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A Complete Convergence Theorem of the Maximum of Partial Sums Under the Sub-Linear Expectations

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Abstract. Let $\{X, X_n; n \geq 0\}$ be a sequence of independent and identically distributed random variables in a sub-linear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. We establish a complete convergence theorem of the maximum of partial sums $\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right|$ under optimal moment condition in a sub-linear expectation space. Our result generalizes and improves the corresponding results.

1. Introduction

In the probability space, let $1 < \alpha \leq 2, \gamma > 0$ and let $\{X, X_n; n \geq 1\}$ be a sequence of negatively associated and identically distributed random variables with $E(X) = 0$. Sung [1] proved that if

$$\begin{cases} E|X|^\gamma < \infty & \text{for } \gamma > \alpha, \\ E|X|^\alpha \log(|X|+2) < \infty & \text{for } \gamma = \alpha, \\ E|X|^\alpha < \infty & \text{for } \gamma < \alpha, \end{cases} \tag{1.1}$$

then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon n^{1/\alpha}\log^{1/\gamma} n\right) < \infty, \tag{1.2}$$

where $\{a_{ni}; 1 \leq i \leq n, n \geq 1\}$ is an array of real numbers satisfying

$$\sup_{n\geq 1}\frac{\sum_{i=1}^{n}|a_{ni}|^\alpha}{n} < \infty. \tag{1.3}$$

Here and thereafter, log denotes the logarithm to the base 2. Chen and Sung [2] proved that $E|X|^\gamma < \infty$ is optimal moment condition for (1.2) when $\gamma > \alpha$ and obtained an almost optimal condition $E|X|^\alpha \log^{1-\alpha/\gamma}(|X|+2) < \infty$ for (1.2) when $\gamma < \alpha$. They put forward an open question of finding optimal moment condition for (1.2) when $\gamma \leq \alpha$.

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In this paper, we provide the necessary and sufficient conditions in a sub-linear expectation space for

$$\sum_{n=1}^{\infty} n^{-1}\mathbb{V}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon n^{1/\alpha}\tilde{L}(n^{1/\alpha})\right) < \infty, \tag{1.4}$$

where $\tilde{L}(.)$ is the de Bruijn conjugate of a slowly varying function $L(.)$ defined on $[A,\infty)$ for some $A > 0$. By letting $L(x) = \log^{-1/\gamma}(x), x \ge 2$, we can obtain optimal moment condition for (1.2) in a sub-linear expectation space. Our result generalizes and improves the corresponding results of Sung [1], Chen and Sung [2].

In the classical probability space, the additivity of the probability and the expectation is assumed. But in practice, such additivity assumption is not feasible in many areas of applications because the uncertain phenomena can not be modeled by using additive probability or additive expectation. To model uncertain phenomena in many areas, such as economics, finance and insurance, Peng [3-4] introduced the general framework of the sub-linear expectation in a general function space. Kuczmaszewska [5], Xi et al. [6] and Feng et al. [7] all studied the complete convergence theorems under the sub-linear expectations. But there are few results about complete convergence theorems of the maximum of partial sums in sub-linear expectation space. We will investigate this aspect.

The sub-linear expectation is a nonlinear expectation. We can not use the additivity of the probability and the linear property of expectation in a sub-linear expectation space. Many powerful methods in the probability space are no longer valid in sub-linear expectation space. For example, "the divergent part" of Borel-Cantelli lemma is no longer valid. When proving the necessary moment condition of the complete convergence, we can not use "the divergent part" of Borel-Cantelli lemma, but need to use a more skilled method to prove it in sub-linear expectation space. There is no perfect Rosenthal inequality in the sub-linear expectation space as that in the probability space. The Rosenthal inequality in the sublinear expectation space contains the upper and lower expectation parts, which need to be handled skillfully when used, and so on. The study of complete convergence theorems of the maximum of partial sums under sub-linear expectations becomes much more complex and challenging.

Throughout this paper, C stands for positive constant which may differ from one place to another and $I(.)$ denotes an indicator function. Let $L(.)$ be a slowly varying function. Then by Theorem 1.5.13 of Bingham et al. [8], there exists a slowly varying function $\tilde{L}(.)$, unique up to asymptotic equivalence, satisfying

$$\lim_{x\to\infty} L(x)\tilde{L}(xL(x)) = 1 \text{ and } \lim_{x\to\infty} \tilde{L}(x)L(x\tilde{L}(x)) = 1. \tag{1.5}$$

The function $\tilde{L}$ is called the de Bruijn conjugate of L, and $(L,\tilde{L})$ is called a (slowly varying) conjugate pair (see, e.g., Bingham et al. [8] p. 29). We can chose $\tilde{L}(x) = 1/L(x)$.

2. Preliminaries

We use the framework and notations of Peng [3]. Let $(\Omega,\mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega,\mathcal{F})$ such that if $X_1,\cdots,X_n \in \mathcal{H}$ then $\varphi(X_1,\cdots,X_n) \in \mathcal{H}$ for each $\varphi \in C_{l.Lip}(\mathbb{R}^n)$, where $C_{l.Lip}(\mathbb{R}^n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y})| \le C(1 + |\boldsymbol{x}|^m + |\boldsymbol{y}|^m)|\boldsymbol{x} - \boldsymbol{y}|, \quad \forall \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^n,$$

for some $C > 0, m \in \mathbb{N}$ depending on φ. $\mathcal{H}$ is considered as a space of "random variables". If X is an element of set $\mathcal{H}$, then we denote $X \in \mathcal{H}$.

Definition 2.1. *(Peng [3]) A sub-linear expectation $\widehat{\mathbb{E}}$ on $\mathcal{H}$ is a function $\widehat{\mathbb{E}} : \mathcal{H} \to \bar{\mathbb{R}}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have*
(a) Monotonicity: If $X \ge Y$ then $\widehat{\mathbb{E}}[X] \ge \widehat{\mathbb{E}}[Y]$;
(b) Constant preserving: $\widehat{\mathbb{E}}[c] = c$;
(c) Sub-additivity: $\widehat{\mathbb{E}}[X+Y] \le \widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$ whenever $\widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;

(d) Positive homogeneity: $\widehat{\mathbb{E}}[\lambda X] = \lambda\widehat{\mathbb{E}}[X], \lambda > 0$.
Here $\bar{\mathbb{R}} = [-\infty, +\infty]$. *The triple* $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ *is called a sub-linear expectation space. Given a sub-linear expectation* $\widehat{\mathbb{E}}$, *let us denote the conjugate expectation* $\widehat{\mathcal{E}}$ *of* $\widehat{\mathbb{E}}$ *by*

$$\widehat{\mathcal{E}}[X] := -\widehat{\mathbb{E}}[-X], \ \ \forall X \in \mathcal{H}.$$

From the definition, we can easily get that $\widehat{\mathcal{E}}[X] \leq \widehat{\mathbb{E}}[X]$, $\widehat{\mathbb{E}}[X + c] = \widehat{\mathbb{E}}[X] + c$, $\widehat{\mathbb{E}}[X - Y] \geq \widehat{\mathbb{E}}[X] - \widehat{\mathbb{E}}[Y]$ and $|\widehat{\mathbb{E}}[X] - \widehat{\mathbb{E}}[Y]| \leq \widehat{\mathbb{E}}[|X - Y|]$. Further, if $\widehat{\mathbb{E}}[|X|]$ is finite, then $\widehat{\mathcal{E}}[X]$ and $\widehat{\mathbb{E}}[X]$ are both finite.

Definition 2.2. *(Peng [3])*
(i) (Identical distribution) Let $\boldsymbol{X_1}$ *and* $\boldsymbol{X_2}$ *be two n-dimensional random vectors defined respectively in sub-linear expectation spaces* $(\Omega_1, \mathcal{H}_1, \widehat{\mathbb{E}}_1)$ *and* $(\Omega_2, \mathcal{H}_2, \widehat{\mathbb{E}}_2)$. *They are called identically distributed, denoted by* $\boldsymbol{X_1} \overset{d}{=} \boldsymbol{X_2}$, *if* $\widehat{\mathbb{E}}_1[\varphi(\boldsymbol{X_1})] = \widehat{\mathbb{E}}_2[\varphi(\boldsymbol{X_2})]$, $\ \forall \varphi \in C_{l.Lip}(\mathbb{R}^n)$, *whenever the sub-expectations are finite.*
(ii) (Independence) In a sub-linear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, *a random vector* $\boldsymbol{Y} = (Y_1, Y_2, \cdots, Y_n)$, $Y_i \in \mathcal{H}$ *is said to be independent to another random vector* $\boldsymbol{X} = (X_1, X_2, \cdots, X_m)$, $X_i \in \mathcal{H}$ *under* $\widehat{\mathbb{E}}$ *if for each test function* $\varphi \in C_{l.Lip}(\mathbb{R}^m \times \mathbb{R}^n)$ *we have* $\widehat{\mathbb{E}}[\varphi(\boldsymbol{X}, \boldsymbol{Y})] = \widehat{\mathbb{E}}[\widehat{\mathbb{E}}[\varphi(\boldsymbol{x}, \boldsymbol{Y})]|_{\boldsymbol{x}=\boldsymbol{X}}]$, *whenever* $\bar{\varphi}(\boldsymbol{x}) := \widehat{\mathbb{E}}[|\varphi(\boldsymbol{x}, \boldsymbol{Y})|] < \infty$ *for all* $\boldsymbol{x}$ *and* $\widehat{\mathbb{E}}[|\bar{\varphi}(\boldsymbol{X})|] < \infty$.
(iii) (IID random variables) A sequence of random variables $\{X_n; n \geq 1\}$ *is said to be independent if* X_{i+1} *is independent to* $(X_1, X_2, \cdots, X_i)$ *for each* $i \geq 1$, *and it is said to be identically distributed if* $X_i \overset{d}{=} X_1$, *for each* $i \geq 1$.

Next, we introduce the capacities corresponding to the sub-linear expectations. Let $\mathcal{G} \subset \mathcal{F}$. A function $V : \mathcal{G} \to [0, 1]$ is called a capacity if

$$V(\phi) = 0, \ V(\Omega) = 1, \ \text{ and } V(A) \leq V(B) \ \ \forall A \subseteq B, A, B \in \mathcal{G}.$$

It is called to be sub-additive if $V(A \cup B) \leq V(A) + V(B)$ for all $A, B \in \mathcal{G}$ with $A \cup B \in \mathcal{G}$.

Let $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ be a sub-linear expectation space, and $\widehat{\mathcal{E}}$ be the conjugate expectation of $\widehat{\mathbb{E}}$. We denote a pair $(\mathbb{V}, \mathcal{V})$ of capacities by

$$\mathbb{V}(A) := \inf\{\widehat{\mathbb{E}}[\xi] : I(A) \leq \xi, \xi \in \mathcal{H}\}, \ \mathcal{V}(A) := 1 - \mathbb{V}(A^c), \ \forall A \in \mathcal{F},$$

where A^c is the complement set of A. It is obvious that $\mathbb{V}$ is sub-additive and

$$\mathbb{V}(A) := \widehat{\mathbb{E}}[I(A)], \ \mathcal{V}(A) := \widehat{\mathcal{E}}[I(A)], \ \text{if } I(A) \in \mathcal{H},$$

$$\widehat{\mathbb{E}}[f] \leq \mathbb{V}(A) \leq \widehat{\mathbb{E}}[g], \ \widehat{\mathcal{E}}[f] \leq \mathcal{V}(A) \leq \widehat{\mathcal{E}}[g], \ \text{if } f \leq I(A) \leq g, f, g \in \mathcal{H}. \tag{2.1}$$

This implies Markov inequality: $\forall X \in \mathcal{H}$,

$$\mathbb{V}(|X| \geq x) \leq \widehat{\mathbb{E}}[|X|^p]/x^p, \ \ \forall x > 0, p > 0$$

from $I(|X| \geq x) \leq |X|^p/x^p \in \mathcal{H}$. By Lemma 4.1 in Zhang [9], we have Hölder inequality: $\forall X, Y \in \mathcal{H}$, $p, q > 1$, satisfying $p^{-1} + q^{-1} = 1$,

$$\widehat{\mathbb{E}}[|XY|] \leq (\widehat{\mathbb{E}}[|X|^p])^{\frac{1}{p}} (\widehat{\mathbb{E}}[|Y|^q])^{\frac{1}{q}},$$

particularly, Jensen inequality:

$$(\widehat{\mathbb{E}}[|X|^r])^{\frac{1}{r}} \leq (\widehat{\mathbb{E}}[|X|^s])^{\frac{1}{s}}, \ \text{ for } 0 < r \leq s.$$

Definition 2.3. *(Zhang [9]) (I) A function* $V : \mathcal{F} \to [0, 1]$ *is called a continuous capacity if it satisfies*
(I1) Continuity from below: $V(A_n) \uparrow V(A)$ *if* $A_n \uparrow A$, *where* $A_n, A \in \mathcal{F}$;
(I2) Continuity from above: $V(A_n) \downarrow V(A)$ *if* $A_n \downarrow A$, *where* $A_n, A \in \mathcal{F}$.

We define the Choquet integrals/expecations $(C_{\mathbb{V}}, C_{\mathcal{V}})$ by

$$C_V(X) := \int_0^\infty V(X \geq x)dx + \int_{-\infty}^0 (V(X \geq x) - 1)dx$$

with V being replaced by $\mathbb{V}$ and $\mathcal{V}$, respectively. If $\lim_{c\to+\infty} \widehat{\mathbb{E}}[(|X| - c)^+] = 0$, then $\widehat{\mathbb{E}}[|X|] \leq C_{\mathbb{V}}(|X|)$ (see Lemma 4.5(iii) of Zhang [9]).

3. Main Results

Theorem 3.1. *Let $1 \leq \alpha < 2$, $\{X, X_n; n \geq 1\}$ be a sequence of independent and identically distributed random variables in a sub-linear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. $\mathbb{V}$ is continuous and $L(.)$ is a slowly varying function defined on $[A, \infty)$ for some $A > 0$. When $\alpha = 1$, we assume further that $L(x) \geq 1$ and is increasing on $[A, \infty)$. Let $b_n = n^{1/\alpha}\tilde{L}(n^{1/\alpha}), n \geq A^\alpha$. If*

$$\widehat{\mathbb{E}}[X] = \widehat{\mathcal{E}}[X] = 0, \quad \widehat{\mathbb{E}}[|X|^\alpha L^\alpha(|X| + A)] \leq C_{\mathbb{V}}[|X|^\alpha L^\alpha(|X| + A)] < \infty \tag{3.1}$$

and for every array of constants $\{a_{ni}, n \geq 1, 1 \leq i \leq n\}$ satisfying

$$\sum_{i=1}^n a_{ni}^2 \leq Cn, \ n \geq 1, \tag{3.2}$$

then for any $\varepsilon > 0$

$$\sum_{n\geq A^\alpha} n^{-1}\mathbb{V}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^j a_{ni}X_i\right| > \varepsilon b_n\right) < \infty, \tag{3.3}$$

$$\sum_{n\geq A^\alpha} n^{-1}\mathbb{V}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^j X_i\right| > \varepsilon b_n\right) < \infty \tag{3.4}$$

and

$$\lim_{n\to\infty} \frac{\max_{1\leq j\leq n}\left|\sum_{i=1}^j X_i\right|}{b_n} = 0 \ \ a.s. \ \mathbb{V}. \tag{3.5}$$

Conversely, if (3.5) holds, then $C_{\mathbb{V}}[|X|^\alpha L^\alpha(|X| + A)] < \infty$.

Remark 3.2 Our Theorem 3.1 is a very general and good result. If we take $L(x) = \log^{-1/\gamma}(x), x \geq 2$ in Theorem 3.1, we can obtain optimal moment condition for (1.2) under the sub-linear expectations. Hence our result generalizes and improves the corresponding results of Sung [1] and Chen and Sung[2].

4. Proof of main result

In order to prove our results, we need the following lemmas. Lemma 4.1 is obvious.

Lemma 4.1. *Let $\alpha, \beta > 0$ and and $L(.)$ be a slowly varying function. Let $f(x) = x^{\alpha\beta}L^\alpha(x^\beta)$ and $h(x) = x^{\frac{1}{\alpha\beta}}\tilde{L}^{\frac{1}{\beta}}(x^{\frac{1}{\alpha}})$. Then*

$$\lim_{x\to\infty} \frac{f(h(x))}{x} = \lim_{x\to\infty} \frac{h(f(x))}{x} = 1. \tag{4.1}$$

Lemma 4.2. *Under the conditions of Theorem 3.1, suppose $X \in \mathcal{H}$ and $b_n = n^{1/\alpha}\tilde{L}(n^{1/\alpha})$. Then for any $c > 0$,*

$$C_{\mathbb{V}}[|X|^{\alpha}L^{\alpha}(|X|+A)] < \infty \Leftrightarrow \sum_{n\geq A^{\alpha}} \mathbb{V}(|X| > cb_n) < \infty \tag{4.2}$$

and

$$C_{\mathbb{V}}[|X|^{\alpha}L^{\alpha}(|X|+A)] < \infty \Longrightarrow \sum_{k\geq k_0} 2^k\mathbb{V}(|X| > b_{2^k}) < \infty, \tag{4.3}$$

where k_0 is some positive integer.

Proof Let $f(x) = x^{\alpha}L^{\alpha}(x)$ and $h(x) = x^{\frac{1}{\alpha}}\tilde{L}(x^{\frac{1}{\alpha}})$. Since L(.) is positive and bounded on finite closed intervals,

$$C_{\mathbb{V}}[|X|^{\alpha}L^{\alpha}(|X|+A)] < \infty \Leftrightarrow C_{\mathbb{V}}[f(|X|+A)] < \infty.$$

By the defination of the Choquet expecations, we have $C_{\mathbb{V}}[|X|] = \int_0^{\infty}\mathbb{V}(|X| > x)dx$. Then $C_{\mathbb{V}}[|X|] < \infty \Leftrightarrow \sum_{n=1}^{\infty}\mathbb{V}(|X| > cn) < \infty$. Then $C_{\mathbb{V}}[f(|X|+A)] < \infty$ is equivalent to

$$\sum_{n=1}^{\infty}\mathbb{V}(f(|X|+A) > cn) < \infty. \tag{4.4}$$

By using Lemma 4.1 with $\beta = 1$, we have $f(h(x)) \sim h(f(x)) \sim x$ as $x \to \infty$. Then by the fact that $f(x)$ and $h(x)$ are increasing on $[A,\infty)$, we get (4.4) is equivalent to

$$\sum_{n\geq A^{\alpha}}\mathbb{V}(|X| > cb_n) < \infty. \tag{4.5}$$

When $C_{\mathbb{V}}[|X|^{\alpha}L^{\alpha}(|X|+A)] < \infty$, there is some positive integer k_0 such that

$$\begin{aligned}\infty &> \sum_{n\geq A^{\alpha}}\mathbb{V}(|X| > cb_n)\\ &\geq \sum_{k=k_0}^{\infty}\sum_{2^{k-1}\leq n\leq 2^k}\mathbb{V}(|X| > b_{2^k})\\ &= C\sum_{k=k_0}^{\infty}2^k\mathbb{V}(|X| > b_{2^k}).\end{aligned}$$

The proof of Lemma 4.2 is completed.

Lemma 4.3. Zhang [9] *Let $\{X_n; n \geq 1\}$ be a sequence of independent random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ and $S_n = \sum_{i=1}^n X_i$. Suppose $p \geq 2$. Then*

$$\begin{aligned}\widehat{\mathbb{E}}\left[\max_{1\leq k\leq n}|S_k|^p\right] &\leq C_p\left\{\sum_{k=1}^n\widehat{\mathbb{E}}[|X_k|^p] + \left(\sum_{k=1}^n\widehat{\mathbb{E}}[X_k^2]\right)^{p/2}\right\}\\ &\quad + C_p\left(\sum_{k=1}^n[(\widehat{\mathcal{E}}[X_k])^- + (\widehat{\mathbb{E}}[X_k])^+]\right)^p.\end{aligned}$$

Proof of Theorem 3.1 For simplicity, we assume that A^{α} is an integer number. For $0 < \mu < 1$, let $g(x) \in C_{l.Lip}(\mathbb{R}), 0 \leq g(x) \leq 1$ for all x, $g(x) = 1$ if $x \leq \mu$, $g(x) = 0$ if $x > 1$ and $g(x) \downarrow$ if $x > 0$. Then

$$I(|x| \leq \mu) \leq g(|x|) \leq I(|x| \leq 1),\ I(|x| > 1) \leq 1 - g(|x|) \leq I(|x| > \mu). \tag{4.6}$$

For $1 \le i \le n, n \ge A^{\alpha}$, let $Y_i = X_i g\left(\frac{|X_i|}{b_n}\right)$. We can easily get

$$\begin{aligned}
&\mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon b_n\right)\\
&\le \mathbb{V}\left(\max_{1\le i\le n}|X_i| > b_n\right) + \mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}Y_i\right| > \varepsilon b_n\right)\\
&\le \sum_{i=1}^{n}\mathbb{V}\left(|X_i| > b_n\right) + \mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}(Y_i - \widehat{\mathbb{E}}[Y_i])\right| > \varepsilon b_n - \max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}\widehat{\mathbb{E}}[Y_i]\right|\right)\\
&\le \sum_{i=1}^{n}\mathbb{V}\left(|X_i| > b_n\right) + \mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}(Y_i - \widehat{\mathbb{E}}[Y_i])\right| > \varepsilon b_n - \sum_{i=1}^{n}\left|a_{ni}\widehat{\mathbb{E}}[Y_i]\right|\right).
\end{aligned} \tag{4.7}$$

We first prove

$$b_n^{-1}\sum_{i=1}^{n}\left|a_{ni}\widehat{\mathbb{E}}[Y_i]\right| \to 0 \text{ as } n \to \infty. \tag{4.8}$$

$\forall 1 \le \gamma \le 2$, by (3.2) and Hölder inequality, we have

$$\sum_{i=1}^{n}|a_{ni}|^{\gamma} \le (\sum_{i=1}^{n}|a_{ni}|^2)^{\frac{\gamma}{2}}(\sum_{i=1}^{n}1)^{1-\frac{\gamma}{2}} \le Cn. \tag{4.9}$$

For $n \ge A^{\alpha}$, by(3.1), (4.6) and (4.9), we have

$$\begin{aligned}
&b_n^{-1}\sum_{i=1}^{n}\left|a_{ni}\widehat{\mathbb{E}}[Y_i]\right|\\
&= b_n^{-1}\sum_{i=1}^{n}|a_{ni}||\widehat{\mathbb{E}}[X_i] - \widehat{\mathbb{E}}[Y_i]|\\
&\le b_n^{-1}\sum_{i=1}^{n}|a_{ni}|\widehat{\mathbb{E}}\left[|X|\left(1 - g\left(\frac{|X|}{b_n}\right)\right)\right]\\
&\le Cnb_n^{-1}\widehat{\mathbb{E}}\left[|X|\left(1 - g\left(\frac{|X|}{b_n}\right)\right)\right].
\end{aligned} \tag{4.10}$$

For n large enough and for $\omega \in (|X| > \mu b_n)$, by (1.5) and the monotonicity of $x^{\alpha-1}L^{\alpha}(x)$, we have

$$\begin{aligned}
\frac{n}{b_n} &= \frac{n^{(\alpha-1)/\alpha}\tilde{L}^{\alpha-1}(n^{1/\alpha})}{\tilde{L}^{\alpha}(n^{1/\alpha})}\\
&= \frac{(n^{1/\alpha}\tilde{L}(n^{1/\alpha}))^{\alpha-1}L^{\alpha}(n^{1/\alpha}\tilde{L}(n^{1/\alpha}))}{\tilde{L}^{\alpha}(n^{1/\alpha})L^{\alpha}(n^{1/\alpha}\tilde{L}(n^{1/\alpha}))}\\
&\le Cb_n^{\alpha-1}L^{\alpha}(b_n) \le C|X(\omega)|^{\alpha-1}L^{\alpha}(X(\omega)).
\end{aligned} \tag{4.11}$$

Combining (4.10), (4.11) and (3.1), we have

$$\begin{aligned}
b_n^{-1}\sum_{i=1}^{n}\left|a_{ni}\widehat{\mathbb{E}}[Y_i]\right| &\le C\widehat{\mathbb{E}}\left[|X|^{\alpha}L^{\alpha}(|X|)\left(1 - g\left(\frac{|X|}{b_n}\right)\right)\right]\\
&\le C\widehat{\mathbb{E}}\left[|X|^{\alpha}L^{\alpha}((|X| + A))\left(1 - g\left(\frac{|X|}{b_n}\right)\right)\right]\\
&\to 0 \text{ as } n \to \infty.
\end{aligned} \tag{4.12}$$

Hence

$$\begin{aligned}
&\sum_{n\geq A^{\alpha}} n^{-1}\mathbb{V}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon b_n\right)\\
&\leq \sum_{n\geq A^{\alpha}} n^{-1}\sum_{i=1}^{n}\mathbb{V}\left(|X_i| > b_n\right) + \sum_{n\geq A^{\alpha}} n^{-1}\mathbb{V}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}(Y_i - \widehat{\mathbb{E}}[Y_i])\right| > \frac{1}{2}\varepsilon b_n\right)\\
&:= I + II.
\end{aligned} \tag{4.13}$$

By (4.2) and (4.6), we have

$$\begin{aligned}
I &= \sum_{n\geq A^{\alpha}} n^{-1}\sum_{i=1}^{n}\mathbb{V}(|X_i| > b_n)\\
&\leq \sum_{n\geq A^{\alpha}} n^{-1}\sum_{i=1}^{n}\widehat{\mathbb{E}}[1 - g(\frac{|X_i|}{b_n})]\\
&= \sum_{n\geq A^{\alpha}} n^{-1}\sum_{i=1}^{n}\widehat{\mathbb{E}}[1 - g(\frac{|X|}{b_n})]\\
&\leq \sum_{n\geq A^{\alpha}} n^{-1}n\mathbb{V}(|X| > \mu b_n) = \sum_{n\geq A^{\alpha}}\mathbb{V}(|X| > \mu b_n) < \infty.
\end{aligned} \tag{4.14}$$

In order to prove (3.3), it remains to show that $II < \infty$. By Lemma 4.3, we have

$$\begin{aligned}
II &\leq C\sum_{n\geq A^{\alpha}} n^{-1}b_n^{-2}\widehat{\mathbb{E}}\left[\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}(Y_i - \widehat{\mathbb{E}}[Y_i])\right|\right)^2\right]\\
&\leq C\sum_{n\geq A^{\alpha}} n^{-1}b_n^{-2}\sum_{i=1}^{n}|a_{ni}|^2\widehat{\mathbb{E}}[|Y_i|^2\\
&\quad + C\sum_{n\geq A^{\alpha}} n^{-1}b_n^{-2}\left(\sum_{i=1}^{n}\left[(\widehat{\mathbb{E}}[a_{ni}Y_i - \widehat{\mathbb{E}}[a_{ni}Y_i]])^{+} + (\widehat{\mathcal{E}}[a_{ni}Y_i - \widehat{\mathbb{E}}[a_{ni}Y_i]])^{-}\right]\right)^2\\
&=: II_1 + II_2.
\end{aligned} \tag{4.15}$$

For $0 < \mu < 1$, let $g_j(x) \in C_{l.Lip}(\mathbb{R})$, $j \geq 1$ such that $0 \leq g_j(x) \leq 1$ for all x and $g_j\left(\frac{|x|}{b_{2^j}}\right) = 1$ if $b_{2^{j-1}} < |x| \leq b_{2^j}$, $g_j\left(\frac{|x|}{b_{2^j}}\right) = 0$ if $|x| \leq \mu b_{2^{j-1}}$ or $|x| > (1+\mu)b_{2^j}$. Then for any $m > 0$

$$g_j\left(\frac{|X|}{b_{2^j}}\right) \leq I(\mu b_{2^{j-1}} < |X| \leq (1+\mu)b_{2^j}),\quad |X|^m g\left(\frac{|X|}{b_{2^k}}\right) \leq 1 + \sum_{j=1}^{k}|X|^m g_j\left(\frac{|X|}{b_{2^j}}\right). \tag{4.16}$$

By (3.2)and (4.16), there exists some positive integer j_0 such that

$$
\begin{aligned}
II_1 &\le C\sum_{n\ge A^\alpha} n^{-1}b_n^{-2}\sum_{i=1}^{n}|a_{ni}|^2\widehat{\mathbb{E}}\left[X^2 g\left(\frac{|X|}{b_n}\right)\right]\\
&\le C\sum_{n\ge A^\alpha} b_n^{-2}\widehat{\mathbb{E}}\left[X^2 g\left(\frac{|X|}{b_n}\right)\right]\\
&\le C\sum_{k\ge k_0}\sum_{2^{k-1}\le n<2^k} b_{2^k}^{-2}\widehat{\mathbb{E}}\left[X^2 g\left(\frac{|X|}{b_{2^k}}\right)\right]\\
&= C\sum_{k\ge k_0} 2^k b_{2^k}^{-2}\sum_{j=j_0}^{k}\widehat{\mathbb{E}}\left[X^2 g_j\left(\frac{|X|}{b_{2^j}}\right)\right]\\
&= C\sum_{j=j_0}^{\infty}\widehat{\mathbb{E}}\left[X^2 g_j\left(\frac{|X|}{b_{2^j}}\right)\right]\sum_{k=j}^{\infty}2^k b_{2^k}^{-2}\\
&\le C\sum_{j=j_0}^{\infty}2^j b_{2^j}^{-2}\widehat{\mathbb{E}}\left[X^2 g_j\left(\frac{|X|}{b_{2^j}}\right)\right]\\
&\le C\sum_{j=j_0}^{\infty}2^j b_{2^j}^{-2}b_{2^j}^{2}\,\mathbb{V}(|X|>\mu b_{2^j})\\
&= C\sum_{j=j_0}^{\infty}2^j\,\mathbb{V}(|X|>\mu b_{2^j})<\infty.
\end{aligned}
\tag{4.17}
$$

Before considering II_2, we estimate $1-g\left(\frac{|X|}{b_{2^k}}\right)$. By the definitions of $g(x)$ and $g_j(x)$, there exists some positive integer $k_0^{'}$ such that

$$
\begin{aligned}
1-g\left(\frac{|X|}{b_{2^k}}\right) &\le I\left(\frac{|X|}{b_{2^k}}>\mu\right)\le I(|X|>b_{2^{k_0^{'}-1}})\\
&\le \sum_{j=k_0^{'}}^{\infty} I(b_{2^{j-1}}<|X|\le b_{2^j})\le \sum_{j=k_0^{'}}^{\infty} g_j\left(\frac{|X|}{b_{2^j}}\right).
\end{aligned}
$$

Now we consider II_2. By the fact $\widehat{\mathbb{E}}[X+C]=\widehat{\mathbb{E}}[X]+C$, then we have $\widehat{\mathbb{E}}[a_{ni}Y_i-\widehat{\mathbb{E}}[a_{ni}Y_i]]=0$. By (4.10), (4.11), (4.12), we have $nb_n^{-1}\widehat{\mathbb{E}}\left[|X|\left(1-g\left(\frac{|X|}{b_n}\right)\right)\right]\to 0, n\to\infty$. Hence, we have

$$
\begin{aligned}
II_2 &= C\sum_{n\ge A^\alpha} n^{-1}b_n^{-2}\left(\sum_{i=1}^{n}\left[(\widehat{\mathcal{E}}[a_{ni}Y_i-\widehat{\mathbb{E}}[a_{ni}Y_i]])^-\right]\right)^2\\
&\le C\sum_{n\ge A^\alpha} n^{-1}b_n^{-2}\left(\sum_{i=1}^{n}\left|-\widehat{\mathbb{E}}[-a_{ni}Y_i+\widehat{\mathbb{E}}[a_{ni}Y_i]]\right|\right)^2\\
&= C\sum_{n\ge A^\alpha} n^{-1}b_n^{-2}\left(\sum_{i=1}^{n}\left|\widehat{\mathbb{E}}[-a_{ni}Y_i]+\widehat{\mathbb{E}}[a_{ni}Y_i]\right|\right)^2\\
&\le C\sum_{n\ge A^\alpha} n^{-1}b_n^{-2}\left(\sum_{i=1}^{n}\left(|\widehat{\mathbb{E}}[-a_{ni}Y_i]|+|\widehat{\mathbb{E}}[a_{ni}Y_i]|\right)\right)^2\\
&\le C\sum_{n\ge A^\alpha} n^{-1}b_n^{-2}\left(\sum_{i=1}^{n}|a_{ni}||\widehat{\mathbb{E}}[-Y_i]|\right)^2+C\sum_{n\ge A^\alpha} n^{-1}b_n^{-2}\left(\sum_{i=1}^{n}|a_{ni}||\widehat{\mathbb{E}}[Y_i]|\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&= C\sum_{n\geq A^\alpha} n^{-1}b_n^{-2}\left(\sum_{i=1}^n |a_{ni}||\widehat{\mathbb{E}}[-X_i]-\widehat{\mathbb{E}}[-Y_i]|\right)^2 + C\sum_{n\geq A^\alpha} n^{-1}b_n^{-2}\left(\sum_{i=1}^n |a_{ni}||\widehat{\mathbb{E}}[X_i]-\widehat{\mathbb{E}}[Y_i]|\right)^2\\
&\leq C\sum_{n\geq A^\alpha} n^{-1}b_n^{-2}\left(\sum_{i=1}^n |a_{ni}|\widehat{\mathbb{E}}[|-X_i-(-Y_i)|]\right)^2 + C\sum_{n\geq A^\alpha} n^{-1}b_n^{-2}\left(\sum_{i=1}^n |a_{ni}|\widehat{\mathbb{E}}[|X_i-Y_i|]\right)^2\\
&\leq C\sum_{n\geq A^\alpha} n^{-1}b_n^{-2}\left(\sum_{i=1}^n |a_{ni}|\widehat{\mathbb{E}}\left[|X|\left(1-g\left(\frac{|X|}{b_n}\right)\right)\right]\right)^2\\
&\leq C\sum_{n\geq A^\alpha} n^{-1}\left(nb_n^{-1}\widehat{\mathbb{E}}\left[|X|\left(1-g\left(\frac{|X|}{b_n}\right)\right)\right]\right)^2\\
&\leq C\sum_{n\geq A^\alpha} n^{-1}nb_n^{-1}\widehat{\mathbb{E}}\left[|X|\left(1-g\left(\frac{|X|}{b_n}\right)\right)\right]\\
&= C\sum_{k=k_0}^\infty \sum_{2^{k-1}\leq n<2^k} b_{2^{k-1}}^{-1}\widehat{\mathbb{E}}\left[|X|\left(1-g\left(\frac{|X|}{b_{2^k}}\right)\right)\right]\\
&\leq C\sum_{k=k_0}^\infty 2^k b_{2^k}^{-1}\sum_{j=k}^\infty \widehat{\mathbb{E}}\left[|X|g_j\left(\frac{|X|}{b_{2^j}}\right)\right]\\
&= C\sum_{j=j_0}^\infty \widehat{\mathbb{E}}\left[|X|g_j\left(\frac{|X|}{b_{2^j}}\right)\right]\sum_{k=k_0}^j 2^k b_{2^k}^{-1}\\
&\leq C\sum_{j=j_0}^\infty 2^j b_{2^j}^{-1}\widehat{\mathbb{E}}\left[|X|g_j\left(\frac{|X|}{b_{2^j}}\right)\right]\\
&\leq C\sum_{j=j_0}^\infty 2^j b_{2^j}^{-1}b_{2^j}\mathbb{V}(|X|>\mu b_{2^{j-1}})\\
&= C\sum_{j=j_0}^\infty 2^j\mathbb{V}(|X|>cb_{2^j})<\infty.
\end{aligned} \tag{4.18}
$$

We complete the proof of (3.3). The implication [(3.3)$\Longrightarrow$ (3.4)] is immediate by letting $a_{ni}=1$. Now, we assume that (3.4) holds. Since

$$
\begin{aligned}
\infty &> \sum_{n\geq A^\alpha} n^{-1}\mathbb{V}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^j X_i\right|>\varepsilon b_n\right)\\
&= C\sum_{k=k_0}^\infty\sum_{2^k\leq n<2^{k+1}} n^{-1}\mathbb{V}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^j X_i\right|>\varepsilon b_n\right)\\
&\geq C\sum_{k=k_0}^\infty\sum_{2^k\leq n<2^{k+1}} \frac{1}{2^{k+1}}\mathbb{V}\left(\max_{1\leq j\leq 2^k}\left|\sum_{i=1}^j X_i\right|>\varepsilon b_{2^{k+1}}\right)\\
&= C\sum_{k=k_0}^\infty \mathbb{V}\left(\max_{1\leq j\leq 2^k}\left|\sum_{i=1}^j X_i\right|>\varepsilon b_{2^{k+1}}\right).
\end{aligned} \tag{4.19}
$$

By Borel-Cantelli Lemma, we have

$$
\mathbb{V}\left(\max_{1\leq j\leq 2^k}\left|\sum_{i=1}^j X_i\right|>\varepsilon b_{2^{k+1}},\ \text{i.o.}\right)=0,
$$

which implies

$$\lim_{k\to\infty}\frac{\max_{1\le j\le 2^{k+1}}\left|\sum_{i=1}^{j}X_i\right|}{b_{2^{k+1}}}=0 \text{ a.s. } \mathbb{V}. \tag{4.20}$$

For any $n\geq A^{\alpha}$, there is k such that $2^k<n\leq 2^{k+1}$,then

$$\frac{\max_{1\le j\le n}\left|\sum_{i=1}^{j}X_i\right|}{b_n}\leq\frac{\max_{1\le j\le 2^{k+1}}\left|\sum_{i=1}^{j}X_i\right|}{b_{2^k}}\to 0 \text{ a.s. } \mathbb{V},\ \ k\to\infty. \tag{4.21}$$

We complete the proof of (3.5).

For the 'converse' part, we assume $C_{\mathbb{V}}[|X|^{\alpha}L^{\alpha}(|X|+A)]=\infty$. Let $g_{\varepsilon}(x)\in C_{l.Lip}(\mathbb{R}), 0\leq g_{\varepsilon}(x)\leq 1$ for all x, $g_{\varepsilon}(x)=1$ if $x>1$, $g_{\varepsilon}(x)=0$ if $x\leq 1-\varepsilon$, where $0<\varepsilon<1$. Then $I(x\geq 1)\leq g_{\varepsilon}(x)\leq I(x>1-\varepsilon)$. So for any $M>0$, by (4.2) we have

$$\begin{aligned}\sum_{j=A^{\alpha}}^{\infty}\widehat{\mathbb{E}}\left[g_{\frac{1}{2}}\left(\frac{|X_j|}{Mb_j}\right)\right]&=\sum_{j=A^{\alpha}}^{\infty}\widehat{\mathbb{E}}\left[g_{\frac{1}{2}}\left(\frac{|X|}{Mb_j}\right)\right]\\&\geq\sum_{j=A^{\alpha}}^{\infty}\mathbb{V}(|X|>Mb_j)=\infty.\end{aligned} \tag{4.22}$$

For any $l\geq 1$, we have

$$\begin{aligned}&\mathcal{V}\left(\sum_{j=A^{\alpha}}^{n}g_{\frac{1}{2}}\left(\frac{|X|}{Mb_j}\right)<l\right)\\&=\mathcal{V}\left(\exp\left\{-\frac{1}{2}\sum_{j=A^{\alpha}}^{n}g_{\frac{1}{2}}\left(\frac{|X|}{Mb_j}\right)\right\}>\exp(-\frac{l}{2})\right)\\&\leq\exp(\frac{l}{2})\widehat{\mathcal{E}}\left[\exp\left\{-\frac{1}{2}\sum_{j=A^{\alpha}}^{n}g_{\frac{1}{2}}\left(\frac{|X|}{Mb_j}\right)\right\}\right]\\&\leq\exp(\frac{l}{2})\prod_{j=A^{\alpha}}^{n}\widehat{\mathcal{E}}\left[\exp\left\{-\frac{1}{2}g_{\frac{1}{2}}\left(\frac{|X|}{Mb_j}\right)\right\}\right].\end{aligned}$$

By the elementary inequality $e^{-x}\leq 1-\frac{1}{2}x\leq e^{-\frac{1}{2}x},\ \forall 0\leq x\leq\frac{1}{2}$, we have

$$\begin{aligned}\widehat{\mathcal{E}}\left[\exp\left\{-\frac{1}{2}g_{\frac{1}{2}}\left(\frac{|X|}{Mb_j}\right)\right\}\right]&\leq\widehat{\mathcal{E}}\left[1-\frac{1}{4}g_{\frac{1}{2}}\left(\frac{|X|}{Mb_j}\right)\right]\\&=1-\frac{1}{4}\widehat{\mathbb{E}}\left[g_{\frac{1}{2}}\left(\frac{|X|}{Mb_j}\right)\right]\leq\exp\left\{-\frac{1}{4}\widehat{\mathbb{E}}\left[g_{\frac{1}{2}}\left(\frac{|X|}{Mb_j}\right)\right]\right\}.\end{aligned}$$

It follows that

$$\begin{aligned}&\mathcal{V}\left(\sum_{j=A^{\alpha}}^{n}g_{\frac{1}{2}}\left(\frac{|X|}{Mb_j}\right)<l\right)\\&\leq\exp(\frac{l}{2})\exp\left\{-\frac{1}{4}\sum_{j=A^{\alpha}}^{n}\widehat{\mathbb{E}}\left[g_{\frac{1}{2}}\left(\frac{|X|}{Mb_j}\right)\right]\right\}\to 0 \text{ as } n\to\infty\end{aligned}$$

by (4.22). That is

$$\mathbb{V}\left(\sum_{j=A^{\alpha}}^{n}g_{\frac{1}{2}}\left(\frac{|X|}{Mb_j}\right)\geq l\right)\to 1 \text{ as } n\to\infty.$$

By continuity of $\mathbb{V}$, for any $M > 0$, we have

$$\begin{aligned}\mathbb{V}\left(\limsup_{n\to\infty}\frac{|X_n|}{b_n} > \frac{M}{2}\right) &= \mathbb{V}\left(\frac{|X_j|}{Mb_j} > \frac{1}{2}, \text{i.o.}\right)\\ &\geq \mathbb{V}\left(\sum_{j=A^\alpha}^{\infty} g_{\frac{1}{2}}\left(\frac{|X_j|}{Mb_j}\right) = \infty\right)\\ &= \lim_{l\to\infty}\mathbb{V}\left(\sum_{j=A^\alpha}^{\infty} g_{\frac{1}{2}}\left(\frac{|X_j|}{Mb_j}\right) > l\right)\\ &= \lim_{l\to\infty}\limsup_{n\to\infty}\mathbb{V}\left(\sum_{j=A^\alpha}^{n} g_{\frac{1}{2}}\left(\frac{|X_j|}{Mb_j}\right) > l\right) = 1.\end{aligned}$$

On the other hand, we have

$$\limsup_{n\to\infty}\frac{|X_n|}{b_n} \leq \limsup_{n\to\infty}\left(\frac{|S_n|}{b_n} + \frac{|S_{n-1}|}{b_n}\right) \leq 2\limsup_{n\to\infty}\frac{|S_n|}{b_n}.$$

It follows that

$$\mathbb{V}\left(\limsup_{n\to\infty}\frac{|S_n|}{b_n} > m\right) = 1, \forall m > 0,$$

that is

$$\mathcal{V}\left(\limsup_{n\to\infty}\frac{|S_n|}{b_n} < m\right) = 0, \forall m > 0,$$

which contradicts $\mathcal{V}\left(\lim_{n\to\infty}\frac{|S_n|}{b_n} = 0\right) = 1$. Therefore, the assumption $C_{\mathbb{V}}[|X|^\alpha L^\alpha(|X| + A)] = \infty$ is incorrect, and so $C_{\mathbb{V}}[|X|^\alpha L^\alpha(|X| + A)] < \infty$. We complete the proof of the theorem.

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