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# Inequalities for the Polar Derivative of a Complex Polynomial

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**Abstract.** Let $P(z) := \sum\limits_{v=0}^{n} a_v z^v$ be a univariate complex coefficient polynomial of degree $n$. Then as a generalization of a well-known classical inequality of Turán [25], it was shown by Govil [7] that if $P(z)$ has all its zeros in $|z| \leq k,\ k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|,$$

whereas, if $P(z) \neq 0$ in $|z| < k,\ k \leq 1$, it was again Govil [6] who gave an extension of the classical Erdös-Lax inequality [13], by obtaining

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|,$$

provided $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, where $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$. In this paper, we obtain several generalizations and refinements of the above inequalities and related results while taking into account the placement of the zeros and extremal coefficients of the underlying polynomial. Moreover, some concrete numerical examples are presented, showing that in some situations, the bounds obtained by our results can be considerably sharper than the ones previously known.

## 1. Introduction

Let $P(z) := \sum\limits_{v=0}^{n} a_v z^v$ be a polynomial of degree $n$ and $P'(z)$ its derivative. The study of comparison inequalities that relate the norm between polynomials on a disk in the plane is a fertile area in analysis, important especially for its applications in the geometric function theory and in the application areas such as physical systems. Various inequalities in both directions relating the norm of the derivative and the underlying polynomial play a key role in the literature for proving the inverse theorems in approximation theory and, of course have their own intrinsic value. The Bernstein and Turán-type inequalities and their various generalizations are very well-known for various norms and for many classes of functions such as polynomials with various constraints, and on various regions of the complex plane. A classical inequality that relates an estimate to the size of the derivative of a polynomial to that of the polynomial itself in the

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uniform-norm on the unit disk in the plane is the famous Bernstein-inequality [4]. It states that, if $P(z)$ is a polynomial of degree $n$, then it is true that

$$\max_{|z|=1}|P'(z)| \leq n \max_{|z|=1}|P(z)|. \tag{1}$$

Equality holds in (1) if and only if $P(z)$ has all its zeros at the origin. It might easily be observed that the restriction on the zeros of $P(z)$ imply an improvement in (1). It turns out that to have any hope of a lower bound or an improved upper bound, one must have some control over the location of the zeros of polynomial $P(z)$. It was conjectured by P. Erdös and later proved by Lax [13] that if $P(z)$ is a polynomial of degree $n$ having no zeros in $|z| < 1$, then

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{2} \max_{|z|=1}|P(z)|. \tag{2}$$

On the other hand, in 1939 (see [25]), Turán obtained a lower bound for the maximum of $|P'(z)|$ on $|z| = 1$, by proving that if $P(z)$ is a polynomial of degree $n$ having all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{2} \max_{|z|=1}|P(z)|. \tag{3}$$

Thus in (2) and (3) equality holds for those polynomials of degree $n$ having all their zeros on $|z| = 1$. The above inequalities are the starting point of a rich literature concerning their extensions, generalizations and improvements in several directions, see, e.g., the papers ([1]-[3], [6]-[10]) to mention only a few. As a generalization of (3), Govil [7] proved that if $P(z)$ is a polynomial of degree $n$ having all its zeros in $|z| \leq k,\ k \geq 1$, then

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{1+k^n} \max_{|z|=1}|P(z)|, \tag{4}$$

whereas, for the class of polynomials not vanishing in $|z| < k,\ k \leq 1$, the precise estimate of maximum of $|P'(z)|$ on $|z| = 1$ does not seem to be known in general, and this problem is still open. However, some special cases in this direction have been considered by many people where some partial extensions of (2) are established. In 1980, it was again Govil [6], who generalized (2) by proving that if $P(z)$ is a polynomial of degree $n$ which does not vanish in $|z| < k,\ k \leq 1$, then

