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Oscillation Criteria for First-Order Nonlinear Differential Equations with Several Delays

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Abstract. In this article, we study a first order nonlinear delay differential equation

$$x'(t) + \sum_{i=1}^{m} p_i(t) f_i\Big(x(\tau_i(t))\Big) = 0, \quad t \geq t_0,$$

where $p_i(t)$ and $\tau_i(t)$ are the functions of nonnegative of real numbers and $\tau_i(t)$ are not necessarily monotone for $1 \leq i \leq m$. Also, we present new sufficient conditions for the oscillatory solutions of this equation. Our results essentially improve the conditions in the literature. Finally, we give examples to illustrate our results.

1. Introduction

In this article, we consider the first-order nonlinear delay differential equation

$$x'(t) + \sum_{i=1}^{m} p_i(t) f_i\Big(x(\tau_i(t))\Big) = 0, \quad t \geq t_0, \tag{1}$$

where the functions f_i, p_i, τ_i satisfy the conditions stated below.
Let $T_0 = \min\{\inf\{\tau_i(t) : t_0 \leq t\}, 1 \leq i \leq m\}\}$. By a solution of (1), we mean a function that is continuous for $t \geq T_0$ and differentiable for $t \geq t_0$. A solution of (1) is called oscillatory if it has arbitrarily large zeros and otherwise, it is called non-oscillatory. A solution is called eventually positive if there exists a t_1 such that $x(t) > 0$ for $t \geq t_1$ (and eventually negative if $x(t) < 0$).
In this article, we use the following conditions and notations.

(H1) $\tau_i \in C(\mathbb{R}, \mathbb{R})$, $\tau_i(t) \leq t$, $\lim_{t\to\infty} \tau_i(t) = \infty$ for $i = 1, 2, \dots, m$.

$$\sigma_i(t) = \sup\{\tau_i(s) : t_0 \leq s \leq t\}, \quad \sigma(t) = \max\{\sigma_i(t) : 1 \leq i \leq m\}$$

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(H2) $p_i \in C([t_0, \infty), \mathbb{R})$, $p_i(t) \geq 0$.

(H3) $f_i \in C(\mathbb{R}, \mathbb{R})$, $x f_i(x) > 0$ for $x \neq 0$ and

$$\tilde{M}_i := \limsup_{x \to 0} \frac{x}{f_i(x)}, \quad 0 < \tilde{M}_i < \infty.$$

In (H1), $\tau_i(t)$ are not necessarily monotonic, $\tau_i(t) \leq \sigma_i(t) \leq \sigma(t) \leq t$ and σ is non-decreasing.
When $f(x) = x$ and $m = 1$, equation (1) becomes the classical linear equation

$$x'(t) + p_1(t)x(\tau_1(t)) = 0, \quad t \geq t_0. \tag{2}$$

The first systematic study for the oscillation of all solutions to this equation was made by Myshkis in 1950. Later, Koplatadze and Chanturija [19], Ladas and Stavroulakis [20], Fukagai and Kusano [17], Ladde et al. [21] and Györi and Ladas [18] studied this equation and obtained well-known oscillation criteria with nondecreasing delay. In 2011, Braverman and Karpuz [3] modified the lim sup condition for not necessarily monotone delay. Moreover, we can see that mathematical modeling with delay differential equations is widely used for analysis and predictions in various areas of life sciences, for example, epidemiology, immunology, neural networks, physiology and population dynamics [24].
Sufficient conditions for the oscillation of solutions to nonlinear equations can be found in [1, 2, 14, 15, 17, 21, 23]. There is a misprint in [23], it should be $\tau(t) = \max\{\tau_i(t) : 1 \leq i \leq m\}$.

2. Preliminaries

Lemma 2.1 ([16, Lemma 2.1.1]). *If* (H1) *and* $\liminf_{t\to\infty} \int_{\tau(t)}^t \sum_{i=1}^m p_i(s)ds > 0$ *hold, then*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{m} p_i(s)ds = \liminf_{t\to\infty} \int_{\sigma(t)}^{t} \sum_{i=1}^{m} p_i(s)ds$$

where $\tau(t) = \max\{\tau_i(t) : 1 \leq i \leq m\}$.

