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Two Theorems Involving Cyclic Generalized Proximal C-Contractive Non-Self Mappings

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Abstract. This paper explores certain best proximity point expansions for a novel class of non-self-mapping $S : P \longrightarrow Q$ and $T : Q \rightarrow P$ called generalized proximal C-contractions of the first and second kinds. We expose many examples to justify our obtained results. Considerable fixed point results are evolved as a consequence of our main theorems.

1. Introduction

The useful classical Banach contraction principle (BCP) states that every contraction mapping on a complete metric space possesses a unique fixed point. It is considered as the limit of Picard sequence. Various nice extensions and generalizations of the BCP have been released in the literature. However, the mappings involved in all these results are self-mappings. So, it is contemplated to give some best proximity point theorems which furnish non-self mappings analogues of the BCP. Consequently, the results established in this work ensure the existence of optimal approximate solutions for some fixed point equations when there is no solution.

Let (M, d) be a metric space. Let (P, Q) be a pair of nonempty subsets on M. A point $x \in M$ is named as a best proximity point of a non self mapping $S : P \longrightarrow Q$ if and only if $d(x, Sx) = d(P, Q)$. Best proximity point theorems are generalizations of fixed point theorems. They provide sufficient conditions that ensure the existence of approximate solutions which are optimal as well. As a first step, a best proximity point theorem was proposed by Fan in [1] for a non-self continuous mapping $S : P \longrightarrow Q$, where P is a nonempty compact convex subset of a Hausdorff locally convex topological vector space. Many generalizations of Fan's theorems were performed in the literature, such that Reich [2], Sehgal and Singh [3] and Prolla

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[4]. Basha [5] made an extension of Banach contraction principle by a best proximity theorem under the hypothesis that N is approximately compact with respect to P. Later on, several best proximity point results were derived (see e.g., [6]-[19]).

The goal of this paper is an extension of the work of Basha [6].

Theorem 1.1. *Let (P,Q) be a pair of nonempty closed subsets of a complete metric space (M,d) such that P_0 and Q_0 are nonvoid. Let $S: P \to Q$ and $T: Q \to P$ satisfy the following conditions:*

1. *S and T are proximal contractions of the first kind;*
2. *$S(P_0) \subset Q_0$ and $T(Q_0) \subset P_0$;*
3. *The pair (S,T) forms a cyclic contraction.*

Then S (resp. T) has a unique best proximity point x in P (resp. y in Q). We have $d(x,y) = d(P,Q)$.

A generalized version of such theorem was initiated on [20] for proximal β-quasi contractive mappings for non-self-mappings $S: P \to Q$ and $T: Q \to P$. Furthermore, some best proximity point developments for a novel class of non-self mappings, called α- proximal Geraghty mappings were investigated in [21].

Our chief aim is to use C- functions introduced first by Ansari in [22]. In Fact, by suggesting the notion of proximal generalized C-contractive non-self mappings, we can generalize the result of Basha [6] for a pair of non-self mappings (S,T) forming cyclic contractions in the context of complete metric spaces. As a conclusion, several fixed point results were derived. We suggest several concrete examples to make our results valid.

2. Preliminaries

Let (P,Q) be a pair of nonempty subsets of a metric space (X,d). We opt the coming notations:

$$\begin{aligned} d(P,Q) &:= \inf\{d(a,b) : a \in P, b \in Q\}; \\ P_0 &:= \{a \in P : \exists\, b \in Q; d(a,b) = d(P,Q)\}; \\ Q_0 &:= \{b \in Q : \exists\, a \in P; d(a,b) = d(P,Q)\}. \end{aligned}$$

Definition 2.1. *[5] Let $S: P \to Q$ be a non-self mapping. A point x_* is called a best proximity point of S if $d(x_*, Sx_*) = d(P,Q)$.*

Definition 2.2. *[6] A non-self mapping $S: P \longrightarrow Q$ is named as a proximal contraction of the first kind if there is $0 < \alpha < 1$ satisfying $d(u_1, Sx_1) = d(P,Q) = d(u_2, Sx_2)$ then $d(u_1,u_2) \leq \alpha d(x_1,x_2)$, $\quad \forall\, x_1, x_2 \in P, u_1, u_2 \in Q$.*

Definition 2.3. *[6] A non-self mapping $S: P \longrightarrow Q$ is said to be a proximal contraction of second kind if there exists $0 < \alpha < 1$ satisfying $d(u_1, Sx_1) = d(P,Q) = d(u_2, Sx_2)$ then $d(Su_1, Su_2) \leq \alpha d(Sx_1, Sx_2)$ for all $x_1, x_2 \in P, u_1, u_2 \in Q$.*

Definition 2.4. *[6] Given non-self-mappings $S: P \to Q$ and $T: Q \to P$. The pair (S,T) is said to form a proximal cyclic contraction if there is $0 < l < 1$ such that:*

$$d(u, Sa) = d(P,Q) \text{ and } d(v, Tb) = d(P,Q) \Longrightarrow d(u,v) \leq l d(a,b) + (1-l) d(P,Q)$$

for all $u, a \in P$ and $v, b \in Q$.

In 2014, the approach of C-class functions was initiated by Ansari [22].

Definition 2.5. *[22] A mapping $J: [0,\infty)^2 \to \mathbb{R}$ is named as a C-class function if it is continuous and satisfies the following assertions:*

1. *$J(u,v) \leq u$;*
2. *$J(u,v) = u \Rightarrow u = 0$ or $v = 0$; for all $u, v \in [0,\infty)$.*

Note for some J, we have that $J(0,0) = 0$. The set of C-class functions is denoted by $\mathcal{C}$.