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1}|P(z)|, \tag{5}$$

provided $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, where $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$. As is easy to see that (4) and (5) become equalities if $P(z) = z^n + k^n$, one would expect that if we exclude the class of polynomials having all zeros on $|z| = k$, then it may be possible to improve the bounds in (4) and (5). In this direction, it was shown by Govil [8] that if $P(z)$ is a polynomial of degree $n$ having all its zeros in $|z| \leq k,\ k \geq 1$, then

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{1+k^n}\left\{\max_{|z|=1}|P(z)| + \min_{|z|=k}|P(z)|\right\}, \tag{6}$$

whereas, if $P(z) \neq 0$ in $|z| < k,\ k \leq 1$, then Aziz and Ahmad [1] showed that

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{1+k^n}\left\{\max_{|z|=1}|P(z)| - \min_{|z|=k}|P(z)|\right\}, \tag{7}$$

provided $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, where $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$. It is topical in the geometric function theory to study the extremal problems of functions of a complex variable and generalizing the classical polynomial inequalities in various directions. The one such generalization is moving from the domain of ordinary derivative to the polar derivative of polynomial. Let us remind that the polar derivative of a polynomial $P(z)$ of degree $n$ with respect to point $\alpha \in \mathbb{C}$ (see [14]) is defined as

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z).$$

Note that $D_\alpha P(z)$ is a polynomial of degree at most $n-1$ and it generalizes the ordinary derivative in the following sense:

$$\lim_{\alpha\to\infty}\left\{\frac{D_\alpha P(z)}{\alpha}\right\} := P'(z),$$

uniformly with respect to $z$ for $|z| \le R,\ R > 0$.
For more information on the polar derivative of polynomials, one can consult the comprehensive books of Marden [14], Milovanonić et al. [15] or Rahman and Schmeisser [24]. The extension of inequalities from ordinary derivative to polar derivative of complex polynomials is a widely studied topic, and for some of the papers in this direction, we refer to a recently published book chapter by Gardner et al. [5] (see also [9], [11], [12], [16]-[23], [26], [27]). In 1998, Aziz and Rather [2] established the polar derivative generalization of (4) by proving that if $P(z)$ is a polynomial of degree $n$ having all its zeros in $|z| \le k,\ k \ge 1$, then for every complex number $\alpha$ with $|\alpha| \ge k$,

$$\max_{|z|=1}|D_\alpha P(z)| \ge n\left(\frac{|\alpha|-k}{1+k^n}\right)\max_{|z|=1}|P(z)|, \tag{8}$$

whereas, the corresponding polar derivative analogue of (5) was recently given by Mir and Breaz [21]. They proved that if $P(z)$ is a polynomial of degree $n$ and $P(z) \neq 0$ in $|z| < k,\ k \le 1$, then for every complex number $\alpha$ with $|\alpha| \ge 1$,

$$\max_{|z|=1}|D_\alpha P(z)| \le n\left(\frac{|\alpha|+k^n}{1+k^n}\right)\max_{|z|=1}|P(z)|, \tag{9}$$

provided $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, where $Q(z) = z^n\overline{P\left(\frac{1}{\bar z}\right)}$. Aziz and Rather [3] further generalized (8) by using a parameter $\beta$ and established that if $P(z)$ is a polynomial of degree $n$ having all its zeros in $|z| \le k,\ k \ge 1$ and $m = \min_{|z|=k}|P(z)|$, then for every complex numbers $\alpha,\ \beta$ with $|\alpha| \ge k$ and $|\beta| \le 1$,

$$\max_{|z|=1}|D_\alpha P(z) + \beta mn| \ge n\left(\frac{|\alpha|-k}{1+k^n}\right)\left\{\max_{|z|=1}|P(z)| + |\beta|m\right\}, \tag{10}$$

whereas, if $P(z) \neq 0$ in $|z| < k,\ k \le 1$, then Mir and Breaz [21] established the following refinement of (9):

