Filomat 36:16 (2022), 5653–5664
https://doi.org/10.2298/FIL2216653A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Sharp Trapezoid Inequality for Quantum Integral Operator

Andrea Aglić Aljinović[a], Domagoj Kovačević[a], Mate Puljiz[a], Ana Žgaljić Keko[a]

[a]*University of Zagreb, Faculty of Electrical Engineering and Computing, Unska 3, 10 000 Zagreb, Croatia*

Abstract. Trapezoid inequality estimates the difference of the integral mean of a function on the finite interval $[a, b]$ and the arithmetic mean of its values at the endpoints a and b. Quantum calculus is the calculus based on finite diference principle or without the concept of limits. Euler-Jackson q-difference operator and q-integral operator are discretization of ordinary derivatives and integrals and they can be generalized to its shifted versions on arbitrary domain $[a, b]$.

In this paper we disprove a trapezoid inequality for shifted quantum integral operator appearing in the literature by giving two counterexamples. We point out some differences between the definite q-integral and Riemann integral to explain why the mistake is made and obtain corrected results. We also prove the sharpness of our new bounds in estimating the value of the quantum integral mean. Further we derive generalized sharp trapezoid inequality in which we point out the case with tightest bounds.

1. Introduction

The well known **trapezoid inequality** [14] which estimates the difference of the integral mean of a function on the finite interval and its values at the endpoints of that interval is given by

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{1}{4}(b-a)\left\|f'\right\|_\infty . \tag{1}$$

It holds for every $f : [a, b] \to \mathbb{R}$ continuous on $[a, b]$ and differentiable on $\langle a, b\rangle$ with derivative $f' : \langle a, b\rangle \to \mathbb{R}$ bounded on $\langle a, b\rangle$ i.e.

$$\left\|f'\right\|_\infty := \sup_{t\in(a,b)} \left|f'(t)\right| < +\infty.$$

In the published paper [17], beside other important integral inequalities, authors obtained the following generalization of the trapezoid inequality for quantum calculus.

2020 *Mathematics Subject Classification*. Primary 05A30; Secondary 26D10, 26D15
Keywords. q-derivative, q-integral, trapezoid inequality
Received: 22 September 2021; Accepted: 17 March 2022
Communicated by Miodrag Spalević
Email addresses: andrea.aglic@fer.hr (Andrea Aglić Aljinović), domagoj.kovacevic@fer.hr (Domagoj Kovačević), mate.puljiz@fer.hr (Mate Puljiz), ana.zgaljic@fer.hr (Ana Žgaljić Keko)

Theorem 1.1. (Incorrect result from [17], Theorem 3.3)
Let $f : [a,b] \to \mathbb{R}$ be a q-differentiable function with $D_q^a f$ continuous on $[a,b]$ and $0 < q < 1$. Then we have

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(qt + (1-q)a)\, d_q^a t \right| \leq \frac{b-a}{2(1+q)} \left\| D_q^a f \right\|_\infty . \tag{2}$$

In this paper, we shall give examples to show that this theorem is not valid, derive the sharp bound for (2) and generalized q-trapezoid inequality.

Quantum calculus is the calculus based on finite difference principle or without the concept of limits. It has two main branches: q-calculus and h-calculus. From q-calculus, by taking $q \to 1$, infinitesimal calculus is obtained. Quantum q-calculus has many significant applications in number theory, combinatorics, fractals, approximation theory, numerical analysis, ordinary and partial difference equations, dynamical systems, quantum groups, quantum algebras, Lie algebras, complex analysis, and also computer science, particle physic and quantum mechanics (see [1–4, 6, 7, 11, 15, 16]).

Among first obtained results concerning integral inequalities in q-calculus was the result of H. Gauchiman (see [8]) for a class of q-integrals called restricted q-integrals. Soon after various quantum analogues of classical integral inequalities were considered (see for instance [5, 8, 12, 13, 16–18]).

The paper is organized as follows. In Section 2 we give preliminaries for quantum calculus. In Section 3 we disprove (2) by giving counterexamples and obtain corrected results. We also prove the sharpness of our new bounds in estimating the value of quantum integral mean and derive generalized sharp trapezoid inequality in which we point out the case with tightest bounds.

