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Regularized Asymptotics of the Solution of Systems of Parabolic Differential Equations

Asan Omuraliev[a], Ella Abylaeva[a]

[a]*Kyrgyz-Turkish Manas University, Kyrgyz Republic, Bishkek*

Abstract. The regularization method for singularly perturbed problems of S. A. Lomov is generalized to constructing the asymptotics of the solution of the first boundary value problem for systems of differential equations of parabolic type with a small parameter at all derivatives.It is shown that the asymptotics of the solution of the problem contains n exponential, $2n$ parabolic and $2n$ angle boundary layer functions.The exponential boundary layer function describes the boundary layer along $t = 0$, the boundary layer along $x = 0$ and $x = 1$ is described by parabolic boundary layer functions.

1. Introduction

The papers [1] - [16] are devoted to the asymptotic solution of singularly perturbed systems of parabolic equations. In [1], a system of two equations is studied when a small parameter tends to zero and the complete degeneration of the differential operators occurs. It is shown that the asymptotics of the solution of this system in the vicinity of the initial and boundary points contains boundary layers. Moreover, in the vicinity of the boundary point $x = 0$, two boundary layers appear depending on a small parameter of different scales. Papers [2], [3] are devoted to the study of systems of two equations of parabolic type with nonsmooth boundary layer layers. The asymptotics of solutions of this problems were constructed using the smoothing procedure.In [4] a system of two singularly perturbed equations of the reaction-diffusion-transfer type is considered in the case of low diffusion and fast reactions. The asymptotics of the solution is constructed by the method of boundary functions using the smoothing procedure. An estimate of the remainder terms of the asymptotics is obtained using barrier functions. For the parabolic system of two singularly perturbed equations [5] which containing different degrees of the small parameter at the derivatives, the asymptotics of the solution is constructed, which characterized by a surge of one or both components of the solution in a neighborhood of a point evolving with time. In the paper [6], an initial-boundary value problem is considered for a singularly perturbed system of two parabolic equations, which degenerates into a system of a finite equation and a first-order differential equation. For a singularly perturbed system of two reaction-diffusion equations, a uniform asymptotics of the solution with respect to the small parameter is constructed in [7]. The proof of the existence of the solution and the estimate of the remainder terms of the asymptotics were carried out using the method of differential inequalities. In [8], conditions are found on the initial functions that ensure the existence of a nonstationary const structure of

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Email addresses: asan.omuraliev@manas.edu.kg (Asan Omuraliev), ella.abylaeva@manas.edu.kg (Ella Abylaeva)

the step type. In all the works described, an asymptotics of the boundary layer type was constructed. In the works described above, systems of two equations of parabolic type are studied and the n-dimensional system, when the required function is a matrix and the second spatial derivative is a scalar function are studied in [9]-[12]. The work [9] is devoted to the construction of the asymptotics of the boundary layer without using the smoothing procedure when the boundary layer function turns out to be nonsmooth. The boundary layer method [10] and the method of regularization of singularly perturbed problems [11] are used to construct the asymptotics of the solution in the case when the limiting equation is unsolvable. The case when the matrix is also at the second spatial derivative is studied in [12].

In this article, continuing the ideas of [12], we construct the asymptotics of the solution of an n-dimensional system of linear differential equations of parabolic type with a small parameter at all derivatives with matrix coefficients at the second spatial derivative. The presence of a small parameter at the time derivative leads to the appearance of an exponential boundary layer along the time axis, as well as angular boundary layers in the vicinity of points (0,0) and (1,0).

2. Statement of the problem

We consider the problem:

$$L_\varepsilon u(x,t,\varepsilon) \equiv \varepsilon\partial_t u - \varepsilon^3 A(x)\partial_x^2 u - D(t)u = f(x,t), \quad (x,t)\in\Omega, \tag{1}$$

$$u|_{t=0} = 0, \;\; u|_{x=0} = u|_{x=1} = 0,$$

where $\varepsilon > 0$ is a small parameter, $\Omega = (0 < x < 1)\times(0 < t \le T)$, $u = u(x,t,\varepsilon) = col(u_1, u_2, ..., u_n)$. Let be:

1. $A(x) \in C^\infty([0,1], C^{n\times n}), f(x,t) \in C^\infty(\overline{\Omega}, C^n)$;
2. Roots of the equation $det(A(x) - \lambda E) = 0$ satisfy the conditions: $Re\lambda_i(x) > 0, \lambda_i(x) \neq \lambda_j(x), \forall i \neq j$, E-identity matrix of n -th order;
3. Real parts of the eigenvalues $\mu_i(t)$, $i = \overline{1,n}$ of the matrix $D(t)$ are non-positive, i.e. $Re\mu_i(t)) \le 0$ moreover $\mu_i(t) \neq \mu_j(t)$, $\forall t \in [0,T]$, $i \neq j$, $i,j = \overline{1,n}$.