The coming functions $J: [0,\infty)^2 \to \mathbb{R}$ belong to $\mathcal{C}$, for all $u, v \geq 0$:

1. $J(u, v) = u - v$, $J(u, v) \Longrightarrow v = 0$;
2. $J(u, v) = mu$, $0 < m < 1$, $J(u, v) = u \Rightarrow u = 0$;
3. $J(u, v) = \frac{u}{(1+v)^r}$; $r \in (0, \infty)$, $J(u, v) = u \Rightarrow u = 0$ or $v = 0$;
4. $J(u, v) = \log(v + a^u)/(1 + v)$, $a > 1$, $J(u, v) = u \Rightarrow u = 0$ or $v = 0$;
5. $J(u, v) = \ln(1 + a^u)/2$, $a > e$, $J(u, v) = s \Rightarrow s = 0$;
6. $J(u, v)) = (u + l)^{1/(1+v)^r} - l$, $l > 1, r \in (0, \infty)$, $J(u, v) = u \Rightarrow v = 0$;
7. $J(u, v) = u \log_{v+a} a$, $a > 1$, $f(u, v) = u \Rightarrow u = 0$ or $v = 0$;
8. $J(u, v) = u - (\frac{1+u}{2+u})(\frac{v}{1+v})$, $J(u, v) = u \Rightarrow v = 0$;
9. $J(u, v) = s\beta(u)$, $\beta : [0, \infty) \to [0, 1)$, and is continuous, $J(u, v) = u \Rightarrow u = 0$;
10. $J(u, v) = u - \frac{v}{k+v}$, $J(u, v) = u \Rightarrow v = 0$;
11. $J(u, v) = u - \varphi(v), J(u, v) = u \Rightarrow u = 0$, here $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\varphi(v) = 0 \Leftrightarrow v = 0$;
12. $J(u, v) = uh(u, v), J(u, v) = u \Rightarrow u = 0$, where $h : [0, \infty) \times [0, \infty) \to [0, \infty)$ is a continuous function such that $h(u, v) < 1$ for all $u, v > 0$;
13. $J(u, v) = u - (\frac{2+v}{1+v})v$, $J(u, v) = u \Rightarrow v = 0$;
14. $J(u, v) = \sqrt[n]{\ln(1 + u^n)}$, $J(u, v) = u \Rightarrow u = 0$;
15. $J(u, v) = \phi(u), f(s, t) = s \Rightarrow s = 0$, where $\phi : [0, \infty) \to [0, \infty)$ is a upper semi-continuous function such that $\phi(0) = 0$, and $\phi(v) < v$ for $v > 0$,
16. $J(u, v) = \frac{u}{(1+u)^r}$; $r \in (0, \infty)$, $J(u, v) = u \Rightarrow u = 0$;
17. $J(u, v) = \vartheta(u)$; $\vartheta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a generalized Mizoguchi-Takahashi type function , $J(u, v) = u \Rightarrow u = 0$;
18. $J(u, v) = \frac{u}{\Gamma(1/2)} \int_0^\infty \frac{e^{-x}}{\sqrt{x}+v}\, dx$, where Γ is the Euler Gamma function.

3. Main developments

In the beginning, we propose the coming concepts:

Definition 3.1. *Let (X, d) be a metric space and (P, Q) be a pair of nonempty subsets of X. A non-self mapping $S : P \to Q$ is called a generalized proximal C-contractive of the first kind, if there exists $J \in C$ such that $d(u_1, Sx_1) = d(P, Q) = d(u_2, Sx_2)$ then,*

$$d(u_1, u_2) \leq J(d(x_1, y_1), d(x_1, y_1)),$$

for all $x_1, y_1, u_1, u_2 \in P$.

Definition 3.2. *Let (X, d) be a metric space and (P, Q) be a pair of nonempty subsets of X. A non-self-mapping $S : P \to Q$ is announced a generalized proximal C-contractive of second kind, if there exists $J \in C$ such that $d(u_1, Sx_1) = d(P, Q) = d(u_2, Sx_2)$, then*

$$d(Su_1, Su_2) \leq J(d(Sx_1, Sy_1), d(Sx_1, Sy_1)),$$

for all $x_1, y_1, u_1, u_2 \in P$.

Now, we are ready to suggest the coming best proximity point theorem.

Theorem 3.3. *Let (X, d) be a complete metric space, (P, Q) be a pair of nonempty closed subsets and (P_0, Q_0) be a pair of nonempty subsets in X. Let $S : P \longrightarrow Q$ and $T : Q \longrightarrow P$ be two non-self mappings satisfying the following assertions:*

1. *$S(P_0) \subset Q_0$ and $T(Q_0) \subset P_0$;*
2. *S and T are generalized proximal C-contractive non-self-mappings of the first kind;*
3. *The pair (S, T) forms a proximal cyclic contraction.*

Then, S has a unique best proximity point $z_* \in P$ and T has a unique best proximity point $t_* \in Q$. These best proximity points verify $d(z_*, t_*) = d(P, Q)$.

Proof. Using the fact P_0 is nonempty, there exist $z_0, z_1 \in P_0$ such that $d(z_1, Sz_0) = d(P, Q)$. As $S(P_0) \subset Q_0$, there exits $z_2 \in P_0$ such that $d(z_2, Sz_1) = d(P, Q)$. Also $S(P_0) \subset Q_0$, there exists $z_3 \in P_0$ such that $d(z_3, Sz_2) = d(P, Q)$.

In same manner, by induction, we can build a sequence $\{z_n\} \subset P_0$ such that

$$d(z_{n+1}, Sz_n) = d(P, Q) \quad \text{and} \quad \text{for all} \quad n \in \mathbb{N} \cup \{0\}. \tag{1}$$

Also, $d(z_n, Sz_{n-1}) = d(P, Q)$ for all $n \in \mathbb{N}$. Using the hypothesis that S is generalized proximal C-contractive of the first kind and the property of J, we obtain that

$$d(z_{n+1}, z_n) \leq J(d(z_n, z_{n-1}), d(z_n, z_{n-1})) \leq d(z_n, z_{n-1}) \quad \text{for all} \quad n \in \mathbb{N}. \tag{2}$$

Thus, the sequence $\{d(z_{n+1}, z_n)\}$ is decreasing. We suppose that there exists $k \in \mathbb{N}$ such that $d(z_k, z_{k+1}) = 0$. In this case, we get $d(z_k, Sz_k) = d(P, Q)$, and consequently S has a best proximity point, which is z_k.