Theorem 2.2 ([23, Theorem 2.1, 2.2]). *Assume* (H1)–(H3), $0 < \tilde{M}_i < \infty$ *and one of the following two conditions hold:*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{m} p_i(s)\,ds > \frac{\tilde{M}^*}{e}, \tag{3}$$

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} \sum_{i=1}^{m} p_i(s)\,ds > \tilde{M}^*. \tag{4}$$

Then, every solution of (1) *is oscillatory, where* $\tau(t) = \max\{\tau_i(t) : 1 \leq i \leq m\}$ *and* $\tilde{M}^* = \max\{\tilde{M}_i : 1 \leq i \leq m\}$.

In [15] the authors studied the oscillation of solutions to

$$x'(t) + \sum_{i=1}^{m} \tilde{f}_i\Big(t, x(\tau_i(t))\Big) = 0\,, \tag{5}$$

where $|\tilde{f}_i(t,x)| \geq p_i(t)g_i(x)$ and g_i satisfies the conditions in (H3) with $\tilde{M}_i < 1$. They also assume that $\int_0^\infty \sum p_i = +\infty$ which we will show that does not need to be explicitly assumed. Our main results are stated as Theorems 3.3 and 3.4 below.

3. Main Results

Lemma 3.1 (Grönwall's inequality). *Assume that $x(t)$ is a positive solution of $x'(t)+\sum_{i=1}^{m} p_i(t)x(t)\le 0$. Then, we have*

$$x(s)\ge x(t)\exp\Big(\int_s^t \sum_{i=1}^{m} p_i(r)\,dr\Big)\quad \text{for } s\le t\,.$$

Further, assume that $x(t)$ is a negative solution of $x'(t)+\sum_{i=1}^{m} p_i(t)x(t)\ge 0$. Then, we have

$$x(s)\le x(t)\exp\Big(\int_s^t \sum_{i=1}^{m} p_i(r)\,dr\Big)\quad \text{for } s\le t\,.$$

Lemma 3.2. *Assume* (H1)–(H3) *hold, and $x(t)$ is an eventually positive solution of* (1). *If*

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t}\sum_{i=1}^{m}\frac{p_i(s)}{M_i}\exp\Big(\int_{\tau_i(s)}^{\sigma(t)}\sum_{j=1}^{m}\frac{p_j(r)}{M_j}\,dr\Big)ds>0, \tag{6}$$

then $\lim_{t\to\infty}x(t)=0$, where M_i are constants with $\tilde{M}_i<M_i$ for $1\le i\le m$.
Also, assume that $x(t)$ is an eventually negative solution of (1). *If* (6) *holds, then $\lim_{t\to\infty}x(t)=0$.*

Proof. Assume that (6) holds. Let $x(t)$ be an eventually positive solution of (1). Then, there exists $t_1>t_0$ such that $x(t)$, $x(\tau_i(t))>0$ for all $t\ge t_1$ and $1\le i\le m$. Thus, from (1), we get

$$x'(t)=-\sum_{i=1}^{m}p_i(t)f_i(x(\tau_i(t)))\le 0$$

for all $t\ge t_1$, which means that $x(t)$ is nonincreasing and has a limit $l>0$ or $l=0$. Now, we claim that $\lim_{t\to\infty}x(t)=0$. Otherwise, $\lim_{t\to\infty}x(t)=l>0$. Then, integrating (1) from $\sigma(t)$ to t, we have

$$x(t)-x(\sigma(t))+\int_{\sigma(t)}^{t}\sum_{i=1}^{m}p_i(s)f_i(x(\tau_i(s)))ds=0. \tag{7}$$

From (H3), we can choose M_i with $\tilde{M}_i<M_i$ for $1\le i\le m$ such that

$$f_i(x(\tau_i(t)))\ge\frac{1}{M_i}x(\tau_i(t)). \tag{8}$$

Using the inequality (8) in (7), we have

$$x(t)-x(\sigma(t))+\int_{\sigma(t)}^{t}\sum_{i=1}^{m}\frac{p_i(s)}{M_i}x(\tau_i(s))\,ds\le 0. \tag{9}$$

Also, using Lemma 3.1 in (9), we get

$$x(t)-x(\sigma(t))+x(\sigma(t))\int_{\sigma(t)}^{t}\sum_{i=1}^{m}\frac{p_i(s)}{M_i}\exp\Big(\int_{\tau_i(s)}^{\sigma(t)}\sum_{j=1}^{m}\frac{p_j(r)}{M_j}\,dr\Big)ds\le 0. \tag{10}$$

Moreover, (6) implies that there exists at least one sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \int_{\sigma(t_n)}^{t_n} \sum_{i=1}^{m} \frac{p_i(s)}{M_i} \exp\Big(\int_{\tau_i(s)}^{\sigma(t_n)} \sum_{j=1}^{m} \frac{p_j(r)}{M_j}\, dr \Big)\, ds > 0, \tag{11}$$

By letting $t \to t_n$ and taking limit $n \to \infty$ in (10), we get

$$l \lim_{n\to\infty} \int_{\sigma(t_n)}^{t_n} \sum_{i=1}^{m} \frac{p_i(s)}{M_i} \exp\Big(\int_{\tau_i(s)}^{\sigma(t_n)} \sum_{j=1}^{m} \frac{p_j(r)}{M_j}\, dr \Big)\, ds \le 0,$$

but this contradicts with (11).
On the other hand, assume that (6) holds. Let $x(t)$ be an eventually negative solution of (1). Then, there exists $t_1 > t_0$ such that $x(t), x(\tau_i(t)) < 0$ for all $t \ge t_1$ and $1 \le i \le m$. Thus, from (1), we get

$$x'(t) = -\sum_{i=1}^{m} p_i(t) f_i(x(\tau_i(t))) \ge 0$$

for all $t \ge t_1$, which means that $x(t)$ is nondecreasing and has a limit $l < 0$ or $l = 0$. Now, we claim that $\lim\limits_{t\to\infty} x(t) = 0$. Otherwise, $\lim\limits_{t\to\infty} x(t) = l < 0$. Then, integrating (1) from $\sigma(t)$ to t, we have

$$x(t) - x(\sigma(t)) + \int_{\sigma(t)}^{t} \sum_{i=1}^{m} p_i(s) f_i(x(\tau_i(s))) ds = 0. \tag{12}$$

From (H3), we can choose M_i with $\tilde{M}_i < M_i$ for $1 \le i \le m$ such that

$$f_i(x(\tau_i(t))) \le \frac{1}{M_i} x(\tau_i(t)). \tag{13}$$

Using the inequality (13) in (12), we have

$$x(t) - x(\sigma(t)) + \int_{\sigma(t)}^{t} \sum_{i=1}^{m} \frac{p_i(s)}{M_i} x(\tau_i(s))\, ds \ge 0. \tag{14}$$

Also, using Lemma 3.1 in (14), we get

$$x(t) - x(\sigma(t)) + x(\sigma(t)) \int_{\sigma(t)}^{t} \sum_{i=1}^{m} \frac{p_i(s)}{M_i} \exp\Big(\int_{\tau_i(s)}^{\sigma(t)} \sum_{j=1}^{m} \frac{p_j(r)}{M_j}\, dr \Big)\, ds \ge 0. \tag{15}$$

Moreover, (6) implies that there exists at least one sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \int_{\sigma(t_n)}^{t_n} \sum_{i=1}^{m} \frac{p_i(s)}{M_i} \exp\Big(\int_{\tau_i(s)}^{\sigma(t_n)} \sum_{j=1}^{m} \frac{p_j(r)}{M_j}\, dr \Big)\, ds > 0, \tag{16}$$

By letting $t \to t_n$ and taking limit $n \to \infty$ in (15), we get

$$l \lim_{n\to\infty} \int_{\sigma(t_n)}^{t_n} \sum_{i=1}^{m} \frac{p_i(s)}{M_i} \exp\Big(\int_{\tau_i(s)}^{\sigma(t_n)} \sum_{j=1}^{m} \frac{p_j(r)}{M_j}\, dr \Big)\, ds \ge 0,$$

but this contradicts with $l < 0$. So, the proof of lemma is completed. $\square$

Theorem 3.3. *Assume* (H1)–(H3) *hold and*

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t}\sum_{i=1}^{m}\frac{p_i(s)}{M_i}\exp\Big(\int_{\tau_i(s)}^{\sigma(s)}\sum_{j=1}^{m}\frac{p_j(r)}{M_j}\,dr\Big)ds>\frac{1}{e}\,. \tag{17}$$

Then all solutions of (1) *are oscillatory, where* $\tau(t)=\max\{\tau_i(t):1\le i\le m\}$ *and* M_i *are constants with* $\tilde{M}_i<M_i$ *for* $1\le i\le m$.