3. Regularization of the problem

We introduce regularizing variables:

$$\tau_i = \frac{1}{\varepsilon}\int_0^t \mu_i(s)ds \equiv \frac{\alpha_i(t)}{\varepsilon}, \;\; \xi_{i,l} = \frac{\varphi_{i,l}(x)}{\varepsilon^2}, \tag{2}$$

$$\varphi_{i,l}(x) = (-1)^{l-1}\int_{l-1}^x \frac{ds}{\sqrt{\lambda_i(s)}}, \quad \eta = \frac{t}{\varepsilon^2}, \;\; i = 1,2,...,n, \;\; l = 1,2,$$

and extended function $\tilde{u}(M,\varepsilon)$, $M = (x,t,\xi,\tau,\eta)$ such that:

$$\tilde{u}(M,\varepsilon)\,|_{\chi=G(x,t,\varepsilon)} \equiv u(x,t,\varepsilon)\;, \tag{3}$$

$$\chi = (\xi,\tau,\eta)\,, \;\; \xi = (\xi_1,\xi_2)\,, \;\; \xi_l = (\xi_{1,l},\xi_{2,l},...,\xi_{n,l})\;,$$

$$\varphi(x) = (\varphi_1,\varphi_2)\,, \varphi_l = (\varphi_{1,l},\varphi_{2,l},...,\varphi_{n,l})\,, \;\; \tau = (\tau_1,\tau_2,...,\tau_n)\,,$$

$$\alpha(t) = (\alpha_1(t),\alpha_2(t),...,\alpha_n(t)), \quad G(x,t,\varepsilon) = \left(\frac{\varphi(x)}{\varepsilon^2}, \frac{\alpha(t)}{\varepsilon}, \frac{t}{\varepsilon^2}\right).$$

Based on (2), we find the derivatives from (3):

$$\partial_t u(x,t,\varepsilon) \equiv \left(\partial_t \tilde{u} + \frac{1}{\varepsilon^2}\partial_\eta \tilde{u} + \frac{1}{\varepsilon}\sum_{i=1}^{n}\mu_i(t)\,\partial_{\tau_i}\tilde{u}\right)_{\chi=G(x,t,\varepsilon)},$$

$$\partial_x u(x,t,\varepsilon) \equiv \left(\partial_x \tilde{u} + \sum_{l=1}^{2}\sum_{i=1}^{n}\left[\frac{\varphi'_{i,l}(x)}{\varepsilon^2}\partial_{\xi_{i,l}}\tilde{u}\right]\right)_{\chi=G(x,t,\varepsilon)},$$

$$\partial_x^2 u(x,t,\varepsilon) \equiv \left(\partial_x^2 \tilde{u} + \sum_{l=1}^{2}\sum_{i=1}^{n}\left[\frac{\left(\varphi'_{i,l}(x)\right)^2}{\varepsilon^4}\partial^2_{\xi_{i,l}}\tilde{u}\right] + \frac{1}{\varepsilon^2}\sum_{i=1}^{n}L_{\xi_i}\tilde{u}\right)_{\chi=G(x,t,\varepsilon)},$$

$$L_{\xi_i} \equiv \sum_{l=1}^{2}(2\varphi'_{i,l}(x)\,\partial^2_{x,\xi_{i,l}} + \varphi''_{i,l}(x)\,\partial_{\xi_{i,l}}).$$

Due to the fact that the mixed derivatives with respect to $\xi_{i,l}$ and $\xi_{j,l}$ do not affect the course of constructing the asymptotics of the solution, they are not included in $\partial_x^2 u$.

To determine $\tilde{u}(M,\varepsilon)$ taking into account (1),(3) and the found derivatives we set the task:

$$\tilde{L_\varepsilon}\tilde{u}(M,\varepsilon) \equiv \frac{1}{\varepsilon}T_1 + T_2\tilde{u} - \varepsilon L_\xi \tilde{u} + \varepsilon\partial_t\tilde{u} - \varepsilon^3 L_x\tilde{u} = f(x,t), \quad M \in Q, \tag{4}$$

$$\tilde{u}|_{t=\tau=\eta=0} = 0, \;\; \tilde{u}|_{x=0,\xi_{i,1}=0} = \tilde{u}|_{x=1,\xi_{i,2}=0} = 0,$$

where-

$$T_1 \equiv \partial_\eta - \sum_{l=1}^{2}\sum_{i=1}^{n}A(x)\left(\varphi'_{i,l}\right)^2\partial^2_{\xi_{i,l}}, \;\; T_2 \equiv \sum_{i=1}^{n}\mu_i(t)\,\partial_{\tau_i} - D(t),$$

$$L_\xi \equiv A(x)\sum_{i=1}^{n}L_{\xi_i}, \;\; L_x \equiv A(x)\,\partial_x^2, \;\; Q = \{M : (x,t) \in \Omega; \xi, \tau, \eta \in (0,\infty)\}.$$

In this case, the following identity holds:

$$\tilde{L_\varepsilon}\tilde{u}(M,\varepsilon)\,|_{\chi=G(x,t,\varepsilon)} \equiv L_\varepsilon u(x,t,\varepsilon). \tag{5}$$

The solution of the extended problem (4) will be defined as a series:

$$\tilde{u}(M,\varepsilon) = \sum_{k=0}^{\infty}\varepsilon^k u_k(M). \tag{6}$$

Substituting (6) into problem (4) and equating the coefficients at the same powers of ε, we obtain the following equations:

$$T_1 u_0 = 0, \;\; T_1 u_1 = -T_2 u_0 + f(x,t), \;\; T_1 u_2 = -T_2 u_1 + L_\xi u_0 - \partial_t u_0,$$

$$T_1 u_k = -T_2 u_{k-1} + L_\xi u_{k-2} - \partial_t u_{k-2} + L_x u_{k-4}, \;\; u_{-1} \equiv 0, \;\; k \geq 3 \;\; M \in Q. \tag{7}$$

The initial and boundary conditions for them are set in the form:

$$u_k|_{t=\tau=\eta=0} = 0, \;\; u_k|_{x=l-1,\xi_l=0} = 0, \;\; l = 1,2, \;\; k \geq 0. \tag{8}$$