2. q-calculus preliminaries

F. H. Jackson [9] in 1908 has defined the **Euler-Jackson q-difference operator** (q-derivative of the function) by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \ x \in \langle 0, b], \ q \in \langle 0, 1\rangle$$
$$D_q f(0) = \lim_{x \to 0} D_q f(x)$$

for an arbitrary function $f : [0,b] \to \mathbb{R}$, $b \in \langle 0, \infty\rangle$. Note that every function $f : [0,b] \to \mathbb{R}$ is q-differentiable for every $x \in \langle 0, b]$ and if $\lim_{x\to 0} D_q f(x)$ exists it is q-differentiable on $[0,b]$. The q-derivative is a discretization of ordinary derivative and if f is differentiable function then

$$\lim_{q \to 1} D_q f(x) = f'(x).$$

F. H. Jackson [10] in 1910 has also defined **q-integral** (or Jackson integral) by

$$\int_0^x f(t)\, d_q t = (1-q)\, x \sum_{k=0}^{\infty} q^k f\left(q^k x\right), \ \ x \in \langle 0, b]. \tag{3}$$

If the series on right hand-side is convergent, then q-integral $\int_0^x f(t)\, d_q t$ exists. If f is continuous on $[0,b]$ as $q \to 1$ the series $(1-q)\, x \sum_{k=0}^{\infty} q^k f\left(q^k x\right)$ tends to the Riemann integral (see [3], [11])

$$\lim_{q \to 1} \int_0^x f(t)\, d_q t = \int_0^x f(t)\, dt.$$

The sufficient condition under which the series $(1-q)\,x\sum_{k=0}^{\infty}q^k f\left(q^k x\right)$ converges to q-antiderivative of f is given in the following theorem from [11]:

Theorem 2.1. *If $\left|f(x)\,x^{\alpha}\right|$ is bounded on the interval $\langle 0,A]$ for some $0<\alpha<1$, then the q-integral defined by (3) converges to q-antiderivative of $f(x)$ on $\langle 0,A]$.*

For $0\le a<b$ we define

$$\int_a^b f(t)\,d_qt=\int_0^b f(t)\,d_qt-\int_0^a f(t)\,d_qt.$$

Theorem 2.2. *(**Fundamental theorem of** q**-calculus** [11]) If $F(x)$ is a q-antiderivative of $f(x)$ and $F(x)$ is continuous at $x=0$, $0\le a<b$ we have*

$$\int_a^b f(t)\,d_qt=F(b)-F(a).$$

Corollary 2.3. *If the ordinary derivative $f'(x)$ exists in a neighborhood of $x=0$ and is continuous at $x=0$, we have*

$$\int_a^b D_qf(t)\,d_qt=f(b)-f(a).$$

Obviously q-integral depends on the values of f outside of the interval of integration. Thus, if $f(t)\ge 0$, $t\in[0,b]$ it is not necessary that $\int_c^x f(t)\,d_qt\ge 0$. If we take, for example, $f(t)=1-t$, $t\in[0,1]$ and $q=\frac{1}{3}$ we have $\int_{\frac{1}{2}}^{1}(1-t)d_{\frac{1}{3}}t=-\frac{1}{16}<0$.

An important difference between the definite q-integral and Riemann integral is that even if we are integrating a function on an interval $[a,b]$, $a>0$ we have to take into account its behavior at $t=0$ as well as its values on $[0,a]$. This is the main reason for mistakes made in Theorem 1.1 (Theorem 3.3. in [17]). For instance, take indicator function of $[\frac{1}{2},1]$

$$\mathbf{1}_{[\frac{1}{2},1]}(t)=\begin{cases}0, & 0\le t<\frac{1}{2},\\ 1, & \frac{1}{2}\le t\le 1.\end{cases}$$

Although for Riemann integral we have

$$\int_{\frac{1}{2}}^{1}dt=\int_{\frac{1}{2}}^{1}\mathbf{1}_{[\frac{1}{2},1]}dt$$

for q-integral this is not the case:

$$\int_{\frac{1}{2}}^{1}1d_qt=\frac{1}{2},$$

but

$$\int_{\frac{1}{2}}^{1} \mathbf{1}_{[\frac{1}{2},1]} d_q t = \int_{0}^{1} \mathbf{1}_{[\frac{1}{2},1]} d_q t - \int_{0}^{\frac{1}{2}} \mathbf{1}_{[\frac{1}{2},1]} d_q t = (1-q)\left(\left(\sum_{k=0}^{\lfloor -\log_q 2 \rfloor} q^k\right) - \frac{1}{2}\right)$$
$$= \left(1 - q^{\lfloor -\log_q 2 \rfloor + 1} - \frac{1-q}{2}\right).$$