Proof. From Lemma 2.1, the definition of limit inferior, and the strict inequality in (17), there exists a constant $c>0$ such that

$$\int_{\sigma(t)}^{t}\sum_{i=1}^{m}\frac{p_i(s)}{M_i}\exp\Big(\int_{\tau_i(s)}^{\sigma(s)}\sum_{j=1}^{m}\frac{p_j(r)}{M_j}\,dr\Big)ds\ge c>\frac{1}{e},\quad \forall t\ge t_1\,. \tag{18}$$

From the continuity of outer integral, there exists $t^*\in(\sigma(t),t)$ such that

$$\int_{\sigma(t)}^{t^*}\sum_{i=1}^{m}\frac{p_i(s)}{M_i}\exp\Big(\int_{\tau_i(s)}^{\sigma(s)}\sum_{j=1}^{m}\frac{p_j(r)}{M_j}\,dr\Big)ds>\frac{1}{2e} \tag{19}$$

and

$$\int_{t^*}^{t}\sum_{i=1}^{m}\frac{p_i(s)}{M_i}\exp\Big(\int_{\tau_i(s)}^{\sigma(s)}\sum_{j=1}^{m}\frac{p_j(r)}{M_j}\,dr\Big)ds>\frac{1}{2e}\,. \tag{20}$$

Assume, for the sake of contradiction, that there exists an eventually positive solution x of (1). Condition (17) implies (6), so as in Lemma 3.2, there exists $t_2\ge t_1$ such that $x(t)>0$, $x(\tau_i(t))>0$ and x is non-increasing for $t\ge t_2$, and $\lim_{t\to\infty}x(t)=0$. From (H3), we can choose $t_3\ge t_2$ and there are M_i with $\tilde{M}_i<M_i$ for $1\le i\le m$ such that

$$f_i(x(\tau_i(t)))\ge\frac{1}{M_i}x(\tau_i(t))\quad\text{for } t\ge t_3. \tag{21}$$

Using this inequality in (1), we have

$$x'(t)+\sum_{i=1}^{m}\frac{p_i(t)}{M_i}x(\tau_i(t))\le 0. \tag{22}$$

Then using that $\tau_i(t)\le t$ and x is non-increasing, we obtain

$$x'(t)+x(t)\sum_{i=1}^{m}\frac{p_i(t)}{M_i}\le 0. \tag{23}$$

Therefore by Lemma 3.1,

$$x(\tau_i(s))\ge x(\sigma(s))\exp\Big(\int_{\tau_i(s)}^{\sigma(s)}\sum_{j=1}^{m}\frac{p_j(r)}{M_j}\,dr\Big). \tag{24}$$

Integrating (22) from $\sigma(t)$ to t^*, and using the inequality above, we have

$$x(\sigma(t))-x(t^*)\ge\int_{\sigma(t)}^{t^*}\sum_{i=1}^{m}\frac{p_i(s)}{M_i}x(\sigma(s))\exp\Big(\int_{\tau_i(s)}^{\sigma(s)}\sum_{j=1}^{m}\frac{p_j(r)}{M_j}\,dr\Big)ds.$$

Since x is non-increasing and σ is non-decreasing, we have

$$x(\sigma(t)) - x(t^*) \geq x(\sigma(t^*)) \int_{\sigma(t)}^{t^*} \sum_{i=1}^{m} \frac{p_i(s)}{M_i} \exp\Big(\int_{\tau_i(s)}^{\sigma(s)} \sum_{j=1}^{m} \frac{p_j(r)}{M_j}\, dr \Big)\, ds\,.$$

Then by (19),

$$x(\sigma(t)) > x(\sigma(t)) - x(t^*) > x(\sigma(t^*)) \frac{1}{2e}\,. \tag{25}$$

Now integrating (22) from t^* to t, we have

$$x(t^*) - x(t) \geq \int_{t^*}^{t} \sum_{i=1}^{m} \frac{p_i(s)}{M_i} x(\sigma(s)) \exp\Big(\int_{\tau_i(s)}^{\sigma(s)} \sum_{j=1}^{m} \frac{p_j(r)}{M_j}\, dr \Big)\, ds.$$

Since x is non-increasing and σ is non-decreasing, we get

$$x(t^*) - x(t) \geq x(\sigma(t)) \int_{t^*}^{t} \sum_{i=1}^{m} \frac{p_i(s)}{M_i} \exp\Big(\int_{\tau_i(s)}^{\sigma(s)} \sum_{j=1}^{m} \frac{p_j(r)}{M_j}\, dr \Big)\, ds\,.$$

