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Approximation by a Family of Summation-Integral Type Operators Preserving Linear Functions

Brijesh Kumar Grewal[a], Meenu Goyal[a]

[a]*Thapar Institute of Engineering and Technology, Patiala- 147004, India.*

Abstract. This article investigates the approximation properties of a general family of positive linear operators defined on the unbounded interval $[0,\infty)$. We prove uniform convergence theorem and Voronovskaya-type theorem for functions with polynomial growth. More precisely, we study weighted approximation *i.e* basic convergence theorems, quantitative Voronovskaya-asymptotic theorems and Grüss Voronovskaya-type theorems in weighted spaces. Finally, we obtain the rate of convergence of these operators via a suitable weighted modulus of continuity.

1. Introduction

For the past several years, many authors have been introducing and studying general families of positive linear operators. The advantage of these families of unified operators is that, instead of investigating the approximation properties of each operator separately, one can study the compact form and draw conclusions about individual operators. Historically in 1980, Mastroianni [1] constructed a class of discrete operators to approximate unbounded functions on $[0,\infty)$. For this purpose, he defined the following operators,

$$L_{n,c}(g,x) = \sum_{k=0}^{\infty} b_{n,k}^{c}(x) g\left(\frac{k}{n}\right),$$

where

$$b_{n,k}^{c}(x) = (-1)^k . \frac{x^k}{k!} g_n^{(k)}(c,x) \ , \ g_n(c,x) = \begin{cases} (1+cx)^{\frac{-n}{c}}, & c>0 \\ e^{-nx}, & c=0. \end{cases}$$

and (g_n) is a sequence of real functions defined on $[0,\infty)$.

A detailed study of the operators $L_{n,c}(g,x)$ and their Kantorovich modification was carried out by Agratini et al. [2], who established new convergence theorems and related results for both operators. Finta and Gupta [3] proposed a generalization of Phillips operators and articulated direct and converse theorems via

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Email addresses: grewalbriz@gmail.com (Brijesh Kumar Grewal), meenu_rani@thapar.edu (Meenu Goyal)

second order Ditizian-Totik modulus of smoothness. Another interesting generalization of these operators is due to Păltănea [4], in which he modified the basis function of Szász Mirakyan and Phillips operators, and formulated approximation results for functions defined on compact intervals by using modulus of continuity. Miheşan [5] presented a sequence $M_n^{\alpha}(g,x)$ of discrete operators by applying gamma transformation to Szász operators. Gupta and Agarwal [6] defined the integral modification of the operators $M_n^{\alpha}(g,x)$, and determined the error of approximation for continuous and bounded functions by means of second order modulus of continuity. They also analyzed the approximation properties of a Bézier variant of the modified operators. The sequence of Durrmeyer type operators which is basically a modification of Ibragimov-Gadjiev operators was studied by Tuncer [7]. He estimated the rate of convergence for functions having derivative of bounded variation. The authors in [8] defined a parametric family of hybrid operators by taking into consideration the generalized basis function of Baskakov and Szasz-Mirakjan operators . Together with the approximation results, they also discussed basic convergence theorems and A-statistical convergence theorem in polynomial weighted spaces.

Since the classical modulus of continuity cannot be used in the linear approximation process (via positive operators) of functions defined on positive real axis. Therefore, researchers have constructed different types of weighted modulus of continuity and studied the well-known approximation results like Korovokin's type theorem, Voronovskaya type and Grüss-Voronovskaya type theorems for various operators.
In this direction, Ulusoy and Acar [9] provided quantitative Voronovskaya type and Grüss-Voronovskaya type results for Baskakov operators using weighted modulus of continuity. