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Inverse Nodal Problem for Conformable Sturm-Liouville Operator with Jump Conditions

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Abstract. We consider a conformable fractional Sturm-Liouville problem with the discontinuous(or jump) condition inside the interval. The asymptotic formulas for the eigenvalues, nodal parameters (nodal points and nodal lengths) of this problem are calculated by the modified Prüfer substitutions. Also, using these asymptotic formulas, an explicit formula for the potential functions is given. After all, we discuss Lipschitz stability for the considered problem.

1. Introduction

1.1. *Presentation of the Problem*

Consider the following conformable fractional Sturm-Liouville problem

$$-D_x^\alpha D_x^\alpha y + q(x)y = \lambda^2 y, \qquad x \in [0, L], \tag{1}$$

$$y(0) = y(L) = 0, \tag{2}$$

with discontinuous conditions

$$y\left(\frac{L}{2}+0\right) = cy\left(\frac{L}{2}-0\right), \; D_x^\alpha y\left(\frac{L}{2}+0\right) = c^{-1} D_x^\alpha y\left(\frac{L}{2}-0\right), \tag{3}$$

where λ is a spectral parameter, $q(x) \in L_\alpha^2(0, L)$, $c > 0$ is real, $c \neq 1$. Also, D_x^α is the conformable derivative of order $\alpha \in (0, 1]$. The problem (1)-(2) is known as conformable fractional Sturm-Liouville problem(CFSLP). We reconsider CFSLP when there is a point of discontinuity inside of the interval $[0, L]$.

This paper is organized as follows: in the present section, we research some studies closely related to the present problem and we will give some basic definitions and properties of the conformable fractional calculus theory. In Section 2, we will redefine Prüfer substitutions on the discontinuous case and present their properties. Moreover, we obtain the asymptotic formulas of the eigenvalues and nodal parameters of the CFSLP with discontinuous conditions (1)-(3). Consequently, the limit form of the potential function q will be given using eigenvalues and nodal length. In Section 3, we study the Lipschitz stability of the CFSLP with discontinuous conditions (1)-(3).

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1.2. Researching of Studies on the Problem

In the mathematical model of physical phenomena, partial differential equations are generally encountered. Wherewithal, some conditions that uniquely determine this process are also needed in order to make a mathematical description of a physical process. Such problems are transformed into ordinary differential equations containing a parameter by some methods. One of these equations is the Sturm-Liouville equation.

In literature, we can encounter two types of inverse problems on Sturm-Liouville problems(SLP) as inverse eigenvalue problems and inverse nodal problems. Reconstructing coefficients of boundary conditions and the operator is obtained from spectral characters such as norming constants, spectrum, eigenfunctions in the first problem (see also[11, 12, 18, 20]), from the eigenvalues together with the roots of the eigenfunctions (nodal points) in the second. While the most important contribution to the development of inverse problems was made by Ambartsumyan [4], McLaughlin [33] has been made the first contribution to inverse nodal problems. After McLaughlin, many researchers have been investigated the inverse nodal problem under different conditions [12–14, 16, 19, 26, 29, 31, 40, 50].

In applied mathematics [15, 44], it can be also appeared discontinuous boundary value problems. A considerable number of authors has been widely studied the inverse problem for the discontinuous case [36, 39]. For example, it has been considered reconstructing two potentials functions on the whole interval depending on the parameters in the boundary and jump conditions in [5]; determining uniquely the potential over half the interval and the other boundary condition by the eigenvalues [21, 24, 37]; uniqueness results for inverse problems on Sturm–Liouville operators with a finite number of discontinuities at interior points of the interval [42, 45]. Moreover, many authors have been studied inverse nodal problems on a finite interval with discontinuity conditions inside the interval (such as uniqueness theorems, a constructive procedure for the solution) [28, 43, 46, 48, 49].

Fractional calculus encountered in different fields of engineering and science extensively with a variety of applications [8, 25] defines a generalization of classical calculus. Almost all the fractional derivatives such as Grünwald-Letnikov, Riemann- Liouville, Caputo and Jumarie, Marchaud and Riesz used in the literature fail to satisfy some basic properties. Thus, we prefer the conformable fractional derivative in the present study. it can be found basic properties and main results in [1, 23] and other results in [6, 9, 10, 32, 34, 38, 47] on conformable fractional derivative. If we take into account the form of the problem (1)-(3), the fractional SLP is obtained by replacing the fractional derivative with the ordinary derivative. In recent years, a fractional generalization of the SLPs has been studied by many authors [3, 7, 27, 41], and a great variety of works [2, 22] on inverse problems for this problem.

Since it is closely related to the present problem, we specifically focused on the studies [30, 35]. Koyunbakan and Mosazadeh [30] considered a discontinuous SLP. They defined new Prüfer substitutions and obtained the asymptotic formulas of eigenvalues and nodal parameters and investigated the inverse nodal problem. On the other hand, Mortazaasl and Akbarfam [35] proved the completeness theorem and an expansion theorem for a CFSLP and investigated the inverse nodal problem for this problem with real-valued coefficients on a finite interval.

1.3. Preliminaries

In this part, we give some basic definitions and properties of the conformable fractional(CF) calculus theory that we will use for the rest of the present study.

Definition 1.1. *[1, 23] Consider the function $f : [0, \infty) \to \mathbb{R}$. Then, CF derivative of f order $\alpha \in (0, 1]$ is defined by:*

$$D_x^\alpha f(x) := \lim_{h\to 0} \frac{f(x + hx^{1-\alpha}) - f(x)}{h}.$$

Here, the symbol D_x^α is CF derivative of order α with respect to x.

If f is $\alpha-$differentiable in some $(0, \alpha)$, and $\lim_{x\to 0^+} D_x^\alpha f(x)$ exits, then define $D_x^\alpha f(0) = \lim_{x\to 0^+} D_x^\alpha f(x)$

If f is usual differentiable, then $D_x^\alpha f(x) = x^{1-\alpha} f'(x)$.