4. Solvability of Iterative Problems

Each of the problems (7) has an infinite number of solutions. Therefore, let us single out a class of functions in which these problems were uniquely solvable. Let us introduce the following classes of functions:

$$U_1 = \left\{ V(x,t): \ V(x,t) = \sum_{i=1}^{n} v_i(x,t)\,\psi_i(t), v_i(x,t) \in C^\infty(\overline{\Omega}) \right\},$$

$$U_2 = \left\{ Y(N): \ Y(N) = \sum_{l=1}^{2}\sum_{i=1}^{n} p_i^l(N_i^l) b_i(x), \ |p_i^l(N_i^l)| < C\exp\left(-\frac{\xi_{i,l}^2}{8\eta}\right) \right\},$$

$$U_3 = \left\{ C(x,t): C(x,t) = \sum_{i,j=1}^{n} c_{i,j}(x,t)\,\psi_i(t)\exp\left(\tau_j\right), c_{i,j}(x,t) \in C^\infty(\overline{\Omega}) \right\},$$

$$U_4 = \left\{ W(M): \ W(M) = \sum_{l=1}^{2}\sum_{i,j=1}^{n} \omega_{i,j}^l(N_i^l) b_i(x)\exp\left(\tau_j\right), \ |\omega_{i,j}^l(N_i^l)| < C\exp\left(-\frac{\xi_{i,l}^2}{8\eta}\right) \right\}.$$

From these classes of functions, we construct a new class as a direct sum:

$$U = U_1 \oplus U_2 \oplus U_3 \oplus U_4.$$

The function $u_k(M) \in U$ is representable in the form:

$$u(M) = \sum_{i=1}^{n} v_{k,i}(x,t)\,\psi_i(t) + \sum_{l=1}^{2}\sum_{i=1}^{n} p_i^{k,l}\left(N_i^l\right) b_i(x) +$$

$$\sum_{i,j=1}^{n}\left[c_{i,j}^k(x,t)\,\psi_i(t) + \sum_{l=1}^{2} \omega_{i,j}^{k,l}\left(N_i^l\right) b_i(x) \right]\exp\left(\tau_j\right), \ N_i^l = (x,t,\xi_{l,i},\eta). \tag{9}$$

A function from the class U_1 describes a regular term, a function from U_3 is an exponential boundary layer along $t = 0$, a function from U_2 is a parabolic boundary layer along $x = 0$ and $x = 1$, a function from U_4 is an angular boundary layer in the vicinity of points (0,0), (1,0).

The vector functions $b_i(x), \psi_i(t)$ in these classes are eigenfunctions of the matrices $A(x)$ and $D(t)$, respectively:

$$A(x)\,b_i(x) = \lambda_i(x)\,b_i(x),\ D(t)\,\psi_i(t) = \mu_i(t)\,\psi_i(t), i = \overline{1,n}. \tag{10}$$

According to condition 1) they are smooth in their arguments.

Along with eigenvectors $b_i(x)$ and $\psi_i(t)$ will be used eigenvectors as $b_i^*(x), d_i^*(t), i = \overline{1,n}$ of the conjugated matrices $A^*(x), D^*(t)$:

$$A^* b_i^* = \overline{\lambda}_i(x)\,b_i^*(x), D^*(t)\,d_i^*(t) = \overline{\mu}_i(t)\,d_i^*(t).$$

Moreover, they are chosen as biorthogonal:

$$\left(b_i(x), b_j^*(x) = \delta_{ij}, \left(\psi_i(t), \psi_i^*(t)\right)\right) = \delta_{i,j}, i,j = \overline{1,n}.$$

By calculating the action of the operators $T_1, T_2, ..., L_\xi, L_x$ on the function $u(M,\varepsilon)$ from (9) , with the index k, taking into account the relations (10) and

$$\varphi_i^{'2}(x) = \frac{1}{\lambda_i(x)}, \ \varphi_{i,l}(x) = (-1)^{l-1}\int_0^x \frac{ds}{\sqrt{\lambda_i(s)}}, \ i = \overline{1,n}, \ l = 1,2,$$

we have:

$$T_1 u_k(M) = \sum_{i=1}^{n} \sum_{l=1}^{2} \left\{ \partial_\eta p_i^{k,l}\left(N_i^l\right) - \partial_{\xi_{i,l}}^2 p_i^{k,l}\left(N_i^l\right) + \right.$$

$$\left. \sum_{j=1}^{n} \left[\partial_\eta \omega_{i,j}^{k,l}\left(N_i^l\right) - \partial_{\xi_{i,l}}^2 \omega_{i,j}^{k,l}\left(N_i^l\right) \right] \exp\left(\tau_j\right) \right\} b_i(x);$$

or in matrix form:

$$T_1 u_k(M) = \sum_{l=1}^{2} B(x) \left\{ \partial_t Y^{k,l}\left(N^l\right) - \partial_{\xi_l}^2 Y^{k,l}\left(N^l\right) + \right.$$

$$\left. \left[\partial_\eta W^{k,l}\left(N^l\right) - \partial_{\xi_l}^2 W^{k,l}\left(N^l\right) \right] \exp(\tau) \right\}.$$

Here $B(x)$ matrix-function (nxn) of which columns are eigenvectors $b_i(x)$ of the matrix $A(x)$.