Then by (20),

$$x(t^*) > x(t^*) - x(t) > x(\sigma(t)) \frac{1}{2e}\,. \tag{26}$$

Substituting (25) in (26), we have

$$1 \leq \frac{x(\sigma(t^*))}{x(t^*)} < (2e)^2\,. \tag{27}$$

Let

$$u := \liminf_{t\to\infty} \frac{x(\sigma(t))}{x(t)}\,. \tag{28}$$

Then $1 \leq u \leq (2e)^2$ because for any t we can find a $t^* \in [\sigma(t), t]$ satisfying (27). Dividing (1) by $x(t)$ and integrating from $\sigma(t)$ to t, we have

$$\int_{\sigma(t)}^{t} \frac{x'(s)}{x(s)}\, ds + \int_{\sigma(t)}^{t} \sum_{i=1}^{m} p_i(s) \frac{f_i(x(\tau_i(s)))}{x(s)} ds = 0\,.$$

Then by (21), we have

$$\begin{aligned}
\ln\Big(\frac{x(\sigma(t))}{x(t)}\Big) &= \int_{\sigma(t)}^{t} \sum_{i=1}^{m} p_i(s) \frac{f_i(x(\tau_i(s)))}{x(\tau_i(s))x(s)} x(\tau_i(s))\, ds \\
&\geq \int_{\sigma(t)}^{t} \sum_{i=1}^{m} \frac{p_i(s)}{M_i x(s)} x(\tau_i(s))\, ds \quad \text{for } t \geq t_3\,.
\end{aligned}$$

By (24),

$$\ln\Big(\frac{x(\sigma(t))}{x(t)}\Big) \geq \int_{\sigma(t)}^{t} \sum_{i=1}^{m} \frac{p_i(s)}{M_i x(s)} x(\sigma(s)) \exp\Big(\int_{\tau_i(s)}^{\sigma(s)} \sum_{j=1}^{m} \frac{p_j(r)}{M_j}\, dr \Big) ds \quad \text{for } t \geq t_3\,.$$

From the mean value theorem for integrals,

$$\ln\Big(\frac{x(\sigma(t))}{x(t)}\Big) \geq \frac{x(\sigma(\zeta))}{x(\zeta)}\Big(\int_{\sigma(t)}^{t}\sum_{i=1}^{m}\frac{p_i(s)}{M_i}\exp\Big(\int_{\tau_i(s)}^{\sigma(s)}\sum_{j=1}^{m}\frac{p_j(r)}{M_j}\,dr\Big)ds$$

for some value $\zeta \in [\sigma(t), t]$. Now we take the limit inferior on both sides of the above inequality and using (17) and (28), we have

$$\liminf_{t\to\infty}\ln\Big(\frac{x(\sigma(t))}{x(t)}\Big) > \liminf_{t\to\infty}\frac{x(\sigma(\zeta))}{x(\zeta)}\frac{1}{e}\,, \tag{29}$$

where we use that $\liminf\big(h(t)k(t)\big) \geq \liminf(h(t))\liminf(k(t))$. Therefore, from (28), (29) and $\ln\Big(\liminf\frac{x(\sigma(t))}{x(t)}\Big) \geq \liminf\Big(\ln\frac{x(\sigma(t))}{x(t)}\Big)$, we have

$$\ln u > u/e,$$

which is not possible for any positive number u. This contradiction indicates that we cannot have a positive solution.
If y is an eventually negative solution we consider $x = -y$ and $\tilde{f}(x) = -f(-x)$ which satisfies conditions (H2) and (H3). Then proceed as above. □

Theorem 3.4. *Assume* (H1)–(H3) *hold and*

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t}\sum_{i=1}^{m}\frac{p_i(s)}{M_i}\exp\Big(\int_{\tau_i(s)}^{\sigma(t)}\sum_{j=1}^{m}\frac{p_j(r)}{M_j}\,dr\Big)ds > 1\,. \tag{30}$$

Then all solutions of (1) *are oscillatory, where M_i are constants with $\tilde{M}_i < M_i$ for $1 \leq i \leq m$.*

Proof. Assume, for the sake of contradiction, that there exists an eventually positive solution x of (1). Since (30) implies (6), by Lemma 3.2, $\lim_{t\to\infty} x(t) = 0$. As the proof of Theorem 3.3, we have Lemma 3.1. So, from Lemma 3.1, we obtain

$$x(\tau_i(s)) \geq x(\sigma(t))\exp\Big(\int_{\tau_i(s)}^{\sigma(t)}\sum_{j=1}^{m}\frac{p_j(r)}{M_j}\,dr\Big). \tag{31}$$

