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An Algorithm for Bi-Objective Integer Linear Programming Problem

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Abstract. In the present paper a bi-objective integer linear programming problem (BILP) is discussed. The main effort in this work is to effectively implement the ϵ-constraint method to produce a complete set of non dominated points. The convergence of the algorithm has been established theoretically. Further a comparative study to some existing algorithm has also been made.

1. Introduction

In mathematical optimization one aims at minimizing or maximizing one or more than one objective functions over some feasible set. In the case when the objective function is single, it is straightforward to define the notion of an optimal solution. A feasible element with the smallest (or largest) objective function value is an optimal solution. The existence of multiple objectives leads to many interesting questions which do not arise in single objective models. The objectives are usually conflicting in nature, so the solution concepts are based on the ideology of compromises among the objectives. From past few decades multi-objective programming has become one of the most favored and challenging area in the field of optimization. However, the research on the multi-objective programming problems with integer variables is limited when compared with continuous variables. The introduction of integer variables makes the problem more difficult even when the objectives are linear.

In past few year there has been some development in the area of multi-objective integer linear programming problem (MILP), still the research in this area is scarce. A ranking approach to generate all integer efficient solution of a (MILP) has been studied by Gupta and Malhotra [5]. Klein and Hannan [8] studied an approach that sequentially generated the set of non dominated points by solving a single objective optimization problem and by taking other objectives as constraints. Each time an efficient point is generated, some additional constraints are added in the previous problem to generate next efficient solution. This is also called ϵ-constraint methodology where one objective will be used as the objective function and the remaining objectives will be treated as constraints using the epsilon value as bound. The trick in this kind of methodology is to decide the best bound so that none of the integer efficient solution is missed. In literature ϵ-constraint technique was first proposed by Haimes [7], later Neumayer [9] implemented this approach to bi-objective transportation problem. Sylva and crema [11] discussed a variation of Klein and Hanna [8] algorithm by using a positive combination of all objective functions instead of optimizing a single criteria problem. Ozlen and Azizoglu [10] proposed a variant of ϵ-constraint algorithm, in which the

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given problem is converted into a single objective optimization problem and other objectives are taken as constraints based on the identification of objective efficiency range. For algorithms and other developments in the field of multi-objective linear optimization one can refer the survey article by Teghem and Knush [12] and by Ehrgott and Gandibleux [2, 3].

Although in literature, methods to solve bi-objective integer linear programming problem (BILP) are available but they do have their limitations. In some cases a complete set of non dominated points is not enumerated, in others the problem becomes very complicated and there are also instances where an explicit formula for the choice of epsilon is not given which makes it difficult to implement practically. In the proposed methodology an efficient ϵ-constraint method for a bi-objective programming problem is proposed, in which we aim at generating the set of all non dominated points of (BILP) and have justified that none of the integer efficient solution is missed. Further there are twofold benefits of the proposed methodology. First the constrained program solved to generate k^{th} non dominated point does not use any previous calculations, like the constraints generated to obtain $(k-1)^{th}$ or any previous non dominated point. Only the $(k-1)^{th}$ point is used to generate k^{th} non dominated point, which makes the algorithm computationally efficient. Second advantage of the proposed methodology is that one can also generate intermediate non dominated points. It is important to obtain intermediate non dominated points, as presence of large number of variables may generate large number of non dominated points. Since in multi-objective optimization problems decision is made on the preference attitude of the decision maker, for instance in a problem like financial investments, decision maker always has in his mind the amount he wants to invest and he is aware of his risk appetite. In such situation it becomes irrelevant to calculate complete set of solutions rather he should be provided with solutions within his desired range that will reduce error as well as save computation time also. We have made a comparative study with some of the algorithms available in literature and have also shown by counter example how these algorithm do not record all non dominated point.

This paper is organized as follows: the following section discusses the problem definition and some basic results. In Section 3 an algorithm and the results pertaining to its convergence are discussed. Section 4 discusses a numerical example to elaborate the procedure. Further comparison with other existing algorithms is also discussed and the last section discusses concluding remarks.

2. Mathematical formulation

Consider the following bi-objective integer linear programming problem

$$(P) \qquad \min_{X\in\Omega} \ (f(X),\ h(X))$$

where

$$f(X) = C^T X + \alpha$$
$$h(X) = D^T X + \beta$$

$\Omega = \{X \in R^n \mid AX = b,\ X \geq 0 \text{ is an integer vector}\}$,

$\bar{\Omega} = \{X \in R^n \mid AX = b,\ X \geq 0\}$,

$C,\ D, \in R^{n\times 1},\ \alpha,\ \beta, \in R,$
$A \in R^{m\times n},\ b \in R^{m\times 1}$

To ensure that there are finite number of integer points in Ω, it is assumed that the set $\bar{\Omega}$ is closed and bounded.