Below, in Section 5, it will be shown that the scalar functions $p_i^{k,l}\left(N_i^l\right)$ and $\omega_{i,j}^{k,l}\left(N_i^l\right)$ are representable in the form:

$$p_i^{k,l}\left(N_i^l\right) = y_i^{k,l}(x,t)\, I_{i,1}^{k,l}(\xi_{i,l},\eta), \quad \omega_{i,j}^{k,l}\left(N_i^l\right) = q_{i,j}^{k,l}(x,t)\, I_{i,2}^{k,l}(\xi_{i,l},\eta),$$

where

$$|I_{i,m}^{k,l}(\xi_{i,l},\eta)| < c exp\left(-\frac{\xi_{i,l}}{8\eta}\right), \quad m = 1,2.$$

Taking into account these representations, we calculate:

$$T_2 u_k(M) = \sum_{j=1}^{n} \mu_j(t)\, \partial_{\tau_j} u_k - D(t)\, u_k = -\sum_{i=1}^{n} v_{k,i}(x,t)\mu_i(t)\psi_i(t) + \tag{11}$$

$$\sum_{i,j=1}^{n} \left(\mu_j(t) - \mu_i(t)\right) c_{i,j}^k(x,t)\psi_i(t) exp(\tau_j) - \sum_{l=1}^{2}\sum_{i=1}^{n} \left[p_i^{k,l}(N_i^l) D(t) b_i(x) + \right.$$

$$\left. \sum_{j=1}^{n} \left(\mu_j(t) b_i(x) - D(t) b_i(x)\right) \omega_{i,j}^{k,l}(x,t) exp(\tau_j) \right],$$

$$L_\xi u_k(M) = \sum_{l=1}^{2} A(x) \sum_{i=1}^{n} \left\{ \left[2\varphi_{i,l}' \left(b_i(x)\, y_i^{k,l}(x,t)\right)_x' + \varphi_{i,l}''(x) \left(b_i(x)\, y_i^{k,l}(x,t)\right) \right] \partial_{\xi_{i,l}} I_{i,1}^{k,l}(\xi_{i,l},\eta) + \right.$$

$$\left. \sum_{j=1}^{n} \left[2\varphi_{i,l}'(x) \left(b_i(x)\, \omega_{i,j}^{k,l}(x,t)\right)_x' + \varphi_{i,l}''(x) \left(b_i(x)\, \omega_{i,j}^{k,l}(x,t)\right) \right] \partial_{\xi_{i,l}} I_{i,2}^{k,l}(\xi_{i,l},\eta) \exp\left(\tau_j\right) \right\}, \tag{12}$$

$$\partial_t u_k = \sum_{i=1}^{n} \left[\partial_t \left(v_{k,i}(x,t)\, \psi_i(t)\right) + \sum_{j=1}^{n} \partial_t \left(c_{i,j}^k(x,t)\psi_i(t)\right) exp(\tau_j) \right] + \tag{13}$$

$$\sum_{l=1}^{2}\sum_{i=1}^{n} \left[\partial_t p_i^{k,l}(N_i^l) + \sum_{j=1}^{n} \partial_t \omega_{i,j}^{k,l}(x,t) exp(\tau_j) \right],$$

$$L_x u_k = A(x)\, \partial_x^2 \left(v_{k,i}(x,t)\, \psi_i(t)\right) + \sum_{l=1}^{n} \partial_x^2 \left(c_{i,j}^k(x,t)\right) \psi_i(t) \exp(\tau_j) +$$

$$\sum_{l=1}^{2}\left[\partial_x^2\left(p_i^{k,l}(N_i^l)b_i(x)\right)+\sum_{j=1}^{n}\partial_x^2\left(\omega_{i,j}^{k,l}(N_i^l)b_i(x)\right)\exp(\tau)\right]$$

With satisfying function $u_k(M)\in U$ to boundary conditions (8) :

$$p_i^{k,l}|_{\xi_{i,l}=0}=y_i^{k,l}(x,t),\quad p_i^{k,l}|_{\eta=0}=0,\ \ p_i^{k,l}|_{\xi_{i,l}=\infty}=0,\ \ l=1,2,$$

$$\omega_{i,j}^{k,l}|_{\eta=0}=0,\ \ \omega_{i,j}^{k,l}(x,t)\,|_{\xi_{i,l}=0}=d_{i,j}^{k,l}(x,t),\ \ \omega_{i,j}^{k,l}|_{\xi_{i,l}=\infty}=0,$$

$$c_{i,i}^{k}(x,0)=-v_{k,i}(x,0)-\sum_{j=1,(j\neq i)}^{n}c_{i,j}^{k}(x,0), \tag{14}$$

$$b_i(x)p_i^{k,l}(N^l)|_{x=l-1,\xi_{i,l}=0}=-v_{k,i}(l-1,t)\psi_i(t),\ \ b_i(x)\omega_{i,j}^{k,l}(N^l)|_{x=l-1,\xi_{i,l}=0}=-c_{i,j}^{k}(l-1,t)\psi_i(t).$$

From the (14) we define:

$$b_i(x)y_i^{k,l}(x,t)|_{x=l-1}=-v_{k,i}(l-1,t)\psi_i(t),\ \ \left(\omega_{i,j}^{k,l}(x,t)b_i(x)\right)|_{x=l-1}=-c_{i,j}^{k}(l-1,t)\psi_i(t). \tag{15}$$

Obtaining these relationships is described in detail in Section 5.

Iterative equations (7) can be written in the form:

$$T_1u_k(M)=H_k(M). \tag{16}$$

Theorem 4.1. *Let the conditions 1)-3) be satisfied and $H_k(M)\in U_2\oplus U_4$, then the equation (16) has a solution $u_k(M)\in U$.*

Proof. Let be:

$$H_k(M)=\sum_{l=1}^{2}\sum_{i=1}^{n}\left[h_i^{k,l}\left(N_i^l\right)+\sum_{j=1}^{n}h_{i,j}^{k,l}\left(N_i^l\right)\exp\left(\tau_j\right)\right]b_i(x).$$

With satisfying function $u_k(M)\in U$ from (9) to the equation (16) and considering calculations (4), with respect to $p_i^{k,l}(N_i^l)$, $\omega_{i,j}^{k,l}(N^l)$ we obtain the equations:

$$\partial_\eta p_i^{k,l}\left(N_i^l\right)=\partial_{\xi_{i,l}}^2p_i^{k,l}\left(N_i^l\right)+h_{k,l}^{i}\left(N_i^l\right),$$

$$\partial_\eta\omega_{i,j}^{k,l}\left(N_i^l\right)=\partial_{\xi_{i,l}}^2\omega_{i,j^i}^{k,l}\left(N_i^l\right)+h_{i,j}^{k,l}\left(N_i^l\right),\ \ i,j=1,2,...,n \tag{17}$$

