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Kähler-Norden Structures on Statistical Manifolds

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Abstract. In this paper, we construct Kähler-Norden statistical structures on pseudo-Riemannian manifolds equipped with a torsion-free linear connection and an almost complex structure. Also, we present some examples and study curvature properties for these structures by using Tachibana operator. Finally, we consider a Norden statistical manifold and study Codazzi coupling of its connection with the almost complex structure on it.

1. Introduction

The notion of an almost Norden manifold was introduced by Norden in 1960 [14]. In the following years, Gribachev, Mekerov and Djelepov studied its geometrical structure and gave the notion of generalized $B-$manifolds [10]. Almost Norden structures are among the most important geometrical structures that can be considered on a manifold. Let M be a $2n-$dimensional differentiable manifold endowed with an almost complex structure J and a pseudo-Riemannian metric g of signature (n, n) such that $g(JX, Y) = g(X, JY)$ for arbitrary vector fields X and Y on M. Then, the pseudo-Riemannian metric g is called a Norden metric. Norden metrics are referred to as anti-Hermitian metrics or $B-$metrics. They find widespread application in mathematics as well as in theoretical physics. In [13], Iscan and Salimov proved that there exist a one-to-one correspondence between Kähler-Norden manifolds and Norden manifolds with a holomorphic metric. Also, they showed that the Riemannian curvature tensor in such manifolds is pure and holomorphic. In addition, the scalar curvature is a locally holomorphic function.

In 1980s, the notion of statistical structure was introduced and began to play an important role to build a very effective branch called information geometry which is a combination of differential geometry and statistics. Its applications can be found in various fields of science. For example, see [1, 3, 7, 8, 18] for some utilization in image processing, physics, computer science and machine learning. A detailed survey on information geometry can be studied in [2].

In this paper, we combine the above mentioned theories in a certain sense, that is, we introduce the notion of Kähler-Norden statistical manifold and study its geometric properties.

The paper is organized as follows. In Section 2, we recall notions of almost Norden manifolds and statistical manifolds. Then, we consider almost Norden statistical manifolds and study a basic property of them. In Section 3, we introduce Kähler-Norden statistical structures. Also, we provide examples

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including these structures. Moreover, we study twin Norden metrics. In Section 4, we present some properties of statistical curvature tensor of Kähler-Norden statistical manifolds by Tachibana operators. Finally, in Section 5, we study Codazzi coupling of the connection of a Norden statistical manifold with the almost complex structure on the manifold.

Throughout this paper, all manifolds, tensor fields and connections are always assumed to be differentiable of class C^∞. Also, we denote by $\Im^p_q(M)$ the set of all tensor fields of type (p,q) on M.

2. Almost Norden statistical manifolds

An isomorphism J of the tangent bundle TM is defined by smooth sections of the bundle $End(TM)$ such that it is invertible everywhere. In the special case when $J^2 = -Id$, J is called an almost complex structure. The manifold M endowed with an almost complex structure J is called an almost complex manifold. Let (M, J) be an almost complex manifold. A pseudo-Riemannian metric g of signature (n, n) is a Norden metric (B-metric) if

$$g(JX, JY) = -g(X, Y)$$

or equivalently $g(JX, Y) = g(X, JY)$ for any $X, Y \in \Im^1_0(M)$. In this case, (M, J, g) is called an almost Norden manifold. Also, an almost Norden manifold (M, J, g) is called a Norden manifold if the almost complex structure J is integrable i.e., the Nijenhuis tensor field N_J vanishes, where

$$N_J[X, Y] = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] \tag{1}$$

for any $X, Y \in \Im^1_0(M)$.

Let ∇ be a torsion-free linear connection on M, and g be a pseudo-Riemannian metric. Denote by $\widehat{\nabla}$ the Levi-Civita connection of g.

Definition 2.1. *A pair (∇, g) is called a statistical structure on M if the cubic tensor field $C = \nabla g$ is totally symmetric; namely the Codazzi equations hold:*

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z), \qquad (= (\nabla_Z g)(X, Y)) \tag{2}$$

for any $X, Y, Z \in \Im^1_0(M)$. Also, the manifold M together with these structures is called a statistical manifold.

In a local coordinate, the cubic tensor field C has the following form

$$C_{ijk} = \partial_k g_{ij} - \Gamma^h_{ik} g_{jh} - \Gamma^h_{jk} g_{ih}, \qquad C_{ijk} = C_{jik} = C_{kij}, \tag{3}$$

where $\partial_i = \frac{\partial}{\partial x^i}$ and Γ^i_{jk} are the Christoffel symbols of the Codazzi connection ∇.

Considering the Levi-Civita connection $\widehat{\nabla}$ of metric g, then the above equation implies

$$\Gamma^r_{ij} = \widehat{\Gamma}^r_{ij} - \frac{1}{2} g^{rk} C_{ijk},$$

where $\widehat{\Gamma}^r_{ij}$ are the connection components of $\widehat{\nabla}$ and given by

$$\widehat{\Gamma}^r_{ij} = \frac{1}{2} g^{rk} \{\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}\}. \tag{4}$$

On a statistical manifold (M, g, ∇), the g-conjugate connection ∇^* of ∇ is introduced by the following formula

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) \tag{5}$$

for any $X, Y, Z \in \mathfrak{I}^1_0(M)$. It can be checked easily that (∇^*, g) is also a statistical structure and

$$C^*(X, Y, Z) = (\nabla^*_X g)(Y, Z) = -C(X, Y, Z).$$

For the statistical structure (∇, g), we set

$$K_X Y = \nabla_X Y - \widehat{\nabla}_X Y. \tag{6}$$

It is easy to see that $K \in \mathfrak{I}^1_2(M)$ satisfies the following:

$$K_X Y = K_Y X, \;\; g(K_X Y, Z) = g(Y, K_X Z) \tag{7}$$

and

$$(\nabla_X g)(Y, Z) = -2g(K_X Y, Z).$$

Conversely, considering a pseudo-Riemannian metric g and $K \in \mathfrak{I}^1_2(M)$ that satisfy (7), it follows that $(\nabla := \widehat{\nabla} + K, g)$ is a statistical structure on M.

Definition 2.2. *An almost Norden manifold (M, J, g) equipped with a statistical structure (∇, g) will be called an almost Norden statistical manifold.*

In the following we get a relation between ∇ and ∇^*. It is remarkable that this relation is presented by Teofilova in [21] without proof.

Proposition 2.3. *Let (∇, g, J) be an almost Norden statistical structure on M. Then we have*

$$g((\nabla_Y J)Z, X) = g(Z, (\nabla^*_Y J)X)$$

for any $X, Y, Z \in \mathfrak{I}^1_0(M)$.

Proof. Since $Yg(JZ, X) = Yg(Z, JX)$ and the pair (∇, g) is a statistical structure, thus (5) implies

$$g(\nabla_Y JZ, X) + g(JZ, \nabla^*_Y X) = g(\nabla_Y Z, JX) + g(Z, \nabla^*_Y JX).$$

The above equation and

$$(\nabla_Y J)X = \nabla_Y JX - J\nabla_Y X$$

complete the proof. □

Let (U, x^i) be a locally coordinate system on (M). Then we have $J\partial_i = J^j_i \partial_j$ and $K_{\partial_i}\partial_j = K^r_{ij}\partial_r$. So, in locally, the Norden statistical structure conditions on the manifold M are expressed as follows

$$\begin{cases} \Gamma^r_{ij} = \Gamma^r_{ji}, \quad C_{ijk} = C_{jik}, \\ J^j_i J^k_j = -\delta^k_i, \quad J^k_i J_{jk} = -g_{ij}, \quad \Gamma^r_{ij} - \widehat{\Gamma}^r_{ij} = K^r_{ij}, \\ J^m_i \partial_m(J^s_j) - J^m_j \partial_m(J^s_i) - \partial_i(J^m_j)J^s_m + \partial_j(J^m_i)J^s_m = 0, \end{cases} \tag{8}$$

where $J_{jk} = J^r_j g_{rk}$.

