Filomat 36:16 (2022), 5615–5629
https://doi.org/10.2298/FIL2216615G

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Gerstewitz Nonlinear Scalar Functional and the Applications in Vector Optimization Problems

Ying Gao[a], Liping Tang[b]

[a]School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, China
[b]National Center for Applied Mathematics in Chongqing, Chongqing Normal University, Chongqing 401331, China

Abstract. In this paper, we study the properties of Gerstewitz nonlinear scalar functional with respect to co-radiant set and radiant set in real linear space. With the help of nonconvex separation theorem with respect to co-radiant set, we first obtain that Gerstewitz nonlinear scalar functional is a special co-radiant(radiant) functional when the corresponding set is a co-radiant(radiant) set. Based on the subadditivity property of this functional with respect to the convex co-radiant set, we calculate its Fenchel(approximate) subdifferential. As the applications, we derive the optimality conditions for the approximate solutions with respect to co-radiant set of vector optimization problem. We also state that this special functional can be used as a coherent measure in the portfolio problem.

1. Introduction

Let Y be a real linear space, the classical Minkowski functional of an absorbing set $A \subset Y$ is defined as

$$P_A(y) = \inf\{t > 0 : y \in tA\}.$$

The convexity of A can guarantee the sublinearity property of this functional. If A is a star set , it also has the following separation result.

$$\{y \in Y : P_A(y) < 1\} \subset A \subset \{y \in Y : P_A(y) \leq 1\}.$$

If Y is a topological vector space and $A \subset Y$ is a closed set with nonempty interiors, then the closure of A equal to the right set, and the interior of A equal to the left set. Especially, when Y is a topological vector space, $D \subset Y$ is a closed convex cone with nonempty interior(that is, the topology interior $\mathrm{int}D \neq \emptyset$). For

2020 *Mathematics Subject Classification*. 90C29; 90C30
Keywords. Gerstewitz nonlinear scalar functional, Co-radiant function, Radiant function, Vector optimization problem, Approximate solutions, Lagrange conditions
Received: 17 October 2021; Revised: 02 May 2022; Accepted: 13 August 2022
Communicated by Dijana Mosić
Corresponding author: Ying Gao
Research supported by the National Natural Science Foundation of China (11991024, 11971084, 12171063, 12171060, 11771064), Foundation of Chongqing Normal University (Grant no. 22XLB006), the Natural Science Foundation of Chongqing (cstc2020jcyj-msxmX0053),Chongqing Science and Technology Research Key Project Nos. KJZD-K202001104,Chongqing University Innovation Research Group Project (Nos. CXQT20014); Chongqing Returned Overseas Students' Entrepreneurship and Innovation Support Program (Nos. cx2020096)
Email addresses: gaoyingimu@163.com (Ying Gao), tanglipings@163.com (Liping Tang)

$e \in \mathrm{int}D$, $A = \{e\} - D$ is a absorbing convex set and $P_A(y) = \inf\{t > 0 : y \in te - D\}$. From the separation results, it is easy to obtain that

$$P_A(y) \leq \lambda \Longleftrightarrow y \in \lambda e - D, \ \ \forall \lambda > 0.$$

Based on the Minkowski functional and the above properties, Gerstewitz and Iwanow [1] first introduced the following Gerstewitz nonlinear scalar functional

$$\varphi_{q,D}(y) = \inf\{t \in \mathbb{R} : y \in tq - D\},$$

where $D \subset Y$ be a nonempty subset, $q \in Y$ and $\inf \emptyset = +\infty$.

Since then, many scholars began to study the properties of Gerstewitz nonlinear scalar functional(see $[2-11]$). Gerth *et al* [2] gave the separation theorem under non-convex sets in linear topological spaces and obtain the scalarazation results for weak (proper) efficient solutions. Gpfert *et al* [3] summarized many properties and non-convex separation theorems for Gerstewitz non-linear scalar functions. Tammer [10] introduced the subdifferential and conjugate functions of nonlinear scalar functions in the convex sense, and points out that this function can theoretically be used as a risk measure function in economy. At then, this kind of nonlinear scalar function was applied to the nonlinear scalarization methods for vector optimization problems($[12-23]$). For example, Gutiérrez [12] derived the sufficient and necessary conditions for approximate effective solutions by nonlinear scalar functions. Bao [18] established new necessary conditions for pareto minimal in the case where the ordering cone has an empty interior. In the meantime, the differential form of nonlinear scalar function is also concerned. Dutta [22] gave the optimality conditions for vector optimization problems by using the nonconvex separation method and the concept of subdifferential, and gives the concrete subdifferential of φ which is a proper convex function.

Inspired by the above literature, we study the properties of the nonlinear scalar function based on co-radiant set and its approximate subdifferential form, and give the optimality conditions of the optimization problem with approximate normal cone. Finally, it is pointed out that the scalar function can be used for risk measurement. The structure of this paper is as follows: Section 2, the basic notations, concepts and results will be introduced. In section 3, the main properties of the nonlinear scalar functional with respect to the co-radiant set are studied in linear spaces, its subdifferential(approximate subdifferential) is also investigated in this part. Through the functional, in section 4 we characterize $\varepsilon-$efficient solutions, involving its necessary conditions, mainly by the approximate normal cone and the subdifferential of the scalarization function in Asplund spaces. The optimality conditions of approximate solutions in vector optimization problems are obtained by the "calculus rules". In section 5, We briefly introduce the theoretical application of scalar function in risk measurement.

2. Preliminaries

Throughout this paper, Y denotes a real linear space. Y^* is the algebraic dual space of Y, which is defined as the set of all linear mappings from Y into R. A function $\langle x, x^*\rangle = \langle x^*, x\rangle$ is a bilinear functional on $Y \times Y^*$. Let A be a nonempty subset of Y, the cone generated by A is the set cone$A := \bigcup_{\lambda \geq 0} \lambda A$. A is said to be a cone if $A = $ cone$A$. A is said to a radiant set if for any $x \in A$ and $\lambda \in (0, 1)$, $\lambda x \in A$. It is said to a co-radiant set if for any $x \in A$ and $\lambda \geq 1$, $\lambda x \in A$. A is proper if it is nonempty and $A \neq Y$, and it is pointed if $A \cap (-A) \subseteq \{0\}$. The positive polar cone of A denoted by

$$A^+ = \{d \in Y^* : \langle d, c\rangle \geqslant 0, \forall c \in A\}.$$

The strict positive polar cone of A denoted by

$$A^{s+} = \{d \in Y^* : \langle d, c\rangle > 0, \forall c \in A \backslash \{0\}\}.$$

Let A be a convex set and $\varepsilon \geq 0$, the normal cone of A and ε-normal cone are defined as

$$N(\bar{x}, A) = \{x^* \in Y^* : \langle x^*, x - \bar{x}\rangle \leq 0, \forall x \in A\}.$$

$$N_\varepsilon(\bar{x}, A) = \{x^* \in Y^* : \langle x^*, x - \bar{x}\rangle \leq \varepsilon, \ \ \forall x \in A\}.$$

Let aff(A) and span(A) denote the affine hull and linear hull of A, respectively. core(A), icr(A) and vcl(A) stand for the algebraic interior, the relative algebraic interior and the vector closure of A, which are defined as

$$\begin{aligned}
\mathrm{core}(A) &= \{y \in Y : \forall v \in Y, \exists \lambda > 0 \ s.t. \ y + [0, \lambda]v \subseteq A\},\\
\mathrm{icr}(A) &= \{y \in A : \forall h \in \mathrm{aff}A - y, \exists \delta > 0, \ \forall t \in [0, \delta], y + th \in A\},\\
\mathrm{vcl}(A) &= \{y \in Y : \exists v \in Y, \ s.t. \ \forall \lambda > 0, \exists \lambda' \in [0, \lambda], y + \lambda' v \in A\}\\
&= \{y \in Y : \exists v \in Y, \lambda_n > 0, \lambda_n \to 0 \ s.t. \ y + \lambda_n v \in A, \forall n \in N\}.
\end{aligned}$$

Moreover, for a given $q \in Y$, we denote vcl_qA the $q-$vector of closure of A (see [12]), which is defined as

$$\mathrm{vcl}_q A = \{y \in Y : \forall \lambda > 0, \exists \lambda' \in [0, \lambda], y + \lambda' q \in A\}.$$

Let A be a nonempty set of Y, $q \in Y\backslash\{0\}$. The nonlinear scalar functional $\varphi_{q,A}(y) : Y \to \bar{R}$ is defined as

$$\varphi_{q,A}(y) = \begin{cases} +\infty & if \ y \notin \mathbb{R}q - A \\ inf\{t \in \mathbb{R} : y \in tq - A\} & otherwise. \end{cases} \tag{1}$$

For a co-radiant set $C \subseteq Y$ and $\varepsilon > 0$, the sets $C(\varepsilon)$ and $C(0)$ are defined as: $C(\varepsilon) = \varepsilon C$ and $C(0) = \bigcup_{\varepsilon>0} C(\varepsilon)$.