$$\max_{|z|=1}|D_\alpha P(z)| \le \frac{n}{1+k^n}\left\{(|\alpha|+k^n)\max_{|z|=1}|P(z)| - (|\alpha|-1)|\min_{|z|=k}|P(z)|\right\}, \tag{11}$$

provided $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, where $Q(z) = z^n\overline{P\left(\frac{1}{\bar z}\right)}$. It is easy to see that the inequalities (10) and (11) sharpen the inequalities (8) and (9) respectively, but both have a drawback that if there is a zero of $P(z)$ on $|z| = k$, then $\min_{|z|=k}|P(z)| = 0$, and so the inequalities (10) and (11) fail to give any improvement over (8) and (9) respectively. Therefore, it is quite natural to ask now, is it possible to obtain better bounds for the polynomial $P(z) = \sum_{v=0}^{n} a_v z^v$ under various restrictions on its zeros, where not all coefficients $a_0, a_1, ...a_n$, are zero and is more informative than the ones given in (8) and (9). Motivated by this, the authors are interested to establish some improved bounds of Bersntein and Turán-type for the derivative and polar derivative of a polynomial. The obtained results produce various refinements of the inequalities (4)-(11) and other related inequalities.

## 2. Main results

In this section, we state our main results. Their proofs are given in the next section. We begin by proving the following result involving the polar derivative of a polynomial having all its zeros in $|z| \leq k,\ k \geq 1$. The obtained result provides a sharpening of (10) and related inequalities.

**Theorem 1.** If $P(z) = \sum_{v=0}^{n} a_v z^v$, is a polynomial of degree $n$ having all its zeros in $|z| \leq k,\ k \geq 1$, then for every complex numbers $\alpha,\ \beta$ with $|\alpha| \geq k,\ |\beta| \leq 1$,

$$\max_{|z|=1} |D_\alpha P(z) + \beta mn| \geq n\left(\frac{|\alpha| - k}{1 + k^n}\right)\left(1 + \frac{L}{n}\right)\left(1 + \frac{M}{2}\right)\left(\max_{|z|=1} |P(z)| + |\beta| m\right), \tag{12}$$

where

$$L = \frac{|a_n|k^n - |a_0| - |\beta|m}{|a_n|k^n + |a_0| + |\beta|m},\ M = \frac{(|a_n|k^n - |a_0| - |\beta|m)(k-1)}{|a_n|k^n + |a_0|k + |\beta|mk}$$

and $m = \min_{|z|=k} |P(z)|$.

If we divide both sides of (12) by $|\alpha|$ and let $|\alpha| \to \infty$, we get the following result.

**Corollary 1.** If $P(z) = \sum_{v=0}^{n} a_v z^v$, is a polynomial of degree $n$ having all its zeros in $|z| \leq k,\ k \geq 1$, then for every complex number $\beta$ with $|\beta| \leq 1$,

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1 + k^n}\left(1 + \frac{L}{n}\right)\left(1 + \frac{M}{2}\right)\left(\max_{|z|=1} |P(z)| + |\beta| m\right), \tag{13}$$

where $L,\ M$ and $m$ are as defined in Theorem 1. Equality in (13) holds for $P(z) = z^n + k^n$.

**Remark 1.** Since $P(z) = \sum_{v=0}^{n} a_v z^v$ has all its zeros in $|z| \leq k,\ k \geq 1$, and if $z_1, z_2, ..., z_n$, are the zeros of $P(z)$, then

$$\left|\frac{a_0}{a_n}\right| = |z_1 z_2 ... z_n| \leq k^n. \tag{14}$$

Here, we show that for $|\beta| \leq 1$,

$$k^n |a_n| \geq |a_0| + |\beta| m, \tag{15}$$

where $m$ is as defined in Theorem 1.