Example 2.4. *Let M be a two-dimension manifold endowed with an almost complex structure J and a local coordinate (x^1, x^2), such that*

$$J = \begin{pmatrix} J^1_1 & J^2_1 \\ J^1_2 & J^2_2 \end{pmatrix}.$$

Since $J^2(\partial_1) = -\partial_1$ and $J^2(\partial_2) = -\partial_2$ where $\partial_i = \frac{\partial}{\partial x^i}, i = 1, 2$, we have

$$\begin{aligned} (J_1^1)^2 + J_1^2 J_2^1 &= -1, \;\; J_1^2(J_1^1 + J_2^2) = 0, \\ J_2^1(J_1^1 + J_2^2) &= 0, \;\; (J_2^2)^2 + J_1^2 J_2^1 = -1. \end{aligned} \tag{9}$$

Now, we consider possible cases for the above equations

Case 1. *$J_2^2 = -J_1^1$.*

In this case, using the first and the fourth equations of (9), we deduce $(J_1^1)^2 + J_1^2 J_2^1 = -1$.

Case 2. *$J_2^2 \neq -J_1^1$.*

In this case, the second and the third equations of (9) imply $J_1^2 = J_2^1 = 0$ and $(J_1^1)^2 = (J_2^2)^2 = -1$. But this is not possible.

Therefore all almost complex structures on M is given by the following matrix presentation:

$$J = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad a^2 + bc = -1.$$

Example 2.5. *Let $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ be a complex structure and a pseudo-Riemannian metric on $M = R^2$ with respect to the standard coordinate system (x_1, x_2) and its associated vector fields $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}$. We consider $(1, 2)$-tensor field $K = K_{ij}^l \frac{\partial}{\partial x^l} \otimes dx^i \otimes dx^j, i, j, l = 1, 2$ on R^2 as follows:*

$$K_{11}^1 = K_{12}^2 = K_{21}^2 = -K_{22}^1 = \alpha, \;\; K_{22}^2 = K_{12}^1 = K_{21}^1 = -K_{11}^2 = \beta,$$

where α, β are functions on R^2. Then K satisfies (7). Also, (2) holds. Hence $(\nabla = \widehat{\nabla} + K, g, J)$ is a Norden statistical structure on R^2.

The normal distribution manifold has a two-dimensional parameter space which is defined as

$$M_1 = \{p(x, \mu, \sigma) | p(x, \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} exp\{-\frac{(x-\mu)^2}{2\sigma^2}, \mu \in \mathbb{R}, \sigma > 0\}\}$$

and hence M_1 may be viewed as a 2-dimensional manifold which has (μ, σ) as a coordinate system.

Example 2.6. *The Fisher metric of the normal distribution manifold M_1 is the following*

$$(g_{ij}) = \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2} \end{bmatrix}. \tag{10}$$

We set $\partial_1 = \frac{\partial}{\partial \mu}$ and $\partial_2 = \frac{\partial}{\partial \sigma}$. The non-zero components $\widehat{\Gamma}_{ij}^k$ of the connection $\widehat{\nabla}$ on M_1 are given by

$$\widehat{\Gamma}_{12}^1 = \widehat{\Gamma}_{21}^1 = -\frac{1}{\sigma}, \;\; \widehat{\Gamma}_{11}^2 = \frac{1}{2\sigma}, \;\; \widehat{\Gamma}_{22}^2 = -\frac{1}{\sigma}.$$

Setting the non-zero components of $(1, 2)$-tensor field K as

$$K_{12}^1 = K_{21}^1 = -\frac{1}{\sigma}, \;\; K_{11}^2 = -\frac{1}{2\sigma}, \;\; K_{22}^2 = -\frac{2}{\sigma},$$

we can see that $(M_1, g, \nabla = \widehat{\nabla} + K)$ is a statistical manifold. But Example 2.4 implies that M_1 with the Fisher metric g doesn't admit an almost Norden statistical structure because the condition $g(J\partial_1, J\partial_1) = -g(\partial_1, \partial_1)$ implies $a^2 + 2b^2 = -1$ and this is not possible. Therefore the quadruple (M_1, ∇, g, J) is never a Norden statistical manifold.

Example 2.7. *Consider the normal distribution manifold M_1. Let $M = M_1 \times M_1$ be a product manifold with a local coordinate $(\sigma, \mu, \eta, \theta)$. We define a Riemannian metric g on M as follows*

$$\begin{bmatrix} \frac{1}{\sigma^2} & 0 & 0 & 0 \\ 0 & \frac{2}{\sigma^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sigma^2} & 0 \\ 0 & 0 & 0 & -\frac{2}{\sigma^2} \end{bmatrix}. \tag{11}$$

Putting $\partial_1 = \frac{\partial}{\partial\sigma}, \partial_2 = \frac{\partial}{\partial\mu}, \partial_3 = \frac{\partial}{\partial\eta}$ and $\partial_4 = \frac{\partial}{\partial\theta}$, the set $= \{\partial_i\}_{i=1}^4$ is a natural basis of TM. Assume K is a $(1,2)$-tensor field on M which is symmetric and satisfies in the following

$$K_{12}^1 = K_{23}^3 = K_{13}^4 = K_{24}^4 = -K_{22}^2 = -K_{44}^2 = -\frac{1}{\sigma},$$
$$K_{11}^2 = -K_{33}^2 = -\frac{1}{2\sigma}, \quad K_{34}^1 = -K_{14}^3 = \frac{2}{\sigma}$$

and other components are zero. On the other hand, non-zero components of the the Levi-Civita $\widehat{\nabla}$ are determined by

$$\widehat{\Gamma}_{12}^1 = \widehat{\Gamma}_{21}^1 = \widehat{\Gamma}_{23}^3 = \widehat{\Gamma}_{32}^3 = \widehat{\Gamma}_{24}^4 = \widehat{\Gamma}_{42}^4 = \widehat{\Gamma}_{22}^2 = \widehat{\Gamma}_{44}^2 = -\frac{1}{\sigma},$$
$$\widehat{\Gamma}_{11}^2 = -\widehat{\Gamma}_{33}^2 = \frac{1}{2\sigma}.$$

According to the above equations we conclude that $(M, g, \nabla = \widehat{\nabla} + K)$ is a statistical manifold. Defining the complex structure J as

$$J\partial_1 = \partial_3, \quad J\partial_2 = \partial_4, \quad J\partial_3 = -\partial_1, \quad J\partial_4 = -\partial_2,$$

it follows $g(J\partial_i, J\partial_j) = -g(\partial_i, \partial_j)$, $i, j = 1,2,3,4$. So $(M, \nabla = \widehat{\nabla} + K, g, J)$ forms a Norden manifold.