Definition 1.2. *[1, 23] Consider the function $f : [0, \infty) \to \mathbb{R}$. Then, CF integral of f order $\alpha \in (0, 1]$ is defined by:*

$$I_\alpha f(x) := \int_0^x f(t) d_\alpha t = \int_0^x t^{\alpha-1} f(t) dt$$

for $x > 0$. Integral to the right of the last equality is the usual Riemann integral.

Theorem 1.3. *[1, 23] Consider two differentiable functions $f, g : [a, b] \to \mathbb{R}$. Then,*

$$\int_a^b f(x) D_x^\alpha g(x) d_\alpha x = fg\Big|_a^b - \int_a^b g(x) D_x^\alpha f(x) d_\alpha x.$$

This formula is called $\alpha-$integration by parts.

Definition 1.4. *[1, 23] Consider $p \in [1, \infty)$, $\alpha > 0$. Denote the space $L_\alpha^p(0, \alpha)$ which consist of all function $f : [0, \alpha) \to \mathbb{R}$ satisfying the condition $\left(\int_0^a |f(x)|^p d_\alpha x \right)^{\frac{1}{p}} < \infty$.*

Theorem 1.5. *[35] Consider the continuous function $f : [x, y] \to \mathbb{R}$. Then, there exists $\xi \in (x, y)$ such that*

$$\int_x^y f(t) d_\alpha t = f(\xi) \left(\frac{y^\alpha - x^\alpha}{\alpha} \right). \tag{4}$$

This formula is called the mean value theorem for the $\alpha-$integrals.

2. Inverse nodal problem

Here, we present the asymptotic expansion of eigenvalues and nodal parameters for the CFSLP with discontinuous conditions (1)-(3). Later, we reconstruct the potential function of this problem as a limit of nodal parameters. To do so, we use the modified Prüfer substitution.

2.1. Modified Prüfer Substitutions and Eigenvalues

In this subsection, since the equation is fractional, we define the similar Prüfer substitution for the discontinuous Sturm-Liouville operator like as [30]. Here, we note that $\Theta(x)$ is not continuous inside the interval. To emphasize the aim of this paper, we took into account some inverse nodal problems [39, 43, 46] for the discontinuous SLP without using this substitutions.

Let us consider the modified Prüfer substitutions for the CFSLP with discontinuous conditions (1)-(3):

$$y(x) = \begin{cases} \rho\left(\frac{x^\alpha}{\alpha}\right) \sin\left(\lambda \Theta(\frac{x^\alpha}{\alpha})\right), & \text{if} \quad 0 < x < \frac{L}{2}, \\ \rho\left(L - \frac{x^\alpha}{\alpha}\right) \sin\left(\lambda \Theta(L - \frac{x^\alpha}{\alpha})\right) & \text{if} \quad \frac{L}{2} < x < L, \end{cases} \tag{5}$$

and

$$D_x^\alpha y(x) = \begin{cases} \lambda \rho\left(\frac{x^\alpha}{\alpha}\right) \cos\left(\lambda \Theta(\frac{x^\alpha}{\alpha})\right), & \text{if} \quad 0 < x < \frac{L}{2}, \\ \lambda \rho\left(L - \frac{x^\alpha}{\alpha}\right) \cos\left(\lambda \Theta(L - \frac{x^\alpha}{\alpha})\right), & \text{if} \quad \frac{L}{2} < x < L, \end{cases} \tag{6}$$

where the functions $\rho(x) := \rho(x,\lambda,\alpha)$ and $\Theta(x) := \Theta(x,\lambda,\alpha)$ defined by a non-zero solution $y(x) := y(x,\lambda,\alpha)$ of the equation (1) are the aptitude and the phase functions, respectively. By applying (5)-(6) to the equation (1), we attain at

$$\begin{cases} D_x^\alpha \Theta(\frac{x^\alpha}{\alpha}) = 1 - \frac{1}{\lambda^2} q(x) \sin^2\left(\lambda \Theta(\frac{x^\alpha}{\alpha})\right), & \text{for} \quad x < \frac{L}{2}, \\ D_x^\alpha \Theta(L - \frac{x^\alpha}{\alpha}) = 1 - \frac{1}{\lambda^2} q(x) \sin^2\left(\lambda \Theta(L - \frac{x^\alpha}{\alpha})\right), & \text{for} \quad x > \frac{L}{2}. \end{cases} \tag{7}$$

The asymptotic expansion of eigenvalues for the CFSLP with discontinuous conditions (1)-(3) is contained in the following theorem.

Theorem 2.1. *Let* $\{\lambda_n\}_{n\geq 1}$ *be eigenvalues of the CFSLP with discontinuous conditions* (1)-(3). *Then, the form*

$$\lambda_n = \frac{n\pi + \gamma_n}{\beta_n} + \frac{\kappa_n}{2n\pi} + O\left(\frac{1}{n^2}\right) \tag{8}$$

holds as $n \longrightarrow \infty$, *where*

$$\beta_n = \begin{cases} \frac{1}{\alpha}\left(\frac{L}{2}\right)^\alpha, & \text{if } n \text{ is even}, \\ \frac{L^\alpha}{\alpha} - \frac{1}{\alpha}\left(\frac{L}{2}\right)^\alpha, & \text{if } n \text{ is odd}, \end{cases} \quad \gamma_n = \begin{cases} \arcsin(\frac{1}{\sqrt{1+c^2}}), & \text{if } n \text{ is even}, \\ -\arcsin(\frac{|c|}{\sqrt{1+c^2}}), & \text{if } n \text{ is odd}, \end{cases}$$

$$\xi = \int_0^L q(x) d_\alpha x, \quad \mu = \int_0^L q(x) d_\alpha x - 2\int_0^{\frac{L}{2}} q(x) d_\alpha x, \quad \kappa_n = \left(\frac{\xi + (-1)^{n-1}\mu}{2}\right).$$