Integrating (22) from $\sigma(t)$ to t using (31), we have

$$x(\sigma(t)) - x(t) \geq x(\sigma(t))\int_{\sigma(t)}^{t}\sum_{i=1}^{m}\frac{p_i(s)}{M_i}\exp\Big(\int_{\tau_i(s)}^{\sigma(t)}\sum_{j=1}^{m}\frac{p_j(r)}{M_j}\,dr\Big)ds\,.$$

Dividing by $x(\sigma(t))$ we have

$$1 - \frac{x(t)}{x(\sigma(t))} \geq \int_{\sigma(t)}^{t}\sum_{i=1}^{m}\frac{p_i(s)}{M_i}\exp\Big(\int_{\tau_i(s)}^{\sigma(t)}\sum_{j=1}^{m}\frac{p_j(r)}{M_j}\,dr\Big)ds,$$

which implies

$$\int_{\sigma(t)}^{t}\sum_{i=1}^{m}\frac{p_i(s)}{M_i}\exp\Big(\int_{\tau_i(s)}^{\sigma(t)}\sum_{j=1}^{m}\frac{p_j(r)}{M_j}\,dr\Big)ds < 1, \quad \forall t \geq t_2\,.$$

Taking the limit superior, this contradicts (30); therefore x can not be eventually positive. If y is an eventually negative solution we consider $x = -y$ and $\tilde{f}(x) = -f(-x)$ which satisfies conditions (H2) and (H3). □

4. Examples and Comparison with the results in literature

In this section, we compare the results which were given before in the literature for the oscillatory solutions of (1). Also, we present examples to illustrate our main results.
First we rewrite (1) as

$$x'(t) + \sum_{i=1}^{m} q_i(t) g_i\Big(x(\tau_i(t))\Big) = 0, \quad t \geq t_0, \tag{32}$$

where g_i satisfies $p_i f_i = q_i g_i$, $x g_i(x) > 0$ for $x \neq 0$, and

$$\tilde{M}_i = \limsup_{x\to 0} \frac{x}{g_i(x)} \leq 1\,. \tag{33}$$

In [15], Eq. (1) was studied and some results were obtained for the oscillation of all solutions of (1). Also, Lemma 1.1 in [15] has the key role to establish these results. But, when we examine the proof of Lemma 1.1 carefully, it can be seen that it is valid when the condition (33) holds. So, the results [15, Theorems 3.1 and 3.2] can be applied for $\tilde{M}_i \leq 1$. They are not applicable when $\tilde{M}_i > 1$. Also, in [15, Theorems 3.1 and 3.2], it was proved that $\lim_{t\to\infty} x(t) = 0$, under the following assumption

$$\int_{t_0}^{\infty} \sum_{i=1}^{m} p_i(s)\, ds = \infty. \tag{34}$$

Moreover, from Theorem 3.3 and Theorem 3.4 given above, it can be seen that it does not need to have (34) in [15, Theorems 3.1 and 3.2]. So, we can restated these results as follows.

Theorem 4.1. *Assume* (H1), (H2), (33), *and one of the conditions hold:*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{m} q_i(s) \exp\Big(\int_{\tau_i(s)}^{\sigma(s)} \sum_{j=1}^{m} q_j(r)\, dr\Big) ds > \frac{1}{e}\,, \tag{35}$$

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} \sum_{i=1}^{m} q_i(s) \exp\Big(\int_{\tau_i(s)}^{\sigma(t)} \sum_{j=1}^{m} q_j(r)\, dr\Big) ds > 1\,. \tag{36}$$

Then all solutions of (32) *are oscillatory.*

Note that by setting $q_i = p_i/M_i$ and $g_i = M_i f_i$, where M_i are constants with $\tilde{M}_i < M_i$ the conditions (17) and (30) imply the conditions (35) and (36), respectively. Hence, when the conditions (35) and (36) hold, we prove that $\lim_{t\to\infty} x(t) = 0$. So, it does not need to have the condition (34) in the proof of Theorem 4.1.