We can assume without loss of generality that $P(z)$ has no zeros on $|z| = k$, for otherwise (15) holds trivially by (14). As in the proof of Theorem 1 (given in next section), we have for every $\beta$ with $|\beta| \leq 1$, the polynomial

$$P(z) + \beta m = (a_0 + \beta m) + \sum_{v=1}^{n} a_v z^v$$

does not vanish in $|z| > k,\ k \geq 1$, hence

$$\left|\frac{a_0 + \beta m}{a_n}\right| \leq k^n. \tag{16}$$

If in (16), we choose the argument of $\beta$ suitably, so that

$$|a_0 + \beta m| = |a_0| + |\beta| m,$$

we get (15).

**Remark 2.** It is clear that, in general for any polynomial $P(z) = a_0 + a_1z + a_2z^2 + ... + a_nz^n$, of degree $n$ having all its zeros in $|z| \leq k$, $k \geq 1$, the inequalities (12) and (13) by virtue of Remark 1, give bounds that are sharper than the bounds obtained from the inequalities (10) and (6) respectively. One can also observe that for $k > 1$, the inequality (12) improves the inequality (10) considerably when $|a_n|k^n - |a_0| - |\beta|m \neq 0$.

**Remark 3.** It may be remarked here that for $\beta \neq 0$, Corollary 1 also improves a result of Govil and Kumar ([9], Corollary 1.3), excepting the case when $P(z)$ has all its zeros on $|z| = k$.

As an application of Corollary 1, we now prove the following result which deals with a subclass of polynomials having no zeros in $|z| < k$, $k \leq 1$. The obtained result and its corollary refines (11), (7) and several other known results in this direction.

**Theorem 2.** Let $P(z) = \sum_{v=0}^{n} a_vz^v$ be a polynomial of degree $n$ having no zeros in $|z| < k$, $k \leq 1$, and $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then for every complex numbers $\alpha$, $\beta$ with $|\alpha| \geq 1$ and $|\beta| \leq 1$, we have

$$\begin{aligned}\max_{|z|=1} &|D_\alpha P(z)| \\ &\leq \frac{n}{1+k^n}\left\{|\alpha|(1+k^n) - k^n(|\alpha|-1)\left(1+\frac{X}{n}\right)\left(1+\frac{Y}{2}\right)\right\}\max_{|z|=1}|P(z)| \\ &- \frac{n(|\alpha|-1)}{1+k^n}\left(1+\frac{X}{n}\right)\left(1+\frac{Y}{2}\right)|\beta|m, \end{aligned} \tag{17}$$

where

$$X = \frac{|a_0| - k^n|a_n| - |\beta|m}{|a_0| + k^n|a_n| + |\beta|m}, \; Y = \frac{(|a_0| - k^n|a_n| - |\beta|m)(1-k)}{k|a_0| + k^n|a_n| + |\beta|m}$$

and $m = \min_{|z|=k}|P(z)|$.

Equality in (17) holds for $P(z) = z^n + k^n$, with real $\alpha \geq 1$.

Dividing both sides of (17) by $|\alpha|$ and let $|\alpha| \to \infty$, we get the following result.

**Corollary 2.** Let $P(z) = \sum_{v=0}^{n} a_vz^v$ be a polynomial of degree $n$ having no zeros in $|z| < k$, $k \leq 1$, and $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then for every complex number $\beta$ with $|\beta| \leq 1$,

$$\begin{aligned}\max_{|z|=1}|P'(z)| \leq &\frac{n}{1+k^n}\left\{(1+k^n) - k^n\left(1+\frac{X}{n}\right)\left(1+\frac{Y}{2}\right)\right\}\max_{|z|=1}|P(z)| \\ &- \frac{n}{1+k^n}\left(1+\frac{X}{n}\right)\left(1+\frac{Y}{2}\right)|\beta|m, \end{aligned} \tag{18}$$

where $X$, $Y$ and $m$ are as defined in Theorem 2. Equality in (18) holds for $P(z) = z^n + k^n$.