Proof. Since (2), we have $\Theta(0) = 0$. Now, lets assume that $\lambda = \lambda_n$. On the other hand, from (3), we arrive at

$$\begin{cases} \Theta\left(\frac{\left(\frac{L}{2}-0\right)^\alpha}{\alpha}\right) = \frac{1}{\lambda_n}\left(2n\pi + Arcsin(\frac{1}{\sqrt{1+c^2}})\right), & n = 0, 1, 2, 3, \cdots, \\ \Theta\left(\frac{\left(\frac{L}{2}+0\right)^\alpha}{\alpha}\right) = \frac{1}{\lambda_n}\left((2n-1)\pi - Arcsin(\frac{|c|}{\sqrt{1+c^2}})\right), & n = 1, 2, 3, \cdots. \end{cases}$$

Taking α-integral both sides of (7) with respect to $x \in \left(0, \frac{L}{2}\right)$ yields

$$\begin{aligned} \Theta\left(\frac{\left(\frac{L}{2}-0\right)^\alpha}{\alpha}\right) &= \int_0^{\frac{L}{2}} \left(1 - \frac{q(x)}{\lambda_n^2}\sin^2\left(\lambda \Theta\left(\frac{x^\alpha}{\alpha}\right)\right)\right) d_\alpha x \\ &= \frac{1}{\alpha}\left(\frac{L}{2}\right)^\alpha - \frac{1}{2\lambda_n^2}\int_0^{\frac{L}{2}} q(x) d_\alpha x + \frac{1}{2\lambda_n^2}\int_0^{\frac{L}{2}} q(x)\cos\left(2\lambda_n \Theta\left(\frac{x^\alpha}{\alpha}\right)\right) d_\alpha x. \end{aligned} \tag{9}$$

Using $\alpha-$ integration by parts into the last $\alpha-$ integral in the equality (9), we see that

$$\int_0^{\frac{L}{2}} q(x)\cos\left(2\lambda_n \Theta\left(\frac{x^\alpha}{\alpha}\right)\right) d_\alpha x = O\left(\frac{1}{\lambda_n}\right).$$

Substituting this result into the equality (9), we have

$$\Theta\left(\frac{\left(\frac{L}{2}-0\right)^\alpha}{\alpha}\right) = \frac{1}{\alpha}\left(\frac{L}{2}\right)^\alpha - \frac{1}{2\lambda_n^2}\int_0^{\frac{L}{2}} q(x) d_\alpha x + O\left(\frac{1}{\lambda_n^3}\right). \tag{10}$$

Similarly, taking α-integral both sides of (7) with respect to $x \in \left(\frac{L}{2}, L\right)$ yields

$$\Theta\left(\frac{\left(\frac{L}{2}+0\right)^{\alpha}}{\alpha}\right) = \frac{L^{\alpha}}{\alpha} - \frac{1}{\alpha}\left(\frac{L}{2}\right)^{\alpha} - \frac{1}{2\lambda_n^2}\left(\int_0^L - \int_0^{\frac{L}{2}}\right) q(x) d_\alpha x + O\left(\frac{1}{\lambda_n^3}\right). \tag{11}$$

If the equalities (10) and (11) are considered together for $n \geq 1$, we arrive at

$$\lambda_n = n\pi \left\{\beta_n - \frac{\gamma_n}{\lambda_n} - \frac{\kappa_n}{2\lambda_n^2} + O\left(\frac{1}{\lambda_n^3}\right)\right\}^{-1}$$

and thus, since $\lambda_n \approx \frac{n\pi}{\beta_n}$ as $n \to \infty$, we get to the asymptotic expansion (8). □

2.2. Nodal Parameters

Here, we present the asymptotic expansion of nodal parameters. Before presenting these expansions, let us talk about denotes.

Let λ_n be the eigenvalues of the CFSLP with discontinuous conditions (1)-(3) and $y_n(x) = y(x, \lambda_n)$ be the eigenfunctions corresponding to these eigenvalues. By Sturm oscillation theorem, it is well known that $y_n(x)$ has exactly n nodal points in $(0, L)$. So, let us denote the set of these nodal points, $y_n(x_n^j) = 0$, of the CFSLP with discontinuous conditions (1)-(3) with $X = \left\{x_n^j \,\middle|\, n \in \mathbb{N}, j = \overline{1, n}\right\}$ and define $z_n^j = \frac{(x_n^j)^{\alpha}}{\alpha}$. Also, these nodal points can be read as the jth zero corresponding to the nth eigenfunction of $y_n(x)$. In addition, $l_n^j = z_n^{j+1} - z_n^j$ be the nodal length of jth nodal domain $I_n^j = \left[z_n^j, z_n^{j+1}\right]$. Also, $j_n(z)$ is be denoted the largest index j such that $z_n^j \in [0, z]$. Then, $j = j_n(z)$ iff $z \in \left[z_n^j \ , z_n^{j+1}\right)$.

Theorem 2.2. *Let us consider the CFSLP with discontinuous conditions* (1)-(3)*. As $n \to \infty$, the asymptotic expansion of nodal points have the forms*

$$z_n^j = \frac{2j\beta_n}{n} + \frac{\beta_n^2}{2n^2\pi^2} \int_0^{x_n^j} q(t) d_\alpha t + O\left(\frac{1}{n^3}\right), \quad x_n^j \in \left(0, \frac{L}{2}\right), \tag{12}$$

$$z_n^j = \frac{2j\beta_n}{n} - \frac{2j\beta_n\gamma_n}{n^2\pi} + \frac{\beta_n^2}{2n^2\pi^2}\left(\int_0^{x_n^j} q(t) d_\alpha t - \int_0^{\frac{L}{2}} q(t) d_\alpha t\right) + O\left(\frac{1}{n^3}\right), \qquad x_n^j \in \left(\frac{L}{2}, L\right) \tag{13}$$

for $j = \overline{1, n}$.