Consider the following single objective integer linear programming problem.

$$(P_1) \qquad \min_{X\in\Omega} C^T X + \alpha$$

Definition (Efficient solution). A solution $X^* \in \Omega$ is said to be an efficient solution if and only if there does not exist another solution $X \in \Omega$ such that

(a) $f(X) \leq f(X^*)$ and
(b) $h(X) \leq h(X^*)$
with strict inequality sign holding for at least one of the relations (a) and (b).

Definition (non dominated point). A pair $(f(X^*), h(X^*))$ corresponding to an efficient solution X^* is called a non dominated point.

Throughout the paper we use following notations:
An optimal solution of the problem (P_1) is denoted by X_1 and if there are m_1 alternates solutions, then the set of alternate optimal solutions is denoted by $X_{1,i},\ i = 1, 2 \dots m_1$. Similarly an optimal solution of the problem (P_1^{k-1}) for $k \geq 2$ is denoted by X_k and if this problem has m_k alternate optimal solutions, then the complete set of optimal solution of (P_1^{k-1}) is denoted as $S_k = \{X_{k,i},\ i = 1, 2 \dots m_k\}$. Further, we will show that $k + 1^{th}$ efficient solution of the problem (P) is an optimal solution of the problem (P_1^k). So we denote X_k as a k^{th} efficient solution of problem (P) and $(f_k, h_k) = (f(X_k), h(X_k))$ denotes the k^{th} non dominated point corresponding to solution X_k.

Remark 2.1. *We know from the theory of Integer linear programming problem (ILP), that if the feasible region of a linear programming problem is a closed and bounded polyhedron, then an optimal solution of the (ILP) must exists and can be obtained by either using branch and bound technique or by using the application of Gomory cut (Gomory 1969).*

As evident from above remark, the problem (P_1) always has a feasible solution, and the set of non dominated points will be nonempty, the following result shows the existence of a non dominated point of the problem (P).

Theorem 2.2. *If the feasible region of the problem* (P) *is nonempty, then set of non dominated points is also non empty.*

Proof: Since the feasible region of problem (P) is nonempty, the problem (P_1) always has an optimal solution. Suppose the set of optimal solutions of the problem (P_1) be $\{X_{1,i},\ i = 1, 2 \dots m_1\}$, yielding optimal value as f^*, i.e., $f(X_{1,i}) = f^*$ for all $i = 1, 2 \dots m_1$. Define $h^* = \min\{h(X_{1,i}),\ i = 1, 2 \dots m_1\}$. Then the pair (f^*, h^*) obtained at the point $X_{1,i}$ for some $i = 1, 2 \dots m_1$, is a non dominated pair, for if $(\tilde{f}, \tilde{h})$ is some other pair which dominates (f_1^*, h^*). It means $\tilde{f} < f_1^*$ and $\tilde{h} \leq h^*$, contradicts the optimality of f_1^*.

For some scalars $a > 0$ and $b > 0$, Consider the problem (P^*) defined as

$$(P^*) \qquad \min_{X \in \Omega} \ (af(X),\ bh(X))$$

The construction of the problem (P^*) is important as it will enable us to convert an objective functions with fractional coefficients to integer coefficients. Then, in the following theorem we show that the set of non dominated points of the problem (P) and (P^*) are same. In the proposed methodology we have made use of objective function cut to generate non dominated points and if the coefficients of the objective function are not integer then this cut may not work.

Theorem 2.3. *There is one to one correspondence between non dominated points of the problem* (P) *and* (P^*)*.*

Proof. Suppose (f, h) is non dominated point of the problem (P), then we need to show that (af, bh) is non dominated point of the problem P^*. Suppose on the contrary, there exist a pair (f^*, h^*) which dominates (af, bh), then $f^* \leq af$ and $h^* \leq bh$ with strict inequality sign holding for at least one place. That means $f^*/a \leq f$ and $h^*/b \leq h$ with strict inequality at at least one place, then $\tilde{f} \leq f$ and $\tilde{h} \leq h$ where $\tilde{f} = f^*/a$ and $\tilde{h} = h^*/a$. Which contradicts the assumption that (f, h) is a non dominated point. Similarly the converse part can be proved.

In order to find all the non dominated points of the problem (P_1), we define the restrictive version of the problem (P), in which the second objective function is treated as constraint. For $k \geq 1$ we define

$$(P_1^k) \qquad \min_{X \in \Omega} f(X) = C^T X + \alpha$$

where X satisfies

$$h(X) < h_k.$$

Here h_k is the value of $h(X)$ at X_k. Related to the problem (P_1^k), following notations are defined:
$S_{k+1} = \{X_{k+1,i},\ i = 1, 2, \dots m_{k+1}\}$is the set of optimal solutions of (P_1^k).
f_{k+1}^* is the optimal value of objective function of (P_1^k).

Theorem 2.4. *An optimal solution of the problem* (P_1^{k-1}) *gives* k^{th} *non dominated point of the problem* (P).