The following lemma states some properties of the co-radiant set.

Lemma 2.1. *Let $C \subseteq Y$ be a proper co-radiant pointed set with nonempty relative algebraic interior, then*
(i) *$C(\varepsilon)$ is a proper pointed co-radiant set, $\forall \varepsilon > 0$.*
(ii) *$C(\varepsilon_2) \subseteq C(\varepsilon_1)$, $\forall \varepsilon_1, \varepsilon_2 > 0, 0 < \varepsilon_1 \leq \varepsilon_2$.*
(iii) *If C is a convex set, then for any $\varepsilon \geq 0$, $C(0) + C(\varepsilon) \subseteq C(\varepsilon)$ and $\mathrm{icr}C(0) + \mathrm{vcl}C(\varepsilon) \subseteq \mathrm{vcl}C(\varepsilon)$.*

Proof. The proofs of (i), (ii) and $C(0) + C(\varepsilon) \subseteq C(\varepsilon)$ were given in [23].
From $C(0) + C(\varepsilon) \subseteq C(\varepsilon)$ and the definition of the vector closure, we can easy to prove that

$$C(0) + \mathrm{vcl}C(\varepsilon) \subseteq \mathrm{vcl}C(\varepsilon).$$

Which implies

$$\mathrm{icr}C(0) + \mathrm{vcl}C(\varepsilon) \subseteq \mathrm{vcl}C(\varepsilon).$$

□

Recently, Gutiérrez *et al*. established the following separation theorem.

Lemma 2.2. (*Theorem* 4, [12]) *If $q \in Y \setminus \{0\}$ and C is a proper subset of Y, then the following equations hold:*
(i) *$\{y \in Y : \ \varphi_{q,C}(y) < 0\} = (-\infty, 0)q - \mathrm{vcl}_q C$.*
(ii) *$\{y \in Y : \ \varphi_{q,C}(y) \leq 0\} = (-\infty, 0]q - \mathrm{vcl}_q C$.*

3. The properties of Gerstewitz nonlinear scalar functional in real linear spaces

In this section, we investigate the properties of Gerstewitz nonlinear scalar functional with respect to co-radiant set and radiant set. First, we derive several properties of the co-radiant set.

Lemma 3.1. *Let C be a co-radiant set with nonempty relative algebraic interior, then we have the following properties.*
(i) *For any $y \in C$,* aff(C) = aff(C) − y *or equivalently* aff(C) = span(C) = aff($C - C$) = span($C - C$) = span(vcl$C$ − vcl$C$).
(ii) icr(C) ⊆ icr$C$(0).
(iii) *If C is a convex set and $q \in \mathrm{icr}(C)$, then $\mathrm{vcl}_q C = \mathrm{vcl}(C)$.*
Moreover,

$$(-\infty, 0]q - \mathrm{vcl}_q(C) = -\mathrm{vcl}(C), \tag{2}$$

$$(-\infty, 0)q - \mathrm{vcl}_q(C) \subseteq -\mathrm{icr}(C). \tag{3}$$

Proof. (i) We only need to prove $0 \in \text{aff}C$. Indeed, take $y \in C$, then $3y \in C$, since C is a co-radiant set. Hence, $0 = \frac{3}{2}y + (-\frac{1}{2})3y \in \text{aff}C$.

And $\text{span}(C - C) = \text{span}(\text{vcl}C - \text{vcl}C)$ have been proved in [14].

(ii) We first prove $\text{aff}(C) = \text{aff}C(0)$. From (i), we have $C(0) \subseteq \text{aff}(C)$ and $\text{aff}C(0) \subseteq \text{aff}(C)$. The converse conclusion is obvious.

From this result, we can easy to see that $\text{icr}(C) \subseteq \text{icr}C(0)$.

(iii) From the definition, we only need to prove $\text{vcl}C \subseteq \text{vcl}_q C$. Take $y \in \text{vcl}C$, then there exist $v \in Y$ and $t_n \in R$ with $t_n > 0$ and $t_n \to 0$ such that $y + t_n v \in C$.

It is easy to see that $v \in \text{aff}(C)$. Since $q \in \text{icr}(C)$, for the vector v, there exists $\alpha > 0$ such that $q - \alpha v \in C$. Put $\bar{q} = q - \alpha v$, then

$$y + t_n v = y + t_n \frac{q - \bar{q}}{\alpha} \in C.$$

From Lemma 2.1(iii), we have

$$y + \frac{t_n}{\alpha} q \in \frac{t_n}{\alpha} \bar{q} + C \subseteq C(0) + C \subseteq C.$$

Note that $\frac{t_n}{\alpha} q \to 0$, we have $y \in \text{vcl}_q C$. Thus, $\text{vcl}(C) = \text{vcl}_q C$.

Moreover, since C is a convex co-radiant set and $q \in \text{icr}(C)$, , for any $\lambda > 0$, we have

$$\lambda q \in \lambda \text{icr}(C) = \text{icr}((\lambda C)) \subseteq \text{icr}((\lambda C)(0)) = \text{icr}(\mathrm{C}(0)).$$

From (ii) and Lemma 2.1(iii), we have

$$(-\infty, 0]q - \text{vcl}_q(C) = (-\infty, 0]q - \text{vcl}(C) \subseteq (-\text{icr}C(0) \cup \{0\}) - \text{vcl}(C) \subseteq -\text{vcl}(C).$$

And it is obvious that $-\text{vcl}(C) \subseteq (-\infty, 0]q - \text{vcl}(C)$, hence (2) holds.

Now we prove (3). Let $k_1 \in (-\infty, 0)q - \text{vcl}_q(C)$, we have $k_1 \in (-\infty, 0)q - \text{vcl}(C)$, then there exist $k' \in \text{vcl}_q(C) = \text{vcl}(C)$ and $\alpha > 0$ such that $k_1 = -\alpha q - k'$. Since $q \in \text{icr}(C)$, we have for any $h \in \text{aff}(C) - q$, there exists $\delta > 0$ such that $q + th \in C$ for any $t \in [0, \delta]$. Thus

$$-k_1 + \alpha th = \alpha q + k' + \alpha th = k' + \alpha(q + th) \subseteq \text{vcl}(C) + \alpha C \subseteq \text{vcl}C.$$

Which means that $k_1 \in -\text{icr}(\text{vcl}C) = -\text{icr}(C)$. □

Similar to the proof of Lemma 2.9 in Qiu [8], we can get the following nonconvex separation theorem.

Lemma 3.2. *Let $D \subseteq Y$ be a proper convex co-radiant set with nonempty relative algebraic interior and $q \in \text{icr}D$, then for any $\lambda \in R$ the following equations hold.*

(i) $\{y \in Y : \varphi_{q,D}(y) < \lambda\} = \lambda q - \text{icr}D$.

(ii) $\{y \in Y : \varphi_{q,D}(y) \leq \lambda\} = \lambda q - \text{vcl}D$.

(iii) $\{y \in Y : \varphi_{q,D}(y) = \lambda\} = \lambda q - \text{ibd}D$.