Example 2.8. *We consider a statistical model M depending on four parameters $\xi^i, i = 1, \cdots, 4$, i.e., the discrete probabitily model given by*

X	x_1	x_2	x_3	x_4	x_5
$P(X = x_k)$	ξ^1	ξ^2	ξ^3	ξ^4	$1 - \xi^1 - \xi^2 - \xi^3 - \xi^4$

with $\xi^i \in (0,1)$. The outcomes $x_i, i = 1, \cdots, 5$ occur with probabilities $\xi^i, i = 1,2,3,4$ and $1 - \xi^1 - \xi^2 - \xi^3 - \xi^4$, respectively. We describe the metric g on M by

$$(g_{ij}) = \begin{pmatrix} \frac{1}{\xi^1} + \frac{1}{1-\xi^1-\xi^2} & \frac{1}{1-\xi^1-\xi^2} & 0 & 0 \\ \frac{1}{1-\xi^1-\xi^2} & \frac{1}{\xi^2} + \frac{1}{1-\xi^1-\xi^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{\xi^1} - \frac{1}{1-\xi^1-\xi^2} & -\frac{1}{1-\xi^1-\xi^2} \\ 0 & 0 & -\frac{1}{1-\xi^1-\xi^2} & -\frac{1}{\xi^2} - \frac{1}{1-\xi^1-\xi^2} \end{pmatrix}.$$

It is easy to see that

$$(g_{ij})^{-1} = \begin{pmatrix} -\xi^1(-1+\xi^1) & -\xi^1\xi^2 & 0 & 0 \\ -\xi^1\xi^2 & -\xi^2(-1+\xi^2) & 0 & 0 \\ 0 & 0 & \xi^1(-1+\xi^1) & \xi^1\xi^2 \\ 0 & 0 & \xi^1\xi^2 & \xi^2(-1+\xi^2) \end{pmatrix}.$$

Also, using (4) and the components g_{ij} and $(g_{ij})^{-1}$, we get

$$\widehat{\Gamma}_{11}^1 = \frac{1}{2}g^{11}\{\partial_1 g_{11}\} + \frac{1}{2}g^{12}\{2\partial_1 g_{12} - \partial_2 g_{11}\} = \frac{-2\xi^1 + \xi^1\xi^2 + 1 - \xi^2}{2\xi^1(1 - \xi^1 - \xi^2)} := \alpha_1.$$

Similarly, we conclude that

$$\widehat{\Gamma}^1_{33} = \widehat{\Gamma}^3_{13} = \alpha_1, \qquad \widehat{\Gamma}^1_{12} = \widehat{\Gamma}^1_{34} = \widehat{\Gamma}^3_{14} = \widehat{\Gamma}^3_{23} = \alpha_2, \quad \widehat{\Gamma}^2_{22} = \widehat{\Gamma}^2_{44} = \widehat{\Gamma}^4_{24} = \alpha_3,$$
$$\widehat{\Gamma}^1_{22} = \widehat{\Gamma}^1_{44} = \widehat{\Gamma}^3_{24} = \alpha_4, \quad \widehat{\Gamma}^2_{12} = \widehat{\Gamma}^2_{34} = \widehat{\Gamma}^4_{14} = \widehat{\Gamma}^4_{23} = \alpha_5, \quad \widehat{\Gamma}^2_{11} = \widehat{\Gamma}^2_{33} = \widehat{\Gamma}^4_{13} = \alpha_6,$$

where $\alpha_2 = \frac{-\xi^1}{2(1-\xi^1-\xi^2)}$, $\alpha_3 = \frac{-\xi^1+\xi^1\xi^2+1-2\xi^2}{2\xi^2(1-\xi^1-\xi^2)}$, $\alpha_4 = \frac{\xi^1(-1+\xi^1)}{2\xi^2(1-\xi^1-\xi^2)}$, $\alpha_5 = \frac{-\xi^2}{2(1-\xi^1-\xi^2)}$ *and* $\alpha_6 = \frac{\xi^2(-1+\xi^2)}{2\xi^1(1-\xi^1-\xi^2)}$. *Defining the non-zero components of the symmetric tensor field* $K = K^l_{ij}\frac{\partial}{\partial\xi^l} \otimes d\xi^i \otimes d\xi^j, i, j, l = 1, 2, 3, 4$ *on* M *as*

$$K^1_{11} = -K^1_{33} = K^3_{13} = \alpha_1, \quad K^1_{12} = -K^1_{34} = K^3_{14} = K^3_{23} = \alpha_2, \quad K^2_{22} = -K^2_{44} = K^4_{24} = \alpha_3,$$
$$K^1_{22} = -K^1_{44} = K^3_{24} = \alpha_4, \quad K^2_{12} = -K^2_{34} = K^4_{14} = K^4_{23} = \alpha_5, \quad K^2_{11} = -K^2_{33} = K^4_{13} = \alpha_6,$$

$(M, \nabla = \widehat{\nabla} + K, g)$ *is a statistical manifold. Considering the complex structure* J *on* M *as*

$$J\frac{\partial}{\partial\xi^1} = \frac{\partial}{\partial\xi^3}, \quad J\frac{\partial}{\partial\xi^2} = \frac{\partial}{\partial\xi^4}, \quad J\frac{\partial}{\partial\xi^3} = -\frac{\partial}{\partial\xi^1}, \quad J\frac{\partial}{\partial\xi^4} = -\frac{\partial}{\partial\xi^2}.$$

One can see $g(J\frac{\partial}{\partial\xi^i}, J\frac{\partial}{\partial\xi^j}) = -g(\frac{\partial}{\partial\xi^i}, \frac{\partial}{\partial\xi^j})$, *for* $i, j = 1, 2, 3, 4$, *i.e.,* g *is a Norden metric. Therefore* $(M, \nabla = \widehat{\nabla} + K, g, J)$ *is a Norden manifold.*

3. Kähler-Norden statistical manifolds

In this section, we shall introduce Kähler-Norden statistical structures. Also, we provide examples including these structures. Moreover, we shall study twin Norden metrics of statistical manifolds.

Now we consider the Φ-operator (or Tachibana operator [20]) applied to the Norden metric g:

$$\begin{aligned}(\Phi_J g)(X, Z_1, Z_2) = {} & JX(g(Z_1, Z_2)) - Xg(JZ_1, Z_2) \\ & + g((L_{Z_1}J)X, Z_2) + g(Z_1, (L_{Z_2}J)X)\end{aligned} \tag{12}$$

for any $Z_1, Z_2 \in \mathfrak{I}^1_0(M)$. If $(\Phi_J g) = 0$, which is equivalent $\widehat{\nabla}J = 0$ [13], then the Norden metric is a holomorphic metric.

Lemma 3.1. *Let* (M, ∇, g, J) *be an almost Norden statistical manifold. Then we have the following formulas:*

i) $(\Phi_J g)(X, Z_1, Z_2) = g((\nabla_{Z_1}J)X, Z_2) + g(X, (\nabla_{Z_2}J)Z_1) - g((\nabla_X J)Z_1, Z_2)$,

ii) $(\Phi_J g)(X, Z_1, Z_2) = g((\nabla^*_{Z_1}J)X, Z_2) + g(X, (\nabla^*_{Z_2}J)Z_1) - g((\nabla^*_X J)Z_1, Z_2)$,

iii) $(\Phi_J g)(X, Z_1, Z_2) + (\Phi_J g)(Z_2, Z_1, X) = 2g((\widehat{\nabla}_{Z_1}J)X, Z_2)$

for any $X, Z_1, Z_2 \in \mathfrak{I}^1_0(M)$.