Proof. Since optimal solution set of the problem (P_1^{k-1}) is denoted by S_k, yielding optimal value as f_k. Define $h_k = \min\{h(X_{k,i}), X_{k,i} \in S_k\} = h(X_{k,p})$, for some $p \in 1, 2, \dots m_k$. then (f_k, h_k) is a non dominated point of the problem (P). For if there exist a pair (f, h) which dominates (f_k, h_k). Then $f \le f_k$ and $h \le h_k$ with strict inequality sign holding for at least one place. Then either $f < f_k$ or if $f = f_k$, then $h < h_k$. Both of these situations are not possible as in first case the optimality of f_k will be contradicted and in latter case the choice of h_k is contradicted. Therefore, (f_k, h_k) is a non dominated point.

Remark 2.5. *It may be noted, if the coefficient vector D in the function* $h(X) = D^T X + \alpha$ *has all the components as integer, then the constraint* $h(X) < h_k$ *in the problem* (P_1^k) *can be replaced with* $h(X) \le h_k - 1$, *as X being an integer vector and none of the integer point will be missed. Further, if D is not an integer vector, then solve the problem* (P^*), *as evident from Theorem 2.3, the problem* (P) *and* (P^*) *have same set of non dominated points. In this case the choice of the scalar b can be suitably made so that bD is an integer vector in the function* $bh(X)$ *and the inequality can be reduced by using the constraint* $bh(X) \le bh(X_k) - 1$.

Remark 2.6. *If some simplex table depicts an optimal solution of an integer linear programming problem with* $z_j - c_j = 0$ *for some nonbasic variable* x_j, *this indicates the presence of alternate optimal solutions. Then corresponding to each such nonbasic variable* x_j, *the alternate optimal integer solutions are calculated by using the formula* $\hat{x}_{B_i} = x_{B_i} - \gamma_j y_{ij}; i \ne r$ *and* $x_{B_r} = \gamma_j$, *where* γ_j *takes integer values such that* $0 \le \gamma_j \le \dfrac{x_{B_r}}{y_{rj}} = \min\limits_i \left\{ \dfrac{x_{B_i}}{y_{ij}} \mid y_{ij} > 0 \right\}$. *Here* $X_B = (x_{B_1}, x_{B_2} \dots, x_{B_m})$ *is a vector of basic variables in the current simplex table and* $y_j = B^{-1} A_j$, *with* A_j *as* j^{th} *column of matrix A. For detailed explanation one may refer Hadley [6]*

3. Algorithm and its convergence

Based on the results proved in previous section, in the following, we have proposed an algorithm which aims at finding all non dominated points of problem (P).

Step 1 Initially take $E = \emptyset$. Solve the problem (P_1) and note all its optimal feasible solutions $X_{1,i}\,; i = 1, 2 \dots m_1$, as discussed in Remark 2.6. Find f_1 and calculate $h_1 = \min\{h(X_{1,i}),\ i = 1, 2 \dots m_1\}$, and update $E = E \cup (f_1, h_1)$. Also calculate $h_l = \min\limits_{X \in \Omega} (D^T X + \beta)$.

Step 2 For $k \ge 2$, construct and solve the problem (P_1^{k-1}) and find the set S_k. Find f_k and $h_k = \min\{h(X), X \in S_k\}$. and go to Step 3.
Step 3 Update $E = E \cup (f_k, h_k)$, If $h_k = h_l$ go to Step 4, else set $k = k + 1$, go to Step 2 .

Step 4 Stop, and note E as the set of non dominated points.

In the following some results pertaining to the convergence of the algorithm are discussed.

Theorem 3.1. *For* $k \ge 1$, (f_k, h_k) *and* (f_{k+1}, h_{k+1}) *are the two points recorded by the procedure described above. Then* $f_k < f_{k+1}$.

Proof. The problem (P_1^k) is obtained from problem (P_1^{k-1}) by appending the constraint $h(X) < h_k$, which implies the set of feasible solutions of the problem (P_1^k) is smaller than (P_1^{k-1}). Therefore the optimal value of the problem (P_1^k) is inferior than the optimal value of problem (P_1^{k-1}), i.e. $f_k \le f_{k+1}$. Further if $f_k = f_{k+1}$, then we can find an $X \in S_{k+1} \cap S_k$ such that $f(X) = f_k = f_{k+1}$ and $h(X) \ge h_k$, which is a contradiction as $h(X) < h_k$ for all $X \in S_{k+1}$. Therefore, $f_{k+1} > f_k$.