Under the appropriate boundary conditions these equations have solutions satisfying the estimates ([18],pp.81):

$$|\omega_{i,j}^{k,l}(N^l)|<C\exp\left(-\frac{\xi_{i,l}^2}{8\eta}\right),\quad |p_i^{k,l}(N^l)|<C\exp\left(-\frac{\xi_{i,l}^2}{8\eta}\right).$$

□

Theorem 4.2. *Let conditions 1)-3) be satisfied, then the equation (16) with additional conditions:*

1. $u|_{t=\tau=\eta=0}=0, u|_{x=l-1,\xi_{i,l}=0}=0, l=\overline{1,2},$
2. $-T_2u_{k-1}-\partial_tu_{k-2}+L_xu_{k-4}\in U_2\oplus U_4,$
3. $L_\xi u_{k-2}(M)=0$

has the only solution in U.

Proof. Satisfying the function $u_k(M) \in U$ with the boundary conditions from (8) we obtain (6) . Based on calculations (11) – (13) condition 2. of the theorem can be written as:

$$F_k(M) \equiv -T_2 u_{k-1} - \partial_t u_{k-2} + L_x u_{k-4} = -\Psi(t)\left[C^{k-1}(x,t)\Lambda(\mu) - \Lambda(\mu)C^{k-1}(x,t)\right]\exp(\tau) -$$

$$\sum_{l=1}^{2}\left[B(x)W^{k-1,l}\left(N^l\right)\Lambda(\mu) + D(t)B(x)W^{k-2,l}\left(N^l\right)\right]\exp(\tau) +$$

$$D(t)V_{k-1}(x,t) + \sum_{l=1}^{2} D(t)B(x)Y^{l,k-1}\left(N^l\right) - \partial_t V_{k-2}(x,t) -$$

$$\sum_{l=1}^{2}\partial_t\left(B(x)Y^{l,k-2}\right) - \partial_t\left(\psi(t)C^{k-2}(x,t)\right)\exp(\tau) -$$

$$\sum_{l=1}^{2}\partial_t\left(B(x)W^{k-2,l}\right)\exp(\tau) +$$

$$A(x)\partial_x^2 V_{k-4}(x,t) + \sum_{l=1}^{2} A(x)\partial_x^2\left(B(x)Y^{l,k-4}\left(N^l\right)\right) +$$

$$A(x)\partial_x^2\left(\Psi(t)C^{k-4}(x,t)\right)\exp(\tau) +$$

$$\sum_{l=1}^{2} A(x)\partial_x^2\left(B(x)W^{k-4,l}\left(N^l\right)\right)\exp(\tau).$$

Hence, ensuring the fulfillment of condition 2) of Theorem 4.2, we set:

$$D(t)\Psi(t)(t)V_{k-1}(x,t) = \partial_t\left(\Psi(t)V_{k-2}(x,t)\right) - A(x)\Psi(t)\partial_x^2 V_{k-4}(x,t), \tag{18}$$

$$\Psi(t)\left[C^{k-1}(x,t)\Lambda(\mu) - \Lambda(\mu)C^{k-1}(x,t)\right] =$$

$$-\partial_t\left(\Psi(t)C^{k-2}(x,t)\right) + A(x)\partial_x^2\left(\psi(t)C^{k-4}(x,t)\right), \tag{19}$$

then the right side will take the form:

$$F_k(M) = -\sum_{l=1}^{2}\left\{\left[B(x)W^{k-1,l}\left(N^l\right)\Lambda(\mu) - D(t)B(x)W^{k-1,l}\left(N^l\right) -\right.\right.$$

$$\partial_t\left(B(x)W^{k-2,l}\left(N^l\right)exp(\tau)+\right.$$

$$D(t)B(x)Y^{l,k-1}\left(N^l\right) - \partial_t\left(B(x)Y^{k-2,l}\left(N^l\right) +\right.$$

$$\left.A(x)\partial_x^2\left(B(x)Y^{k-4,l}\left(N^l\right)\right)\right\} \in U_2 \oplus U_4.$$

By Theorem 4.1, the equation $T_1 u_k(M) = F_k(M)$ has a solution $u_k(M) \in U$.

Ensuring condition 3) of the theorem, we put in (12):

$$2\varphi'_{i,l}(x)\left(b_i(x)y_i^{k,l}(x,t)\right)'_x + \varphi''_{i,l}(x)\left(b_i(x)y_i^{k,l}(x,t)\right) = 0,$$

$$2\varphi'_{i,l}(x)\left(b_i(x)\omega_{i,j}^{k,l}(x,t)\right)'_x + \varphi''_{i,l}(x)\left(b_i(x)\omega_{i,j}^{k,l}(x,t)\right) = 0. \tag{20}$$

Solving these equations under the initial condition (15), we uniquely define $y_i^{k,l}(x,t)$ and $\omega_{i,j}^{k,l}(x,t)$.

The system (19) is solvable if the diagonal elements of the right-hand side satisfy the following relations:

$$\partial c_{i,i}^{k-2}(x,t) + \gamma_{i,i}(t)\, c_{i,i}^{k-2}(x,t) =$$

$$\sum_{k=1}^{n} \beta_{k,i}(x,t)\, \partial_x^2 c_{k,i}^{k-4}(x,t) - \sum_{k=1,k\neq i}^{n} \gamma_{k,i}(t)\, c_{ki}^{k-2}(x,t), \tag{21}$$

$$\gamma_{i,k}(t) = \left(\psi_i'(t), \psi_k^*(t)\right), \;\; \beta_{i,k}(x,t) = \left(A(x)\,\psi_i(t), \psi_k^*(t)\right).$$

From the equation (21) with the initial condition from (15) we define $c_{i,i}^{k-2}(x,t)$. This ensures the decidability of the system (19).