Now, we suppose that $d(z_n, Sz_{n-1}) > 0$ for all $n \in \mathbb{N}$. So the sequence $\{d(z_{n+1}, z_n)\}$ is decreasing and of non-negative terms, then there exists $\sigma \geq 0$ such that
$\sigma = \lim_{n \longrightarrow \infty} d(z_n, z_{n+1})$. If we suppose that $\sigma > 0$, then by letting $n \longrightarrow +\infty$ on (2) and using the continuity of f, we obtain that

$$\sigma \leq J(\sigma, \sigma) \leq \sigma.$$

Consequently, we find that $J(\sigma, \sigma) = \sigma$, so $\sigma = 0$, which is an absurdity.

Now, we declare that the sequence $\{z_n\}$ is a Cauchy sequence. Suppose it not the case. Then there exist $\alpha > 0$ and subsequences $\{z_{m_p}\}$ and $\{z_{n_p}\}$ such that for all positive integers p with $m(p) > n(p) > p$, we have $d(z_{m(p)}, z_{n(p)}) \geq \alpha$ and $d(z_{m(p)}, z_{n(p)-1}) < \alpha$. Using the triangular inequality, we get

$$\begin{aligned} \alpha \leq d(z_{m(p)}, z_{n(p)}) \quad &\leq d(z_{m(p)}, z_{m(p)-1}) + d(z_{m(p)-1}, z_{n(p)}) \\ &\leq \alpha + d(z_{m(p)}, z_{m(p)-1}). \end{aligned} \tag{3}$$

Taking limit as $p \longrightarrow +\infty$ in the overhead inequality (3), we get

$$\lim_{p \longrightarrow +\infty} d(z_{m(p)}, z_{n(p)}) = \alpha > 0. \tag{4}$$

Using additionally the triangular inequality, one writes

$$d(z_{m(p)}, z_{n(p)}) \leq d(z_{m(p)}, z_{m(p)-1}) + d(z_{m(p)-1}, z_{n(p)-1}) + d(z_{n(p)-1}, z_{n(p)}). \tag{5}$$

If we apply the limit on both sides and using (4), we obtain

$$\lim_{p \longrightarrow +\infty} d(z_{m(p)-1}, z_{n(p)-1}) = \alpha. \tag{6}$$

Using that S is a proximal generalized C-proximal non-self mapping of the first kind, we obtain for all $p \geq 0$,

$$d(z_{m(p)}, z_{n(p)}) \quad \leq J(d(z_{m(p)-1}, z_{n(p)-1}), d(z_{m(p)-1}, z_{n(p)-1})) \leq d(z_{m(p)-1}, z_{n(p)-1}). \tag{7}$$

Letting $p \longrightarrow +\infty$ in the above inequality (7), we have

$$\alpha \leq f(\alpha, \alpha) \leq \alpha.$$

Thus, $\alpha = 0$ which contradicts (4). Therefore, the sequence $\{z_n\}$ is a Cauchy sequence. Using the closeness of the subset P of the complete metric space (X, d), there exists $z_* \in P$ such that the sequence $\{z_n\}$ converges to z_*.

In view of the fact $T(Q_0) \subset Q_0$, it is guaranteed that there exists a sequence $\{t_n\} \in Q_0$ such that $d(t_{n+1}, Tt_n) = d(P, Q)$. Since T is proximal generalized C- contractive of the first kind, there exists $J_1 \in C$ such that

$$d(t_{n+1}, t_n) \leq J_1(d(t_n, t_{n-1}), d(t_n, t_{n-1})) \leq d(t_n, t_{n-1}).$$

Therefore, the sequence $\{t_n\}$ is Cauchy and so it converges to some $t_* \in Q$. Using the hypothesis that (S, T) forms a proximal cyclic contraction, we get $d(z_{n+1}, t_{n+1}) \leq kd(z_n, t_n) + (1 - k)d(P, Q)$. Letting $n \longrightarrow +\infty$, we get $d(z_*, t_*) = d(P, Q)$. Thus, we guarantee that $z_* \in P_0$ and $t_* \in Q_0$. Since $S(P_0) \subset Q_0$ and $T(Q_0) \subset P_0$, there exist $z \in P$ and $t \in Q$ such that $d(z, Sz_*) = d(t, Tt_*) = d(P, Q)$. Since S is generalized Proximal C-contractive of the first kind, we get

$$d(z, z_{n+1}) \leq f(d(z_*, z_n), d(z_*, z_n)) \leq d(z_*, z_n).$$

Letting $n \longrightarrow +\infty$, we get that $z = z_*$. In a similar fashion, we can demonstrate that $t = t_*$. Therefore, $d(z_*, Sz_*) = d(t_*, Tt_*) = d(P, Q)$. We conclude that z_* and t_* are best proximity points respectively for S and T.

Concerning the uniqueness, we assume there are two different best proximity points for S such that $d(z_*, S_*) = d(P, Q) = d(x_*, Sx_*)$. Let $r = d(z_*, x_*) > 0$. Taking in account that S is generalized proximal C-contractive of the first kind, then $r \leq J(r, r) \leq r$. It results that $r = 0$, which is a contradiction. □

Theorem 3.4. *Let (X, d) be a complete metric space, (P, Q) be a pair of nonempty closed subsets and (P_0, Q_0) be a pair of nonvoid subsets in X. Let $S : P \longrightarrow Q$ satisfy the following assertions:*

1. *$S(P_0) \subset Q_0$;*
2. *S is generalized proximal C-contractive of the first and second kinds.*

Then, S has a unique best proximity point, say $u_* \in P$.