Proof. Using $[X, Y] = \nabla_X Y - \nabla_Y X$, firstly, we give the following

$$\begin{aligned}(L_{Z_1}J)X &= (L_{Z_1}JX) - J(L_{Z_1}X) \\ &= (\nabla_{Z_1}JX) - \nabla_{JX}Z_1 - J(\nabla_{Z_1}X - \nabla_X Z_1) \\ &= (\nabla_{Z_1}J)X - \nabla_{JX}Z_1 + J\nabla_X Z_1.\end{aligned}$$

Similarly we get

$$(L_{Z_2}J)X = (\nabla_{Z_2}J)X - \nabla_{JX}Z_2 + J\nabla_X Z_2.$$

By means of the above equations, when we applied the Tachibana operator to the Norden metric g, we get from (12)

$$\begin{aligned}(\Phi_J g)(X, Z_1, Z_2) = {} & (\nabla_{JX}g)(Z_1, Z_2) - (\nabla_X g)(JZ_1, Z_2) + g((\nabla_{Z_1}J)X, Z_2) \\ & + g(Z_1, (\nabla_{Z_2}J)X) - g((\nabla_X J)Z_1, Z_2).\end{aligned} \tag{13}$$

Since of being a statistical structure (∇, g), the equation (13) transforms into

$$(\Phi_J g)(X, Z_1, Z_2) = (\nabla_{Z_2} g)(Z_1, JX) - (\nabla_{Z_2} g)(JZ_1, X) - g((\nabla_X J)Z_1, Z_2) \\ + g(Z_1, (\nabla_{Z_2} J)X) + g((\nabla_{Z_1} J)X, Z_2). \tag{14}$$

We find also

$$(\nabla_{Z_2} g)(Z_1, JX) = (\nabla_{Z_2} g)(JZ_1, X) \\ + g(((\nabla_{Z_2} J)Z_1, X) - g(Z_1, (\nabla_{Z_2} J)X). \tag{15}$$

Substituting (15) into (14), we obtain

$$(\Phi_J g)(X, Z_1, Z_2) = g((\nabla_{Z_1} J)X, Z_2) + g(X, (\nabla_{Z_2} J)Z_1) - g((\nabla_X J)Z_1, Z_2),$$

which gives (i). Similarly, we conclude (ii) and (iii). □

An almost complex manifold equipped with a holomorphic metric is called a Kähler-Norden manifold. With this in hand, we will derive a Kähler-Norden statistical manifold as an almost Norden statistical manifold equipped with a holomorphic metric.

Theorem 3.2. *An almost Norden statistical manifold (M, ∇, g, J) is a Kähler-Norden statistical manifold if $\nabla J = 0$, in addition the formula*

$$K_X JY = JK_X Y \tag{16}$$

holds for any $X, Y \in \mathfrak{I}^1_0(M)$, where $K \in \mathfrak{I}^1_2(M)$ is given as (6).

Proof. Setting $\nabla J = 0$ in the part (i) of Proposition 2.3, it immediately follows that $(\Phi_J g) = 0$, i.e., the Norden metric g is a holomorphic metric. Hence, the almost Norden statistical manifold (M, ∇, g, J) is a Kähler-Norden statistical manifold. In addition, the equation (6) with $Y \to JY$ becomes

$$K_X JY = \nabla_X JY - \widehat{\nabla}_X JY \\ = (\nabla_X J)Y + J\nabla_X Y - (\widehat{\nabla}_X J)Y - J\widehat{\nabla}_X Y.$$

But $\nabla J = 0$ implies $\Phi_J g = 0$ and consequently $\widehat{\nabla} J = 0$. So, the above equation reduces to

$$K_X JY = JK_X Y,$$

which completes the proof. □

In [9], the second author and his collaborator proved that if $\nabla J = 0$, then $\nabla^* J = 0$. Hence, we have the following.

Proposition 3.3. *Let g and ∇ be a pseudo-Riemannian metric and a torsion-free linear connection on the manifold M, respectively. If ∇^* is the g-conjugate connection of ∇, and J is an almost complex structure on M, then the following statements are equivalent:*
a) (∇, g, J) is Kähler-Norden statistical structure,
b) (∇^*, g, J) is Kähler-Norden statistical structure.

Example 3.4. *Equip $M = \mathbb{R}^4$ by the pseudo-Riemannian metric $g = \sum_{i=1}^{2}(dx^i \otimes dx^{i+2} + dx^{i+2} \otimes dx^i)$ and the complex structure $J(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^{i+2}}, i = 1, 2$ and $J(\frac{\partial}{\partial x^i}) = -\frac{\partial}{\partial x^{i-2}}, i = 3, 4$, where (x_1, x_2, x_3, x_4) is the standard coordinate* *system and* $\{\frac{\partial}{\partial x^i}\}_{i=1}^4$ *are its associated vector fields. For functions* $\alpha_i, i = 1, \cdots, 8$ *on* $\mathbb{R}^4$*, define* $(1,2)$*-tensor field* $K = K_{ij}^l \frac{\partial}{\partial x^l} \otimes dx^i \otimes dx^j, i, j, l = 1,2,3,4$ *on* $\mathbb{R}^4$ *by*

$$
\begin{aligned}
&K_{11}^1 = -K_{33}^1 = K_{13}^3 = \alpha_1,\\
&K_{12}^1 = -K_{34}^1 = K_{11}^2 = -K_{33}^2 = K_{14}^3 = K_{23}^3 = K_{13}^4 = \alpha_2,\\
&K_{13}^1 = -K_{11}^3 = K_{33}^3 = \alpha_3,\\
&K_{14}^1 = K_{23}^1 = K_{13}^2 = -K_{12}^3 = K_{34}^3 = -K_{11}^4 = K_{33}^4 = \alpha_4,\\
&K_{22}^2 = -K_{44}^2 = K_{24}^4 = \alpha_5,\\
&K_{22}^1 = -K_{44}^1 = K_{12}^2 = -K_{34}^2 = K_{24}^3 = K_{14}^4 = K_{23}^4 = \alpha_6,\\
&K_{24}^2 = -K_{22}^4 = K_{44}^4 = \alpha_7,\\
&K_{24}^1 = K_{14}^2 = K_{23}^2 = -K_{22}^3 = K_{44}^3 = -K_{12}^4 = K_{34}^4 = \alpha_8,
\end{aligned}
$$

where $K_{ij}^l = K_{ji}^l$ *and other components of the tensor K are zero. So K satisfies the conditions in Definition 2.1, and we get a Kähler-Norden statistical manifold* $(\mathbb{R}^4, \nabla = \widehat{\nabla} + K, g, J)$*.*

Example 3.5. *Let M be a manifold with a local coordinate* (x_1, x_2, x_3, x_4) *such that* $\{\partial_i := \frac{\partial}{\partial x^i}\}_{i=1}^4$ *is a natural basis of TM with the dual* $\{dx^i\}_{i=1}^4$*. We define a pseudo-Riemannian metric g on M as follows*

$$
\begin{pmatrix}
0 & 0 & \frac{1}{C^2} & 0\\
0 & 0 & 0 & \frac{2}{C^2}\\
\frac{1}{C^2} & 0 & 0 & 0\\
0 & \frac{2}{C^2} & 0 & 0
\end{pmatrix}, \tag{17}
$$

where C is constant. Assume $K = K_{ij}^l \partial_l \otimes dx^i \otimes dx^j, i, j, l = 1,2,3,4$ *is a* $(1,2)$*-tensor field on M. Using (7), we get* $K_{ij}^l = K_{ji}^l$ *and*

$$
\begin{aligned}
&K_{11}^1 = K_{13}^3,\ K_{12}^1 = K_{23}^3 = 2K_{13}^4,\ K_{13}^1 = K_{33}^3,\ K_{14}^1 = 2K_{13}^2 = K_{34}^3,\\
&K_{22}^1 = 2K_{23}^4,\ K_{23}^1 = 2K_{33}^4,\ K_{24}^1 = 2K_{23}^2 = 2K_{34}^4,\ K_{34}^1 = 2K_{33}^2,\\
&K_{44}^1 = 2K_{34}^2,\ K_{11}^2 = \frac{1}{2}K_{14}^3,\ K_{12}^2 = K_{14}^4 = \frac{1}{2}K_{24}^3,\ K_{14}^2 = \frac{1}{2}K_{44}^3,\\
&K_{22}^2 = K_{24}^4,\ K_{24}^2 = K_{44}^4,\ K_{12}^3 = 2K_{11}^4,\ K_{22}^3 = 2K_{12}^4.
\end{aligned}
$$

We define the complex structure J by $J(\partial_i) = \partial_{i+2}, i = 1,2$ *and* $J(\partial_i) = -\partial_{i-2}, i = 3,4$*. Hence (16) implies*

$$
\begin{aligned}
&K_{11}^1 = -K_{33}^1,\ K_{12}^1 = K_{23}^3 = K_{14}^3 = -K_{34}^1,\ K_{13}^1 = -K_{11}^3,\ K_{14}^1 = K_{23}^1 = -K_{12}^3,\\
&K_{22}^1 = K_{24}^3 = -K_{44}^1,\ K_{24}^1 = K_{44}^3 = -K_{22}^3,\ K_{11}^2 = -K_{33}^2 = K_{13}^4,\\
&K_{12}^2 = -K_{34}^2 = K_{23}^4,\ K_{13}^2 = -K_{11}^4 = K_{33}^4,\ K_{14}^2 = K_{23}^2 = -K_{12}^4 = K_{34}^4,\\
&K_{24}^2 = -K_{22}^4 = K_{44}^4,\ K_{44}^2 = -K_{24}^4.
\end{aligned}
$$

For functions $f_i = f_i(\mu, \sigma), i = 1, \cdots, 8$ *on M, we set*

$$
\begin{aligned}
&K_{11}^1 = f_1,\ K_{12}^1 = f_2,\ K_{13}^1 = f_3,\ K_{14}^1 = f_4,\\
&K_{22}^1 = f_5,\ K_{24}^1 = f_6,\ K_{22}^2 = f_7,\ K_{24}^2 = f_8.
\end{aligned}
$$

Using $\nabla_{\partial_i}\partial_j = \widehat{\nabla}_{\partial_i}\partial_j + K_{\partial_i}\partial_j$, *we have*

$$
\begin{aligned}
\nabla_{\partial_1}\partial_1 &= f_1e_1 + \frac{1}{2}f_2e_2 - f_3e_3 - \frac{1}{2}f_4e_4,\\
\nabla_{\partial_1}\partial_2 &= f_2e_1 + \frac{1}{2}f_5e_2 - f_4e_3 - \frac{1}{2}f_6e_4,\\
\nabla_{\partial_1}\partial_3 &= f_3e_1 + \frac{1}{2}f_4e_2 + f_1e_3 + \frac{1}{2}f_2e_4,\\
\nabla_{\partial_1}\partial_4 &= f_4e_1 + \frac{1}{2}f_6e_2 + f_2e_3 + \frac{1}{2}f_5e_4,\\
\nabla_{\partial_2}\partial_2 &= f_1e_1 + \frac{1}{2}f_2e_2 - f_3e_3 - \frac{1}{2}f_4e_4,\\
\nabla_{\partial_2}\partial_3 &= f_2e_1 + \frac{1}{2}f_5e_2 - f_4e_3 - \frac{1}{2}f_6e_4,\\
\nabla_{\partial_2}\partial_4 &= f_6e_1 + f_7e_2 + f_5e_3 + f_8e_4,\\
\nabla_{\partial_3}\partial_3 &= -f_1e_1 - \frac{1}{2}f_2e_2 + f_3e_3 + \frac{1}{2}f_4e_4,\\
\nabla_{\partial_3}\partial_4 &= -f_2e_1 - \frac{1}{2}f_5e_2 + f_4e_3 + \frac{1}{2}f_6e_4,\\
\nabla_{\partial_4}\partial_4 &= -f_5e_1 - f_8e_2 + f_6e_3 + f_7e_4.
\end{aligned}
$$

It is easy to see that ∇ *satisfies (2). Therefore* (M, ∇, g, J) *is a Kähler-Norden statistical manifold.*

Example 3.6. *We consider a* 4*-dimensional manifold* M *with a local coordinate* (x^1, x^2, x^3, x^4) *such that* $\{\partial_i = \frac{\partial}{\partial x^i}\}_{i=1}^4$ *is a natural basis of tangent space* TM *with the dual basis* $\{dx^i\}_{i=1}^4$. *Let* (J, g) *be a Norden-Walker structure on* M*, that is*

$$J\partial_1 = \partial_2, J\partial_2 = -\partial_1, J\partial_3 = -\partial_4, J\partial_4 = \partial_3 \tag{18}$$

and

$$g = (g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix},$$

where $a = -b, c \in \mathbb{R}$ *(see, [16, 17]). Considering functions* A_1 *and* A_2 *on* M*, we describe non-zero components of* $(1, 2)$*-tensor field* K *on* M *as*

$$K_{33}^1 = -K_{34}^2 = -K_{43}^2 = -K_{44}^1 = A_5, \quad K_{44}^2 = -K_{34}^1 = -K_{43}^1 = -K_{33}^2 = A_6.$$

From the above equations, (7) holds. Also, setting $\nabla_{\partial_i}\partial_j = \widehat{\nabla}_{\partial_i}\partial_j + K_{\partial_i}\partial_j$, *we have*

$$C_{333} = -2A_5, \quad C_{334} = C_{343} = C_{433} = 2A_6, \quad C_{444} = -2A_6, \quad C_{344} = C_{443} = 2A_5.$$

Thus (M, ∇, g) *forms a statistical manifold. Moreover, it easy to see that* $K_{\partial_i}J\partial_j = JK_{\partial_i}\partial_j, i = 1, 2, 3, 4,$ *so* (∇, g, J) *is a Kähler-Norden statistical structure on* M.

Now, we restrict ourselves to the case that g is a Norden metric on an almost complex manifold (M, J). So we can apply J to obtain a new Norden metric G associated with the Norden metric g of the almost Norden manifold (M, J, g) defined by

$$G(X, Y) = g(JX, Y)$$

for any $X, Y \in \mathfrak{I}_0^1(M)$. In this case, the pure tensor field G is called the twin metric of g.

Proposition 3.7. *Let (∇, g, J) be a Kähler-Norden statistical structure on M. Then (∇, G, J) is a Kähler-Norden statistical structure on M.*

Proof. We assume that the condition (16) holds, and we will show that

$$(\nabla_X G)(Y, Z) = (\nabla_Y G)(X, Z).$$

The action of ∇ on a $(0, 2)$−tensor field G is defined as

$$(\nabla_X G)(Y, Z) = XG(Y, Z) - G(\nabla_X Y, Z) - G(Y, \nabla_X Z), \tag{19}$$

which gives

$$(\nabla_X G)(Y, Z) = Xg(JY, Z) - g(J\nabla_X Y, Z) - g(JY, \nabla_X Z). \tag{20}$$

Using (6) and (16) in the above equation, we get

$$(\nabla_X G)(Y, Z) = (\widehat{\nabla}_X g)(Y, JZ) - g(K_X Y, JZ) - g(Y, K_X JZ).$$

Since $(\widehat{\nabla}_X g)(Y, JZ) = 0$, the last equation and (7) imply

$$(\nabla_X G)(Y, Z) = -2g(K_X Y, JZ).$$

We derive, analogously,

$$(\nabla_Y G)(X, Z) = -2g(K_Y X, JZ).$$

Hence, (7) and the last two equations give the result. □

4. Statistical curvature tensor on Kähler-Norden statistical manifolds

For a torsion-free linear connection ∇, the curvature tensor R^∇ is defined as

$$R^\nabla(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

for any $X, Y, Z \in \mathfrak{I}_0^1(M)$. Let (∇, g) be a statistical structure on M. We denote the curvature tensor field of ∇ by R^∇ or R for short, and denote R^{∇^*} by R^* in the similar fashion. The following formulas hold

$$g(R(X, Y)Z, W) = -g(R(Y, X)Z, W),$$

$$g(R^*(X, Y)Z, W) = -g(R^*(Y, X)Z, W),$$

$$g(R(X, Y)Z, W) = -g(R^*(X, Y)W, Z).$$

We set

$$S(X, Y)Z = \frac{1}{2}\{R(X, Y)Z + R^*(X, Y)Z\} \tag{21}$$

for any $X, Y, Z \in \mathfrak{I}_0^1(M)$ and call $S \in \mathfrak{I}_3^1(M)$ the statistical curvature tensor field of (∇, g). We set $S(X, Y, Z, W) = g(S(X, Y)Z, W)$, then S satisfies

$$S(X, Y, Z, W) = -S(Y, X, Z, W),$$

$$S(X, Y, Z, W) = -S(X, Y, W, Z),$$

$$S(X, Y, Z, W) = S(Z, W, X, Y).$$

Assume that (M, ∇, g, J) is a Kähler-Norden statistical manifold. Let R denote the curvature tensor field of (∇, g). The condition $\nabla J = 0$ leads to

$$JR(X, Y)Z = R(X, Y)JZ.$$

Also, Proposition 3.3 implies

$$JR^*(X, Y)Z = R^*(X, Y)JZ.$$

According to the above equations, one deduce that

$$JS(X, Y)Z = S(X, Y)JZ.$$

Hence, we have

$$S(JX, Y, Z, W) = S(X, JY, Z, W),$$

$$S(X, Y, JZ, W) = S(X, Y, Z, JW),$$

i.e., S is pure with respect to X and Y, and also pure with respect to Z and W.

Denote by $\widehat{R}$ the Riemannian curvature tensor of $\widehat{\nabla}$. It is well known that [12]

$$S(X, Y, Z, W) = 2\widehat{R}(X, Y, Z, W) - g(K_Y K_X Z - K_X K_Y Z, W).$$

On the other hand, using (7) and Theorem 3.2, we get

$$K_{JX} Y = K_X JY = JK_X Y.$$

Using this and the purity of the Riemannian curvature tensor, we obtain

$$\begin{aligned} S(X, JY, Z, W) &= 2\widehat{R}(X, JY, Z, W) - g(K_{JY} K_X Z - K_X K_{JY} Z, W) \\ &= 2\widehat{R}(X, Y, Z, JW) - g(J(K_Y K_X Z - K_X K_Y Z), W) \\ &= 2\widehat{R}(X, Y, Z, JW) - g(K_Y K_X Z - K_X K_Y Z, JW) \\ &= S(X, Y, Z, JW), \end{aligned}$$

which means that $S(X, Y, Z, W)$ is pure with respect to Y and W. Thus, the statistical curvature tensor S is pure.

Proposition 4.1. *The statistical curvature tensor field S of a Kähler-Norden statistical manifold is pure.*

If a torsion-free linear connection ∇ which preserve the almost complex structure J ($\nabla J = 0$) satisfies the condition $\nabla_{JX} Y = J\nabla_X Y$, then ∇ is called a holomorphic connection (p. 185 of [19]) (see also [4, 5, 6, 11, 13]). As it is known, the Levi-Civita connection $\widehat{\nabla}$ of Kähler-Norden manifold is a holomorphic connection. Then

$$K_{JX} Y = \nabla_{JX} Y - \widehat{\nabla}_{JX} Y,$$

which gives us

$$J(K_X Y - \widehat{\nabla}_X Y) = \nabla_{JX} Y$$

or

$$J\nabla_X Y = \nabla_{JX} Y.$$

From $\nabla^*_X Y = \widehat{\nabla}_X Y - K_X Y$ and $\nabla^* J = 0$, similarly we get $J\nabla^*_X Y = \nabla^*_{JX} Y = \nabla^*_X JY$. Therefore, we have the following result.

Corollary 4.2. *In a Kähler-Norden statistical manifold (M, ∇, g, J), the torsion-free linear connection ∇ and its g-conjugation ∇^* are holomorphic.*

Any tensor field A of arbitrary type which is pure with respect to an complex structure J is holomorphic if $\Phi_J A = 0$ (fore details, see [15])

Theorem 4.3. *In a Kähler-Norden statistical manifold (M, ∇, g, J), the statistical curvature tensor field S is holomorphic.*

Proof. In the proof, we use several times the techniques of [13]. The Tachibana operator applied to the $(0, 4)$-tensor field S is as follow [13, 15]:

$$\begin{aligned}(\Phi_J S)(X, Y, Z, W, T) = &(JX)(S(Y, Z, W, T)) - X(S(JY, Z, W, T)) + S((L_Y J)X, Z, W, T)\\ &+ S(Y, (L_Z J)X, W, T) + S(Y, Z, (L_W J)X, T) + S(Y, Z, W, (L_T J)X).\end{aligned}$$

From (21), it follows

$$\begin{aligned}(\Phi_J S)(X, Y, Z, W, T) &= \frac{1}{2}((JX)((R + R^*)(Y, Z, W, T)) - X((R + R^*)(JY, Z, W, T))\\ &\quad + (R + R^*)((L_Y J)X, Z, W, T) + (R + R^*)(Y, (L_Z J)X, W, T)\\ &\quad + (R + R^*)(Y, Z, (L_W J)X, T) + (R + R^*)(Y, Z, W, (L_T J)X))\\ &= \frac{1}{2}((\Phi_J R)(X, Y, Z, W, T) + (\Phi_J R^*)(X, Y, Z, W, T)).\end{aligned}$$

On the other hand

$$\begin{aligned}&(\Phi_J R)(X, Y, Z, W, T) = (JX)(R(Y, Z, W, T)) - X(R(JY, Z, W, T))\\ &+ R(\nabla_Y JX - \nabla_{JX} Y - J\nabla_Y X + J\nabla_X Y, Z, W, T) + R(Y, \nabla_Z JX - \nabla_{JX} Z - J\nabla_Z X + J\nabla_X Z, W, T)\\ &+ R(Y, Z, \nabla_W JX - \nabla_{JX} W - J\nabla_W X + J\nabla_X W, T) + R(Y, Z, W, \nabla_T JX - \nabla_{JX} T - J\nabla_T X + J\nabla_X T).\end{aligned}$$

From $\nabla J = 0$, the last equation implies

$$\begin{aligned}(\Phi_J R)(X, Y, Z, W, T) = &(JX)(R(Y, Z, W, T)) - X(R(JY, Z, W, T))\\ &+ R(-\nabla_{JX} Y + J\nabla_X Y, Z, W, T) + R(Y, -\nabla_{JX} Z + J\nabla_X Z, W, T)\\ &+ R(Y, Z, -\nabla_{JX} W + J\nabla_X W, T) + R(Y, Z, W, -\nabla_{JX} T + J\nabla_X T).\end{aligned}$$

Since ∇ (resp. ∇^*) is a holomorphic connection, its curvature tensor field R (resp. R^*) is pure (see [13]). Hence, from $\nabla J = 0$, it follows

$$\begin{aligned}&-X(R(JY, Z, W, T)) + R(J\nabla_X Y, Z, W, T)\\ &+R(Y, J\nabla_X Z, W, T) + R(Y, Z, J\nabla_X W, T) + R(Y, Z, W, J\nabla_X T)\\ &= -X(R(JY, Z, W, T) + R(\nabla_X JY, Z, W, T)\\ &+R(JY, \nabla_X Z, W, T) + R(JY, Z, \nabla_X W, T) + R(JY, Z, W, \nabla_X T)\\ &= -(\nabla_X R)(JY, Z, W, T)\end{aligned}$$

and

$$\begin{aligned}&(JX)(R(Y, Z, W, T)) - R(\nabla_{JX} Y, Z, W, T) - R(Y, \nabla_{JX} Z, W, T)\\ &-R(Y, Z, \nabla_{JX} W, T) - R(Y, Z, W, \nabla_{JX} T)\\ &= (\nabla_{JX} R)(Y, Z, W, T).\end{aligned}$$

Thus

$$(\Phi_J R)(X, Y, Z, W, T) = -(\nabla_X R)(JY, Z, W, T) + (\nabla_{JX} R)(Y, Z, W, T).$$

We get

$$(\nabla_X R)(JY, Z, W, T) = (\nabla_X g)(R(JY, Z)W, T) + g((\nabla_X R)(JY, Z, W), T).$$

The last two equations, yield

$$(\Phi_J R)(X, Y, Z, W, T) = -(\nabla_X g)(R(JY, Z)W, T) - g((\nabla_X R)(JY, Z, W), T) \\ + (\nabla_{JX} g)(R(Y, Z)W, T) + g((\nabla_{JX} R)(Y, Z, W), T). \tag{22}$$

Since $\nabla J = 0$ and using the Bianchi's 2nd identity, we conclude

$$\begin{aligned} &-g((\nabla_X R)(JY, Z, W), T) + g((\nabla_{JX} R)(Y, Z, W), T) = -g(J(\nabla_X R)(Y, Z, W), T) \\ &-g((\nabla_Y R)(Z, JX, W), T) - g((\nabla_Z R)(JX, Y, W), T) \\ &= g(J(\nabla_X R)(Y, Z, W), T) - g(J(\nabla_Y R)(Z, X, W), T) - g(J(\nabla_Z R)(X, Y, W), T) = 0. \end{aligned}$$

Also, since (∇, g) is a statistical structure, we obtain

$$(\nabla_X g)(R(JY, Z)W, T) = (\nabla_{JX} g)(R(Y, Z)W, T).$$

Setting two last equations in (22), we deduce $(\Phi_J R)(X, Y, Z, W, T) = 0$. In the same way, we get $(\Phi_J R^*)(X, Y, Z, W, T) = 0$. Therefore, we have the assertion. □

5. Codazzi Coupling

In this section, we consider a Norden statistical manifold (M, ∇, g, J) and study Codazzi coupling of the connection ∇ with the almost complex structure J. Also, we conclude that the connection ∇ is Codazzi coupled with the twin metric G if and only if ∇ reduces to the Levi-Civita connection $\widehat{\nabla}$ or J is parallel under ∇.

Definition 5.1. *Let (∇, g, J) be a Norden statistical structure on the manifold M. The almost complex structure J and ∇ is called Codazzi coupled if we have*

$$(\nabla_X J)Y = (\nabla_Y J)X,$$

for any $X, Y \in \Im_0^1(M)$.

Lemma 5.2. *In a Norden statistical manifold (M, ∇, g, J), if (16) holds, then the Tachibana operator to the Norden metric g satisfies the following identity*

$$(\Phi_J g)(X, Y, Z) = (\Phi_J g)(Z, Y, X) \tag{23}$$

if and only if

$$(\nabla_X J)Y = (\nabla_Y J)X \tag{24}$$

for any $X, Y \in \Im_0^1(M)$.

Proof. If (23) holds, then using (6), we have

$$(\nabla_X J)Y = \nabla_X JY - J\nabla_X Y = \widehat{\nabla}_X JY - J\widehat{\nabla}_X Y + K_X JY - JK_X Y = (\widehat{\nabla}_X J)Y + K_X JY - JK_X Y.$$

Setting the above equation in the part (i) of Lemma 3.1, we have

$$(\Phi_J g)(X, Y, Z) = g((\widehat{\nabla}_Y J)X, Z) + g(X, (\widehat{\nabla}_Z J)Y) - g((\widehat{\nabla}_X J)Y, Z) \\ + g(K_Y JX - JK_Y X, Z) + g(X, K_Z JY - JK_Z Y) - g(K_X JY - JK_X Y, Z),$$

which using (16) gives

$$(\Phi_J g)(X,Y,Z) = g((\widehat{\nabla}_Y J)X, Z) + g(X, (\widehat{\nabla}_Z J)Y) - g((\widehat{\nabla}_X J)Y, Z).$$

Similarly, it follows

$$(\Phi_J g)(Z,Y,X) = g((\widehat{\nabla}_Y J)Z, X) + g(Z, (\widehat{\nabla}_X J)Y) - g((\widehat{\nabla}_Z J)Y, X).$$

As $g((\widehat{\nabla}_Y J)Z, X) = g((\widehat{\nabla}_Y J)X, Z)$, the last two equations imply $g((\widehat{\nabla}_X J)Y, Z) = g((\widehat{\nabla}_Y J)X, Z)$. So, we deduce

$$(\widehat{\nabla}_X J)Y = (\widehat{\nabla}_Y J)X.$$

Applying $K_X JY = K_Y JX$ in the above equation, we obtain $(\nabla_X J)Y = (\nabla_Y J)X$. In the similar way, we can prove the converse of the Lemma. □

Example 5.3. *Considering the Norden statistical manifold (M, ∇, g, J) in Example 2.7, we obtain*

$$(\nabla_{\partial_i} J)\partial_j = 0 = (\nabla_{\partial_j} J)\partial_i, \quad i, j = 1,2,3,4,$$

except

$$(\nabla_{\partial_1} J)\partial_1 = -\frac{1}{\sigma}\partial_4, \quad (\nabla_{\partial_2} J)\partial_2 = -\frac{2}{\sigma}\partial_4, \quad (\nabla_{\partial_1} J)\partial_3 = -\frac{1}{\sigma}\partial_2 = (\nabla_{\partial_3} J)\partial_1,$$
$$(\nabla_{\partial_3} J)\partial_3 = \frac{1}{\sigma}\partial_4, \qquad (\nabla_{\partial_4} J)\partial_4 = \frac{2}{\sigma}\partial_4 \qquad (\nabla_{\partial_2} J)\partial_4 = -\frac{2}{\sigma}\partial_2 = (\nabla_{\partial_4} J)\partial_2.$$

Thus the pair (∇, J) is Codazzi coupled. On the other hand, it follows

$$(\Phi_J g)(\partial_i, \partial_j, \partial_k) = 0, \quad i, j = 1,2,3,4,$$

unless

$$(\Phi_J g)(\partial_2,\partial_1,\partial_3) = -\frac{2}{\sigma^3}, \quad (\Phi_J g)(\partial_2,\partial_2,\partial_4) = -\frac{4}{\sigma^3}, \quad (\Phi_J g)(\partial_2,\partial_3,\partial_1) = -\frac{2}{\sigma^3}, \quad (\Phi_J g)(\partial_2,\partial_4,\partial_2) = -\frac{4}{\sigma^3},$$
$$(\Phi_J g)(\partial_4,\partial_1,\partial_1) = \frac{2}{\sigma^3}, \quad (\Phi_J g)(\partial_4,\partial_2,\partial_2) = \frac{4}{\sigma^3}, \quad (\Phi_J g)(\partial_4,\partial_3,\partial_3) = -\frac{2}{\sigma^3}, \quad (\Phi_J g)(\partial_4,\partial_4,\partial_4) = -\frac{4}{\sigma^3}.$$

So Lemma 5.2 doesn't hold. But considering σ = constant and defining a new $(1,2)$-tensor field K on M such that $K^1_{11} = K^3_{13} = K^3_{31} = -K^1_{33}$ and other components of the tensor K are zero, we conclude that $(M, \nabla = \widehat{\nabla} + K, g, J)$ is a Norden manifold. It is easy to see that $K_{\partial_i} J\partial_j = K_{\partial_j} J\partial_i$ and $(\Phi_J g)(\partial_i, \partial_j, \partial_k) = 0, i, j = 1,2,3,4$. Therefore Lemma 5.2 holds.

Now, considering a Norden statistical manifold (M, ∇, g, J), we study Codazzi coupling of ∇ with the twin metric G. In order we present the following theorem:

Theorem 5.4. *Let (M, ∇, g, J) be a Norden statistical manifold. Then we have*

$$(\nabla_X G)(Y,Z) - (\nabla_Y G)(X,Z) = g((\nabla^*_Z J - \nabla_Z J)Y, X) + g((\nabla_X J)Y - (\nabla_Y J)X, Z) \tag{25}$$

for any $X, Y, Z \in \mathfrak{I}^1_0(M)$.

Proof. Applying (19) and (20), we can write

$$(\nabla_X G)(Y,Z) = (\nabla_X g)(JY, Z) + g((\nabla_X J)Y, Z) = C(X, JY, Z) + g((\nabla_X J)Y, Z). \tag{26}$$

Similarly, we have

$$(\nabla_Y G)(X,Z) = C(JX, Y, Z) + g((\nabla_Y J)X, Z). \tag{27}$$

On the other hand, we obtain

$$C(X, JY, Z) - C(JX, Y, Z) = g((\nabla^*_Z J - \nabla_Z J)Y, X). \tag{28}$$

From (26)-(28), we conclude (5.5). □

Corollary 5.5. *Let (M,∇,g,J) be a Norden statistical manifold. If the pair (∇,J) is Codazzi coupled, then*

$$(\nabla_X G)(Y,Z)-(\nabla_Y G)(X,Z)=g((\nabla^*_Z J-\nabla_Z J)Y,X)$$

for any $X,Y,Z\in\mathfrak{I}^1_0(M)$.

Corollary 5.6. *In a Norden statistical manifold (M,∇,g,J), the condition Codazzi coupling of ∇ with G, i.e.,*

$$(\nabla_X G)(Y,Z)=(\nabla_Y G)(X,Z) \tag{29}$$

is equivalent to

$$(\nabla_X J)Y=(\nabla_Y J)X \tag{30}$$

for any $X,Y,Z\in\mathfrak{I}^1_0(M)$, if and only if one of the following assertion holds:
i) $\nabla=\nabla^*$, i.e., ∇ reduces to the Levi-Civita connection $\widehat{\nabla}$.
ii) J is parallel under ∇, i.e., $\nabla J=0$.

Example 5.7. *The twin metric G of g in Example 5.3 is determined by*

$$(G_{i,j})=\begin{bmatrix}0&0&-\frac{1}{\sigma^2}&0\\0&0&0&-\frac{2}{\sigma^2}\\-\frac{1}{\sigma^2}&0&0&0\\0&-\frac{2}{\sigma^2}&0&0\end{bmatrix}.$$

On the other hand, it follows that $(\nabla_{\partial_i}G)(\partial_j,\partial_k)-(\nabla_{\partial_j}G)(\partial_i,\partial_k)=0=g((\nabla^*_{\partial_k}J-\nabla_{\partial_k}J)\partial_j,\partial_i)$ for $i,j,k=1,2,3,4$, except

$$\begin{aligned}
(\nabla_{\partial_1}G)(\partial_4,\partial_1)-(\nabla_{\partial_4}G)(\partial_1,\partial_1)&=\frac{2}{\sigma^3}=g((\nabla^*_{\partial_1}J-\nabla_{\partial_1}J)\partial_4,\partial_1),\\
(\nabla_{\partial_2}G)(\partial_3,\partial_1)-(\nabla_{\partial_3}G)(\partial_2,\partial_1)&=\frac{2}{\sigma^3}=g((\nabla^*_{\partial_1}J-\nabla_{\partial_1}J)\partial_3,\partial_2),\\
(\nabla_{\partial_2}G)(\partial_4,\partial_2)-(\nabla_{\partial_4}G)(\partial_2,\partial_2)&=\frac{8}{\sigma^3}=g((\nabla^*_{\partial_2}J-\nabla_{\partial_2}J)\partial_4,\partial_2),\\
(\nabla_{\partial_1}G)(\partial_2,\partial_3)-(\nabla_{\partial_2}G)(\partial_1,\partial_3)&=-\frac{2}{\sigma^3}=g((\nabla^*_{\partial_3}J-\nabla_{\partial_3}J)\partial_2,\partial_1),\\
(\nabla_{\partial_3}G)(\partial_4,\partial_3)-(\nabla_{\partial_4}G)(\partial_3,\partial_3)&=-\frac{2}{\sigma^3}=g((\nabla^*_{\partial_3}J-\nabla_{\partial_3}J)\partial_4,\partial_3).
\end{aligned}$$

So Corollary 5.5 holds.

Example 5.8. *We consider the Norden statistical structure (∇,g,J) in Example 3.6. According to (18), we obtain $(\nabla_{\partial_i}J)\partial_j=0, i,j=1,2,3,4$, i.e., the pair structure (∇,J) is Codazzi coupling. The twin metric G of the Norden-Walker metric g is given by*

$$G=(G_{ij})=\begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&-1&-c&a\\1&0&a&c\end{pmatrix}.$$

As $(\nabla_{\partial_i}G)(\partial_j,\partial_k)=0=(\nabla_{\partial_j}G)(\partial_i,\partial_k), i,j,k=1,2,3,4$, except for

$$\begin{aligned}
&(\nabla_{\partial_3}G)(\partial_3,\partial_3)=-2A_2, &&(\nabla_{\partial_3}G)(\partial_4,\partial_3)=-2A_1=(\nabla_{\partial_4}G)(\partial_3,\partial_3),\\
&(\nabla_{\partial_4}G)(\partial_4,\partial_4)=2A_1, &&(\nabla_{\partial_3}G)(\partial_4,\partial_4)=2A_2=(\nabla_{\partial_4}G)(\partial_3,\partial_4),
\end{aligned}$$

it follows that G is Codazzi Coupled with ∇. Hence Corollary 5.6 holds.

We can define in analogous to g-conjugation of a connection, for any connection ∇, the G-conjugate connection $\nabla^\dagger$ by setting

$$XG(Y,Z) = G(\nabla_X Y, Z) + G(Y, \nabla^\dagger_X Z)$$

for any $X, Y, Z \in \mathfrak{I}^1_0(M)$. It is easy to see that $\nabla^\dagger$ is a connection, and $(\nabla^\dagger)^\dagger = \nabla$. Since the metric G is symmetric, the above equation can be written as

$$XG(Y,Z) = G(\nabla^\dagger_X Y, Z) + G(Y, \nabla_X Z).$$

Using the definition of ∇G, we have

$$(\nabla_X G)(Y,Z) = G(\nabla^\dagger_X Y - \nabla_X Y, Z),$$

and since $(\nabla^\dagger)^\dagger = \nabla$ it follows that

$$(\nabla_X G)(Y,Z) = -(\nabla^\dagger_X G)(Y,Z).$$

According to the above discussions we have the following proposition:

Proposition 5.9. *In a Norden statistical manifold (M, ∇, g, J), the following statements are equivalent:*
i) $\nabla G = 0$,
ii) $\nabla = \nabla^\dagger$.

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