Remark 4.2. *Let M_i be constants with $\tilde{M}_i < M_i$ for $1 \leq i \leq m$. Then, because of the definition of $\tilde{M}^* = \max\{\tilde{M}_i : 1 \leq i \leq m\}$ in the condition* (3) *and* (4), *the left part of the conditions* (17) *and* (30) *are greater than* (3) *and* (4), *respectively. So we have*

$$\int_{\tau(t)}^{t} \sum_{i=1}^{m} \frac{p_i(s)}{M_i} \exp\Big(\int_{\tau_i(s)}^{\sigma(s)} \sum_{j=1}^{m} \frac{p_j(r)}{M_j}\, dr\Big) ds \geq \int_{\tau(t)}^{t} \sum_{i=1}^{m} \frac{p_i(s)}{\tilde{M}^*}\, ds$$

and

$$\int_{\sigma(t)}^{t} \sum_{i=1}^{m} \frac{p_i(s)}{M_i} \exp\Big(\int_{\tau_i(s)}^{\sigma(t)} \sum_{j=1}^{m} \frac{p_j(r)}{M_j}\, dr\Big) ds \geq \int_{\sigma(t)}^{t} \sum_{i=1}^{m} \frac{p_i(s)}{\tilde{M}^*}.$$

Hence, the conditions (17) *and* (30) *are better results than* (3) *and* (4), *respectively.*
On the other hand, when $\tilde{M}_i > 1$ for $1 \le i \le m$, Theorem 4.1 can not be applied. It is valid only when $\tilde{M}_i \le 1$ for $1 \le i \le m$. Also, using same facts as above, we have

$$\int_{\tau(t)}^{t} \sum_{i=1}^{m} \frac{p_i(s)}{M_i} \exp\Big(\int_{\tau_i(s)}^{\sigma(s)} \sum_{j=1}^{m} \frac{p_j(r)}{M_j}\, dr\Big)\, ds \ge \int_{\tau(t)}^{t} \sum_{i=1}^{m} p_i(s) \exp\Big(\int_{\tau_i(s)}^{\sigma(s)} \sum_{j=1}^{m} p_j(r)\, dr\Big)\, ds$$

and

$$\int_{\sigma(t)}^{t} \sum_{i=1}^{m} \frac{p_i(s)}{M_i} \exp\Big(\int_{\tau_i(s)}^{\sigma(t)} \sum_{j=1}^{m} p_j(r)\, dr\Big)\, ds \ge \int_{\sigma(t)}^{t} \sum_{i=1}^{m} p_i(s) \exp\Big(\int_{\tau_i(s)}^{\sigma(t)} \sum_{j=1}^{m} p_j(r)\, dr\Big)\, ds.$$

Therefore, the conditions (17) *and* (30) *are better than* (35) *and* (36), *respectively.*

Now, we give examples to illustrate these results.

Example 4.3. In this example the conditions of Theorem 3.3 are satisfied, while the conditions of Theorem 2.2 are not. Consider equation (1) with

$$\tau_1(t) = t - 1 + \frac{1}{6}\sin(t), \quad \tau_2(t) = t - \frac{1}{2} + \frac{1}{6}\sin(t),$$
$$f_1(x) = x(\frac{1}{5} + x^2), \quad f_2(x) = x(\frac{1}{4} + x^4), \quad p_1(t) = 1, \quad p_2(t) = 2.5\,.$$

Then, $\tau(t) = \max\{\tau_1(t), \tau_2(t)\} = \tau_2(t)$.

$$\tilde{M}_1 = \lim_{x\to 0} \frac{x}{f_1(x)} = 5, \quad \tilde{M}_2 = \lim_{x\to 0} \frac{x}{f_2(x)} = 4, \quad \tilde{M}^* = 5\,.$$

Condition (3) is not satisfied because

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} (1 + 2.5)\, ds = \liminf_{t\to\infty} \Big(\frac{1}{2} - \frac{1}{6}\sin(t)\Big)(3.5) = \frac{3.5}{3} \approx 1.6667$$
$$< \frac{\tilde{M}^*}{e} \approx 1.8394\,.$$

However, we observe that condition (17) is satisfied when $M_1 = 5.01$ and $M_2 = 4.01$. Note that

$$\sum_{j=1}^{2} \frac{p_j}{M_j} = \frac{1}{5.01} + \frac{2.5}{4.01} \approx 0.8230\,,$$

and that

$$\int_{\tau_1(s)}^{\sigma(s)} 0.8230\, dr \ge \int_{\tau_1(s)}^{\tau(s)} 0.8230\, dr = \frac{1}{2} 0.8230, \qquad \int_{\tau_2(s)}^{\sigma(s)} 0.8230\, dr \ge \int_{\tau_2(s)}^{\tau(s)} dr = 0\,.$$

Then,

$$\sum_{i=1}^{2} \frac{p_i(s)}{M_i} \exp\Big(\int_{\tau_i(s)}^{\sigma(s)} \sum_{j=1}^{2} \frac{p_j(r)}{M_j}\, dr\Big) \ge \frac{1}{5.01} e^0 + \frac{2.5}{4.01} e^{0.4115} \approx 1.1404\,.$$

In the limit,

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} 1.1404\, ds = \liminf_{t\to\infty} \Big(\frac{1}{2} - \frac{1}{6}\sin(t)\Big) 1.1404 \approx 0.3801 > \frac{1}{e} \approx 0.3678\,.$$

Therefore all the conditions of Theorem 3.3 are satisfied. So, all solutions of this equation are oscillatory. Also, since $\tilde{M}_i > 1$, Theorem 4.1 can not be applied for this example.