Proof. Consider the nodal points $x_n^j \in \left(0, \frac{L}{2}\right)$. Then, $\alpha-$integrating the first equality of (7) on $\left(0, x_n^j\right)$, and taking $\lambda = \lambda_n$ yields that

$$\frac{2j\pi}{\lambda_n} = z_n^j - \frac{1}{2\lambda_n^2} \int_0^{x_n^j} q(t) d_\alpha t + \frac{1}{2\lambda_n^2} \int_0^{x_n^j} q(t) \cos\left(2\lambda_n \Theta\left(\frac{t^{\alpha}}{\alpha}\right)\right) d_\alpha t. \tag{14}$$

On the other hand, we arrive at

$$\begin{cases} \frac{1}{\lambda_n} = \frac{\beta_n}{n\pi} - \frac{\beta_n\gamma_n}{n^2\pi^2} - \frac{\beta_n^2\kappa_n}{2n^3\pi^3} + O\left(\frac{1}{n^4}\right), \\ \frac{1}{\lambda_n^2} = \frac{\beta_n^2}{n^2\pi^2} + O\left(\frac{1}{n^3}\right) \end{cases} \tag{15}$$

form Theorem 2.1. So, substituting the equalities (15) into the equality (14), we obtain

$$z_n^j = \frac{2j\beta_n}{n} + \frac{\beta_n^2}{2n^2\pi^2}\int_0^{x_n^j} q(t)d_\alpha t - \frac{\beta_n^2}{2n^2\pi^2}\int_0^{x_n^j} q(t)\cos\left(2\lambda_n\Theta\left(\frac{t^\alpha}{\alpha}\right)\right)d_\alpha t + O\left(\frac{1}{n^3}\right).$$

Since as $n \to \infty$

$$\int_0^{x_n^j} q(t)\cos\left(2\lambda_n\Theta\left(\frac{t^\alpha}{\alpha}\right)\right)d_\alpha t \to 0$$

from Riemann-Lebesgue lemma in last equality, (12) is obtained.

Similarly, for $x_n^j \in \left(\frac{L}{2}, L\right)$, $\alpha-$integrating the second equality of (7) on $\left(\frac{L}{2}, x_n^j\right)$, and taking $\lambda = \lambda_n$ yields that

$$z_n^j = \frac{2j\beta_n}{n} - \frac{2j\beta_n\gamma_n}{n^2\pi} + \frac{\beta_n^2}{2n^2\pi^2}\left(\int_0^{x_n^j} q(t)d_\alpha t - \int_0^{\frac{L}{2}} q(t)d_\alpha t\right)$$
$$- \frac{\beta_n^2}{2n^2\pi^2}\left(\int_0^{x_n^j} q(t)\cos\left(2\lambda_n\Theta\left(\frac{t^\alpha}{\alpha}\right)\right)d_\alpha t - \int_0^{\frac{L}{2}} q(t)\cos\left(2\lambda_n\Theta\left(\frac{t^\alpha}{\alpha}\right)\right)d_\alpha t\right) + O\left(\frac{1}{n^3}\right).$$

Since as $n \to \infty$

$$\int_0^{x_n^j} q(t)\cos\left(2\lambda_n\Theta\left(\frac{t^\alpha}{\alpha}\right)\right)d_\alpha t - \int_0^{\frac{L}{2}} q(t)\cos\left(2\lambda_n\Theta\left(\frac{t^\alpha}{\alpha}\right)\right)d_\alpha t \to 0$$

from Riemann-Lebesgue lemma in last equality, (13) is obtained and this completes the proof. □

Lemma 2.3. *Let us consider the CFSLP with discontinuous conditions* (1)-(3). *As $n \to \infty$, the nodal lengths have the forms*

$$\ell_n^j = \frac{2\beta_n}{n} + \frac{\beta_n^2}{2n^2\pi^2}\int_{x_n^j}^{x_n^{j+1}} q(t)d_\alpha t + o\left(\frac{1}{n^2}\right), \quad x_n^j \in \left(0, \frac{L}{2}\right), \tag{16}$$

$$\ell_n^j = \frac{2\beta_n}{n} - \frac{2\beta_n\gamma_n}{n^2\pi} + \frac{\beta_n^2}{2n^2\pi^2}\int_{x_n^j}^{x_n^{j+1}} q(t)d_\alpha t + o\left(\frac{1}{n^2}\right), \quad x_n^j \in \left(\frac{L}{2}, L\right). \tag{17}$$

Proof. *By using the definition of nodal length $\ell_n^j = z_n^{j+1} - z_n^j$, it is not difficult to complete proof.* □

2.3. Potential Function

The aim of this subsection is to construct the solution of the inverse nodal problem of the CFSLP with discontinuous conditions (1)-(3) from nodal lengths without the need for asymptotic knowledge of eigenfunctions. That is, the limit form of the potential function q will be given.

Firstly, we will prove the uniqueness of the potential function by nodal points.