Theorem 3.2. *The algorithm records only non dominated points.*

Proof. It may be observed that the first point obtained by the algorithm (f_1, h_1) is a non dominated point (by using Theorem 2.2), then we claim that (f_2, h_2) cannot be dominated by (f_1, h_1). Since $f_2 > f_1$ (using Theorem 3.1) and $h_2 = \min\{h(X_{i,j}) \mid X_{i,j} \in S_2\} < h_1$ (as defined in the algorithm). Therefore, (f_2, h_2) is also a non dominated point. Proceeding on the same lines, using (f_2, h_2) as a non dominated point the next point obtained (f_3, h_3) is also non dominated. Then by using induction it is easy to see if (f_i, h_i), $i = 1 \ldots k-1$ are non dominated points previously obtained, then (f_k, h_k) is also a non dominated point. Hence every point recorded by the algorithm is a non dominated point.

Since the non dominated points are generated according to the decreasing value of second objective function, the following theorem is useful in obtaining the stopping criteria of the algorithm, i.e., as soon as the minimum of second objective function is reached the process terminates.

Theorem 3.3. *If $(f_l,\ h_l)$ is the last non dominated point for the problem (P), then $h_l = \min\limits_{X\in S} h(X)$.*

Proof. Suppose $(f_l,\ h_l)$ is the last non dominated point for the problem (P). Assume that h_l is not the minimum of $h(X)$ and $\hat{h} = \min\limits_{X\in\Omega} h(X) = h(\hat{X})$ (say), this implies $h_l > \hat{h}$. Then either $f_l \geq \hat{f}$ or $f_l < \hat{f}$, where $\hat{f} = f(\hat{X})$. In the first case, when $f_l \geq \hat{f}$, then $(f_l,\ h_l)$ is dominated by $(\hat{f}, \hat{h})$, which is not true. In the latter case where $f_l < \hat{f}$ as $h_l > \hat{h}$, then the solution of problem (P_1^l) will exist and $(f_l,\ h_l)$ will not be the last non dominated point recorded by the procedure. Hence, the assumption is wrong.

The following theorem justifies that none of the non dominated solution is missed and is useful in obtaining the convergence of the algorithm.

Theorem 3.4. *The algorithm described above records all the non dominated points for problem (P).*

Proof. Let $(\bar{f}, \bar{h})$ with the corresponding solution $\bar{X}$ for problem (P), be a point which is not recorded by the algorithm. Then we have $\bar{f} \geq f_1$, as f_1 being optimal value of $f(X)$. If $\bar{f} = f_1$ and the point $(\bar{f}, \bar{h})$ is not recorded, then $\bar{h} > \min\{h(X_{i,j}),\ X_{i,j} \in S_1\} = h_1$, where S_1 is the index set of all the optimal solutions for the problem (P_1). This implies that $(\bar{f}, \bar{h})$ is dominated by (f_1, h_1). If $\bar{f} > f_1$, then there exists some k for which $\bar{f} \geq f_k$. Again we have two possibilities, either $\bar{f} = f_k$ or $\bar{f} > f_k$. If $\bar{f} = f_k$ then $(\bar{f}, \bar{h})$ is dominated by (f_k, h_k), if $\bar{f} > f_k$, in this case again either of the two subcases arises viz. $\bar{f} = f_{k+1}$ or $\bar{f} > f_{k+1}$. Continuing in this way we find that either $(\bar{f}, \bar{h})$ is a dominated point or $\bar{f} > f_l$, where $(\bar{f}, \bar{h})$ is the last non dominated point . Also $\bar{h} \geq h_l$ as $h_l = \min\limits_{X\in S} h(X)$, then again $(\bar{f}, \bar{h})$ is a dominated point.

Remark 3.5. *It may be noted that there are finite number of non dominated points because the feasible region is assumed to be closed and bounded. So the algorithm will terminate in a finite number of steps as it records only non dominated points.*

Remark 3.6. *The problem (P_1^k) is an integer linear programming problem with a modified linear constraint as described in Remark 2.5. Its optimal solution can be obtained by using simplex technique along with application of Gomory's cut (Gomory 1969), by using branch and bound technique or by any of the method that solve integer linear programming problem.*

An extended (BILP)

Suppose the decision maker wants to explore his options to take his decision, when his decision parameters with one of objectives lying within certain range. In this case, suppose the decision maker want to find the set of non dominated points only in the range (γ, δ), $\gamma, \delta \in R$, of second objective function . Then the the problem (P) can be redefined as follows:

$(\bar{P})$ $$\min_{X\in\Omega}(f(X), h(X))$$

Subject to $$\gamma \leq h(X) \leq \delta$$

Then the algorithm discussed Section 3, to obtain non dominated pairs of the problem (P) can be generalized in the following way. The modified algorithm discussed below will yield only those non dominated points for which second objective function lies within γ and δ.

Algorithm

Step 1 Initially take $E = \emptyset$. Solve the problem $(\bar{P}_2)$ defined as follows,

$$(\bar{P}_2) \qquad \min_{X\in\Omega} f(X) \text{ subject to } \gamma \leq h(X) \leq \delta,$$

and note all its optimal feasible solutions $X_{1,i}$; $i = 1, 2 \ldots m_1$. Find f_1 and calculate $h_1 = \min\{h(X_{1,i}),\ i = 1, 2 \ldots m_1\}$, and update $E = E \cup (f_1, h_1)$.