Proof. As in Theorem 3.3, using the hypothesis (1) such that $S(P_0) \subset Q_0$, we can build a sequence $\{u_n\}$ in P_0 such that $d(u_{n+1}, Su_n) = d(P, Q)$ for all $n \in \mathbb{N} \cup \{0\}$. Since S is generalized proximal C-contractive of the first kind, there exits $J \in C$ such that

$$d(u_{n+1}, u_n) \leq J(d(u_n, u_{n-1}), d(u_n, u_{n-1})) \leq d(u_n, u_{n-1}) \quad \text{for all} \quad n \in \mathbb{N}.$$

Now, we prove that $\{u_n\}$ is a Cauchy sequence in the closed subset P. Then there exists u_* such that $u_n \longrightarrow u_*$. Again, by considering that S is generalized proximal C-contractive of second kind, we get

$$d(Su_{n+1}, Su_n) \leq J(d(Su_n, Su_{n-1}), d(Su_n, Su_{n-1})) \leq d(Su_n, Su_{n-1}) \quad \text{for all} \quad n \in \mathbb{N}.$$

In a similar way, we can demonstrate that $\{Su_n\}$ is a Cauchy sequence in the closed set Q. Therefore, there exits $v_* \in Q$, such that $Su_n \longrightarrow v_*$. We write

$$d(u_*, v_*) = \lim_{n \longrightarrow \infty} d(u_{n+1}, Su_n) = d(P, Q). \tag{8}$$

From (8), we obtain that $u_* \in P_0$. Since $S(P_0) \subset Q_0$, there is $c \in P_0$ such that $d(c, Su_*) = d(P, Q)$.

Therefore, $d(u_{n+1}, Su_n) = d(c, Su_*) = d(P, Q)$. Again, using the fact that S is generalized proximal C-contractive of the first kind, it results that there exits $h \in C$ such that

$$d(u_{n+1}, c) \leq h(d(u_n, u_*), d(u_n, u_*)) \leq d(u_n, u_*). \tag{9}$$

Letting $n \longrightarrow \infty$ in the above inequality (9), we conclude that $u_* = c$, and so u_* is a best proximity point for S.

For the uniqueness, we proceed in a similar manner as in proof of Theorem 3.3. □

4. Illustrative examples

This section is devoted to present non-trivial examples to support and strengthen the theoretical results.

Example 4.1. *Consider the complete Euclidean space $X = \mathbb{R}^2$ with the metric $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$. Let $P = \{(z, 0), z \in [2, 3]\}$ and $Q = \{(b, 0), t \in [\frac{1}{3}, \frac{1}{2}]\}$. Consider the non-self mappings $S : P \longrightarrow Q$ and $T : Q \longrightarrow P$ such that $S(z, 0) = (\frac{1}{z}, 0)$ and $T(t, 0) = (\frac{1}{t}, 0)$. Note that $S(P) = Q$, $T(Q) = P$ and P and Q are closed subset in the complete space $(\mathbb{R}^2, d)$. We have $d(S(2, 0), (2, 0)) = |\frac{1}{2} - 2| = \frac{3}{2} = d(P, Q)$. Then, it is to see that $P_0 = \{(2, 0)\}$ and $Q_0 = \{(\frac{1}{2}, 0)\}$. Therefore, $S(P_0) \subset Q_0$ and $T(N_0) \subset P_0$. We show the hypothesis (2) in Theorem 3.3, that is, T is generalized proximal C-contractive of the first kind.*

Let $(u, 0), (v, 0), (z, 0), (z', 0) \in P = [2, 3] \times \{0\}$ be such that

$$\begin{aligned} d((u, 0), S(z, o)) &= d((v, 0), S(z', 0)) \\ &= \frac{3}{2} = d(P, Q). \end{aligned}$$

This means that $(u, 0) = (z, 0) = (v, 0) = (2, 0) \in P$. Therefore,

$$d((u, 0), (v, 0)) = 0 \leq \frac{d((z, 0), (z', 0))}{1 + d((z, 0), (z', 0))}.$$

Thus, the hypothesis (2) in Theorem 3.3 is fulfilled for the function $J(u, v) = \frac{u}{1+v}$. It is the same for T.

Now, it is remaining to prove that the couple (S, T) forms a proximal cyclic contraction. Here, $d(u, Sa) = d(P, Q) = \frac{3}{2}$ means that $u = a = (2, 0) \in P$ and $d(v, Tb) = d(P, Q) = \frac{3}{2}$ implies that $v = b = (\frac{1}{2}, 0) \in Q$. Since $d(u, Sa) = d((2, 0), S(2, 0)) = \frac{3}{2} = d(P, Q)$ and $d(v, Tb) = d(\frac{1}{2}, 0), T(\frac{1}{2}, 0))) = d(2, 1) = \frac{3}{2} = d(P, Q)$, one considers

$$\begin{aligned} \frac{3}{2} = d(u, v) &= d\Big((2, 0), \Big(\frac{1}{2}, 0\Big)\Big) \\ &\leq kd\Big((2, 0), \Big(\frac{1}{2}, 0\Big)\Big) + (1 - k)d(P, Q) \\ &= \frac{3}{2}\Big(k + (1 - k)\Big) = \frac{3}{2}. \end{aligned}$$

Hence, the pair (S, T) is a proximal cyclic contraction for any $0 \leq k < 1$. Therefore,

$$d(z_*, t_*) = d\Big((2, 0), \Big(\frac{1}{2}, 0\Big)\Big) = \frac{3}{2} = d(P, Q).$$

Remark 4.2. *The mapping S is generalized proximal C-contractive of second kind. In fact, $d((u, 0), S(z, o)) = d((v, 0), S(z', 0)) = \frac{3}{2} = d(P, Q)$. This means that $(u, 0) = (z, 0) = (v, 0) = (2, 0) = (z', 0) \in P$. Therefore, we have*

$$d(S(u, 0), S(v, 0)) = 0 \leq \frac{d(S(z, 0), S(z', 0))}{1 + d(S(z, 0), S(z', 0))} = 0.$$

Thus, S is generalized proximal C-contractive of second kind with $g(u, v) = \frac{u}{1+v}$. Therefore, according to Theorem 3.4, S has a unique best proximity point, which is eventually in our case the point $(2, 0) \in M$.