It is not hard to check that the latter expression is not equal to $\frac{1}{2}$ for all $q \in \langle 0, 1 \rangle$. In fact, the two coincide only if

$$q = \frac{1}{\sqrt[n]{2}}, \; n \in \mathbb{N}.$$

Previous definitions and results for $f : [0, b] \to \mathbb{R}$ can easily be generalized for $f : [a, b] \to \mathbb{R}$ (see [16]).

If we have a function $f : [a, b] \to \mathbb{R}$ then "shifted" q-derivative for $q \in \langle 0, 1 \rangle$ can be defined as

$$D_q^a f(x) = \frac{f(x) - f(a + q(x-a))}{(1-q)(x-a)}, \text{ if } x \in \langle a, b], \\ D_q^a f(a) = \lim_{x \to a} D_q^a f(x).$$

If $\lim_{x \to a} D_q^a f(x)$ exists, $f : [a, b] \to \mathbb{R}$ is said to be q-differentiable. "Shifted" q-integral is defined by

$$\int_a^x f(t) \, d_q^a t = (1-q)(x-a) \sum_{k=0}^{\infty} q^k f\left(a + q^k (x-a)\right), \; x \in [a, b].$$

If the series on right hand-side is convergent, then q-integral $\int_a^x f(t) \, d_q^a t$ exists and $f : [a, b] \to \mathbb{R}$ is q-integrable on $[a, x]$. If $c \in \langle a, x \rangle$, q-integral is defined by

$$\int_c^x f(t) \, d_q^a t = \int_a^x f(t) \, d_q^a t - \int_a^c f(t) \, d_q^a t.$$

If $f(x)$ and $g(x)$ are two real functions defined on $[a, b]$ whose ordinary derivatives exist in a neighborhood of $x = a$ and are continuous at $x = a$, then we have q-integration by parts formula

$$\int_a^b f(t) \, D_q^a g(t) \, d_q^a t = f(b) g(b) - f(a) g(a) - \int_a^b g(qt + (1-q)a) \, D_q^a f(t) \, d_q^a t.$$

Remark 2.4. *Since definitions of "shifted" q-derivative and "shifted" q-integral when $a = 0$ coincide with definitions of Jackson q-derivative and Jackson integral (3) in the rest of the paper we shall omit superscript zero when $a = 0$.*

3. q-Trapezoid inequality

Here and hereafter the symbol $\|\cdot\|_\infty^{\langle a,b]}$ denotes the norm

$$\|f\|_\infty^{\langle a,b]} = \sup_{t \in \langle a,b]} |f(t)|.$$

Remark 3.1. *In the proof of Theorem 1.1 (Theorem 3.3. in [17]) the above mentioned mistake was made:*

$$\int_{\frac{a+b}{2}}^{b}\left|t-\frac{a+b}{2}\right|d_q^a t=\int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right)d_q^a t$$
$$=\int_{a}^{b}\left(t-\frac{a+b}{2}\right)d_q^a t-\int_{a}^{\frac{a+b}{2}}\left(t-\frac{a+b}{2}\right)d_q^a t$$

but $\left|t-\frac{a+b}{2}\right|\neq t-\frac{a+b}{2}$ *for* $t\in[a,b]$ *or for* $t\in\left[a,\frac{a+b}{2}\right]$.

Next examples will show that Theorem 1.1 is not valid for all q-differentiable function f with $D_q^a f$ continuous on $[a,b]$, and all $0<q<1$.