Equations with respect to $Y^{k,l}\left(N^l\right)$ and $W^{k,l}\left(N^l\right)$ with free term $F_k(M)$, by Theorem 4.1 they are solvable and their solutions under the boundary conditions from (14) are representable in the form:

$$p_i^{k,l}\left(N_i^l\right) = y_i^{k,l}(x,t)\, erfc\left(\frac{\xi_{i,l}}{2\sqrt{\eta}}\right) + I_i^{k,l}\left(N_i^l\right),$$

$$\omega_{i,j}^{k,l}\left(N_i^l\right) = \omega_{i,j}^{k,l}(x,t)\, erfc\left(\frac{\xi_{i,l}}{2\sqrt{\eta}}\right) + I_{i,j}^{k,l}\left(N_i^l\right),$$

$$I_i^{k,l}\left(N_i^l\right) = \frac{1}{2\sqrt{\pi}}\int_0^{\eta}\int_0^{\infty} \frac{H_{k,i}^{1,1}(\cdot)}{\sqrt{\eta-z}}\left[\exp\left(-\frac{(\xi_{i,l}-s)^2}{4(\eta-z)}\right) - \exp\left(-\frac{(\xi_{il}+s)^2}{4(\eta-z)}\right)\right] ds dz,$$

$$I_{i,j}^{k,l}\left(N_i^l\right) = \frac{1}{2\sqrt{\pi}}\int_0^{\eta}\int_0^{\infty} \frac{H_{i,j}^{2,1}(\cdot)}{\sqrt{\eta-z}}\left[\exp\left(-\frac{(\xi_{i,l}-s)^2}{4(\eta-z)}\right) - \exp\left(-\frac{(\xi_{il}+s)^2}{4(\eta-z)}\right)\right] ds dz.$$

In this case the following estimates are valid [14] pp.81:

$$\left|I_i^{k,l}\left(N_i^l\right)\right| < c\exp\left(-\frac{\xi_{il}^2}{8\eta}\right),$$

$$\left|I_{ij}^{k,l}\left(N_i^l\right)\right| < c\exp\left(-\frac{\xi_{il}^2}{8\eta}\right),$$

$$\left|erfc\left(\frac{\xi_{i,l}^2}{2\sqrt{\eta}}\right)\right| < c\exp\left(-\frac{\xi_{il}^2}{8\eta}\right).$$

Thus, the function is uniquely determined $u_k(M) \in U$. □

5. Solution of iterative problems

The iterative equation (7) at $k = 0$ is homogeneous, therefore, by Theorem 4.1, it has a solution $u_0(M) \in U$. With substituting $u_0(M)$ from (9) into equation (7) at $k = 0$, based on calculations (15), with respect to $p_i^{0,l}\left(N^l\right)$ and $\omega_{i,j}^{0,l}\left(N^l\right)$, $l = 1,2$, $i,j \geq 1,2,...,n$ we obtain scalar equations:

$$\partial_\eta p_i^{0,l}(N^l) = \partial_{\xi_{i,l}}^2 p_i^{0,l}(N^l),$$

$$\partial_\eta \omega_{i,j}^{0,l}(N^l) = \partial_{\xi_{i,l}}^2 \omega_{i,j}^{0,l}(N^l),$$

solving the above equations under the boundary conditions:

$$p_i^{0,l}(N^l)|_{\eta=0} = 0, p_i^{0,l}(N^l)|_{\xi_{i,l}=0} = y_i^{0,l}(x,t),$$

$$\omega_{i,j}^{0,l}(N^l)|_{\eta=0} = 0, \omega_{i,j}^{0,l}(N^l)|_{\xi_{i,l}=0} = q_{i,j}^{0,l}(x,t),$$

we find:

$$p_i^{0,l}(N^l) = y_i^{0,l}(x,t)\, erfc\left(\frac{\xi_{i,l}}{2\sqrt{\eta}}\right),$$

$$\omega_{i,j}^{0,l}(N^l) = q_{i,j}^{0,l}(x,t)\, erfc\left(\frac{\xi_{i,l}}{2\sqrt{\eta}}\right).$$

To uniquely determine $u_0(M)$ we apply conditions 1) -3) of Theorem 4.2. The condition 3), i.e. $L_\xi u_0(M) = 0$ is equivalent to solution of the problem (20) , (15), whence we define:

$$b_i(x)y_i^{0,l}(x,t) = v_{i,0}(l-1,t)\nu_{i,l}(x)\psi_i(t),$$

$$b_i(x)\omega_{i,j}^{0,l}(x,t) = \nu_{i,l}(x)c_{i,j}^0(l-1,t)\psi_i(t), \quad \nu_{i,l}(x,t) = -\sqrt{\frac{\varphi_{i,l}'(0)}{\varphi_{i,l}'(x)}}.$$