**Remark 4.** Since $P(z) \neq 0$ in $|z| < k$, $k \leq 1$, the polynomial $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$ of degree $n$ has all its zeros in $|z| \leq \frac{1}{k}$, $\frac{1}{k} \geq 1$. On applying inequality (15) of Remark 1 to $Q(z)$, we get for $|\beta| \leq 1$,

$$\frac{|a_0|}{k^n} \geq |a_n| + |\beta|m', \tag{19}$$

where

$$m' = \min_{|z|=\frac{1}{k}}|Q(z)| = \min_{|z|=\frac{1}{k}}\left|z^n\overline{P\left(\frac{1}{\bar{z}}\right)}\right| = \frac{1}{k^n}\min_{|z|=k}|P(z)| = \frac{m}{k^n}.$$

The above inequality (19) is equivalent to

$$|a_0| \geq k^n|a_n| + |\beta|m,$$

which further implies that $X \geq 0,\ Y \geq 0$ for $k \leq 1$. Thus the bounds obtained in (17) and (18) are sharper than the bounds obtained from the inequalities (11) and (7) respectively.
As remarked before, Theorem 1 in general gives the bound sharper than the bound obtained from (11), in some cases the improvement can be considerably significant, and this we show by means of the following example.
**Example 1.** Let $P(z) = z^4 - 2z^3 + 4z - 4$, with all zeros $\{-\sqrt{2}, \sqrt{2}, 1-i, 1+i\}$ on the circle $|z| = \sqrt{2}$. For this polynomial, we find that $\max_{|z|=1}|P(z)| = 9.614$ (approximately) and $m = \min_{|z|=k}|P(z)| = (k^2-2)[(k-1)^2+1]$, $k \geq \sqrt{2}$. If we take $k = 2$, so that $P(z)$ has all its zeros in $|z| \leq k = 2$. Taking $\alpha = 3 + i\sqrt{7}$, so that $|\alpha| = 4$. By (10), we obtain for $\beta = 1$,

$$\max_{|z|=1}|D_\alpha P(z) + mn| \geq 6.404,$$

while as Theorem 1 yields

$$\max_{|z|=1}|D_\alpha P(z) + mn| \geq 7.80,$$

showing that (12) gives a considerable improvement over the bound obtained from (10).
In the same way Theorem 2 in general provides much better information than (11) regarding $\max_{|z|=1}|D_\alpha P(z)|$. We illustrate this by means of the following example.
**Example 2.** Consider $P(z) = z^3 - z^2 + z - 1$, which is a polynomial of degree 3. Clearly $P(z)$ has all its zeros $\{1, i, -i\}$ which all lie on $|z| = 1$. We take $k = \frac{1}{2}$, so that $P(z) \neq 0$ in $|z| < k = \frac{1}{2}$. We find numerically that $\max_{|z|=1}|P(z)| = 4$ and $\min_{|z|=\frac{1}{2}}|P(z)| = \frac{5}{8}$. Taking $\alpha \in \mathbb{C}$ with $|\alpha| = 3$, we obtain by (11), that

$$\max_{|z|=1}|D_\alpha P(z)| \leq 30,$$

while as Theorem 2 yields

$$\max_{|z|=1}|D_\alpha P(z)| \leq 26.39,$$

showing that (17) gives a considerable improvement over the bound obtained from (11).

It is easy to see that Theorem 2 also provides a refinement of the following result due to Kumar and Dhankhar ([12], Theorem 4).

**Theorem A.** Let $P(z) = \sum_{v=0}^{n} a_v z^v$ be a polynomial of degree $n$ having no zeros in $|z| < k$, $k \leq 1$, and $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then for every complex number $\alpha$ with $|\alpha| \geq 1$, we have

$$\max_{|z|=1}|D_\alpha P(z)| \leq \left(\frac{n(|\alpha| + k^n)}{1 + k^n} - \frac{n(|\alpha| - 1)k^n(|a_0| - k^n|a_n|)(1-k)}{2(1+k^n)(k|a_0| + k^n|a_n|)}\right)\max_{|z|=1}|P(z)|.$$

## 3. Auxiliary results

In order to prove our main results, we need the following lemmas.