Example 3.2. *In order to find functions which do not satisfy statement of the Theorem 1.1 we have implemented q-derivative defined by Euler-Jackson difference operator and q-integral by using definition (3) with first* 10^4 *elements of the sum. One possible example of the function for which the Theorem 1.1 is not valid is*

$$f(x)=134x^2-63x-1, \tag{4}$$

when $a=0, b=1$ *and* $q=\frac{1}{10}$. *Therefore, using standard Python libraries NumPy and Matplotlib we easily calculate and show graphically the* $\frac{1}{10}$*−derivative of this function. In the Figure 1 we show graph of this function together with the graph of its q-derivative. Numerically we have obtained, that the left hand side of the inequality (2) is*

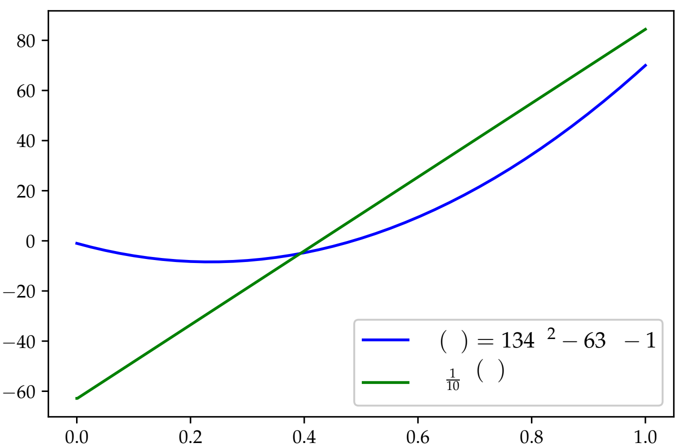

Figure 1: Graph of $f(x)$ and its $\frac{1}{10}$−derivative

approximately 40.02, *while the right hand side is* 38.36.

Now, we will calculate the same analytically, using standard formulas taken from [11] as follows:

$$D_q x^n=[n]x^{n-1},\ n\in\mathbb{N}$$

and

$$\int x^n d_q x=\frac{x^{n+1}}{[n+1]},\quad n\neq -1,$$

where $[n]$ *is a q-analog of n defined as:*

$$[n] = \frac{q^n - 1}{q - 1}.$$

Now, we easily calculate the true values of q-integral and q-derivative of the function given by (4), for $a = 0$, $b = 1$, $q = \frac{1}{10}$ *as follows:*

$$D_{\frac{1}{10}} f(x) = 134 \cdot [2]x - 63 \cdot [1] = 134 \cdot \frac{\frac{1}{100} - 1}{\frac{1}{10} - 1} x - 63 = 147.4x - 63.$$

We easily see that $\|D_q f(x)\|_\infty = 147.4 - 63 = 84.4$, *so the right hand side of (1) equals*

$$\frac{1}{2(1 + \frac{1}{10})} \cdot 84.4 \approx 38.36.$$

Furthermore,

$$\int f(\frac{1}{10}x) d_{\frac{1}{10}} x = \frac{134}{100} \int x^2 d_{\frac{1}{10}} x - \frac{63}{10} \int x d_{\frac{1}{10}} x - \int d_{\frac{1}{10}} x = \frac{134}{111} x^3 - \frac{63}{11} x^2 - x.$$

From Theorem 2.2 we have

$$\int_0^1 f(\frac{1}{10}x) d_{\frac{1}{10}} x = \frac{134}{111} - \frac{63}{11} - 1.$$

Values obtained analytically coincides with those obtained using Python.

Next we give another example of a class of functions for which (2) doesn't hold.

Example 3.3. *We will show that for*

$$f_q(t) = \left| \frac{1}{q} t - 1 \right|, \ t \in [0, 1]$$

inequality

$$\left| \frac{f_q(0) + f_q(1)}{2} - \int_0^1 f_q(qt) \, d_q t \right| \leq \frac{1}{2(1 + q)} \left\| D_q f_q \right\|_\infty \tag{5}$$

doesn't hold for every $q \in \langle 0, 1 \rangle$. *Denote* $m = \lfloor -\log_q 2 \rfloor$, *so* $q^{m+1} < \frac{1}{2} \leq q^m$ *and we have*

$$\int_0^1 f_q(qt) \, d_q t = (1 - q) \sum_{k=0}^{\infty} q^k f_q\left(q^{k+1}\right) = (1 - q) \left(0 + \sum_{k=1}^{\infty} q^k \left(1 - q^k\right) \right) = \frac{q}{1 + q}.$$

Since $f_q(0) = 1$, $f_q(1) = \frac{1-q}{q}$ *and*

$$D_q f_q(t) = -\tfrac{1}{q}, \qquad 0 \leq t \leq \tfrac{1}{q},$$

$$D_q f_q(t) \in \left\langle -\tfrac{1}{q}, \tfrac{1}{q} \right\rangle \quad \tfrac{1}{q} < t < \tfrac{1}{q^2},$$

$$D_q f_q(t) = \tfrac{1}{q}, \qquad \tfrac{1}{q^2} \leq t \leq 1,$$

we have $\left\|D_q f_q\right\|_\infty^{\langle 0,1]} = \frac{1}{q}$. *In this case inequality (5) reduces to*

$$\left|\frac{1}{2q} - \frac{q}{1+q}\right| \leq \frac{1}{2(1+q)q},$$

that is

$$\left|\frac{1+q-2q^2}{2(1+q)q}\right| \leq \frac{1}{2(1+q)q},$$

after multiplying by $2(1+q)q$ *it can be rewriten as*

$$\left|1+q-2q^2\right| \leq 1.$$

Last inequality is invalid for every $q \in \left\langle 0, \frac{1}{2}\right\rangle$.

Next we give q-Trapezoid inequality and its generalization and prove that both inequalities are sharp.

Theorem 3.4. *(**Trapezoid inequality for** q**-calculus**) Let* $f : [a,b] \to \mathbb{R}$ *be a function whose ordinary derivative exist in a neighborhood of* $x = a$ *and is continuous at* $x = a$. *Then for* $q \in \langle 0, 1\rangle$ *we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(qt+(1-q)a)\, d_q^a t\right| \leq (b-a)\left(\frac{1+q\left(2q^m-1\right)\left(1-2q^{m+1}\right)}{2(1+q)}\right)\left\|D_q^a f\right\|_\infty \tag{6}$$

where $m = \left\lfloor \log_q \frac{1}{2}\right\rfloor$.

Proof. The q-integration by parts formula gives

$$\int_a^b \left(t - \frac{a+b}{2}\right) D_q^a f(t)\, d_q^a t = (b-a)\left(\frac{f(a)+f(b)}{2}\right) - \int_a^b f(qt+(1-q)a)\, d_q^a t$$

and using generalized triangle inequality for q-calculus (see [3]) we obtain

$$\left|(b-a)\frac{f(a)+f(b)}{2} - \int_a^b f(qt+(1-q)a)\, d_q^a t\right| = \left|\int_a^b \left(t - \frac{a+b}{2}\right) D_q^a f(t)\, d_q^a t\right|$$
$$\leq \int_a^b \left|t - \frac{a+b}{2}\right| \left|D_q^a f(t)\right| d_q^a t \leq \left\|D_q^a f\right\|_\infty \int_a^b \left|t - \frac{a+b}{2}\right| d_q^a t$$

and

$$\int_a^b \left|t - \frac{a+b}{2}\right| d_q^a t = (1-q)(b-a)\sum_{k=0}^{\infty} q^k \left|a + q^k(b-a) - \frac{a+b}{2}\right|$$
$$= (1-q)(b-a)^2 \sum_{k=0}^{\infty} q^k \left|q^k - \frac{1}{2}\right|.$$

Since $m = \lfloor -\log_q 2 \rfloor$ we have

$$m \leq -\log_q 2 < m + 1$$

$$q^{m+1} < \frac{1}{2} \leq q^m$$

and

$$\begin{aligned}\sum_{k=0}^{\infty} q^k \left| q^k - \frac{1}{2} \right| &= \sum_{k=0}^{m} q^k \left(q^k - \frac{1}{2} \right) + \sum_{k=m+1}^{\infty} q^k \left(\frac{1}{2} - q^k \right) \\ &= \frac{1 - q^{2m+2}}{1 - q^2} - \frac{1}{2} \frac{1 - q^{m+1}}{1 - q} + \frac{1}{2} q^{m+1} \frac{1}{1 - q} - q^{2m+2} \frac{1}{1 - q^2} \\ &= \frac{1 - 2q^{2m+2}}{1 - q^2} + \frac{q^{m+1} - \frac{1}{2}}{1 - q}.\end{aligned}$$

Thus

$$\begin{aligned}\int_a^b \left| t - \frac{a+b}{2} \right| d_q^a t &= (b-a)^2 \left(\frac{1 - 2q^{2m+2}}{1+q} + q^{m+1} - \frac{1}{2} \right) \\ &= (b-a)^2 \left(\frac{1 + q\left(2q^m - 1\right)\left(1 - 2q^{m+1}\right)}{2(1+q)} \right).\end{aligned}$$

The proof is done after dividing the first inequality by $(b - a)$. $\square$

Remark 3.5. *Since we have $q^{m+1} < \frac{1}{2} \leq q^m$ and also $2q^{m+1} < 1 \leq 2q^m$, it is obvious that for any $q \in \langle 0, 1 \rangle$ we have*

$$\frac{1}{2(1+q)} \leq \frac{1 + q\left(2q^m - 1\right)\left(1 - 2q^{m+1}\right)}{2(1+q)}$$

and the equality holds for $m = -\log_q 2$.