Remark 3.1. (i) *When D is a convex cone, Lemma 3.2 will reduce to Theorem 3.1 in [4].*

(ii) *Qiu [8] established the same result with Lemma 3.1*(iii), *when D is a convex cone and $q \in \text{core}(D)$(see Proposition 2.3). Notice that,* $\text{core}(D) \subseteq \text{icr}(D)$. *And for several convex set* $\text{icr}(D)$ *may be nonempty when* $\text{core}(D) = \emptyset$. *For example, if $Y = R^2$, consider $A = \{(x_1, x_2) \in R^2 : x_1 = 0, x_2 \geq 0\}$. Moreover, a cone is a co-radiant set, but the converse may not be true.*

(iii) *We have proved the condition* $(0, \infty)q + \text{vcl}(D) \subseteq \text{icr}(D)$ *in Lemma 3.1, it satisfies the assumption B in [7], we still can get the result in Lemma 3.2.*

Theorem 3.1. *Consider $\emptyset \neq A \subseteq Y$ and $q \in Y \setminus \{0\}$. We have the following properties.*

(i) *If* $\text{vcl}_q A$ *is a co-radiant set, then $\varphi_{q,A}$ is a co-radiant function, that is*

$$\varphi_{q,A}(\lambda y) \leq \lambda \varphi_{q,A}(y), \quad \forall y \in Y, \lambda \geq 1, \tag{4}$$

or equivalently,

$$\varphi_{q,A}(\lambda y) \geq \lambda\varphi_{q,A}(y), \quad \forall y \in Y, \lambda \in (0,1].$$

Moreover, if

$$\mathrm{vcl}_q(A) + (0,\infty)q \subseteq \mathrm{vcl}_q(A), \tag{5}$$

then the converse result of (i) *holds.*

(ii) $\varphi_{q,A}$ *is a co-radiant function if and only if the epigraph* epi$\varphi_{q,A}$ *is a co-radiant set.*

(iii) *If A is a convex co-radiant set, then $\varphi_{q,A}$ is a subadditive function.*

Proof. (i) Since $\varphi_{q,A} = \varphi_{q,\mathrm{vcl}_q A}$(see Lemma 3 in [12]), we only need to prove $\varphi_{q,\mathrm{vcl}_q A}$ is a co-radiant function.

For any $\lambda \geq 1$ and $y \in Y$, if $\varphi_{q,\mathrm{vcl}_q A}(\lambda y) < +\infty$, then the definition of $\varphi_{q,\mathrm{vcl}_q A}$ implies that

$$\begin{aligned}\varphi_{q,\mathrm{vcl}_q A}(\lambda y) = \inf\{s \in R : \lambda y \in sq - \mathrm{vcl}_q A\} &= \inf\{s \in R : y \in \frac{s}{\lambda}q - \frac{1}{\lambda}\mathrm{vcl}_q A\} \\ &= \lambda \inf\{\frac{s}{\lambda} \in R : y \in \frac{s}{\lambda}q - \frac{1}{\lambda}\mathrm{vcl}_q A\}.\end{aligned} \tag{6}$$

Since $\mathrm{vcl}_q A$ is a co-radiant set, for any $\lambda \geq 1$, we have $\mathrm{vcl}_q A \subseteq \frac{1}{\lambda}\mathrm{vcl}_q A$. Hence,

$$\varphi_{q,A}(\lambda y) = \varphi_{q,\mathrm{vcl}_q A}(\lambda y) \leq \lambda \inf\{s \in R : y \in sq - \mathrm{vcl}_q A\} = \lambda\varphi_{q,\mathrm{vcl}_q A}(y) = \lambda\varphi_{q,A}(y).$$

If $\varphi_{q,\mathrm{vcl}_q A}(\lambda y) = +\infty$,then the definition of $\varphi_{q,\mathrm{vcl}_q A}$ implies that $\lambda y \notin \mathbb{R}q - \mathrm{vcl}_q A$. Since $\lambda \geq 1$ and $\mathrm{vcl}_q A$ is a co-radiant set, $y \notin Rq - \mathrm{vcl}_q A$. In this case, we have

$$\varphi_{q,A}(\lambda y) = \lambda\varphi_{q,A}(y) = +\infty.$$

If $\lambda \in (0,1]$, the proof is omitted.

Conversely, assume that (4) holds. Suppose to the contradiction that $\mathrm{vcl}_q A$ is not a co-radiant set, then there exist $\lambda > 1$ and $y \in \mathrm{vcl}_q A$ such that $\lambda y \notin \mathrm{vcl}_q A$. Since condition (5) is equivalent to

$$\{y \in Y : \varphi_{q,A}(y) \leq 0\} = -\mathrm{vcl}_q A,$$

$\lambda y \notin \mathrm{vcl}_q A$ implies $\varphi_{q,A}(-\lambda y) > 0$ and $\lambda\varphi_{q,A}(-y) \leq 0$. Which contradicts with (4). Therefore, the converse result of (i) holds.

(ii) Take any $(x,\alpha) \in \mathrm{epi}\varphi_{q,A}$, then for any $\lambda > 1$, the inequality (4) implies $\varphi_{q,A}(\lambda x) \leq \lambda\varphi_{q,A}(x) \leq \lambda\alpha$. Thus, $(\lambda x, \lambda\alpha) \in \mathrm{epi}\varphi_{q,A}$, and $\mathrm{epi}\varphi_{q,A}$ is a co-radiant set.

Conversely, since $(x, \varphi_{q,A}(x)) \in \mathrm{epi}\varphi_{q,A}$, if $\mathrm{epi}\varphi_{q,A}$ is a co-radiant set, then for any $\lambda > 1$, we have $(\lambda x, \lambda\varphi_{q,A}(x)) \in \mathrm{epi}\varphi_{q,A}$, which implies (4) holds. And the proof is completed.

(iii) For any $y_1, y_2 \in Y$, we have

$$\varphi_{q,A}(y_1) = \inf\{s \in R : y_1 \in sq - A\}, \varphi_{q,A}(y_2) = \inf\{s \in R : y_2 \in sq - A\}.$$

According to the definition of infimum, for any $\varepsilon > 0$, we have $\varphi_{q,A}(y_1) > s_1 - \varepsilon$, $\varphi_{q,A}(y_2) > s_2 - \varepsilon$.

Since A is a convex co-radiant set, $y_1 + y_2 \in (s_1 + s_2)q - 2A \subseteq (s_1 + s_2)q - A$, that is

$$\varphi_{q,A}(y_1 + y_2) \leq s_1 + s_2 < \varphi_{q,A}(y_1) + \varphi_{q,A}(y_2) + 2\varepsilon.$$

For the randomicity of ε,

$$\varphi_{q,A}(y_1 + y_2) \leq \varphi_{q,A}(y_1) + \varphi_{q,A}(y_2).$$

□

Remark 3.2. (i) *For $\lambda < 1$, $\varphi_{q,A}(\lambda y) \leq \lambda\varphi_{q,A}(y)$ may not be true. See Example 3.2.*

(ii) *In [24], Rubinov considered a special abstract convex function, which is called ICR function, (that is, increasing function f defined on Y such that $f(\alpha x) \geq \alpha f(x)$ for all $x \in Y$ and $\alpha \in [0,1]$.) The properties of this special function has been studied in recent years(see [25-27]).*

According to the Theorem 8 in [12], we have that $\varphi_{q,A}$ is A-nondecreasing if and only if

$$vcl_q A + A \subseteq [0,+\infty)q + vcl_q A. \tag{7}$$

Hence, it is easy to see that if A is a convex co-radiant set, then $\varphi_{q,A}$ is A-nondecreasing convex functional. And even A is not a convex set, (7) may be hold. See the following examples.

Example 3.1. (i) *Let $D \subseteq Y$ be a proper pointed convex cone with nonempty relative algebraic interior, $q \in \text{icr}D$, $A = D \cap Y \setminus (q - D)$. Then, it is easy to check that A is a co-radiant set, and A satisfies (7). Therefore, $\varphi_{q,A}$ is A-nondecreasing. But, A may not be a convex set.*

(ii) *Let $Y = R^2$ and $A = \{(x,y) \in R^2 : x^2 + y^2 \geq 1, x \geq 0, y \geq 0\}$, then A is a co-radiant set and the inclusion (4) holds. Thus, $\varphi_{q,A}$ is a ICR function. But we note that A is not a convex set and it follows from Lemma 2.1 that $\varphi_{q,A}$ is not a convex function.*

Example 3.2. *Let $Y = R^2, A = \{(x,y) \in R^2 : x + y \geq 1, x \geq 0, y \geq 0\}$, and $q = (1,2)^T, y = (-3,-2)^T, \lambda = \frac{1}{3}$. It is easy to check that $\varphi_{q,A}(\lambda y) = -\frac{2}{9} > -\frac{1}{3} = \lambda\varphi_{q,A}(y)$. If $\lambda = 3, q = (1,2)^T, y = (-1,-2)^T$, we have $\varphi_{q,A}(\lambda y) = -\frac{8}{3} < -2 = \lambda\varphi_{q,A}(y)$.*

Theorem 3.2. *Let A be a nonempty subset of Y and $q \in Y \setminus \{0\}$. If $\text{vcl}_q(A) + (0,\infty)q \subseteq \text{vcl}_q(A)$, the following statements are equivalent.*

(i) vcl*A is a radiant set.*

(ii) *$\varphi_{q,A}$ is radiant function, that is*

$$\varphi_{q,A}(\lambda y) \leq \lambda\varphi_{q,A}(y), \quad \forall y \in Y, \lambda \in [0,1]. \tag{8}$$

(iii) *The epigraph* epi$\varphi_{q,A}$ *is radiant set.*

Proof. Since vcl$A$ is a radiant set, for any $\lambda \in (0,1)$ we have vcl$A \subseteq \frac{1}{\lambda}vclA$. Hence, it is analogous to the proof of Theorem 3.1, we can complete the proof. □

From Theorem 3.1 we know that $\varphi_{q,A}$ with respect to co-radiant set may not a positively homogeneous functional.