Theorem 2.4. *Consider the dense sub-nodal sets X and $\widetilde{X}$ corresponding to $q(x)$ and $\widetilde{q}(x)$, in problem (1)-(3), respectively. If $X = \widetilde{X}$ such that for all k, then there exist n_k and $\widetilde{n}_k$ such that $X_{n_k} = \widetilde{X}_{\widetilde{n}_k}$. That is, $n_k = \widetilde{n}_k$ for all large k. Furthermore, $q(x) = \widetilde{q}(x)$ on $(0, L)$ almost everywhere.*

Proof. Consider the sufficiently large n_k and $\widetilde{n}_k$ such that $n_k \geq \widetilde{n}_k$. Since $X_{n_k} = \widetilde{X}_{\widetilde{n}_k}$, there exist $\widetilde{m}_1$ and $\widetilde{m}_2$ such that $z_{n_k}^{j_k} = \widetilde{z}_{\widetilde{n}_k}^{\widetilde{m}_1}$ and $z_{n_k}^{j_k+1} = \widetilde{z}_{\widetilde{n}_k}^{\widetilde{m}_2}$. By using the (12) for sufficiently large k, we get

$$z_{n_k}^{j_k+1} - z_{n_k}^{j_k} = \frac{2\beta_{n_k}}{n_k} + o\left(\frac{1}{n_k^2}\right) \qquad \text{or} \qquad \widetilde{z}_{\widetilde{n}_k}^{\widetilde{m}_2} - \widetilde{z}_{\widetilde{n}_k}^{\widetilde{m}_1} = \frac{2\left(\widetilde{m}_2 - \widetilde{m}_1\right)\beta_{n_k}}{\widetilde{n}_k} + o\left(\frac{1}{\widetilde{n}_k^2}\right).$$

From these equalities, as $n_k, \widetilde{n}_k \to \infty$, we obtain $\frac{\widetilde{n}_k}{n_k(\widetilde{m}_2 - \widetilde{m}_1)} = 1 + o(1)$. Namely, $n_k = \widetilde{n}_k$, $\widetilde{m}_2 = \widetilde{m}_1 + 1$. Therefore, $z_{n_k}^{j_k} = \widetilde{z}_{n_k}^{\widetilde{m}_1}$ and $z_{n_k}^{j_k+1} = \widetilde{z}_{n_k}^{\widetilde{m}_1+1}$.

If (12) is taken into account again, the equalities

$$z_{n_k}^{j_k} = \frac{2 j_k \beta_{n_k}}{n_k} + o\left(\frac{1}{n_k^2}\right) \qquad \text{or} \qquad \widetilde{z}_{\widetilde{n}_k}^{\widetilde{m}_1} = \frac{2\widetilde{m}_1 \beta_{n_k}}{n_k} + o\left(\frac{1}{n_k^2}\right)$$

hold. Namely, since $z_{n_k}^{j_k} = \widetilde{z}_{\widetilde{n}_k}^{\widetilde{m}_1}$, $j_k = \widetilde{m}_1$. Hereby, $z_{n_k}^{j_k} = \widetilde{z}_{n_k}^{j_k}$.

On the other hand, using the asymptotic formula (8) and from [17, 35], as $n \to \infty$, the nth eigenfunction

$$y_n(x) = \cos\left(\frac{n\pi}{\alpha\beta_n} x^\alpha\right) + O\left(\frac{1}{n}\right), \qquad x \in \left(0, \frac{L}{2}\right),$$

$$y_n(x) = b_1 \cos\left(\frac{n\pi}{\alpha\beta_n} x^\alpha\right) + b_2 \cos\left(\frac{n\pi}{\alpha\beta_n}(\alpha L - x^\alpha)\right) + O\left(\frac{1}{n}\right), \qquad x \in \left(\frac{L}{2}, L\right)$$

hold, where $b_1 = \frac{c + c^{-1}}{2}$ and $b_2 = \frac{c - c^{-1}}{2}$.

Let's proof the uniqueness of the potential function for the case $x \in \left(0, \frac{L}{2}\right)$, the other case can be proved in a similar way.

Consider $\left\{x_{n_k}^{j_k}\right\}$ be a subsequence of X that convergences to x for $\forall x \in \left(0, \frac{L}{2}\right)$. From the $\alpha-$Green's formula [3], the Riemann-Lebesgue lemma and

$$y_n(x)\widetilde{y}_n(x) = \frac{1}{2}\left(1 + \cos\left(2\frac{n\pi}{\alpha\beta_n} x^\alpha\right)\right) + O\left(\frac{1}{n}\right),$$

we obtain

$$0 = \int_0^{x_{n_k}^{j_k}} \left(\left\{D_x^\alpha D_x^\alpha y_{n_k}(x)\right\} \widetilde{y}_{n_k}(x) - y_{n_k}(x) D_x^\alpha D_x^\alpha \widetilde{y}_{n_k}(x)\right) d_\alpha x = \int_0^{x_{n_k}^{j_k}} \left(q(x) - \widetilde{q}(x) - \lambda_{n_k}^2 + \widetilde{\lambda}_{n_k}^2\right) y_{n_k}(x) \widetilde{y}_{n_k}(x) d_\alpha x$$

$$= \frac{1}{2} \int_0^{x_{n_k}^{j_k}} \left(q(x) - \widetilde{q}(x) - \frac{\kappa_{n_k}^2 - \widetilde{\kappa}_{n_k}^2}{2 n_k \pi}\right) d_\alpha x + o(1).$$

That is, as $n_k \to \infty$, we get

$$\int_0^x \left(q(t) - \widetilde{q}(t)\right) d_\alpha t = 0, \qquad \forall x \in \left(0, \frac{L}{2}\right).$$

Consequently, $q(x) = \widetilde{q}(x)$ on $\left(0, \frac{L}{2}\right)$ almost everywhere. So, the proof is completed. □

Theorem 2.5. *Given the nodal set X, then the potential function $q \in L^1_\alpha[0,\pi]$ for the CFSLP with discontinuous conditions* (1)-(3) *can be reconstructed by the following formulas:*

$$q(x) = \lim_{n\to\infty} \frac{2n^2\pi^2}{\beta_n^2}\left(\frac{n}{2\beta_n}\ell_n^j - 1\right), \qquad x \in \left(0, \frac{L}{2}\right)$$

$$q(x) = \lim_{n\to\infty} \frac{2n^2\pi^2}{\beta_n^2}\left(\frac{n}{2\beta_n}\ell_n^j - 1 + \frac{\gamma_n}{n\pi}\right), \qquad x \in \left(\frac{L}{2}, L\right)$$

for $j = j_n(x) = \max\left\{j : z_n^j < x\right\}$.