The propositions below are used in the next examples.

Proposition 4.3. *Assume that $g : [0, \infty) \to [0, \infty)$ is a function described by $g(\tau) = \ln(1 + \tau)$, then the inequality below holds*

$$g(r) - g(s) \leq g(|r - s|), \text{ for all } r, s \in [0, \infty).$$

Proof. The inequality holds directly if $r_1 = r_2$. Let $r > s$, then we get

$$\frac{1+r}{1+s} = \frac{1+r+s-s}{1+s} = 1 + \frac{r-s}{1+s} < 1 + |r-s|.$$

It follows that

$$\ln(1+r) - \ln(1+s) < \ln(1+|r-s|).$$

The same result is satisfied if we take $r < s$. □

Proposition 4.4. *For each $a, b \in \mathbb{R}$, the inequality below is fulfilled*

$$\frac{1}{(1+|a|)(1+|b|)} \leq \frac{1}{1+|a-b|}.$$

Proof. We have

$$\begin{aligned} 1+|a-b| &\leq 1+|a|+|b| \\ &\leq 1+|a|+|b|+|ab| \\ &= (1+|a|)+(1+|a|). \end{aligned}$$

□

Example 4.5. *Let $(\mathbb{R}^2, d)$ be a complete metric space, where d is Euclidean metric. Consider*

$$P = \{(0,r) : r \in \mathbb{R}\} \text{ and } Q = \{(2,s) : s \in \mathbb{R}\}.$$

Then $d(P,Q) = 2$. Define the mappings $S : P \to Q$ and $T : Q \to P$ by

$$S((0,r)) = (2, \ln(1+|r|)) \text{ and } T((2,s)) = (0, \ln(1+|s|)).$$

Then $P_0 = P$, $Q_0 = Q$, $S(P) \subset Q$ and $T(Q) \subset P$.

Assume that $(0,r_1), (0,r_2), (0,t_1)$, and $(0,t_2)$ are elements in P verifying

$$d((0,t_1), S(0,r_1)) = d(P,Q) = 2 \text{ and } d((0,t_2), S(0,r_2)) = d(P,Q) = 2.$$

Then we get $t_i = \ln(1+|r_i|)$ for $i = 1,2$. If $r_1 = r_2$, we have done. Let $r_1 \neq r_2$, then by Proposition 4.3 and the fact that $g(\tau) = \ln(1+\tau)$ is increasing, we get

$$\begin{aligned} d((0,t_1),(0,t_2)) &= d((0,\ln(1+|r_1|),(0,\ln(1+|r_2|)) \\ &= |\ln(1+|r_1|) - \ln(1+|r_2|)| \\ &\leq |\ln(1+||r_1|-|r_2||)| \\ &\leq |\ln(1+|r_1-r_2|)| \\ &= \frac{|\ln(1+|r_1-r_2|)|}{|r_1-r_2|}|r_1-r_2| \\ &= md((0,r_1),(0,r_2)), \end{aligned}$$

where $m = \frac{|\ln(1+|r_1-r_2|)|}{|r_1-r_2|} \leq 1$. Thus S is generalized proximal C−contractive of the first kind with $J(u,v) = mu$. We get a similar result if we consider the mapping T.

Now, we prove that S is not a proximal contraction of the first kind. So, assume the contrary, that is S is a proximal contractive of the first kind, then for each $(0,p^*), (0,q^*), (0,k^*), (0,l^*) \in P$, verifying

$$d((0,p^*), S(0,k^*)) = d(P,Q) = 2 \text{ and } d((0,q^*), S(0,l^*)) = d(P,Q) = 2, \tag{10}$$

there is $\ell \in [0,1)$ so that

$$d\left((0,p^*),(0,q^*)\right) \le \ell d\left((0,k^*),(0,l^*)\right).$$

Based on (10), we have $p^* = \ln(1+|k^*|)$ *and* $q^* = \ln(1+|l^*|)$ *and so*

$$\begin{aligned} |\ln(1+|k^*|) - \ln(1+|l^*|)| &= d\left((0,p^*),(0,q^*)\right) \\ &\le \ell d\left((0,k^*),(0,l^*)\right) \\ &= \ell\,|k^* - l^*|. \end{aligned}$$

Setting $l^* = 0$, *we get*

$$1 = \lim_{|k^*|\to 0^+} \frac{|\ln(1+|k^*|)|}{|k^*|} \le \ell < 1,$$

a contradiction. Therefore, S is not a proximal contraction of the first kind. By the same method, we conclude that T is not a proximal contraction of the first kind.