First, we consider the so-called ICR functions. Let $\varphi_{q,A}$ be a function defined on the co-radiant set $A \subseteq Y$. A function $\hat{\varphi}_{q,A}(x,t)$ defined on Y^*, which is defined as

$$Y^* = \{(x,t) \in Y \times R_{++} : x \in Y, t > 0\}.$$

Define the positively homogenous extension of the $\varphi_{q,A}$ as $\hat{\varphi}_{q,A}(x,t) = t\varphi_{q,A}(\frac{x}{t})$ on Y^*. Then, we have the following relations between $\varphi_{q,A}$ and its positively homogenous extension function.

Theorem 3.3. *Suppose that A is a co-radiant set, then the following statements are true.*

(i) *$\hat{\varphi}_{q,A}(x,t)$ is positively homogenous function. And $\varphi_{q,A}$ is nondecreasing (that is, if $y_1 - y_2 \in A$, then $\varphi_{q,A}(y_1) \geq \varphi_{q,A}(y_2)$) if and only if $\hat{\varphi}_{q,A}(x,\lambda)$ is nondecreasing in both variables x and λ.*

(ii) *$\varphi_{q,A}(x) \leq 0$ if and only if for any $\lambda \in (0,1]$, $\hat{\varphi}_{q,A}(x,\lambda) \leq 0$.*

(iii) *$\varphi_{q,A}(x) \geq 0$ if and only if for any $\lambda \geq 1$, $\hat{\varphi}_{q,A}(x,\lambda) \geq 0$.*

(iv) *$\varphi_{q,A}$ is subadditive if and only if for any $\lambda \in (0,1]$, $\hat{\varphi}_{q,A}(x,\lambda)$ is subadditive.*

Proof. (i) The proof is similar to the corresponding proof in Theorem 3.1 of [24], and we omit the proof. The positive homogenously of $\hat{\varphi}_{q,A}(x,t)$ is obvious.

(ii) For any $\lambda \in (0,1]$, it follows from the definition of $\hat{\varphi}_{q,A}$ and Theorem 3.1(i), we have that $\hat{\varphi}_{q,A}(x,\lambda) = \lambda\varphi_{q,A}(\frac{x}{\lambda}) \leq \varphi_{q,A}(x)$. Thus, if $x \in Y$ satisfies $\varphi_{q,A}(x) \leq 0$, then $\hat{\varphi}_{q,A}(x,\lambda) \leq 0$.

Conversely, when $\lambda = 1$, then we have the result.

(iii) For any $\lambda \geq 1$, we have $\hat{\varphi}_{q,A}(x, \lambda) = \lambda\varphi_{q,A}(\frac{x}{\lambda}) \geq \varphi_{q,A}(x)$. Thus, if $x \in Y$ satisfies $\varphi_{q,A}(x) \geq 0$, then $\hat{\varphi}_{q,A}(x, \lambda) \geq 0$.

Conversely, when $\lambda = 1$, then we have the result.

(iv) Assume that $\varphi_{q,A}$ is subadditive, for any $\lambda \in (0, 1]$, it follows from the definition of $\hat{\varphi}_{q,A}$ and Theorem 3.1 (i), we have

$$\hat{\varphi}_{q,A}(y_1 + y_2, \lambda) = \lambda\varphi_{q,A}(\frac{y_1 + y_2}{\lambda}) \leq \lambda\varphi_{q,A}(\frac{y_1}{\lambda}) + \lambda\varphi_{q,A}(\frac{y_2}{\lambda}) = \hat{\varphi}_{q,A}(y_1, \lambda) + \hat{\varphi}_{q,A}(y_2, \lambda),$$

which means that $\hat{\varphi}_{q,A}$ is subadditive. The converse is omitted. $\square$

Lemma 3.3. (i) *If $A \subseteq Y$ is a radiant set and $A + A \subseteq A$, then A is a convex set.*
(ii) *If $A \subseteq Y$ is a convex co-radiant set, then $A + A \subseteq A$.*

Definition 3.1. *Let $\varepsilon \geq 0$, f be a convex function defined on X, $x_0 \in dom f$. The ε−subdifferential of f at x_0 is the set*

$$\partial_\varepsilon f(x_0) := \{x^* \in X^* : f(x) \geq f(x_0) + \langle x^*, x - x_0\rangle - \varepsilon, \forall x \in X\}.$$

Theorem 3.4. *Let $A \subset Y$ be a closed proper convex co-radiant set and $\bar{x} \in Y$. We have the following properties.*
(i) *If* icrA $\neq \emptyset$, *then $\partial\varphi_{q,A}(\bar{x}) \subseteq A^+ \setminus \{0\}$.*
(ii) *$\partial_\varepsilon\varphi_{q,A}(\bar{x}) \subseteq \{\xi : \langle\xi, u\rangle \leq \varepsilon, \forall u \in -A\}$.*
(iii) *If $0 \in A$, then $\partial_\varepsilon\varphi_{q,A}(\bar{x}) = \{\xi \in \partial\varphi_{q,A}(0) : \langle\xi, \bar{x}\rangle \geq \varphi_{q,A}(\bar{x}) - \varphi_{q,A}(0) - \varepsilon\}$.*

Proof. (i) For any $\xi \in \partial\varphi_{q,A}(\bar{x})$, we have $\langle\xi, x - \bar{x}\rangle \leq \varphi_{q,A}(x) - \varphi_{q,A}(\bar{x})$, $\forall x \in Y$. Let $u \in -A$ and $x = \bar{x} + u$. It's obvious that $\langle\xi, \bar{x} + u - \bar{x}\rangle \leq \varphi_{q,A}(\bar{x} + u) - \varphi_{q,A}(\bar{x})$, $\forall u \in -A$. Using the (iii) in Theorem 3.1 in [29], we have the result

$$\langle\xi, u\rangle \leq \varphi_{q,A}(\bar{x}) + \varphi_{q,A}(u) - \varphi_{q,A}(\bar{x}) = \varphi_{q,A}(u) \leq 0, \ \forall u \in -A.$$

Which means that $\partial\varphi_{q,A}(\bar{x}) \subseteq A^+$.

Next, we prove that $0 \notin \partial\varphi_{q,A}(\bar{x})$, we suppose that $0 \in \partial\varphi_{q,A}(\bar{x})$, then

$$\varphi_{q,A}(x) \geq \varphi_{q,A}(\bar{x}), \forall x \in Y.$$

Also since $\varphi_{q,A}$ is subadditive, $\varphi_{q,A}(x - \bar{x}) \geq 0, \forall x \in Y$. That is, for any $y \in Y, \varphi_{q,A}(y) \geq 0$. According to Lemma 3.2(i), we have $Y = \text{vcl}A$, it contradicts with A is a closed proper set. Hence, $\partial\varphi_{q,A}(\bar{x}) \subseteq A^+ \setminus \{0\}$.

(ii) For any $\xi \in \partial_\varepsilon\varphi_{q,A}(\bar{x})$, according to the definition we have $\langle\xi, x - \bar{x}\rangle \leq \varphi_{q,A}(x) - \varphi_{q,A}(\bar{x}) + \varepsilon$, $\forall x \in Y$. Let $x = \bar{x} + u, \forall\ u \in -A$, then

$$\langle\xi, u\rangle \leq \varphi_{q,A}(\bar{x} + u) - \varphi_{q,A}(\bar{x}) + \varepsilon \leq \varphi_{q,A}(u) + \varepsilon \leq \varepsilon, \ \forall u \in -A,$$

which means that $\partial_\varepsilon\varphi_{q,A}(\bar{x}) \subset \{\xi : \langle\xi, u\rangle \leq \varepsilon, \forall u \in -A\}$.