Taking into account the last relations, we calculate:

$$F_1(M) = -T_2u_0(M) + f(x,t) = -\sum_{j=1}^{n}\mu_j(t)\partial_{\tau_j}u_0 + D(t)u_0 =$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left[c_{i,j}^0(x,t)\left(\mu_i(t)-\mu_j(t)\right)\psi_i(t) - \sum_{l=1}^{2}\left(\mu_j(t)b_i(x)\omega_{i,j}^{0,l}(x,t) + D(t)b_i(x)\omega_{i,j}^{0,l}(x,t)\right)erfc\left(\frac{\xi_{i,l}}{2\sqrt{\eta}}\right)\right]\times$$

$$\exp(\tau_j) + \sum_{i=1}^{n}(\mu_i(t)\psi_i(t)v_{i0}(x,t)) + \sum_{l=1}^{2}\sum_{i=1}^{n}D(t)b_i(x)y_i^{0,l}(x,t)erfc\left(\frac{\xi_{il}}{2\sqrt{\eta}}\right) = \quad (22)$$

$$-\sum_{i,j=1}^{n}\left[\left(\mu_j(t)-\mu_i(t)\right)c_{i,j}^0(x,t) + \sum_{l=1}^{2}\left(\mu_j(t)-\mu_i(t)\right)c_{i,j}^0(l-1,t)\nu_{i,l}(x)erfc\left(\frac{\xi_{i,l}}{2\sqrt{\eta}}\right)\right]\times$$

$$\psi_i(t)\exp(\tau_j) + \sum_{i=1}^{n}\mu_i(t)\psi_i(t)v_{i,0}(x,t) + \sum_{l=1}^{2}\sum_{i=1}^{n}\mu_i(t)\nu_{i,l}(x)v_{i,0}(l-1,t)\psi_i(t)erfc\left(\frac{\xi_{i,l}}{2\sqrt{\eta}}\right) +$$

$$+\sum_{i=1}^{n}\left(f(x,t),\psi_i^*(t)\right)\psi_i(t).$$

To ensure that this function belongs to the class $U_2 \otimes U_4$, we set $\left(\mu_j(t)-\mu_i(t)\right)c_{i,j}^0(x,t) = 0$, $v_{i,0}(x,t) = -\frac{1}{\mu_i(t)}\left(f(x,t),\psi_i^*(t)\right)$. From here we define that: $c_{i,j}^0(x,t) = 0, \forall i \neq j$, and the function $c_{i,i}^0(x,t)$ is still arbitrary and it will be defined in the next step. Taking into account that $L_\xi u_0(M) = 0$, the free term of the iterative equation (7) for $k = 2$ can be represented in the following form:

$$F_2(M) = -T_2u_1 - \partial_t u_0 =$$

$$-\sum_{i,j=1}^{n}\left[(\mu_j(t)-\mu_i(t))c_{i,j}^1(x,t) + \sum_{l=1}^{2}(\mu_j(t)-\mu_i(t))\nu_{i,l}(x)c_{i,j}^1(l-1,t)erfc\left(\frac{\xi_{i,l}}{2\sqrt{\eta}}\right)\right]\times$$

$$\psi_i(t)\exp(\tau_j)+\sum_{i=1}^{n}\mu_i(t)\psi_i(t)v_{i,1}(x,t)+$$

$$\sum_{l=1}^{2}\sum_{i=1}^{n}\mu_i(t)\nu_{i,l}(x)v_{i,1}(l-1,t)\psi_i(t)erfc\left(\frac{\xi_{i,l}}{2\sqrt{\eta}}\right)-$$

$$\sum_{i=1}^{n}\left\{\partial_t v_{i,0}(x,t)+\left[\partial_t c_{i,i}^{0}(x,t)\exp(\tau_i)+\sum_{j=1}^{n}\left(\psi_j'(t),\psi_i^*(t)\right)\left(v_{j,0}(x,t)+\exp(\tau_j)c_{j,j}^{0}(x,t)\right)\right]\right\}\psi_i(t)-$$

$$\sum_{l=1}^{2}\sum_{i=1}^{n}\left[\partial_t p_i^{0,l}(N^l)+\sum_{j=1}^{n}\partial_t\omega_{i,j}^{0,l}(N^l)\exp(\tau_j)\right]b_i(x)erfc\left(\frac{\xi_{i,l}}{2\sqrt{\eta}}\right).$$

By assuming

$$\left(\mu_j(t)-\mu_i(t)\right)c_{i,j}^{1}(x,t)=\sum_{j\neq i}(\psi_j'(t),\psi_i(t))c_{j,j}^{0}(x,t),$$

$$\mu_j(t)v_{i1}(x,t)=-\partial_t v_{i0}(x,t)-\sum_{m=1}^{n}\left(\psi_m'(t),\psi_i^*(t)\right)v_{m0}(x,t),$$

$$\partial_t c_{ii}^{0}-\left(\psi_i'(t),\psi_i^*(t)\right)c_{i,i}^{0}(x,t)=0$$

we ensure that the function belongs to $F_2(M)\in U_2\otimes U_4$. From these equations we define $c_{ij}^{1}(x,t)=0,\forall i\neq j, v_{i1}(x,t)$, the function $c_{ii}^{1}(x,t)$ is arbitrary. We solve the last equation under the initial condition:

$$c_{i,i}^{0}(x,t)|_{t=0}=-v_{i,0}(x,0).$$

By this completely the main term of the asymptotics is uniquely determined.