**Lemma 1.** If $P(z)$ is a polynomial of degree $n$, and $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$, then on $|z| = 1$,

$$|P'(z)| + |Q'(z)| \leq n\max_{|z|=1}|P(z)|.$$

The above lemma is due to Govil and Rahman [10].

**Lemma 2.** If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree $n$ having all its zeros in $|z| \leq k$, $k \geq 1$, then for every complex number $\alpha$ with $|\alpha| \geq k$, we have

$$\max_{|z|=1} |D_\alpha P(z)| \geq n\left(\frac{|\alpha|-k}{1+k^n}\right)\left(1+\frac{A}{n}\right)\left(1+\frac{B}{2}\right)\max_{|z|=1}|P(z)|,$$

where $A = \dfrac{|a_n|k^n - |a_0|}{|a_n|k^n + |a_0|}$ and $B = \dfrac{(|a_n|k^n - |a_0|)(k-1)}{|a_n|k^n + k|a_0|}$.

The above lemma is due to Mir and Breaz [21].

## 4. Proofs of the main results

**Proof of Theorem 1.** Recall that $P(z) = \sum_{v=0}^{n} a_v z^v$ has all its zeros in $|z| \leq k,\ k \geq 1$. If $P(z)$ has a zero on $|z| = k$, then $m = \min_{|z|=k} |P(z)| = 0$, and the result follows from Lemma 2 in this case. Henceforth, we suppose that $P(z)$ has all its zeros in $|z| < k,\ k \geq 1$. Let $H(z) = P(kz)$ and $G(z) = z^n\overline{H(\frac{1}{\bar{z}})} = z^n\overline{P(\frac{k}{\bar{z}})}$. Then all the zeros of $G(z)$ lie in $|z| > 1$ and $|H(z)| = |G(z)|$ for $|z| = 1$. This gives

$$\left|z^n\overline{P\left(\frac{k}{\bar{z}}\right)}\right| = |P(kz)| \geq m \text{ for } |z| = 1.$$

It follows by the Minimum Modulus Principle, that

$$\left|z^n\overline{P\left(\frac{k}{\bar{z}}\right)}\right| \geq m \ \text{ for } \ |z| \leq 1.$$

Replacing $z$ by $\frac{1}{\bar{z}}$, it implies that

$$|P(kz)| \geq m|z|^n \ \text{ for } \ |z| \geq 1,$$

or

$$|P(z)| \geq m\left|\frac{z}{k}\right|^n \ \text{ for } \ |z| \geq k. \tag{20}$$

Now, consider the polynomial $F(z) = P(z) + \beta m$, where $\beta$ is a complex number with $|\beta| \leq 1$, then all the zeros of $F(z)$ lie in $|z| \leq k$. Because, if for some $z = z_1$ with $|z_1| > k$, we have $F(z_1) = P(z_1) + \beta m = 0$, then

$$|P(z_1)| = |\beta m| \leq m < m\left|\frac{z_1}{k}\right|^n,$$

which contradicts (20). Hence, for every complex number $\beta$ with $|\beta| \leq 1$, the polynomial $F(z) = P(z) + \beta m = (a_0 + \beta m) + \sum_{v=1}^{n} a_v z^v$, has all its zeros in $|z| \leq k$, where $k \geq 1$. Applying Lemma 2 to the polynomial $F(z)$, we get for every complex number $\alpha$ with $|\alpha| \geq k$ and $|z| = 1$,

$$\begin{aligned}\max_{|z|=1} |D_\alpha(P(z) + \beta m)| \geq \frac{1}{2}\left(\frac{|\alpha|-k}{1+k^n}\right)&\left(n + \frac{k^n|a_n| - |a_0+\beta m|}{k^n|a_n| + |a_0+\beta m|}\right)\\ &\times\left(2 + \frac{(k^n|a_n| - |a_0+\beta m|)(k-1)}{k^n|a_n| + k|a_0+\beta m|}\right)|P(z) + \beta m|.\end{aligned} \tag{21}$$