Example 4.4. *We consider the following first order nonlinear delay differential equation.*

$$x'(t)+\frac{0.5}{e}x\left(\tau_1(t)\right)\ln(|x\left(\tau_1(t)\right)|+5)+\frac{0.2}{e}x\left(\tau_2(t)\right)\ln(|x\left(\tau_2(t)\right)|+4)=0,\ t\geq 0, \tag{37}$$

where

$$\tau_1(t)=\begin{cases} t-1, & t\in[3k,\ 3k+1] \\ -3t+12k+3, & t\in[3k+1,\ 3k+2] \\ 5t-12k-13, & t\in[3k+2,\ 3k+3] \end{cases},\quad k\in\mathbb{N}_0$$

$$\tau_2(t)=\tau_1(t)-2$$

and

$$\sigma_1(t):=\sup_{s\leq t}\{\tau_1(s)\}=\begin{cases} t-1, & t\in[3k,\ 3k+1] \\ 3k, & t\in[3k+1,\ 3k+2.6] \\ 5t-12k-13, & t\in[3k+2.6,\ 3k+3] \end{cases},\qquad k\in\mathbb{N}_0,$$

$$\sigma_2(t)=\sigma_1(t)-2,$$

then,

$$\tau(t)=\max_{1\leq i\leq m}\{\tau_i(t)\}=\tau_1(t).$$

Also, we find

$$\tilde{M}_1=\limsup_{x\to 0}\frac{x\left(\tau_1(t)\right)}{x\left(\tau_1(t)\right)\ln(|x\left(\tau_1(t)\right)|+5)}=\frac{1}{\ln 5}\cong 0.62133$$

and

$$\tilde{M}_2=\limsup_{x\to 0}\frac{x\left(\tau_2(t)\right)}{x\left(\tau_2(t)\right)\ln(|x\left(\tau_2(t)\right)|+4)}=\frac{1}{\ln 4}\cong 0.72134,$$

then,

$$\tilde{M}^*=\max_{1\leq i\leq m}\{\tilde{M}_i\}=\tilde{M}_1\cong 0.72134.$$

So,

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t}\sum_{i=1}^{m}p_i(s)=\liminf_{t\to\infty}\int_{t-1}^{t}\frac{0.7}{e}ds\cong 0.25751<\frac{\tilde{M}^*}{e}\cong 0.26536$$

and

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t}\sum_{i=1}^{m}p_i(s)\exp\left\{\int_{\tau_i(s)}^{\sigma(s)}\sum_{j=1}^{m}p_j(r)dr\right\}ds$$

$$=\liminf_{t\to\infty}\int_{t-1}^{t}\left[\frac{0.5}{e}\exp\left\{\int_{s-1}^{s-1}\frac{0.7}{e}dr\right\}+\frac{0.2}{e}\exp\left\{\int_{s-3}^{s-1}\frac{0.7}{e}dr\right\}\right]ds$$

$$\tilde{=} 0.30706 < \frac{1}{e} \tilde{=} 0.36,$$

that is, (3) *and* (35) *are not satisfied, respectively.*
However, if we take $M_1 = 0.63$ *and* $M_2 = 0.73$*, we obtain*

$$= \liminf_{t\to\infty} \int_{t-1}^{t} \left[\frac{0.5}{e(0.63)} \exp\left\{ \int_{s-1}^{s-1} \Big(\frac{0.5}{e(0.63)} + \frac{0.2}{e(0.73)} \Big) dr \right\} + \frac{0.2}{e(0.73)} \exp\left\{ \int_{s-3}^{s-1} \Big(\frac{0.5}{e(0.63)} + \frac{0.2}{e(0.73)} \Big) dr \right\} \right] ds$$

$$\tilde{=} 0.51301 > \frac{1}{e}$$

then, the condition (17) *is satisfied. So, all the solutions of* (37) *are oscillatory.*

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