(iii) If $\xi \in \partial\varphi_{q,A}(0)$, we have $\langle\xi, x\rangle \leq \varphi_{q,A}(x) - \varphi_{q,A}(0)$, $\forall x \in Y$.

$$\langle\xi, x - \bar{x}\rangle \leq \varphi_{q,A}(x) - \varphi_{q,A}(\bar{x}) + \varepsilon, \ \forall x \in Y,$$

hence $\xi \in \partial_\varepsilon\varphi_{q,A}(\bar{x})$.

If $\xi \in \partial_\varepsilon\varphi_{q,A}(\bar{x})$, so $\langle\xi, x - \bar{x}\rangle \leq \varphi_{q,A}(x) - \varphi_{q,A}(\bar{x}) + \varepsilon$. Let $x = 0$, that is

$$\langle\xi, \bar{x}\rangle \geq \varphi_{q,A}(\bar{x}) - \varphi_{q,A}(0) - \varepsilon.$$

Now let us prove $\xi \in \partial\varphi_{q,A}(0)$. Let $x = \bar{x} + ty(t \geq 1), \forall y \in Y$, we have

$$t\langle\xi, y\rangle \leq \varphi_{q,A}(\bar{x} + ty) - \varphi_{q,A}(\bar{x}) + \varepsilon \leq \varphi_{q,A}(ty) + \varepsilon \leq t\varphi_{q,A}(y) + \varepsilon.$$

Assume $t \to \infty$, thus $\langle\xi, y\rangle \leq \varphi_{q,A}(y) = \varphi_{q,A}(y) - \varphi_{q,A}(0)$. It means that $\xi \in \partial\varphi_{q,A}(0)$. $\square$

The following example illustrate converse inclusion relation in (i) may not hold.

Example 3.3. *Consider the co-radiant set $A = [1, +\infty), Y = R$, let $q = 1, \bar{x} = 1$. Then $\varphi_{q,A}(x) = x + 1, \varphi_{q,A}(\bar{x}) = 2$. If $\xi \in \partial\varphi_{q,A}(\bar{x})$, from the definition we have*

$$\langle \xi, x - \bar{x} \rangle \leq \varphi_{q,A}(x) - \varphi_{q,A}(\bar{x}),\ \forall x \in Y.$$

So we get $\partial\varphi_{q,A}(\bar{x}) = 1$, and obviously, $A^+ \setminus \{0\} = (0, +\infty)$, that is $\partial\varphi_{q,A}(\bar{x}) \subseteq A^+ \setminus \{0\}$, while $A^+ \setminus \{0\} \not\subseteq \partial\varphi_{q,A}(\bar{x})$.

Remark 3.3. (i) *Since A^+ is a closed convex cone, $A^+ \setminus \{0\} = (clconeA)^+ \setminus \{0\}$. Therefore $\partial\varphi_{q,A}(\bar{x}) \subseteq A^+ \setminus \{0\} =$* $N(\text{clcone}(-A), 0) \setminus \{0\}$.

(ii) *If A is a convex cone, we get that $\varphi_{q,A}$ is sublinear function. And Theorem 2.4.14 in [30] implies that*

$$\partial_\varepsilon\varphi_{q,A}(0) = \partial\varphi_{q,A}(0).$$

(iii) *If A is a convex cone, $\partial_\varepsilon\varphi_{q,A}(\bar{x}) \subseteq -N_\varepsilon(0, A)$.*

Theorem 3.5. *Let $A \subset Y$ be a closed proper convex co-radiant set and $q \in \text{icr}A$, for every $y \in Y$ there exists $t \in R$ such that $y + tq \notin A$. Let $\bar{x} \in Dom\varphi_{q,A}$, for every $\lambda \in R$ and $x \in Y$,*

$$Dom\varphi_{q,A} = \{x \in Y : \varphi_{q,A}(x) \leq \lambda\} = \lambda q - A.$$

Then

$$\partial_\varepsilon\varphi_{q,A}(\bar{x}) = \{\xi \in Y^* : \langle \xi, q \rangle = 1,\ \langle \xi, a \rangle + \langle \xi, \bar{x} \rangle - \varphi_{q,A}(\bar{x}) + \varepsilon \geq 0, \forall a \in A\}.$$

Proof. The assumptions ensure that $\varphi_{q,A}$ is convex and proper. An element $\xi \in \partial_\varepsilon\varphi_{q,A}(\bar{x})$ iff

$$\varphi_{q,A}(x) \geq \langle \xi, x \rangle - \langle \xi, \bar{x} \rangle + \varphi_{q,A}(\bar{x}) - \varepsilon, \forall x \in Y.$$

This means that for all $x \in Dom\varphi_{q,A}$ and $\lambda \in R$ with $\lambda \geq \varphi_{q,A}(x)$, we have

$$\lambda \geq \langle \xi, x \rangle - \langle \xi, \bar{x} \rangle + \varphi_{q,A}(\bar{x}) - \varepsilon.$$

Consequently, for all $x \in \lambda q - A$ one has $\lambda \geq \varphi_{q,A}(x)$, $\lambda \geq \langle \xi, x \rangle - \langle \xi, \bar{x} \rangle + \varphi_{q,A}(\bar{x}) - \varepsilon$. This implies that

$$\lambda \geq \lambda\langle \xi, q \rangle - \langle \xi, a \rangle - \langle \xi, \bar{x} \rangle + \varphi_{q,A}(\bar{x}) - \varepsilon, \forall a \in A.$$

Since $\langle \xi, q \rangle = 1$, we have

$$\langle \xi, a \rangle + \langle \xi, \bar{x} \rangle - \varphi_{q,A}(\bar{x}) + \varepsilon \geq 0, \forall a \in A.$$

For the converse, take $\xi \in Y^*$ such that $\langle \xi, a \rangle + \langle \xi, \bar{x} \rangle - \varphi_{q,A}(\bar{x}) + \varepsilon \geq 0, \forall a \in A$ and $\langle \xi, q \rangle = 1$. Fix $x \in Dom\varphi_{q,A}$ and take $\lambda = \varphi_{q,A}(x)$. Then there exists $a \in A$ such that $x = \lambda q - a$. Accordingly,

$$\langle \xi, x \rangle = \lambda\langle \xi, q \rangle - \langle \xi, a \rangle \leq \lambda + \langle \xi, \bar{x} \rangle - \varphi_{q,A}(\bar{x}) + \varepsilon.$$

Since $\lambda = \varphi_{q,A}(x)$ is arbitrarily chosen, one has

$$\langle \xi, x \rangle \leq \varphi_{q,A}(x) + \langle \xi, \bar{x} \rangle - \varphi_{q,A}(\bar{x}) + \varepsilon, \forall x \in Y,$$

which implies $\xi \in \partial_\varepsilon\varphi_{q,A}(\bar{x})$. □

Remark 3.4. (i) *When $\varepsilon = 0$, the conclusion in Theorem 3.5 can reduce to Theorem 2.2 in [22].*

(ii) *If $\bar{x} = 0$, we have*

$$\partial_\varepsilon\varphi_{q,A}(0) = \{\xi \in Y : \langle \xi, q \rangle = 1,\ \langle \xi, a \rangle \leq \varepsilon - \delta, \forall a \in -A\},\ \textit{where}\ \delta \in [0, \varepsilon).$$

In fact, from Theorem 3.5 we have $\partial_\varepsilon\varphi_{q,A}(0) = \{\xi \in Y : \langle \xi, q \rangle = 1,\ \langle \xi, a \rangle - \varphi_{q,A}(0) + \varepsilon \geq 0, \forall a \in A\}$. According to the result in [31], we have $\varphi_{q,A}(0) = \delta \in [0, \varepsilon)$. Hence, $\langle \xi, a \rangle \geq -\varepsilon + \delta, \forall a \in A$.

If A is a closed proper convex cone, we have

$$\partial_\varepsilon\varphi_{q,A}(0) = \partial\varphi_{q,A}(0) = \{\xi \in -N(0; A) : \langle \xi, q \rangle = 1\}.$$

(iii) *We can find that Theorem 3.4*(iii) *is closely related to the result of Theorem 3.5. Actually, from Theorem 3.5 we have $\partial\varphi_{q,A}(0) = \{\xi \in Y : \langle \xi, q \rangle = 1,\ \langle \xi, a \rangle \geq \varphi_{q,A}(0), \forall a \in A\}$, together with the result of Theorem 3.4*(iii)*, that is*

$$\partial_\varepsilon\varphi_{q,A}(\bar{x}) = \{\xi \in Y : \langle \xi, q \rangle = 1,\ \langle \xi, a \rangle + \langle \xi, \bar{x} \rangle - \varphi_{q,A}(\bar{x}) + \varepsilon \geq 0, \forall a \in A\},$$

which is the result of Theorem 3.5.