Proof. From (4), applying the mean value theorem for the $\alpha-$integrals to (16), with fixed n, there exists $\xi \in \left(x_n^j, x_n^{j+1}\right)$ such that $\int\limits_{x_n^j}^{x_n^{j+1}} q(t)d_\alpha t = q(\xi)\ell_n^j$. Hereby, from (16) we arrive at

$$\ell_n^j = \frac{2\beta_n}{n} + \frac{\beta_n^2 q(\xi)\ell_n^j}{2n^2\pi^2} \quad \text{or} \quad q(\xi) = \frac{4n\pi^2}{\beta_n\ell_n^j}\left(\frac{n}{2\beta_n}\ell_n^j - 1\right) + o(1), \quad x_n^j \in \left(0, \frac{L}{2}\right).$$

Using the same method used above, we get

$$\ell_n^j = \frac{2\beta_n}{n} - \frac{2\beta_n\gamma_n}{n^2\pi} + \frac{\beta_n^2 q(\xi)\ell_n^j}{2n^2\pi^2} \quad \text{or} \quad q(\xi) = \frac{4n\pi^2}{\beta_n\ell_n^j}\left(\frac{n}{2\beta_n}\ell_n^j - 1 + \frac{\gamma_n}{n\pi}\right), \quad x_n^j \in \left(\frac{L}{2}, L\right)$$

from (17). Finally, as $n \to \infty$, we arrive at end of the proof. □

3. Stability of Inverse Nodal Problem

The aim of this section is to investigate the stability of the considered problem. What we mean by stability is the Lipschitz stability [31, 49]. It is taken into account that we will give Lipschitz stability for CFSLP with discontinuous conditions (1)-(3). Besides, for $\alpha = 1$, the results convert to results of classical case (i.e, SLP with discontinuous conditions) [30].

Definition 3.1. *Let* $W_1 = [0\ , \frac{L}{2})$, $W_2 = \left(\frac{L}{2}\ , L\right]$ *and* $N_1 = \{2, 3, 4, \cdots\}$.

(i) *For* $k = 1, 2$, Ω_k *is the set of all potentials* $q \in L^1_\alpha(W_k)$ *such that the limits* $q\left(\frac{L}{2} \pm 0\right) = \lim\limits_{x\to\frac{L}{2}\pm 0} q(x)$ *are finite, and* $\Omega := \Omega_1 \cup \Omega_2$.

(ii) $\mathcal{S}_1$ *is the set of all sequences* $X^{(1)} = \left\{\left(X_j^{(n,1)}\right)^\alpha\right\}$, $j = 1, 2, ..., n-1$, $n \in N_1$, *such that* $0 < \left(X_1^{(n,1)}\right)^\alpha < ... < \left(X_{n-1}^{(n,1)}\right)^\alpha < \frac{L}{2}$,
and $\mathcal{S}_2$ *is the set of all sequences* $X^{(2)} = \left\{\left(X_j^{(n,2)}\right)^\alpha\right\}$, $j = 1, 2, ..., n-1$, $n \in N_1$, *such that* $\frac{L}{2} < \left(X_1^{(n,2)}\right)^\alpha < ... < \left(X_{n-1}^{(n,2)}\right)^\alpha < L$.

(iii) *For* $X^{(k)} = \left\{\left(X_j^{(n,k)}\right)^\alpha\right\} \in \mathcal{S}_k$, $k = 1, 2$, $\quad I_j^{(n,k)} := \left(\left(X_j^{(n,k)}\right)^\alpha, \left(X_{j+1}^{(n,k)}\right)^\alpha\right)$, $\quad L_j^{(n,k)} := \left(X_{j+1}^{(n,k)}\right)^\alpha - \left(X_j^{(n,k)}\right)^\alpha$.

Definition 3.2. *For* $k = 1, 2$, *suppose* $X^{(k)}, \overline{X}^{(k)} \in \mathcal{S}_k$ *with* $L_j^{(n,k)}$ *and* $\overline{L}_j^{(n,k)}$ *as their respective grid lengths. Let*

$$S_{n,k}\left(X^{(k)}, \overline{X}^{(k)}\right) = \frac{n^2\pi^2}{2\beta_n^3}\sum_{j=0}^{n-1}\left|L_j^{(n,k)} - \overline{L}_j^{(n,k)}\right|.$$

Definition 3.3. *Denote d and $d_{\mathcal{S}}$ as two metrics on $\mathcal{S}_k$ the forms*

$$d(X^{(k)}, \overline{X}^{(k)}) = \overline{\lim_{n\to\infty}} S_{n,k}(X^{(k)}, \overline{X}^{(k)}),$$

$$d_{\mathcal{S}}(X^{(k)}, \overline{X}^{(k)}) = \overline{\lim_{n\to\infty}} \frac{S_{n,k}(X^{(k)}, \overline{X}^{(k)})}{1 + S_{n,k}(X^{(k)}, \overline{X}^{(k)})} = \frac{d(X^{(k)}, \overline{X}^{(k)})}{1 + d(X^{(k)}, \overline{X}^{(k)})},$$

respectively.

For $k = 1, 2$, we can also obtain

$$d(X^{(k)}, \overline{X}^{(k)}) = \frac{d_{\mathcal{S}}(X^{(k)}, \overline{X}^{(k)})}{1 - d_{\mathcal{S}}(X^{(k)}, \overline{X}^{(k)})}, \quad d_{\mathcal{S}}(X^{(k)}, \overline{X}^{(k)}) \leq d(X^{(k)}, \overline{X}^{(k)}).$$

Definition 3.4. *Denote $\sim$ as a relation on $\mathcal{S}_k$ the form*

$$X^{(k)} \sim \overline{X}^{(k)} \iff d_{\mathcal{S}}(X^{(k)}, \overline{X}^{(k)}) = 0.$$

Hereby, $\sim$ is an equivalence relation on $\mathcal{S}_k$. From here, $d_{\mathcal{S}}$ would be a metric for the partition set $\mathcal{S}_k^* := \mathcal{S}_k/\sim$.