For every $\beta \in \mathbb{C}$, we have

$$|a_0 + \beta m| \leq |a_0| + |\beta| m,$$

and since the functions

$$x \to \frac{k^n|a_n| - x}{k^n|a_n| + x} \text{ and } x \to \frac{(k^n|a_n| - x)(k-1)}{k^n|a_n| + kx}, \ (x \geq 0)$$

are both non-increasing for every $k \geq 1$, it follows from (21) that for every $|\beta| \leq 1$ and $|z| = 1$,

$$\max_{|z|=1} |D_\alpha P(z) + \beta mn| \geq \frac{1}{2}\left(\frac{|\alpha| - k}{1 + k^n}\right)\left(n + \frac{k^n|a_n| - |a_0| - |\beta|m}{k^n|a_n| + |a_0| + |\beta|m}\right) \times \left(2 + \frac{(k^n|a_n| - |a_0| - |\beta|m)(k-1)}{k^n|a_n| + k|a_0| + |\beta|km}\right)|P(z) + \beta m|. \tag{22}$$

Choosing the argument of $\beta$ on the right hand side of (22) such that

$$|P(z) + \beta m| = |P(z)| + |\beta|m,$$

we obtain from (22) for $|z| = 1$, that

$$\max_{|z|=1} |D_\alpha P(z) + \beta mn| \geq \frac{1}{2}\left(\frac{|\alpha| - k}{1 + k^n}\right)\left(n + \frac{k^n|a_n| - |a_0| - |\beta|m}{k^n|a_n| + |a_0| + |\beta|m}\right) \times \left(2 + \frac{(k^n|a_n| - |a_0| - |\beta|m)(k-1)}{k^n|a_n| + k|a_0| + |\beta|km}\right)(|P(z)| + |\beta|m),$$

which in particular gives (12) and this completes the proof of Theorem 1.

**Proof of Theorem 2.** Let $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$. Since $P(z) \neq 0$ in $|z| < k$, $k \leq 1$, the polynomial $Q(z)$ of degree $n$ has all its zeros in $|z| \leq \frac{1}{k}$, $\frac{1}{k} \geq 1$. On applying Corollary 1 to $Q(z)$, and using the fact that $\max_{|z|=1} |Q(z)| = \max_{|z|=1} |P(z)|$, we get for $|\beta| \leq 1$,

$$\max_{|z|=1} |Q'(z)| \geq \frac{n}{1 + \frac{1}{k^n}}\left(1 + \frac{L'}{n}\right)\left(1 + \frac{M'}{2}\right)\left(\max_{|z|=1} |P(z)| + |\beta|m'\right), \tag{23}$$

where

$$m' = \min_{|z|=\frac{1}{k}} |Q(z)| = \min_{|z|=\frac{1}{k}} \left|z^n \overline{P\left(\frac{1}{\bar{z}}\right)}\right| = \frac{1}{k^n} \min_{|z|=k} |P(z)| = \frac{m}{k^n},$$

$$L' = \frac{\frac{|a_0|}{k^n} - |a_n| - |\beta|\frac{m}{k^n}}{\frac{|a_0|}{k^n} + |a_n| + |\beta|\frac{m}{k^n}} = \frac{|a_0| - k^n|a_n| - |\beta|m}{|a_0| + k^n|a_n| + |\beta|m} = X$$

$$\text{and } M' = \frac{\left(\frac{|a_0|}{k^n} - |a_n| - |\beta|\frac{m}{k^n}\right)\left(\frac{1}{k} - 1\right)}{\frac{|a_0|}{k^n} + \frac{|a_n|}{k} + |\beta|\frac{m}{k^{n+1}}} = \frac{(|a_0| - k^n|a_n| - |\beta|m)(1-k)}{k|a_0| + k^n|a_n| + |\beta|m} = Y.$$