Step 2 For $k \geq 1$, construct the problem $(\bar{P}_1^k)$ defined as follows:

$$(\bar{P}_1^k) \qquad \min_{X\in\Omega} f(X) \text{ subject to } h(X) \leq h_k - 1,$$

and call its optimal solution set as S_k. Find $f_k = f(X);\ X \in S_k$ and $h_k = \min\{h(X), X \in S_k\}$. and go to Step 3.
Step 3 Update $E = E \cup (f_k, h_k)$, If $h_k = h_l$, where $h_l = \min\limits_{X\in\Omega,\ h(X)\geq\gamma} h(X)$, go to Step 4, else set $k = k + 1$, goto Step 2.

Step 4 Stop, and note E as the set of non dominated points of the problem $(\bar{P})$.

It can be easily seen that an optimal solution of the problem $(\bar{P}_1^k)$, will yield (f_k, h_k) as the k^{th} non dominated pair of the problem $(\bar{P})$. The convergence of this algorithm can be discussed on the same lines as done earlier in this Section.

4. Numerical Example

In this section we have discussed a numerical example to elaborate the procedure for finding non dominated points of the problem (P). All the calculations are done by using simplex algorithm and Gomory's cutting plane technique for integer programming problem as discussed in Bazaraa et al. [1] and Hadley [6].
Example 1 Consider the following bi-objective integer linear programming problem

$$(\tilde{P}) \qquad \min\ (x_1 + x_2/2,\ -1/3x_1 + 2/3x_2)$$

Subject to $3x_1 + 2x_2 \geq 6,\ 4x_1 + 5x_2 \leq 20,\ x_1, x_2 \geq 0$ and integers. It may be noted that by taking $a = 2$ and $b = 3$ we get the following (BILP).

$$(P) \qquad \min\ (2x_1 + x_2,\ -x_1 + 2x_2)$$

Subject to $3x_1 + 2x_2 \geq 6,\ 4x_1 + 5x_2 \leq 20,\ x_1, x_2 \geq 0$ and integers.

Then by virtue of Theorem 2.3, it is equivalent to solve the problem (P) instead of problem $(\tilde{P})$. Therefore, corresponding the problem (P), we have following single objective integer linear programming problem (P_1) given as:

$$(P_1) \qquad \min\ f = 2x_1 + x_2$$

Subject to $3x_1 + 2x_2 \geq 6,\ 4x_1 + 5x_2 \leq 20,\ x_1, x_2 \geq 0$ and integers.

An optimal feasible solution of the problem (P_1) is given in the Table 1.

Table 1

		c_j	2	1	0	0
c_B	B	X_B	y_1	y_2	y_3	y_4
1	x_2	3	3/2	1	−1/2	0
0	x_4	5	−7/2	0	5/2	1
	$f = 3$	$Z_j - c_j$	−1/2	0	−1/2	0

The optimal solution is given by $X_1 = (0, 3)$, with $f(0, 3) = 3$ and $h(0, 3) = 6$ and by virtue of Theorem 2.2, we have our first non dominated point $(f_1, h_1) = (3, 6)$, with the corresponding efficient solution as $X_1 = (0, 3)$. Now for $k = 1$, construct the problem (P_1^k), by adding second objective as an additional constraint. By using Remark 2.5, this additional constraint is given by $-x_1 + 2x_2 \leq 5$, and an optimal integer solution of the problem (P_1^1) is given in the Table 2.

Table 2

		c_j	2	1	0	0	0	0
c_B	B	X_B	y_1	y_2	y_3	y_4	y_5	y_6
1	x_2	0	0	1	−2	0	0	3
0	x_4	12	0	0	6	1	0	−7
2	x_1	2	1	0	1	0	0	−2
0	x_5	7	0	0	5	0	1	−8
	$f = 4$	$Z_j^1 - c_j$	0	0	0	0	0	−1

Since $z_3 - c_3 = 0$ It is evident from this table that the problem has alternate optimal solutions. Since $0 \leq \gamma_3 \leq 7/5$, the integer solutions are obtained by giving $\gamma_3 = 0, 1$ in the formula $\hat{x}_{B_i} = x_{B_i} - \gamma_j y_{i,j},\ x_3 = \gamma_j$. The set of optimal integer solutions of the problem (P_1^1) obtained are $\{X_{2,1} = (2, 0), X_{2,2} = (1, 2)\}$, with $f_2 = 4$ and $h_2 = \min\{\ h(2, 0), h(1, 2)\} = -2$ and second non dominated point obtained is $(f_2, h_2) = (4, -2)$, with corresponding efficient solution as $X_{2,1} = (2, 0)$. Further construct the problem (P_1^2) by appending the constraint $h(X) \leq -3$ and proceed as above. Continuing on the lines of algorithm we have obtained all the non dominated points of the problem (P), and the correspondingly for the problem $(\tilde{P})$. The following Table lists all the non dominated points of both the problems (P) as well as $(\tilde{P})$ set of non dominated points.