4. Application to Vector Optimization Problems

In this section, we establish the Lagrange multiplier rules for ε-efficient solutions of vector optimization problem.

Let X be a *Asplund* space with dual X^*. A Banach space X is called an Asplund space if every continuous convex function defined on an open convex subset A of X is Fréchet differentiable at each point of a dense subset of A. Let $F : X \rightrightarrows X^*$, the supremum of F is defined as follows.

$$\limsup_{x\to\bar{x}} F(x) = \{x^* \in X^* : \exists x_k \to \bar{x}, x_k^* \xrightarrow{w^*} x^*, x_k^* \in F(x_k), \forall k \in N\}.$$

Definition 4.1. [32] *Let $\varepsilon \geq 0$, Ω be a nonempty subset of X and $x \in cl\Omega$.*
(i) *The Frechet $\varepsilon-$normal to Ω at x is defined by*

$$\hat{N}_\varepsilon(x, \Omega) = \{x^* \in X^* : \limsup_{u\xrightarrow{\Omega}x} \frac{\langle x^*, u - x\rangle}{\|u - x\|} \leq \varepsilon\}.$$

When $\varepsilon = 0$, the set of $\hat{N}(x, \Omega)$ is called the Fréchet normal cone to Ω at x.
(ii) *The basic (or limiting, or Mordukhovich) normal cone to Ω at $\bar{x}$ is*

$$N_L(\bar{x}, \Omega) = \limsup_{x\xrightarrow{\Omega}\bar{x}} \hat{N}_\varepsilon(x, \Omega),$$

which means that $\exists\ \varepsilon_n \downarrow 0, x_n \xrightarrow{\Omega} \bar{x}, x_n^* \xrightarrow{w^*} x^*, x_n^* \in \hat{N}_{\varepsilon_n}(x_n, \Omega)$, $\forall n \in N$, where $u \xrightarrow{\Omega} \bar{x}$ means $u \in \Omega$ and $u \to \bar{x}$. It is important to note that if X is an Asplund space, we have

$$N_L(\bar{x}, \Omega) = \limsup_{x\xrightarrow{\Omega}\bar{x}} \hat{N}(x, \Omega),$$

If Ω is a convex set, then $\hat{N}_\varepsilon(\bar{x}, \Omega)$ and $(\hat{N}(\bar{x}, \Omega))$ coincide with the normal cone in the usual sense of convex analysis, that is,

$$\hat{N}_\varepsilon(\bar{x}, \Omega) = \{x^* \in X^* : \langle x^*, x - \bar{x}\rangle \leq \varepsilon\|x - \bar{x}\|,\ \forall x \in \Omega\}.$$

$$N(\bar{x}, \Omega) = \{x^* \in X^* : \langle x^*, x - \bar{x}\rangle \leq 0,\ \ \forall x \in \Omega\}.$$

Definition 4.2. [32] *Let $\varphi :\ X \to R \cup \{+\infty\}$ be a given function and $x_0 \in dom\varphi$. The set*

$$\partial_L\varphi(\bar{x}) = \{x^* \in X^* \mid (x^*, -1) \in N_L((\bar{x}, \varphi(\bar{x})); epi\varphi)\}$$

is called the Mordukhovich subdifferential of φ at $\bar{x}$.

Definition 4.3. [32] *Let $\varphi : X \to R \cup \{+\infty\}$ be a lower-semicontinuous function. Then for $\varepsilon \geq 0$ the limiting $\varepsilon-$subdifferential is given as*

$$\partial_\varepsilon^L\varphi(\bar{x}) = \limsup_{x\xrightarrow{\varphi}\bar{x}} \partial_\varepsilon^F\varphi(x),$$

where $\partial_\varepsilon^F\varphi(\bar{x})$ denotes the Frechet $\varepsilon-$subdifferential of φ at $\bar{x} \in dom\varphi$ is given as follows

$$\partial_\varepsilon^F\varphi(x) = \{x^* \in X^* : \liminf_{u\to x} \frac{\varphi(u) - \varphi(x) - \langle x^*, u - x\rangle}{\|u - x\|} \geq -\varepsilon\}.$$

Note that if φ is a lower semicontinuous convex function, then $\partial_\varepsilon^F\varphi(\bar{x}) = \partial_\varepsilon^L\varphi(\bar{x}) = \partial\varphi(\bar{x}) + \varepsilon B_{X^*}$. Furthermore, from the Theorem 1.93 in [32], if $\varphi : X \to R$ is convex and finite at $\bar{x}$, we have

$$\partial_\varepsilon^F\varphi(\bar{x}) = \{x^* \in X^* : \langle x^*, x - \bar{x}\rangle \leq \varphi(x) - \varphi(\bar{x}) + \varepsilon\|x - \bar{x}\|,\ \ \forall x \in X\}.$$

Combining with proof of Theorem 3.4(ii), it concludes that

$$\partial_\varepsilon^F\varphi(\bar{x}) \subseteq \{\xi : \langle \xi, u\rangle \leq \varepsilon\|u\|, \forall u \in -A\}. \tag{9}$$

Lemma 4.1. [33] *Let X, Z be Asplund space, $f, g : X \to R \cup \{+\infty\}$ be two lower semicontinuous proper functions, one of them being locally Lipschitzian.*

(i) *For each $x \in domf \cap domg$ and each $\varepsilon > 0$, then*

$$\partial^L_\varepsilon(f+g)(x_0) \subseteq \partial^L_\varepsilon f(x_0) + \partial^L_\varepsilon g(x_0).$$

(ii)*Let $F : X \to Z$ be a strictly Lipschitzian at x_0, then*

$$\partial^L_\varepsilon(g \circ F)(x_0) \subseteq \bigcup_{y^* \in \partial_L g(F(x_0))} \partial^L_\varepsilon(y^* \circ F)(x_0).$$

Definition 4.4. [23] *Let $\varepsilon \geq 0$, $\Omega \subset X$, C be a co-radiant set of X.*

(i) *It is said that $\bar{y} \in \Omega$ is an $\varepsilon-$efficient point with respect to C, if*

$$\Omega \cap (\bar{y} - C(\varepsilon)) \subseteq \{\bar{y}\}.$$

(ii) *It is said that $\bar{y} \in \Omega$ is a weakly ε-efficient point with respect to C, if*

$$\Omega \cap (\bar{y} - intC(\varepsilon)) \subseteq \{\bar{y}\}.$$

(iii) *It is said that $\bar{y} \in \Omega$ is a properly ε-efficient point with respect to C, if*

$$\text{clcone}(\Omega + C(\varepsilon) - \bar{y}) \cap -C(\varepsilon) \subseteq \{0\}.$$

The ε-efficient point, weakly ε-efficient point, properly ε-efficient point of Ω are respectively denoted by $AE[\Omega, C(\varepsilon)]$, $WAE[\Omega, C(\varepsilon)], PAE[\Omega, C(\varepsilon)]$.

Consider the following optimization problem

$$Min\ \Phi(x) \quad s.t. \quad x \in \Omega,$$

where, $\emptyset \neq \Omega \subseteq X$ and $\Phi : \Omega \to R$.

Let $\varepsilon \geq 0, \bar{x} \in \Omega$. $\bar{x}$ is called $\varepsilon-$minimizer of the above optimization problem, if

$$\Phi(x) \geq \Phi(\bar{x}) - \varepsilon, \quad \forall x \in \Omega.$$

In the following results, we will use the approximate normal cone of nonconvex set to obtain the optimality conditions for approximate points.

We consider the following set

$$D_{\eta,e} = cone(B(e,\eta)) = \{t \cdot z : t \geq 0, z \in B(e,\eta)\},\ \eta \in (0, \|e\|),$$

where $B(e,\eta)$ is a closed ball with the center e and the radius η.