Since $t_i = \ln(1+|r_i|) \le |r_i|$ *for* $i = 1,2$, *we can write*

$$\begin{aligned} d\left(S(0,t_1),S(0,t_2)\right) &= d\left((2,\ln(1+|t_1|),(2,\ln(1+|t_2|)\right) \\ &= |\ln(1+|t_1|) - \ln(1+|t_2|)| \\ &\le |\ln(1+||t_1|-|t_2||)| \\ &\le |\ln(1+|t_1-t_2|)| \\ &\le |\ln(1+||r_1|-|r_2||)| \\ &\le |\ln(1+|r_1-r_2|)| \\ &= \frac{|\ln(1+|r_1-r_2|)|}{|r_1-r_2|}|r_1-r_2| \\ &= md\left((0,r_1),(0,r_2)\right). \end{aligned}$$

This proves that S is a generalized proximal C−contractive of the first kind with $J(u,v) = mu$, *for* $m = \frac{|\ln(1+|r_1-r_2|)|}{|r_1-r_2|} \le 1$. *Analogously respect to the mapping T.*

Finally, we shall show that the pair (S,T) *is a proximal cyclic contraction. Indeed, consider* $(0,u),(0,x) \in P$ *and* $(2,v),(2,y) \in Q$ *be that*

$$d\left((0,u),S(0,x)\right) = d(P,Q) = 2 \text{ and } d\left((2,v),T(2,y)\right) = d(P,Q) = 2,$$

Then, we get

$$u = \ln(1+|x|) \text{ and } v = \ln(1+\left|y\right|),$$

the case of $x = y$ *is trivial. Let* $x \ne y$, *then we get*

$$\begin{aligned} d((0,u),(2,v)) &= 2 + |u-v| \\ &= 2 + \left|\ln(1+|x|) - \ln(1+\left|y\right|)\right| \\ &= 2 + \left|\ln\left(\frac{1+|x|}{1+\left|y\right|}\right)\right| \\ &= 2 + \left|\ln\left(\frac{1+|x|-\left|y\right|+\left|y\right|}{1+\left|y\right|}\right)\right| \\ &= 2 + \left|\ln\left(1+\frac{|x|-\left|y\right|}{1+\left|y\right|}\right)\right| \end{aligned}$$

$$
\begin{aligned}
&\leq 2+\frac{\left||x|-|y|\right|}{1+|y|}\\
&\leq 2+\left(\frac{1}{1+|y|}\right)|x-y|\\
&= 2(1-l)+l\left(|x-y|+2\right)\\
&= ld((0,x),(0,y))+(1-l)d(P,Q),
\end{aligned}
$$

where $l=\left(\frac{1}{1+|y|}\right)<1$. *Then the pair* (S,T) *is a proximal cyclic contraction, therefore all assertions of Theorem 3.3 and Theorem 3.4 are fulfilled. Moreover,* $(0,0)\in P$ *and* $(2,0)\in Q$ *are the unique elements so that*

$$d((0,0),S(0,0))=d((2,0),T(2,0))=d((0,0),(2,0))=d(P,Q)=2,$$

this means that $(0,0)$ *and* $(2,0)$ *are best proximity points of* S *and* T.

Example 4.6. *Let* $(\mathbb{R}^2,d)$ *be a complete metric space, where d is described as*

$$d\left((r_1,r_2),(s_1,s_2)\right)=|r_1-s_1|+|r_2-s_2|\,,\ \textit{for all}\ \left(r_1,r_2\right),\left(s_1,s_2\right)\in\mathbb{R}^2.$$

Define

$$P=\{(0,r):r\in\mathbb{R}\}\ \textit{and}\ Q=\{(2,s):s\in\mathbb{R}\}.$$

Then $d(P,Q)=2$. *Define the two mappings* $S:P\to Q$ *and* $T:Q\to P$ *by*

$$S((0,r))=\left(2,\frac{|r|}{2(1+|r|)}\right)\ \textit{and}\ T((2,s))=(0,\frac{|s|}{2(1+|s|)}).$$

Then $P_0=P$, $Q_0=Q$, $S(P)\subset Q$ *and* $T(Q)\subset P$.

Next, we prove that S and T are generalized proximal $C-$*contractive of the first kind. Let* $(0,r_1),(0,r_2),(0,t_1),(0,t_2)\in P$, *justifying*

$$d\left((0,t_1),S(0,r_1)\right)=d(P,Q)=2\ \textit{and}\ d\left((0,t_2),S(0,r_2)\right)=d(P,Q)=2.$$

Then we get

$$t_i=\frac{|r_i|}{2(1+|r_i|)}\ \textit{for}\ i=1,2.$$

If $r_1=r_2$, *nothing is to prove. Let* $r_1\neq r_2$, *then by Proposition 4.4, we get*

$$
\begin{aligned}
d\left((0,t_1),(0,t_2)\right) &= d\left((0,\frac{|r_1|}{2(1+|r_1|)}),(0,\frac{|r_2|}{2(1+|r_2|)})\right)\\
&= \left|\frac{|r_1|}{2(1+|r_1|)}-\frac{|r_2|}{2(1+|r_2|)}\right|\\
&= \left|\frac{|r_1|-|r_2|}{2(1+|r_1|)(1+|r_2|)}\right|\\
&\leq \left|\frac{r_1-r_2}{(1+|r_1|)(1+|r_2|)}\right|\\
&\leq \left(\frac{1}{(1+|r_1-r_2|)}\right)|r_1-r_2|\\
&= md\left((0,r_1),(0,r_2)\right), \qquad (11)
\end{aligned}
$$

where $m = \frac{1}{(1+|r_1-r_2|)} < 1$. *Thus* S *is generalized proximal* $C-$*contractive of the first kind with* $J(u,v) = mu$. *Similarly, we can show that* T *is generalized proximal* $C-$*contractive of the first kind.*

Now, we shall show that S *and* T *are generalized proximal* $C-$*contractive of the second kind, then by (11), and* $t_i = \frac{|r_i|}{2(1+|r_i|)} \leq |r_i|$, *for* $i = 1, 2$, *one can write*

$$\begin{aligned} d\left(S(0,t_1), S(0,t_2)\right) &= d\left((2, \frac{|t_1|}{2(1+|t_1|)}), (2, \frac{|t_2|}{2(1+|t_2|)})\right) \\ &= \left|\frac{|t_1|}{2(1+|t_1|)} - \frac{|t_2|}{2(1+|t_2|)}\right| \\ &= \left|\frac{|t_1| - |t_2|}{2(1+|t_1|)(1+|t_2|)}\right| \\ &\leq \left|\frac{t_1 - t_2}{(1+|t_1|)(1+|t_2|)}\right| \\ &\leq \left(\frac{1}{(1+|t_1-t_2|)}\right)|t_1 - t_2| \\ &\leq \left(\frac{1}{(1+|t_1-t_2|)}\right)||r_1| - |r_2|| \\ &\leq \left(\frac{1}{(1+|t_1-t_2|)}\right)|r_1 - r_2| \\ &= md\left((0,r_1),(0,r_2)\right). \end{aligned}$$