Table 3

S.no	1	2	3	4	5
Efficient solutions	(0,3)	(2,0)	(3,0)	(4,0)	(5,0)
non dominated points of (P)	(3,6)	(4,-2)	(6,-3)	(8,-4)	(10,-5)
non dominated points of $(\tilde{P})$	(3/2,2)	(2,-2/3)	(3,-1)	(4,-4/3)	(5,-5/3)

Remark 4.1. *Consider a (BILP) with one of the objective function has fractional coefficient given as:*

$$\min\ (2x_1 + x_2, -1/3x_1 + 2/3x_2)$$

If the coefficients of second objective function are not converted into integer by suitably chosen integer b(=3 in this case) as suggested in Remark 2.5, then the algorithm may not generate all non dominated points. As in this case only three non dominated points will be recorded by the algorithm namely $(3, 2), (4, -2/3)$ *and (10,-1.6). But after changing the second objective function to* $-x_1 + 2x_3$*, a complete set of non dominated points is generated, which include (6,-1) and (8,-1.3) in addition to three already generated.*

Remark 4.2. *It may be observed in the above example there are eleven integer points in* Ω*, out of which only five are efficient and are obtained by the proposed algorithm and none other integer points is evaluated. Indeed some alternate optimal solutions are scanned as and when required, but to evaluate these integer solutions no extra simplex iterations are required which makes the proposed methodology more efficient.*

Example 2 Consider a bi-objective integer linear programming problem with four variables

$$(\tilde{P}) \qquad \min\ (x_1 - 2x_2 + x_3 + 2x_4,\ -x_1 + 2x_2 - 2x_3 + x_4)$$

Subject to $x_1 + 2x_2 + x_3 + 4x_4 \leq 8,\ -x_1 + x_2 + 2x_3 - x_4 \geq 4,\ x_1, x_2, x_3, x_4 \geq 0$ and integers. First we solve the following single objective integer linear programming problem (P_1) given as:

$$(P_1) \qquad \min\ f = x_1 - 2x_2 + x_3 + 2x_4$$

$$x_1 + 2x_2 + x_3 + 4x_4 \leq 8,\ -x_1 + x_2 + 2x_3 - x_4 \geq 4,\ x_1, x_2, x_3, x_4 \geq 0 \text{ and integers.}$$

The following table depicts an optimal feasible solution of this problem.

Table 4

		c_j	1	-2	1	2	0	0
c_B	B	X_B	y_1	y_2	y_3	y_4	y_5	y_6
0	x_6	0	3/2	0	$-3/2$	3/2	1/2	1
-2	x_2	4	1/2	1	1/2	1/2	1/2	0
	$f = 6$	$Z_j - c_j$	0	0	0	-1	-1	0

The optimal solution is given by $X_1 = (0,4,0,0)$, with $f(0,4,0,0) = -8$ and $h(0,4,0,0) = 8$. So the first non dominated point is $(f_1, h_1) = (-8, 8)$, with the corresponding efficient solution as $X_1 = (0,4,0,0)$. Now by adding second objective as an additional constraint, we construct the problem (P_1^1) given as:

(P_1^1) $\min\ f = x_1 - 2x_2 + x_3 + 2x_4$

subject to $x_1 + 2x_2 + x_3 + 4x_4 \leq 8,\ -x_1 + x_2 + 2x_3 - x_4 \geq 4, -x_1 + 2x_2 - 2x_3 + x_4 \leq 7\ x_1, x_2, x_3, x_4 \geq 0$ and integers.

An optimal integer solution of the problem (P_1^1) is $(0,3,1,0)$ with $f(0,3,1,0) = -5$, which is an efficient point. The second non dominated vector obtained at this point is $(f_2, h_2) = (-5, 4)$. Now continuing on the same lines we get the following efficient points.

Table 5

S.no	1	2	3	4	5	6
X_k	(0,4,0,0)	(0,3,1,0)	(0,3,2,0)	(0,2,2,0)	(0,2,3,0)	(0,2,4,0)
(f_k, h_k)	(-8,8)	(-5,4)	(-4,2)	(-2,0)	(-1,-2)	(0,-4)
S.no	7	8	9	10	11	12
X_k	(0,1,4,0)	(0,1,5,0)	(0,1,6,0)	(0,0,6,0)	(0,0,7,0)	(0,0,8,0)
(f_k, h_k)	(2,-6)	(3,-8)	(4,-10)	(6,-12)	(7,-14)	(8,-16)

It may be noted that the optimal solution of the second objective function over Ω is $(0,0,8,0)$ with objective function value as $h_l = -16$. Since optimal value of the second objective function is attained, the algorithm terminates. The complete set of efficient points is given in above table.