The above inequality (23) is equivalent to

$$\max_{|z|=1} |Q'(z)| \geq \frac{nk^n}{1 + k^n}\left(1 + \frac{X}{n}\right)\left(1 + \frac{Y}{2}\right)\left(\max_{|z|=1} |P(z)| + |\beta|\frac{m}{k^n}\right). \tag{24}$$

Since $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, we have

$$\max_{|z|=1} (|P'(z)| + |Q'(z)|) = \max_{|z|=1} |P'(z)| + \max_{|z|=1} |Q'(z)|. \tag{25}$$

On combining (24), (25) and Lemma 1, we get

$$n \max_{|z|=1} |P(z)| \geq \max_{|z|=1} |P'(z)| + \frac{nk^n}{1 + k^n}\left(1 + \frac{X}{n}\right)\left(1 + \frac{Y}{2}\right)\left(\max_{|z|=1} |P(z)| + |\beta|\frac{m}{k^n}\right),$$

which after simplification gives

$$\max_{|z|=1}|P'(z)| \le n\left\{1-\frac{k^n}{1+k^n}\left(1+\frac{X}{n}\right)\left(1+\frac{Y}{2}\right)\right\}\max_{|z|=1}|P(z)| \\ -\frac{n}{1+k^n}\left(1+\frac{X}{n}\right)\left(1+\frac{Y}{2}\right)|\beta|m. \tag{26}$$

Since $Q(z)=z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$, it is easy to verify that for $|z|=1$,

$$|Q'(z)| = |nP(z)-zP'(z)|. \tag{27}$$

Also, for any complex number $\alpha$ with $|\alpha|\ge 1$, we have by (27) and $|z|=1$, that

$$\begin{aligned}|D_\alpha P(z)| &= |nP(z)+(\alpha-z)P'(z)| \\ &\le |nP(z)-zP'(z)|+|\alpha||P'(z)| \\ &= |Q'(z)|+|\alpha||P'(z)| \\ &= |Q'(z)|+|P'(z)|-|P'(z)|+|\alpha||P'(z)| \\ &\le n\max_{|z|=1}|P(z)|+(|\alpha|-1)|P'(z)| \quad \text{(by Lemma 1)} \\ &\le n\max_{|z|=1}|P(z)|+(|\alpha|-1)\max_{|z|=1}|P'(z)|. \end{aligned} \tag{28}$$

Inequality (28) in conjunction with inequality (26), gives for $|\alpha|\ge 1$ and $|z|=1$, that

$$\begin{aligned}&\max_{|z|=1}|D_\alpha P(z)| \\ &\le n\max_{|z|=1}|P(z)|+n(|\alpha|-1)\left\{1-\frac{k^n}{1+k^n}\left(1+\frac{X}{n}\right)\left(1+\frac{Y}{2}\right)\right\}\max_{|z|=1}|P(z)| \\ &-\frac{n(|\alpha|-1)}{1+k^n}\left(1+\frac{X}{n}\right)\left(1+\frac{Y}{2}\right)|\beta|m,\end{aligned}$$

from which we can obtain (17). This completes the proof of Theorem 2.

**Conclusion:** Studying the extremal problems of functions of a complex variable and gereralizing the classical polynomial inequalities is topical in geometric function theory. In the past few years, a series of papers related to the Bernstein and Turán-type inequalities has been published and significant advances have been achieved. This type of inequalities are of interest both in mathematics and in the application areas such as physical systems. In this paper, we continue the study of this type of inequalities for a certain class of polynomials, following up on a study started by various authors in the recent past. More precisely, the authors establish for a certain class of polynomials some new bounds for the derivative and polar derivative of a polynomial on the unit disk while taking into account the placement of the zeros and extremal coefficients of the underlying polynomial.

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