Hence, S *is generalized proximal* $C-$*contractive of the second kind with* $J(u,v) = mu$, *for* $m = \frac{1}{(1+|t_1-t_2|)} < 1$. *Analogously, we consider the mapping* T. *Ultimately, we will prove that the pair* (S,T) *is a proximal cyclic contraction. Assume that* $(0,u), (0,x) \in P$ *and* $(2,v), (2,y) \in Q$ *are so that*

$$d\left((0,u), S(0,x)\right) = d(P,Q) = 2 \text{ and } d\left((2,v), T(2,y)\right) = d(P,Q) = 2.$$

Then, we get

$$u = \frac{|x|}{2(1+|x|)} \text{ and } v = \frac{|y|}{2(1+|y|)}.$$

The case of $x = y$ *is easy. Let* $x \neq y$, *then we have*

$$\begin{aligned} d((0,u),(2,v)) &= |u-v| + 2 \\ &= \left|\frac{|x|}{2(1+|x|)} - \frac{|y|}{2(1+|y|)}\right| + 2 \\ &= \left|\frac{|x| - |y|}{2(1+|x|)(1+|y|)}\right| + 2 \\ &\leq \frac{|x-y|}{2(1+|x|)(1+|y|)} + 2 \\ &\leq \frac{1}{2}|x-y| + 2 \\ &= l\left(|x-y| + 2\right) + 2(1-l) \\ &= ld((0,x),(0,y)) + (1-l)d(P,Q), \end{aligned}$$

where $l = \frac{1}{2} < 1$. *Then the pair* (S,T) *is a proximal cyclic contraction. Hence, all hypotheses of Theorem 3.3 and 3.4 are satisfied. Here,* $(0,0) \in P$ *and* $(2,0) \in Q$ *are best proximity points of* S *and* T.

5. Some related results

As a consequence of our principal theorems, we suggest the coming two results of Basha.

Corollary 5.1. *[6] Let (X, d) be a complete metric space, (P, Q) be a pair of nonempty closed subsets and (P_0, Q_0) be a pair of nonempty subsets in X. Let $S : P \to Q, T : Q \to P$ satisfy the following assertions:*

1. *S and T are proximal contractions of the first kind;*
2. *$S(P_0) \subset Q_0$ and $T(Q_0) \subset P_0$;*
3. *The pair (S, T) forms a cyclic contraction.*

Then, there exist a unique element x_* in P and a unique element y_* in Q satisfying the following conditions that $d(x_*, Sx_*) = d(y_*, Ty_*) = d(P, Q)$ and $d(x_*, y_*) = d(P, Q)$.

Proof. The fact that S and T are proximal contractions of the first kind means that S and T are generalized proximal C-contractive where $J(u, v) = ku$ with $k \in (0, 1)$ and $J_1(u, v) = \alpha u$ with $\alpha \in (0, 1)$. □

Corollary 5.2. *[6] "Let (X, d) be a complete metric space, (P, Q) a pair of nonempty closed subsets and (P_0, Q_0) a pair of nonempty subsets in X. Let $S : P \to Q$ satisfy the following assertions:*

1. *S is a proximal contraction of the first and second kind;*
2. *$S(P_0) \subset Q_0$.*

Then, there exists a unique element x_* in P and such that $d(x_*, Sx_*) = d(P, Q)$. Further, for any fixed element $x_0 \in P$, the sequence $\{x_n\}$ defined by $d(x_{n+1}, Sx_n) = d(P, Q)$, converges to the best proximity x of S.

Proof. It is a clear consequence of Theorem 3.4. Take into the account that S is a proximal contraction of the first and second kinds, so S is generalized proximal generalized C-contractive with $J(u, v) = ku$, where $k \in (0, 1)$. □

In the following, we can deduce many fixed points results that are known in the literature. A natural definition of a generalized proximal C-contraction of the first kind for the case of self-mappings is the following:

Definition 5.3. *Let (X, d) a metric space. A self-mapping $S : X \longrightarrow X$ is called C-contractive if there exists $J \in C$ such that*

$$d(Sx, Sy) \leq J(d(x, y), d(x, y))\, \forall x, y \in X.$$

As a consequence of Theorem 3.3, we state the following fixed point result.

Corollary 5.4. *Let (X, d) be a complete metric space and $S : X \to X$ be a C- contractive mapping. Then S has a unique fixed point.*

Proof. It is a consequence of Theorem 3.3 by suggesting $P = Q = X$ and taking (S, S) as a cyclic contraction. □

From this corollary, we propose the following fixed points results.

Corollary 5.5. *Let (X, d) be a metric space and $S, T : X \longrightarrow X$ be two C- contractive mappings such that*

$$d(Sx, Ty) \leq kd(x, y), \ \forall\, x, y \in X,$$

where $0 < k < 1$. Then S and T have a unique common fixed point.

Proof. Also, this is a direct consequence of Theorem 3.3 by letting $P = Q = X$ and taking the pair (S, T) as a cyclic contraction. It means that $d(Sx, Ty) \leq kd(x, y)$. □

Our first consequence is the Banach contraction principle [23].

Corollary 5.6. *Let (X,d) be a complete metric space. Let $S : X \to X$ satisfy the following condition:*

$$d(Sx,Sy) \leq kd(x,y), \ \text{for each} \ x,y \in X, \ \text{where} \ k \in (0,1).$$

Then T has a unique fixed point.