5. Comparison with some existing techniques

In the following we have made a comparative analysis of the proposed methodology with some of the available algorithms in literature. These algorithms are either failed to generate a complete set of non dominated points or in case they are able to generate, they are computationally very expensive.

1. Ranking method of Gupta and Malhotra [5] The ranking algorithm proposed by Gupta and Malhotra [5] requires to calculate all integer points of the feasible region. At each stage new cuts are appended in the previous simplex table to find next best integer feasible solution. These integer solutions are obtained in ascending order of one of the objective function, then at each ranked solution k tuple is evaluated and the non dominated set updated at each stage after removing dominated points.

The proposed methodology is much better than the ranking algorithm discussed by Gupta and Malhotra [5], as it evaluates only non dominated points, unlike ranking method which scan all the integer points of the feasible region. For instance, if the ranking algorithm is applied to the Example 1 above, it will require at least eleven simplex iterations (as some alternates are available), where as proposed methodology terminates in five simplex calculations. Another advantage of the proposed algorithm over ranking algorithm is that it requires to solve an integer programming problem (P_1^k) only, which is obtained by appending a constraint $h(X) < h_k$ in the problem (P_1). Whereas, ranking method requires repeated application of ranking cuts as well as Gomory cuts to scan all integer feasible solutions. So if a problem has eleven integer solutions, ranking algorithm requires at least eleven ranking cuts in addition to at least eleven Gomory cuts to be appended in a given problem, which is computationally very difficult to solve.

2. A modified ϵ-constraint technique of Özlen and Azizoğlu [10] The algorithm studied by Özlen and Azizoğlu [10] is a modified version of classical ϵ constraint method, which aims at generating complete set of non dominated of a multi-objective linear programming problem. For a two objective case this algorithm requires to solve a weighted single objective integer programming problem (CWSOIP) defined as

$$\min f(x) + wh(x), \text{ Subject to } h \le l_2,\ x \in X$$

where X is the set of feasible solutions, w is some weight function and the bound l_2 is updated at each stage. We applied their algorithm to a two objective problem given in Remark 4.1 and after making the calculations of w as given by Özlen and Azizoğlu [10], obtained the problem (CWSOIP) as below

$$\min_{x_1,x_2\in\Omega} \frac{151}{78}x_1 + \frac{44}{39}x_2 \text{ Subject to } -\frac{1}{3}x_1 + \frac{2}{3}x_2 \le l_2$$

Initially the l_2 is taken as 2.6 and the non dominated pair obtained is (3,2), then l_2 is updated to $l_2 = 1$ the non dominated pair obtained is $(4, \frac{-2}{3})$ and further l_2 is updated to $l_2 = \frac{-5}{3}$ and the non dominated pair obtained is (10,-1.6). Here it missed two non dominated points $(6, -1)$ and $(8, -1.3)$ and these points will not be recovered by the algorithm at any later stage.

3. Weighted function approach of Sylva and Crema [11] In this algorithm, authors proposed a weighted function approach to find non dominated points of a multi-objective integer linear programming problem. At each stage the algorithm constructs a sequence of progressively constrained integer programming problem to generate non dominated points. The main drawback of a weighted approach is the choice of weights, which in many practical situations difficult to obtain, though useful in obtaining theoretical results. In the example discussed by Sylva and Crema [11], we will show that if the choice of lambda is changed the results may also change.

In this algorithm, the first non dominated point is obtained by solving following integer programming problem for $\lambda > 0$

$$(P_0) \qquad \max\{\lambda^t Cx : Ax = b, x \ge 0, x \in Z^n\}$$

and after obtaining l^{th} efficient solution x^l a new problem $(P_l)j$ is constructed after adding the constraint

$$(Cx)_k \ge ((Cx^l)_k + 1)y_k^l - M_k(1 - y_k^l), \text{ for } k = 1, 2 \ldots p$$
$$\textstyle\sum_{k+1}^{p} y_k^l \ge 1,\ y_k^l \in \{0, 1\}, \text{ for } k = 1, 2, \ldots p,$$

to the previously generated problem (P_{l-1}). Here $-M_k$ is a lower bound on k^{th} objective function. In order to demonstrate the algorithm, following example was discussed by Sylva and Crema [11].

$$\max(x_1 - 2x_2, -x_1 + 3x_2) \text{ Subject to } \quad x_1 - 2x_2 \le 0,$$

Authors obtained a set of non dominated points by taking $\lambda = (4, 3)$, as they claimed for any $\lambda > 0$, the algorithm will work, though it is not the case. We changed the value of λ to $(1, 2)$ and got the following single objective integer programming problem

$$(P_0) \qquad \max\ z = -x_1 + 4x_2 \text{ Subject to } x_1 - 2x_2 \le 0,\ x_1, x_2 \in \{0, 1, 2\}$$