Now, let $\mathcal{S}_{k,1} \subset \mathcal{S}_k$ be the subspace of all asymptotically equivalent nodal sequences, and let $\mathcal{S}_{k,1}^* := \mathcal{S}_{k,1}/\sim$, $\mathcal{S}^* = \mathcal{S}_{1,1}^* \cup \mathcal{S}_{2,1}^*$.

Lemma 3.5. *Let $X^{(k)}, \overline{X}^{(k)} \in \mathcal{S}_{k,1}$. Then,*

(i) *The interval $I_j^{(n,k)}$ between the points $X_j^{(n,k)}$ and $\overline{X}_j^{(n,k)}$ has the length $O\left(\frac{1}{n^2}\right)$.*

(ii) *For all $x \in W_k$, $|j_{n,k}(x) - \overline{j}_{n,k}(x)| \leq 1$ for sufficiently large n.*

Proof. Since $X^{(k)}, \overline{X}^{(k)} \in \mathcal{S}_{k,1}$, we arrive at

$$\left|I_j^{(n,k)}\right| \leq \left|\left(X_j^{(n,k)}\right)^\alpha - \frac{2j\beta_n}{n}\right| + \left|\frac{2j\beta_n}{n} - \left(\overline{X}_j^{(n,k)}\right)^\alpha\right| = O\left(\frac{1}{n^2}\right).$$

Hence, we obtain (i).

Now, fix $x \in W_k$ and suppose that $j_k = j_{n,k}(x)$, $\overline{j}_k = \overline{j}_{n,k}(x)$. Since

$$\frac{2j_k\beta_n}{n} + O\left(\frac{1}{n^2}\right) = \left(X_{j_k}^{(n,k)}\right)^\alpha \leq x \leq \left(X_{j_k+1}^{(n,k)}\right)^\alpha = \frac{2\,(j_k+1)\,\beta_n}{n} + O\left(\frac{1}{n^2}\right),$$

$$\frac{2\overline{j}_k\beta_n}{n} + O\left(\frac{1}{n^2}\right) = \left(\overline{X}_{\overline{j}_k}^{(n,k)}\right)^\alpha \leq x \leq \left(\overline{X}_{\overline{j}_k+1}^{(n,k)}\right)^\alpha = \frac{2\left(\overline{j}_k+1\right)\beta_n}{n} + O\left(\frac{1}{n^2}\right),$$

we get $j_k \leq \overline{j}_k + 1$ and $\overline{j}_k \leq j_k + 1$ for sufficiently large n. In other words, $|j_k - \overline{j}_k| \leq 1$. We arrive at (*ii*), and this completes the proof. □

Theorem 3.6. *Let $X^{(k)}, \overline{X}^{(k)} \in \mathcal{S}_{k,1}$, be asymptotically nodal to q and $\overline{q}$ in Ω_k, for $k = 1, 2$, respectively. Then, $\|q - \overline{q}\|_{L_\alpha^1} \leq 2d(X^{(k)}, \overline{X}^{(k)})$, where $\|.\|_{L_\alpha^1} = \int_a^b |.|d_\alpha x < \infty$.*

Proof. The proof is planned to be done only for $k = 1$. The proof for the other case is similar. In that case, from (8) and Theorem 2.5, we arrive at

$$\begin{aligned} q(x) - \overline{q}(x) &= \lim_{n\to\infty} \frac{2n^2\pi^2}{\beta_n^2}\left(\frac{n}{2\beta_n} L^{(n,1)}_{j_{n,1}(x)} - 1\right) - \lim_{n\to\infty} \frac{2n^2\pi^2}{\beta_n^2}\left(\frac{n}{2\beta_n} \overline{L}^{(n,1)}_{\overline{j}_{n,1}(x)} - 1\right) \\ &= \lim_{n\to\infty} \frac{n^3\pi^2}{\beta_n^3}\left(L^{(n,1)}_{j_{n,1}(x)} - \overline{L}^{(n,1)}_{\overline{j}_{n,1}(x)}\right) \end{aligned}$$

on W_1. Hereby, by Fatou's lemma we have

$$\begin{aligned} \int_0^{\frac{\pi}{2}} |q(x) - \overline{q}(x)| d_\alpha x &\le \underline{\lim}_{x\to\infty} \frac{n^3\pi^2}{\beta_n^3} \int_0^{\frac{\pi}{2}} \left|\overline{L}^{(n,1)}_{\overline{j}_{n,1}(x)} - L^{(n,1)}_{j_{n,1}(x)}\right| d_\alpha x \\ &\le \frac{\pi^2}{\beta_n^3} \overline{\lim_{n\to\infty}} \left\{ n^3 \int_0^{\frac{\pi}{2}} \left|\overline{L}^{(n,1)}_{\overline{j}_{n,1}(x)} - \overline{L}^{(n,1)}_{j_{n,1}(x)}\right| d_\alpha x + n^3 \int_0^{\frac{\pi}{2}} \left|\overline{L}^{(n,1)}_{j_{n,1}(x)} - L^{(n,1)}_{j_{n,1}(x)}\right| d_\alpha x \right\}. \end{aligned} \tag{18}$$