Theorem 3.6. *(**Generalized trapezoid inequality for** q**-calculus**) Let $f : [a, b] \to \mathbb{R}$ be a function whose ordinary derivative exist in a neighborhood of $t = a$ and is continuous at $t = a$. Then for $q \in \langle 0, 1 \rangle$, $x \in \langle a, b]$ we have*

$$\begin{aligned}&\left| \frac{x-a}{b-a} f(a) + \frac{b-x}{b-a} f(b) - \frac{1}{b-a} \int_a^b f(qt + (1-q)a)\, d_q^a t \right| \\ &\leq (b-a) \left(\frac{1 - 2q^{2m+2}}{1+q} + \frac{x-a}{b-a} \left(2q^{m+1} - 1 \right) \right) \left\| D_q^a f \right\|_{\infty} \qquad (7)\end{aligned}$$

where $m = \left\lfloor \log_q \frac{x-a}{b-a} \right\rfloor$.

Proof. The q-integration by parts formula gives

$$\int_a^b (t - x) D_q^a f(t)\, d_q^a t = (x - a) f(a) + (b - x) f(b) - \int_a^b f(qt + (1-q)a)\, d_q^a t.$$

and using generalized triangle inequality (see [3]) we obtain

$$\left|(x-a)\,f(a)+(b-x)\,f(b)-\int_a^b f(qt+(1-q)\,a)\,d_q^a t\right|=\left|\int_a^b (t-x)\,D_q^a f(t)\,d_q^a t\right|$$
$$\leq\int_a^b |t-x|\left|D_q^a f(t)\right|d_q^a t\leq\left\|D_q^a f\right\|_\infty\int_a^b |t-x|\,d_q^a t$$

and

$$\int_a^b |t-x|\,d_q^a t=(1-q)\,(b-a)\sum_{k=0}^{\infty}q^k\left|a+q^k(b-a)-x\right|$$
$$=(1-q)\,(b-a)^2\sum_{k=0}^{\infty}q^k\left|q^k-\frac{x-a}{b-a}\right|.$$

Since $m=\left\lfloor\log_q\frac{x-a}{b-a}\right\rfloor$ we have

$$m\leq\log_q\frac{x-a}{b-a}<m+1$$
$$q^{m+1}<\frac{x-a}{b-a}\leq q^m$$

and

$$\sum_{k=0}^{\infty}q^k\left|q^k-\frac{x-a}{b-a}\right|=\sum_{k=0}^{m}q^k\left(q^k-\frac{x-a}{b-a}\right)+\sum_{k=m+1}^{\infty}q^k\left(\frac{x-a}{b-a}-q^k\right)$$
$$=\frac{1-q^{2m+2}}{1-q^2}-\left(\frac{x-a}{b-a}\right)\frac{1-q^{m+1}}{1-q}+\left(\frac{x-a}{b-a}\right)q^{m+1}\frac{1}{1-q}-q^{2m+2}\frac{1}{1-q^2}$$
$$=\frac{1-2q^{2m+2}}{1-q^2}+\left(\frac{x-a}{b-a}\right)\frac{2q^{m+1}-1}{1-q}.$$

Thus

$$\int_a^b |t-x|\,d_q^a t=(b-a)^2\left(\frac{1-2q^{2m+2}}{1+q}+\frac{x-a}{b-a}\left(2q^{m+1}-1\right)\right)$$

The proof is done after dividing the first inequality by $(b-a)$. □

Theorem 3.7. *Let $q\in\langle 0,1\rangle$. Trapezoid inequality for q-calculus (7) is sharp for every $x\in\langle 0,1]$.*

Proof. In order to simplify notations we set $a=0$ and $b=1$. For every fixed $q\in\langle 0,1\rangle$ and any $x\in\langle 0,1]$ we will show that equality in

$$\left|xf(0)+(1-x)\,f(1)-\int_0^1 f(qt)\,d_q t\right|\leq\left(\frac{1-2q^{2m+2}}{1+q}+x\left(2q^{m+1}-1\right)\right)\left\|D_q f\right\|_\infty \tag{8}$$

is obtained for function

$$f_x(t)=\left|t-q^{m+1}\right|,\ t\in[0,1].$$

It is obvious that

$$D_q f_x(t) = -1, \qquad 0 \le t \le q^{m+1},$$

$$D_q f_x(t) \in \langle -1, 1 \rangle \quad q^{m+1} < t < q^m,$$

$$D_q f_x(t) = 1, \qquad q^m \le t \le 1,$$

so $\left\| D_q f_x \right\|_\infty = 1$. For $m = \left\lfloor \log_q x \right\rfloor$ we have $q^{m+1} < x \le q^m$. Further $f_x(0) = q^{m+1}$, $f_x(1) = 1 - q^{m+1}$ and

$$x f_x(0) + (1-x) f_x(1) = 1 - q^{m+1} + x\left(2q^{m+1} - 1\right).$$

Also

$$\begin{aligned}
\int_0^1 f_x(qt)\, d_q t &= (1-q)\sum_{k=0}^{\infty} q^k f_x\left(q^{k+1}\right) \\
&= (1-q)\left(\sum_{k=0}^{m-1} q^k\left(q^{k+1} - q^{m+1}\right) + \sum_{k=m+1}^{\infty} q^k\left(q^{m+1} - q^{k+1}\right)\right) \\
&= (1-q)\left(q\frac{1-q^{2m}}{1-q^2} - q^{m+1}\frac{1-q^m}{1-q} + q^{m+1}\frac{q^{m+1}}{1-q} - q^{2m+3}\frac{1}{1-q^2}\right) \\
&= \frac{q + 2q^{2m+2} - q^{m+1} - q^{m+2}}{1+q}
\end{aligned}$$

and (8) reduces to

$$\begin{aligned}
&\left| 1 - q^{m+1} + x\left(2q^{m+1} - 1\right) - \frac{q + 2q^{2m+2} - q^{m+2} - q^{m+1}}{1+q} \right| \\
&\le \left(\frac{1 - 2q^{2m+2}}{1+q} + x\left(2q^{m+1} - 1\right)\right).
\end{aligned}$$

After short calculation we can see that

$$1 - q^{m+1} - \frac{q + 2q^{2m+2} - q^{m+2} - q^{m+1}}{1+q} = \frac{1 - 2q^{2m+2}}{1+q}$$

so in (8) equality is obtained. □

Remark 3.8. *In the case when $x = \frac{a+b}{2}$, and for $q \in \langle 0, 1 \rangle$, (7) reduces to (6). Thus (6) is sharp since we have proved that (7) is sharp for every $x \in \langle 0, 1 \rangle$.*

Corollary 3.9. *Suppose all assumptions from Theorem 3.6 hold. Then the following inequality holds*

$$\left| f(a) - \frac{1}{b-a}\int_a^b f(qt + (1-q)a)\, {}_a d_q t \right| \le \frac{q(b-a)}{1+q}\left\| {}_a D_q f \right\|_\infty .$$

Proof. Take $x = b$ in (7). □

Corollary 3.10. *Suppose all assumptions from Theorem 3.6 hold. Then the following inequality holds*

$$\left| f(b) - \frac{1}{b-a}\int_a^b f(qt + (1-q)a)\, {}_a d_q t \right| \le \frac{(b-a)}{1+q}\left\| {}_a D_q f \right\|_\infty . \tag{9}$$

Proof. Following the proof of Theorem 3.6 we have for $x = a$

$$\left| (b-a) f(b) - \int_a^b f(qt + (1-q)a) \, d_q^a t \right| \leq \left\| D_q^a f \right\|_\infty \int_a^b |t-a| \, d_q^a t$$

and from

$$\int_a^b |t-a| \, d_q^a t = (1-q)(b-a)^2 \sum_{k=0}^{\infty} q^{2k} = \frac{(b-a)^2}{1+q}.$$

we obtain (9).

□

Next we prove that the tightest bound in (7) is achieved for $m = \lfloor \log_q \frac{1}{2} \rfloor$ and $x = a + (b-a)q^m$.

Corollary 3.11. *Suppose all assumptions from Theorem 3.6 hold. The bound* (7) *is minimal for* $m = \lfloor \log_q \frac{1}{2} \rfloor$ *and* $x = a + (b-a)q^m$.

Proof. Let us find m (and x) such that the right hand side of (7) is minimal. Hence, we have to find the minimal value of the function

$$f(m, x) = \frac{1 - 2q^{2m+2}}{1+q} + \frac{x-a}{b-a}(2q^{m+1} - 1).$$

The expression $2q^{m+1} - 1$ can be positive or negative. If it is positive, then the fraction $\frac{x-a}{b-a}$ has to be minimal. If our expression is negative, then this fraction has to be maximal. We have two cases. Let $m_q = \left\lfloor \log_q \frac{1}{2} \right\rfloor$, that gives

$$q^{m_q+1} < \frac{1}{2} \leq q^{m_q}.$$

Now, $2q^{m_q} - 1 \geq 0$ and $2q^{m_q+1} - 1 < 0$.

If $2q^{m+1} - 1 \geq 0$ (for $m \leq m_q - 1$) then $\frac{x-a}{b-a}$ has to be minimal. Since $m = \left\lfloor \log_q \frac{x-a}{b-a} \right\rfloor$, $q^{m+1} < \frac{x-a}{b-a} \leq q^m$. The fraction $\frac{x-a}{b-a}$ cannot attain the minimal value. However we will put $\frac{x-a}{b-a} = q^{m+1}$ even though it is not correct. It would be more correct to put $\frac{x-a}{b-a} = q^{m+1} + \varepsilon$ for some $\varepsilon > 0$, but we prefer this sloppy notation. The corresponding x will be denoted by x_m. We claim that for $m \leq m_q - 1$ it is valid $f(m, x_m) \leq f(m-1, x_{m-1})$, that is

$$\frac{1 - 2q^{2m+2}}{1+q} + q^{m+1}(2q^{m+1} - 1) \leq \frac{1 - 2q^{2m}}{1+q} + q^m(2q^m - 1).$$

Multiplication by $1 + q$ and rearrangement give

$$2q^{2m} - 2q^{2m+2} + (1+q)(2q^{2m+2} - 2q^{2m}) \leq (1+q)(q^{m+1} - q^m)$$

and it follows

$$2q^{m+1}(q^2 - 1) \leq (1+q)(q-1).$$

This is valid for $m \leq m_q - 1$. Hence, the function f attains minimum at $m = m_q - 1$ for $m \in \{0, 1, \ldots m_q - 1\}$.

If $2q^{m+1} - 1 < 0$ (for $m \geq m_q$) then $\dfrac{x-a}{b-a}$ has to be maximal. Hence, $\dfrac{x-a}{b-a} = q^m$. We claim that for $m \geq m_q$ it is valid $f(m+1, x_{m+1}) \geq f(m, x_m)$, that is

$$\frac{1-2q^{2m+4}}{1+q} + q^{m+1}(2q^{m+2} - 1) \geq \frac{1-2q^{2m+2}}{1+q} + q^m(2q^{m+1} - 1).$$

Multiplication by $1+q$ and rearrangement give

$$-2q^{2m+4} + 2q^{2m+2} + (1+q)(2q^{2m+3} - 2q^{2m+1}) \geq (1+q)(q^{m+1} - q^m)$$

and it follows

$$2q^{2m+1}(q^2 - 1) \geq q^m(1+q)(q-1).$$

This is valid for $m \geq m_q$. Hence, the function f attains minimum at $m = m_q$ when $m \geq m_q$.

It remains to find the minimum of the function f since there are two candidates: $m_q - 1$ and m_q. It turns out that $f(m_q - 1, x_{m_q - 1}) = f(m_q, x_{m_q})$, or,

$$\frac{1-2q^{2m_q}}{1+q} + q^{m_q}(2q^{m_q} - 1) = \frac{1-2q^{2m_q+2}}{1+q} + q^{m_q}(2q^{m_q+1} - 1).$$

It is an easy exercise for the careful reader. □

Acknowledgments

Domagoj Kovačević was supported by the QuantiXLie Centre of Excellence, a project co financed by the Croatian Government and European Union through the European Regional Development Fund - the Competitiveness and Cohesion Operational Programme (Grant KK.01.1.1.01.0004).

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