An optimal solution of this problem is $(0, 2)$ yielding the first efficient pair as $(-4, 6)$. Then the problem (P_1) is constructed as follows

$$(P_1) \qquad \max\ z = -x_1 + 4x_2$$

Subject to $\quad x_1 - 2x_2 \ge y_1^1 - 4,\ -x_1 + 3x_2 \ge 9y_2^1 - 2,\ y_1^1 + y_2^1 \ge 1,\ x_1, x_2 \in \{0, 1, 2\},\ y_1^1, y_2^1 \in \{0, 1\},$

An optimal solution of this problem is $X^2 = (1, 1)$, with yielding $z = 3$ the second non dominated pair as $(-1, 2)$. In this process an efficient pair (2,2) will never be recorded by the algorithm as it is infeasible for the current problem, but this pair was recorded with earlier choice of $\lambda = (4, 3)$.

Now we will implement this algorithm on Example 1 also. So we take given problem in maximization form given as

$$\max(-2x_1 - x_2,\ x_1 - 2x_2)$$

Subject to $3x_1 + 2x_2 \geq 6,\ 4x_1 + 5x_2 \leq 20,\ x_1, x_2 \geq 0$ and integers.

Now construct the problem (P_0) using the positive weights $\lambda_1 = 1$ and $\lambda_2 = 4$. We get the following integer linear programming problem:

(P_0) $$\max\ z = 2x_1 - 9x_2$$

Subject to $3x_1 + 2x_2 \geq 6,\ 4x_1 + 5x_2 \leq 20,\ x_1, x_2 \geq 0$ and integers.

An optimal feasible solution of this problem is $x_1 = 5, x_2 = 0$, which is also an efficient solution of the given problem. To find second efficient point, we need to construct the problem (P_1). For this we calculate the values of $M_1 = 10$ and $M_2 = 8$ ($-M_1$ and $-M_2$ are lower bounds on the corresponding objective functions). Therefore the problem (P_1) becomes

(P_1) $$\max\ z = 2x_1 - 9x_2$$

Subject to
$3x_1 + 2x_2 \geq 6,\ 4x_1 + 5x_2 \leq 20,\ 2x_1 + x_2 + y_1^1 \leq 10, -x_1 + 2x_2 + 14y_2^1 \leq 8, y_1 + y_2^1 \geq 1\ x_1, x_2 \geq 0, y_1^1, y_2^1 \in \{0, 1\}$

It may be observed that to find second efficient point we have to add three more constraints and two more variables. Further at every subsequent stage we need to add $3k$ more constraints and $2k$ more variables in addition to the existing ones, which makes this problem very complex. In the proposed methodology only one constraint is added to the original set of constraints even at the subsequent stages same constraint is modified. So the proposed methodology is more effective and easier to implement.

Computational Results The method was programmed in Matlab 2018 and tested on a randomly generated multi-objective integer linear programming problem with up to 10 constraints and 40 variables. The program was run on Intel Core i5 processor and we have observed with the increasing number of variables and constrains the running time was also increased. The program also returned a very large number of efficient points. The Table 6 shows the computational experiments with different values of m and n.

6. Concluding Remarks

In the proposed methodology, an epsilon constraint method for bi-objective integer programming problem is discussed. In literature various authors have used ϵ-constraint method for multi-objective programming problems. The main difficulty lies in the choice of epsilon, for which the single objective problem is solved. In general it is very difficult to choose the correct value of epsilon, and can be efficiently done only in certain class of problems. In the proposed methodology we are able to choose epsilon for a $(BILP)$, and have generated a complete set of non dominated points. Further, after generating the pair (f_k, h_k), a new problem (P_1^k) is generated, which does not use any previous calculations, so there is a least chance of rounding off error. Moreover, only one constraint $h(X) < h_k$, is added to the problem (P_1^k) which makes this problem computationally better than those algorithm having a number of constraints added at restricted problem. The efficiency of the proposed methodology depends upon the Gomory cut/ branch and bound technique used to solve the problem (P_1^k).

The idea proposed in this method can be generalized to bounded variable linear programming problem and multi-objective integer linear programming problems. Further, to find and efficient epsilon for linear fractional programming problem and for mixed integer linear/linear fractional programming problems is still open to work upon, and the proposed methodology can be extended to these problems.

Table 6

No.of Constraints (m)	Number of variables (n)	Efficient points (k)	Running time(s) (seconds)
6	20	2285	337.14
5	7	429	29.84
1	20	1876	178.88
9	12	665	36.77
1	14	2306	87.76
2	4	229	8.77
10	39	6742	840.74
10	6	1	0.85
5	7	844	49.2
7	4	107	6.67
2.	6	1259	69.61
3	4	1	9.73
3	6	130	6.16
7	10	963	108.19
2	10	657	30
2	2	107	4.81
5	7	560	33.61

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