,\alpha}^{\delta,\gamma}(x)\right)^{\frac{\sigma}{2}},$$

where $\mu_{n,s,\alpha}^{\delta,\gamma}(x) = \mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(\tau_2(t);x)$.

Proof. For any $f \in Lip_K^{(\beta_1,\beta_2)}(\sigma)$ and $\sigma \in (0,1]$, first we will check that the statement holds for $\sigma = 1$. We can write

$$\begin{aligned}|\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(f;x)-f(x)| &\leq |\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(|f(t)-f(x)|;x)| + f(x)\,|\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(e_0(t);x)-1|\\ &\leq \mathcal{D}_{n,s,\alpha}^{\delta,\gamma}\left(\left|f(t)-f(x)\right|;x\right)\\ &\leq K\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}\left(\frac{|t-x|}{(\beta_1 x^2+\beta_2 x+t)^{\frac{1}{2}}};x\right).\end{aligned}$$

For any $\beta_1,\ \beta_2 \geq 0$, we obtain by using the inequality $(\beta_1 x^2+\beta_2 x+t)^{-1/2} \leq (\beta_1 x^2+\beta_2 x)^{-1/2}$ and applying the well-known Cauchy-Schwarz inequality that

$$\begin{aligned}|\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(f;x)-f(x)| &\leq K(\beta_1 x^2+\beta_2 x)^{-1/2}\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(|t-x|;x)\\ &= K(\beta_1 x^2+\beta_2 x)^{-1/2}|\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(t-x;x)|\\ &\leq K(\beta_1 x^2+\beta_2 x)^{-1/2}\left|\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}((t-x)^2;x)\right|^{1/2}\end{aligned}$$

which proves that it is true for $\sigma = 1$. Now, we want to show the statement is valid for $\sigma \in (0,1)$. Applying the monotonicity property to operators $\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}$ and using the Hölder's inequality two times with $c = 2/\sigma$ and $d = 2/(2-\sigma)$, we can write here

$$\begin{aligned}\left|\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(f;x)-f(x)\right| &\leq \mathcal{D}_{n,s,\alpha}^{\delta,\gamma}\left(\left|f(t)-f(x)\right|;x\right)\\ &\leq \left(\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}\left(\left|f(t)-f(x)\right|\right)\right)^{\frac{\sigma}{2}}\left(\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(e_0(t);x)\right)^{\frac{2-\sigma}{2}}\\ &\leq K\left(\frac{\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(\tau_2(t);x)}{t+\beta_1 x^2+\beta_2 x}\right)^{\frac{\sigma}{2}}\\ &\leq K(\beta_1 x^2+\beta_2 x)^{-\sigma/2}\left\{\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(\tau_2(t);x)\right\}^{\frac{\sigma}{2}}\\ &\leq K(\beta_1 x^2+\beta_2 x)^{-\sigma/2}\left[\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(\tau_2(t);x)\right]^{\frac{\sigma}{2}}.\end{aligned}$$

This completes the proof. □

For any $\sigma \in (0,1]$ and $f \in C[0,1+s]$, one can define the Lipschitz maximal function of order σ [30] by

$$\omega_{\sigma}(f;x) = \sup_{\xi_1,\xi_2\in[0,1]}\frac{|f(\xi_1)-f(\xi_2)|}{|\xi_1-\xi_2|^{\sigma}}, \qquad \xi_1 \neq \xi_2.$$

Theorem 4.4. *For all $f \in C[0, 1+s]$, we obtain*

$$\left|\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(f;x) - f(x)\right| \leq \omega_\sigma(f;x)\left(\mu_{n,s,\alpha}^{\delta,\gamma}(x)\right)^{\frac{\sigma}{2}}.$$

Proof. From the well-known Hölder inequality, it is easy to conclude that

$$\begin{aligned}\left|\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(f;x) - f(x)\right| &\leq \mathcal{D}_{n,s,\alpha}^{\delta,\gamma}\left(\left|f(t) - f(x)\right|;x\right) \\ &\leq \omega_\sigma(f;x) \mid \mathcal{D}_{n,s,\alpha}^{\delta,\gamma}\left(|t-x|^\sigma;x\right) \\ &\leq \omega_\sigma(f;x)\left(\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(e_0(t);x)\right)^{\frac{2-\sigma}{2}}\left(\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(|t-x|^2;x)\right)^{\frac{\sigma}{2}} \\ &= \omega_\sigma(f;x)\left(\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(\tau_2(t);x)\right)^{\frac{\varsigma}{2}}.\end{aligned}$$

This completes the proof of Theorem 4.4. □

5. Statistical approximation

Gadjiev and Orhan [15] studied the Koronkin approximation theorem by using the idea of statistical convergence [13] while for the classical Korovkin theorem, we refer to [29]. Recently, Korovkin-type theorems via some convergence methods have been studied in [4, 17, 34, 35, 44] and reference therein.