Since

$$n^3 \int_0^{\frac{\pi}{2}} \left|\overline{L}^{(n,1)}_{\overline{j}_{n,1}(x)} - \overline{L}^{(n,1)}_{j_{n,1}(x)}\right| d_\alpha x = o(1),$$

the first term in the right side of (18) will vanish after taking the limit supremum. Moreover, for $1 \le j \le n-2$,

$$L^{(n,1)}_j = \frac{2\beta_n}{n} + \frac{\beta_n^2}{2n^2\pi^2} \int_{X^{(n,1)}_j}^{X^{(n,1)}_{j+1}} q(\tau) d_\alpha \tau + +o\left(\frac{1}{n^2}\right),$$

which implies that

$$\left|L^{(n,1)}_j - \overline{L}^{(n,1)}_j\right| = \frac{\beta_n^2}{2n^2\pi^2} \left| \int_{X^{(n,1)}_j}^{X^{(n,1)}_{j+1}} q(\tau) d_\alpha \tau - \int_{\overline{X}^{(n,1)}_j}^{\overline{X}^{(n,1)}_{j+1}} \overline{q}(\tau) d_\alpha \tau \right| + o\left(\frac{1}{n^3}\right). \tag{19}$$

Thus, $\left|L^{(n,1)}_j - \overline{L}^{(n,1)}_j\right| = o\left(\frac{1}{n^2}\right)$. Hereby, by virtue of (18) we obtain

$$\begin{aligned} \int_0^{\frac{\pi}{2}} |q(x) - \overline{q}(x)| d_\alpha x &\le \frac{\pi^2}{\beta_n^3} \overline{\lim_{n\to\infty}} \sum_{j=0}^{n-1} n^3 L^{(n,1)}_j \left|L^{(n,1)}_j - \overline{L}^{(n,1)}_j\right| \\ &\le \frac{\pi^2}{\beta_n^3} \overline{\lim_{n\to\infty}} \sum_{j=0}^{n-1} n^2 \left|L^{(n,1)}_j - \overline{L}^{(n,1)}_j\right| = 2d(X^{(1)}, \overline{X}^{(1)}). \end{aligned}$$

This completes the proof. □

Corollary 3.7. *Let the asymptotically nodal points X and $\overline{X}$ to q and $\overline{q}$ in Ω respectively. Then, the following relation holds:*

$$\|q - \overline{q}\|_{L^1_\alpha} \le 2d(X, \overline{X})$$

from Theorems 3.6.

Theorem 3.8. *Let $X^{(k)}$ and $\overline{X}^{(k)}$ be asymptotically nodal to q and $\overline{q}$ in Ω_k, for $k = 1, 2$, respectively. Then, the relation $d(X^{(k)}, \overline{X}^{(k)}) \leq \frac{1}{4\beta_n}\|q - \overline{q}\|_{L^1_\alpha}$ holds.*

Proof. We prove the case when $k = 1$. For $k = 2$, the proof is similar.

By virtue of (19) and Definition 3.2, we arrive at

$$
\begin{aligned}
S_{n,1}\left(X^{(1)}, \overline{X}^{(1)}\right) &= \frac{1}{4\beta_n}\sum_{j=0}^{n-1}\left|\int_{X_j^{(n,1)}}^{X_{j+1}^{(n,1)}} q(\tau)d_\alpha\tau - \int_{\overline{X}_j^{(n,1)}}^{\overline{X}_{j+1}^{(n,1)}} \overline{q}(\tau)d_\alpha\tau\right| + o(1) \\
&\leq \frac{1}{4\beta_n}\sum_{j=0}^{n-1}\left\{\left|\int_{X_j^{(n,1)}}^{X_{j+1}^{(n,1)}} (q(\tau) - \overline{q}(\tau))d_\alpha\tau\right| + \left|\left(\int_{X_j^{(n,1)}}^{X_{j+1}^{(n,1)}} - \int_{\overline{X}_j^{(n,1)}}^{\overline{X}_{j+1}^{(n,1)}}\right)\overline{q}(\tau)d_\alpha\tau\right|\right\} + o(1) \\
&\leq \frac{1}{4\beta_n}\|q - \overline{q}\|_{L^1_\alpha} + \sum_{j=0}^{n-1}\int_{\widetilde{I}_j^{(n,1)}} |\overline{q}(\tau)|d_\alpha\tau + o(1),
\end{aligned}
$$

where $\widetilde{I}_j^{(n,1)} = I_j^{(n,1)} \cup I_{j+1}^{(n,1)}$, and $I_j^{(n,1)}$ is the interval bounded by $X_j^{(n,1)}$ and $\overline{X}_j^{(n,1)}$. Let $I^{(n,1)} = \cup_{j=0}^{n}\widetilde{I}_j^{(n,1)}$. By Lemma 3.5, we get

$$
|I^{(n,1)}| = \sum_{j=0}^{n} |\widetilde{I}_j^{(n,1)}| = O\left(\frac{1}{n}\right).
$$

Consequently,

$$
\overline{\lim_{x\to\infty}} \int_{I^{(n,1)}} |q(\tau)|d_\alpha\tau = 0,
$$

and thus $d(X^{(1)}, \overline{X}^{(1)}) \leq \frac{1}{4\beta_n}\|q - \overline{q}\|_{L^1_\alpha}$. □

Theorem 3.9. *The metric spaces $(\Omega, \|.\|_{L^1_\alpha})$ and $(\mathcal{S}^*, d_\mathcal{S})$ are homeomorphic to each other.*

Proof. Corollary 3.7 and Theorem 3.8 yield that $d_\mathcal{S}(X, \overline{X}) = 0$ iff $q = \overline{q}$. Hence, the partition $\mathcal{S}_1^*$ is in one-one correspondence with $\Omega = \Omega_1 \cup \Omega_2$. Let $d_\mathcal{S}(X, \overline{X}) < \frac{1}{2}$. It follows from $d = \frac{d_\mathcal{S}}{1-d_\mathcal{S}}$ and Corollary 3.7 that

$$
\|q - \overline{q}\|_{L^1_\alpha} \leq \frac{1}{2\beta_n} d_\mathcal{S}(X, \overline{X}). \tag{20}
$$

On the other hand, it follows from Theorem 3.8 that if $\|q - \overline{q}\|_{L^1_\alpha}$ is small, then $d_\mathcal{S}(X, \overline{X})$ is also small. This and (20) complete the proof. □

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