Theorem 4.1. *Let $\varepsilon \geq 0$, $0 \notin C$ be a convex co-radiant set. Assume that the epigraphical set of Ω with respect to $C(\varepsilon)$ and $\Omega + C(\varepsilon)$ are locally closed at $\bar{z}$. For $\bar{z} \in C(\varepsilon) \cap (\Omega + C(\varepsilon))$ and every $e \in C(\varepsilon) \setminus \{0\}$ satisfying*

$$-e \notin clcone(\Omega + C(\varepsilon) - \bar{z}),$$

there exist $z_0 \in C(\varepsilon) \cap (\Omega + C(\varepsilon))$ and $z^* \in X^*$ such that $\|z_0 - \bar{z}\| \leq 1$ and

$$-z^* \in N_L(z_0; \Omega) \cap \hat{N}_\epsilon(0; D_{\eta,e}).$$

Proof. Since $-e \notin clcone(\Omega + C(\varepsilon) - \bar{z})$, there exists $\eta \in (0, \|e\|)$ such that

$$clcone(\Omega + C(\varepsilon) - \bar{z}) \cap -B(e,\eta) \subseteq \{0\},$$

that is

$$clcone(\Omega + C(\varepsilon) - \bar{z}) \cap -D_{\eta,e} \subseteq \{0\}.$$

Which means that $\Omega + C(\varepsilon) - \bar{z} \nsubseteq -D_{\eta,e}$. And $\varphi_{e,D_{\eta,e}}(p + e - \bar{z}) \geq 0,\ \forall p \in \Omega,\ e \in C(\varepsilon)$.

Since $\varphi_{e,D_{\eta,e}}$ is sublinear, we have

$$0 \leq \varphi_{e,D_{\eta,e}}(p + e - \bar{z}) \leq \varphi_{e,D_{\eta,e}}(p - \bar{z}) + \varphi_{e,D_{\eta,e}}(e),\ \ \forall p \in \Omega, e \in C(\varepsilon).$$

According to the definition of $D_{\eta,e}$, there exists $q \in D_{\eta,e} \cap C(\varepsilon)$ such that $\varphi_{e,D_{\eta,e}}(q) \geq 0$ and $\varphi_{e,D_{\eta,e}}(p - \bar{z}) \geq -\varphi_{e,D_{\eta,e}}(q)$. Take $\epsilon = \varphi_{e,D_{\eta,e}}(q)$, we have

$$\varphi_{e,D_{\eta,e}}(p - \bar{z}) \geq -\epsilon,\ \ \forall p \in \Omega.$$

Hence, $\bar{z}$ is an $\epsilon-$minimizer of the scalarzation function $\varphi_{e,D_{\eta,e}}(z - \bar{z})$ over Ω.

According to the optimality conditions of Theorem 3.1 in [34], there exists $z_0 \in C(\varepsilon) \cap \Omega + C(\varepsilon)$ such that $||z_0 - \bar{z}|| \leq 1$ and

$$0 \in \partial^L_\epsilon \varphi_{e,D_{\eta,e}}(\cdot - \bar{z})(z_0) + N_L(z_0, \Omega) = \partial^F_\epsilon \varphi_{e,D_{\eta,e}}(z_0 - \bar{z}) + N_L(z_0, \Omega).$$

From (9) we have

$$\partial^F_\epsilon \varphi_{e,D_{\eta,e}}(z_0 - \bar{z}) \subset \{z^* \in X^* : \langle z^*, z\rangle \leq \epsilon||z||, \forall z \in -D_{\eta,e}\} = -\hat{N}_\epsilon(0; D_{\eta,e}).$$

Which implies $z^* \in \partial^F_\epsilon \varphi_{e,D_{\eta,e}}(z_0 - \bar{z})$ and $-z^* \in N_L(z_0; \Omega) \cap \hat{N}_\epsilon(0; D_{\eta,e})$.

□

The following example illustrates $-e \notin clcone(\Omega + C(\varepsilon) - \bar{z})$ is necessary.

Example 4.1. *Considering the set* $\Omega := \{(x, y)^T \in R^2 : y \geq -x^2 + \frac{1}{2}\}$, $C = \{0\} \times [1, +\infty)$. *Taking* $\varepsilon = \frac{1}{2}$, $\bar{z} = (0, 1)$, *we have* $\bar{z}$ *is not the properly* $\varepsilon-$*efficient point, since*

$$clcone(\Omega + C(\varepsilon) - \bar{z}) \cap -C(\varepsilon) \nsubseteq \{0\}.$$

Let $z_0 = (0, \frac{3}{2})$, *we can check that* $||z_0 - \bar{z}|| \leq 1$. *According to the definition of Mordukhovich normal cone we have*

$$-z^* = (-5, 0) \in N_L(z_0, \Omega).$$

Take $q = (0, 1) \in D_{\eta,e}$ *and* $\epsilon = 1$. *It is obvious that* $-e \in clcone(\Omega + C(\varepsilon) - \bar{z})$ *for any* $e \in C(\varepsilon)$ *and* $-z^* \notin \hat{N}_\epsilon(0; D_{\eta,e})$. *Thus the necessary condition in Theorem 4.1 is not applicable to this example.*

Remark 4.1. (i) *If the cone*$(\Omega + C(\varepsilon) - \bar{z})$ *is closed at the origin, every* $\varepsilon-$*efficient point with respect to* $C(\varepsilon)$ *is a properly* $\varepsilon-$*efficient point in the sense that*

$$clcone(\Omega + C(\varepsilon) - \bar{z}) \cap -C(\varepsilon) \subseteq \{0\}.$$

(ii) *The closedness of the cone*$(\Omega + C(\varepsilon) - \bar{z})$ *and the set itself are different. Let* $C = \{(x, y)^T \in R^2 \mid x \geq 1, y \geq 1\}$, $\Omega := \{(x, y)^T \in R^2 : y \geq x^2)\}$. *Taking* $\varepsilon = \frac{1}{2}$, $\bar{z} = (\frac{1}{2}, \frac{1}{2})$, *we have that* $\Omega + C(\varepsilon) - \bar{z}$ *is a closed set, while cone*$(\Omega + C(\varepsilon) - \bar{z})$ *is not a closed set. If* $\Omega := \{(x, y)^T \in R^2 : x \geq (y - 1)^2 - 1, x < y\} \cup \{0\}$, *then* Ω *is not a closed set. But cone*$(\Omega + C(\varepsilon) - \bar{z})$ *is a closed set.*

Lemma 4.2. *Let* $\varepsilon \geq 0$, $0 \notin C$ *be a convex co-radiant set,* $\bar{z} \in \Omega + C(\varepsilon)$. *If* $\bar{z} \in AE[\Omega, C(\varepsilon)]$, *then* $\bar{z} \in AE[\Omega + C(\varepsilon), C(\varepsilon)]$.

Proof. Suppose to the contrary that $\bar{z} \notin AE[\Omega + C(\varepsilon), C(\varepsilon)]$, we have

$$(\Omega + C(\varepsilon) - \bar{z}) \cap -C(\varepsilon) \nsubseteq \{0\}.$$

There exists $p \in -C(\varepsilon)\backslash\{0\}$ such that $p \in \Omega + C(\varepsilon) - \bar{z}$. That is, there exist $e \in \Omega$ and $q \in C(\varepsilon)$ such that $p = e + q - \bar{z}$. Hence, $0 \neq e - \bar{z} = p - q \in -C(\varepsilon) - C(\varepsilon) \subseteq -C(\varepsilon)$. Which is a contradiction to $\bar{z} \in AE[\Omega, C(\varepsilon)]$. □

Next,we will improve Theorem 4.1 to obtain the following result.

Corollary 4.1. *Let $\epsilon \geq 0, \bar{z} \in \Omega + C(\varepsilon)$ and $\bar{z} \in AE[\Omega, C(\varepsilon)]$. Assume that $\Omega + C(\varepsilon)$ is locally closed at $\bar{z}$ and $cone(\Omega + C(\varepsilon) - \bar{z})$ is closed, then for every $e \in C(\varepsilon) \setminus \{0\}$, there exist $z_0 \in C(\varepsilon) \cap \Omega + C(\varepsilon), z^* \in X^*$ such that $\|z_0 - \bar{z}\| \leq 1$ and*

$$-z^* \in N_L(z_0; \Omega) \cap \hat{N}_\epsilon(0; D_{\eta,e}).$$

Proof. If $\bar{z} \in \Omega + C(\varepsilon)$ and $\bar{z} \in AE[\Omega, C(\varepsilon)]$, Lemma 4.2 implies that $\bar{z} \in AE[\Omega + C(\varepsilon), C(\varepsilon)]$. Therefore,

$$(\Omega + C(\varepsilon) - \bar{z}) \cap -C(\varepsilon) \subseteq \{0\}.$$

Which implies $cone(\Omega + C(\varepsilon) - \bar{z}) \cap -C(\varepsilon) \subseteq \{0\}$. Since the $cone(\Omega + C(\varepsilon) - \bar{z})$ is closed, then for any $e \in C(\varepsilon) \setminus \{0\}$, we have $-e \notin clcone(\Omega + C(\varepsilon), \bar{z})$. By using Theorem 4.1 we can get the result. □

Let Z be a Asplund space. We consider the following vector optimization problem.

$$(\text{VP}) \begin{cases} \text{minimize} & f(x) \\ \text{s.t.} & x \in S \end{cases}$$

where S be a nonempty subset of X , $f : S \to Z$.