Proof. Consider in Corollary 5.4, $J(u,v) = ku$ where $k \in (0,1)$. □

Corollary 5.7. *Let (X,d) be a complete metric space. Suppose that there exists a continuous function $\beta : [0,\infty) \longrightarrow [0,1)$ such that the mapping $S : X \longrightarrow X$ is such that*

$$d(Sx,Sy) \leq \beta(d(x,y))d(x,y), \ \text{for each} \ x,y \in X.$$

Then S has a unique fixed point.

Proof. Take the function $J(u,v) = u\beta(v)$ in Corollary 5.4. □

Corollary 5.8. *Let (X,d) be a complete metric space. Let $S : X \to X$ satisfy the following condition:*

$$d(Sx,Sy) \leq \frac{d(x,y)}{(1+d(x,y))^r}, \ \text{for each} \ x,y \in X, \ \text{where} \ r > 0.$$

Then S has a unique fixed point.

Proof. By suggesting $J(u,v) = \frac{u}{(1+v)^r}$ (where $r \in (0,\infty)$) in Corollary 5.4. □

Corollary 5.9. *Let (X,d) be a complete metric space. Let $S : X \to X$ satisfy the following condition:*

$$d(Sx,Sy) \leq \frac{d(x,y)+a^{d(x,y)}}{1+d(x,y)}, \ \text{for each} \ x,y \in X, \ \text{where} \ a \in (1,\infty).$$

Then S has a unique fixed point.

Proof. We simply take the function $J(u,v) = \frac{log(v+a^u)}{1+v}$ (where $a \in (1,\infty)$) in Corollary 5.4. □

Corollary 5.10. *Let (X,d) be a complete metric space. Let $S : X \to X$ satisfy the following condition:*

$$d(Sx,Sy) \leq \frac{\ln(2+a^{d(x,y)})}{2}, \ \text{for each} \ x,y \in X, \ \text{where} \ a \in (e,\infty).$$

Then S has a unique fixed point. Here, $e = \exp(1)$.

Proof. Let consider $J(u,v) = \frac{\ln(1+a^u)}{2}$ (where $a \in (e,\infty)$) in Corollary 5.4. □

Corollary 5.11. *Let (X,d) be a complete metric space. Let $S : X \to X$ satisfy the following condition:*

$$d(Sx,Sy) \leq d(x,y)log_{d(x,y)+a}(a), \ \text{for each} \ x,y \in X, \ \text{where} \ a \in (1,\infty).$$

Then S has a unique fixed point.

Proof. Take $J(u,v) = ulog_{v+a}a$ (where $a \in (1,\infty)$) in Corollary 5.4. □

Let $\Phi = \{\varphi : [0,\infty) \longrightarrow [0,\infty)$is a continuous function such that$\varphi(v) = 0 \Leftrightarrow v = 0\}$

Corollary 5.12. *Let (X,d) be a complete metric space. Let $S : X \to X$ satisfy the following condition:*

$$d(Sx,Sy) \leq d(x,y) - \varphi(d(x,y)), \ \text{for each} \ x,y \in X, \ \text{where} \ \varphi \in \Phi.$$

Then S has a unique fixed point.

Proof. Take $J(u, v) = u - \varphi(u)$ (where $\varphi \in \Phi$) in Corollary 5.4. □

Corollary 5.13. *Let (X, d) be a complete metric space. Let $S : X \to X$ satisfy the following condition:*

$$d(Sx, Sy) \leq \sqrt[n]{\ln\Big(1 + (d(x, y))^n\Big)}, \text{ for each } x, y \in X.$$

Then S has a unique fixed point.

Proof. Take $J(u, v) = \sqrt[n]{1 + u^n}$ in Corollary 5.4. □

Corollary 5.14. *Let (X, d) be a complete metric space. Let $S : X \to X$ satisfy the following condition:*

$$d(Sx, Sy) \leq \phi(d(x, y)), \text{ for each } x, y \in X,$$

where $\phi : [0, \infty) \longrightarrow [0, \infty)$ is an upper semi-continuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$. Then S has a unique fixed point.

Proof. By considering $J(u, v) = \phi(u)$ in Corollary 5.4. □

Corollary 5.15. *Let (X, d) be a complete metric space. Let $T : X \to X$ satisfy the following condition:*

$$d(Tx, Ty) \leq \theta(d(x, y)), \text{ for each } x, y \in X,$$

where $\theta : [0, \infty) \times [0, \infty) \longrightarrow \mathbb{R}$ is a generalized Mizoguchi-Takahachi Type function. Then T has a unique fixed point.

Proof. Let consider $J(u, v) = \theta(u)$ in Corollary 5.4. □

Corollary 5.16. *Let (X, d) be a complete metric space. Let $S : X \to X$ satisfy the following condition:*

$$d(Sx, Sy) \leq \frac{d(x, y)(d(x, y) + 1)}{2 + d(x, y)}, \text{ for each } x, y \in X.$$

Then S has a unique fixed point.

Proof. We suggest $J(u, v) = u - (\frac{1+u}{2+u})(\frac{v}{1+v})$ in Corollary 5.4. □

Corollary 5.17. *Let (X, d) be a complete metric space. Let $S : X \to X$ satisfy the following condition:*

$$d(Sx, Sy) \leq \frac{d(x, y)}{\Gamma(\frac{1}{2})} \int_0^\infty \frac{e^{-w}}{\sqrt{w} + d(x, y)} dw, \text{ for each } x, y \in X.$$

Then S has a unique fixed point

Proof. We propose $J(u, v) = \dfrac{u}{\Gamma(\frac{1}{2})} \displaystyle\int_0^\infty \frac{e^{-w}}{\sqrt{w} + v} dw$ in Corollary 5.4. □

6. Conclusion

We ensured the existence of best proximity points for a pair of non-self mappings forming cyclic contractions involving C-class functions in the context of complete metric spaces. The obtained results are illustrated by some examples. Several related corollaries have been presented. Further results in same direction could be derived using other auxiliary functions.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

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