Then the process is repeated, but when solving the iterative equations (7) in free terms $F_k(M)$ should be switched from $\nu_{il}(x)c_{ij}^{k}(l-1,t)\psi_i(t)$ and $\nu_{il}(x)v_{ik}(l-1,t)\psi_i(t)$ to $b_i(x)y_i^{k,l}(x,t)$ and $b_i(x)q_{ij}^{k,l}(x,t)$. Thus, we construct all the members of the partial sum:

$$u_{\varepsilon n}(M)=\sum_{k=0}^{n}\varepsilon^k u_k(M).$$

6. Estimation of the remainder term

Let's substitute the function

$$u(M,\varepsilon)=u_{\varepsilon n}(M)+\varepsilon^{n+1}R_{\varepsilon n}(M,\varepsilon)=\sum_{k=0}^{n+1}\varepsilon^k u_k(M)-\varepsilon^{n+1}u_{n+1}(M)+\varepsilon^{n+1}R_{\varepsilon_n}(M,\varepsilon) \tag{23}$$

into the extended equation (4) . Based on (7) with respect to the remainder $R_{\varepsilon n}(M,\varepsilon)$ we obtain the equation:

$$\tilde{L}_\varepsilon R_{\varepsilon n}(M)=-T_2u_{n+1}(M)-\sum_{l=1}^{2}\varepsilon^{l-1}\partial_t u_{n+l-1}+\sum_{l=1}^{4}\varepsilon^{l-1}L_x u_{n+l-3}+\tilde{L}_\varepsilon u_{n+1}(M)\equiv g_n(M,\varepsilon). \tag{24}$$

In the equation (24) we make a narrowing through regularizing functions, then introducing the notation:

$$R_{\varepsilon,n}(x,t,\varepsilon)\equiv R_{\varepsilon,n}(M,\varepsilon)\big|_{\chi=G(x,t,\varepsilon)},\quad g_n(x,t,\varepsilon)\equiv g_n(M,\varepsilon)\big|_{\chi=G(x,t,\varepsilon)}$$

and based on (5), we get:

$$L_\varepsilon R_{\varepsilon,n}(x,t,\varepsilon) = g_n(x,t,\varepsilon),\ \ R_{\varepsilon,n}|_{t=0} = R_{\varepsilon,n}|_{x=l-1} = 0,\ \ l = 1,2.$$

Using the representation:

$$u_{n+1} = \sum_{i=1}^{n}\left[v_{n+1,i}(x,t) + \sum_{j=1}^{n} c_{i,j}^{n+1}(x,t)\exp\left(\frac{\alpha_j(t)}{\varepsilon}\right)\right]\psi_i(t)+$$

$$\sum_{l=1}^{2}\sum_{i=1}^{n}\left[p_i^{n+1,l}(x,t,\frac{\varphi_{i,l}(x)}{\varepsilon^2},\frac{t}{\varepsilon^2}) + \sum_{j=1}^{n}\omega\left(x,t,\frac{\varphi_{i,l}(x)}{\varepsilon^2},\frac{t}{\varepsilon^2}\right)\exp\left(\frac{\alpha_j(t)}{\varepsilon}\right)\right]b_i(x)$$

we calculate the action of the operator $L_\varepsilon u_{n+1}$. The functions $v_{n+1,i}(x,t), c_{i,j}^{n+1}(x,t)$ included in the first sums are determined from equations (18), (19) under corresponding initial conditions from (14). Based on conditions 1), 2) they are bounded $\forall(x,t)\in\bar{\Omega}, i,j = 1,2,..,n$. The functions $p_i^{n+1}(\bullet), \omega_{i,j}^{n+1,l}(\bullet)$ included in the second sums are defined in Theorem 4.2 and they contain the following integrals which bounded in the domain $\bar{\Omega}$:

$$\left|\int_0^\infty \frac{\varepsilon^3 H(.)}{\sqrt{\left(\frac{t}{\varepsilon^2}-z\right)^3}}\left(\frac{\varphi_{i,l}(x)}{\varepsilon^2}-s\right)\frac{\varphi'_{i,l}(x)}{\varepsilon^2}\exp\left(-\frac{\left(\frac{\varphi_{i,l}(x)}{\varepsilon^2}-s\right)^2}{4\left(\frac{t}{\varepsilon^2}-z\right)}\right)dsdz\right| < c\varepsilon\exp\left(-\frac{\varphi_{i,l}^2(x)}{8\varepsilon^2 t}\right),$$

$$\left|\int_0^\infty \frac{\varepsilon H(.)}{\sqrt{\left(\frac{t}{\varepsilon^2}-z\right)^3}}\exp\left(-\frac{\left(\frac{\varphi_{i,l}(x)}{\varepsilon^2}-s\right)^2}{4\left(\frac{t}{\varepsilon^2}-z\right)}\right)dsdz\right| < c\varepsilon\exp\left(-\frac{\varphi_{i,l}^2(x)}{8\varepsilon^2 t}\right).$$

By analogy with [19] (pp. 72), in this problem we pass to the Euclidean norms, we obtain the scalar equation:

$$\varepsilon\partial_t r(x,t,\varepsilon) - \varepsilon^3 a(x)\partial_x^2 r(x,t,\varepsilon) - d(t)r(x,t,\varepsilon) = |g_n(x,t,\varepsilon)|,$$

$$r(x,t,\varepsilon) \equiv |R_{\varepsilon,n}(x,t,\varepsilon)|,\ \ a(x) \equiv \|A(x)\|,\ \ d(t) \equiv \|D(t)\|.$$

Further, repeating the arguments of Theorem 2.1 from [21], we obtain an estimate of the form (2.12) from [21]. In view of the above estimates and the form of the constructed asymptotics, we have: $|g_n(x,t,\varepsilon)| < c$, taking into account estimate (2.12) from [21], we obtain the following estimate:

$$|r(x,t,\varepsilon)| < c\ \ \forall(x,t)\in\overline{\Omega},\ \ \varepsilon > 0.$$

This inequality can be proved by the method of [20].

Theorem 6.1. *Let condition 1)-3) be satisfied, then for the solution of the system (24) for any $x,t\in\overline{\Omega}$ and sufficiently small $\varepsilon > 0$ the following estimate is valid:*

$$\|u(x,t,\varepsilon) - u_{\varepsilon,n}(x,t,\varepsilon)\| < \varepsilon^{n+1}\|R_{\varepsilon n}(x,t,\varepsilon)\| < c\varepsilon^{n+1}.$$

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