Definition 4.5. *Let $\bar{x} \in S, \epsilon \geq 0$ and $C \subseteq Z$ be a convex co-radiant set.*
(i) *$\bar{x}$ is an $\varepsilon-$efficient solutions of* (VP)*, if*

$$f(\bar{x}) \in AE[f(S), C(\varepsilon)].$$

(ii) *$\bar{x}$ is a weakly $\varepsilon-$efficient solutions of* (VP)*, if*

$$f(\bar{x}) \in WAE[f(S), C(\varepsilon)].$$

(iii) *$\bar{x}$ is a properly $\varepsilon-$efficient solutions of* (VP) *, if*

$$f(\bar{x}) \in PAE[f(S), C(\varepsilon)].$$

Theorem 4.2. *Assume that $\bar{x} \in S$, $\epsilon \geq 0$, $f(x)$ is Lipschitz continuous at $\bar{x}$, S is locally closed around $\bar{x}$. If $\bar{x}$ is an $\varepsilon-$efficient solutions of* (VP)*, then for every $e \in C(\varepsilon) \setminus \{0\}$ satisfying $-e \notin clcone(f(S) + C(\varepsilon) - f(\bar{x}))$, there exist $x \in B(\bar{x}, \sqrt{\epsilon}) \cap S$ and $z^* \in -N(0, D_{\eta,e})$ such that*

$$0 \in \partial_L(z^* \circ f)(x) + N_L(x, S) + \sqrt{\epsilon} B_{X^*}.$$

Proof. Similar to the proof of Theorem 4.1, we have

$$clcone(f(S) + C(\varepsilon) - f(\bar{x})) \cap -D_{\eta,e} \subseteq \{0\}.$$

Therefore, we have

$$\varphi_{e,D_{\eta,e}}(f(x) + e - f(\bar{x})) \geq 0, \ \forall x \in S, \ e \in C(\varepsilon).$$

Let $G(x) := f(x) - f(\bar{x})$. According to the proof of Theorem 4.1, there exists $\epsilon > 0$ such that

$$\varphi_{e,D_{\eta,e}}(z) \geq -\epsilon, \ \forall z \in G(x).$$

Which implies that $\bar{x}$ is an $\epsilon-$minimal of

$$Min(\varphi_{e,D_{\eta,e}} \circ G)(x), \ \ s.t. \ x \in S.$$

Using the Ekeland variational principle for $\varphi_{e,D_{\eta,e}} \circ G$ on S, we can get $x \in B(\bar{x}, \sqrt{\epsilon}) \cap S$. Which is a minimum solution of the function $\varphi_{e,D_{\eta,e}} \circ G(\cdot) + \sqrt{\epsilon} \parallel \cdot - x \parallel$ on S. And from Theorem 3.2 in [36] we have

$$
\begin{aligned}
0 &\in \partial_L(\varphi_{e,D_{\eta,e}} \circ G(\cdot) + \sqrt{\epsilon} \parallel \cdot - x \parallel + N_L(x,S) \\
&\subseteq \partial_L(\varphi_{e,D_{\eta,e}} \circ G)(x) + \sqrt{\epsilon} B_{X^*} + N_L(x,S) \\
&\subseteq \bigcup_{z^* \in \partial_L \varphi_{e,D_{\eta,e}}(f(x)-f(\bar{x}))} \partial_L(z^* \circ f)(x) + \sqrt{\epsilon} B_{X^*} + N_L(x,S).
\end{aligned}
$$

Since $\varphi_{e,D_{\eta,e}}$ is a convex function,

$$\partial_L \varphi_{e,D_{\eta,e}}(f(x) - f(\bar{x})) = \partial \varphi_{e,D_{\eta,e}}(f(x) - f(\bar{x})) \subset -N(0, D_{\eta,e}).$$

Therefore, there exists $z^* \in -N(0, D_{\eta,e})$ such that

$$0 \in \partial_L(z^* \circ f)(x) + N_L(x,S) + \sqrt{\epsilon} B_{X^*}.$$

□

5. Risk Measure

Risk management is an important branch in the financial field, its basic work is to select appropriate risk metrics and scientific calculation methods to measure risk. So far, there are numerous results on risk measurement. In 1952, Markowitz pioneered the theory of mean-variance portfolio, but due to the poor fitting effect of the model, scholars further improved the model to mean-semivariance and even more general forms. As research advances, scholars have found that investors are paying more attention to downstream risk measurement such as VaR and CVaR. Therefore, the relevant theories have developed rapidly. However, VaR is not meeting sub-additives, that is inconsistent with the basic principle that portfolio investment will reduce risk. This prompts scholars to seek better risk metrics.

In 1999, Artzner et al. [35] introduced a coherent risk measure function and gave a reasonable explanation in the economic sense.

Definition 5.1. *The risk measure satisfies the following four axioms as a coherent risk measure $\rho(y)$:*
(P1) $\rho(y_1) \le \rho(y_2)$ if $y_1 \ge y_2$, (Monotonicity)
(P2) $\rho(\lambda y) = \lambda \rho(y)$ for all $y \in Y$ and $\lambda \ge 0$, (Positive Homogeneity)
(P3) $\rho(y_1 + y_2) \le \rho(y_1) + \rho(y_2)$ for all $y_1, y_2 \in Y$, (Subadditivity)
(P4) $\rho(y + tq) = \rho(y) - t$ for $q \in Y \setminus \{0\}$ (Translation Invariance).

Let Λ be a linear space of random variables, Ω be a set of elementary events. Then a future payment of an investment is a random variable $y : \Omega \to R$, positive payment in the future are wins, negative ones are loses. But modern people are no longer satisfied with this, investment is bound to want more money, so we stipulate that $y(w) \ge m (w \in \Omega)$ is a better return, and $0 < y(w) < m$ is not a very desirable income, if no investment is being done, then $y(w)$ takes on the value zero. In order to evaluate such investments, we introduce an ordering relation which defined by a set Λ and acceptable set Γ.

(i) $\{y : y(w) \ge m, w \in \Omega\} \subset \Gamma, m \in R^+$,
(ii) $\{y : y(w) < m, w \in \Omega\} \cap \Gamma = \emptyset, m \in R^+$,
(iii) Γ is a convex co-radiant set.

In particular, if we take $\Lambda = D + \varepsilon\{e\}$, where D is a convex cone and $e := (1, \cdots, 1)^T$, then
(i) $\{y : y(w) \ge \varepsilon, w \in \Omega\} \subset \Gamma$,
(ii) $\{y : y(w) < \varepsilon, w \in \Omega\} \cap \Gamma = \emptyset$,

For this case, we can introduce the following weak risk measure to investigate the risk measurement.

Definition 5.2. *If a real-valued function $\mu : \Lambda \to \mathbb{R}$ satisfy the following properties*
(P1) $\mu(y_1) \le \mu(y_2)$ if $y_1 \ge y_2$, (Monotonicity)
(P2') $\mu(\lambda y) \le \lambda \mu(y)$ for all $y \in \Lambda$ and $\lambda \ge 1$,
(P3) $\mu(y_1 + y_2) \le \mu(y_1) + \mu(y_2)$ for all $y_1, y_2 \in \Lambda$, (Subadditivity)
(P4) $\mu(y + tq) = \mu(y) - t$ for $q \in \Lambda \setminus \{0\}$ (Translation Invariance),
then $\mu : \Lambda \to \mathbb{R}$ is a weak risk measure.

6. Conclusion

In this paper, the nonlinear scalarization function defined on a cone is extended to co-radiant in linear spaces, and its corresponding properties and subdifferential forms are obtained. we also find the relationship between scalar function and co-radiant(radiant) function. Consequently, the optimality condition of the $\varepsilon-$efficient point is studied by the approximate normal cone and the nonlinear scalarization function in multiobjective optimization. Ultimately, we get the optimality conditions of $\varepsilon-$efficient solutions by use of the "calculus rules" in vector optimization. In the following work, we can consider the application of the nonlinear scalar function in the actual risk measurement.

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