Let $A = (a_{jn})$ be an infinite matrix. For a given sequence $x = (x_n)$, the A-transform of x, denoted by $Ax = ((Ax)_j)$ holds by $Ax : (Ax)_n$, is defined as

$$(Ax)_j = \sum_{k=1}^{\infty} a_{jn}x_n,$$

provided that the series converges for each $j \in \mathbb{N}$. An infinite matrix $A = (a_{jn})$ is said to be regular [16] if

$$\lim_{j\to\infty}(Ax)_j = L \quad \text{whenever} \quad \lim_{j\to\infty} x_j = L.$$

If $A = (a_{jn})$ is a non-negative regular matrix, then $x = (x_n)$ is said to be A-statistically convergent to a number L, provided that, for every $\epsilon > 0$,

$$\lim_j \sum_{n:|x_n - L|\geq\epsilon} a_{jn} = 0.$$

In this case, we write $st_A - \lim x = L$ [14] (see also [11, 28]).

We prove the following theorem.

Theorem 5.1. *Let $A = (a_{jn})$ be a non-negative regular matrix. Then, we have*

$$st_A - \lim_n \left\|\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(f;x) - f\right\| = 0 \tag{17}$$

for any $f \in C[0, 1+s]$ and $x \in [0, 1]$.

Proof. Consider the sequence of function $e_j(x) = x^j$. To prove (17), it is sufficient to show that

$$st_A - \lim_n \left\|\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(e_j(t);x) - e_j\right\| = 0$$

for $j \in \{0, 1, 2\}$. From Lemma 2.2, it is obvious that

$$st_A - \lim_n \left\|\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}(e_0(t);x) - e_0\right\| = 0$$

holds. Again, from Lemma 2.2, we can write

$$\begin{aligned}\left\|\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(e_1(t);x)-e_1\right\| &= \sup_{x\in[0,1]}\left|\left(\frac{s-\gamma-1}{n+\gamma+1}\right)x+\frac{2\delta+1}{n+\gamma+1}\right| \\ &\leq \frac{s-\gamma-1}{n+\gamma+1}+\frac{2\delta+1}{n+\gamma+1}. \end{aligned} \tag{18}$$

For given $\epsilon > 0$, let us define the following sets

$$\begin{aligned} J &= \left\{n : \left\|\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(e_1(t);x)-x\right\| \geq \epsilon\right\}, \\ J_1 &= \left\{n : \frac{s-\gamma-1}{n+\gamma+1} \geq \frac{\epsilon}{2}\right\}, \\ J_2 &= \left\{n : \frac{2\delta+1}{n+\gamma+1} \geq \frac{\epsilon}{2}\right\}. \end{aligned}$$

It follows from (18) that $J \subseteq J_1 \cup J_2$. Then, for each $j \in \mathbb{N}$, we have

$$\sum_{n\in J} a_{jn} \leq \sum_{n\in J_1} a_{jn} + \sum_{n\in J_2} a_{jn}. \tag{19}$$

We can see that

$$st_A - \lim \frac{s-\gamma-1}{n+\gamma+1} = 0$$

and

$$st_A - \lim \frac{2\delta+1}{n+\gamma+1} = 0$$

Using these facts and taking the limit $j \to \infty$, we get

$$\lim_j \sum_{n\in J} a_{jn} = 0$$

which guarantees that

$$st_A - \lim \left\|\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(e_1(t);x)-e_1\right\| = 0.$$

Similarly, one can show that

$$st_A - \lim \left\|\mathcal{D}^{\delta,\gamma}_{n,s,\alpha}(e_2(t);x)-e_2\right\| = 0.$$

This completes the proof. □

6. Conclusion and observation

We constructed the Stancu-type α-Schurer-Kantorovich operators $\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}$ (6) and studied uniform convergence theorem. We also studied order of approximation and rate of convergence by means of suitable modulus of continuity and Peetre's K-functional, respectively, including some approximation results involving the idea of Lipschitz-type function. Finally, in this last section, we studied the approximation result using the notion of A-statistical convergence, where $A = (a_{jn})$ is a non-negative regular matrix.

If we choose $\delta = \gamma = 0$, the operators $\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}$ reduced to α-Schurer-Kantorovich operators $\mathcal{K}_{n,s}^{\alpha}$ defined in [42], in addition, if $\alpha = 1$ then $\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}$ reduces to classical Schurer Kantorovich. Also, for the choice $s = 0$, the operators $\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}$ reduced to Stancu-type α-Bernstein-Kantorovich operators defined in [37], in addition, if $\gamma = \delta = 0$ then the operators $\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}$ reduced to α-Bernstein-Kantorovich operators defined in [32]. Moreover, if we take $\alpha = 1$ and $s = \delta = \gamma = 0$, then the operators $\mathcal{D}_{n,s,\alpha}^{\delta,\gamma}$ reduced to Bernstein-Kantorovich operators [24]. So, we conclude that (6) is a nontrivial generalization of some linear positive operators existing in the literature and so our results as well.

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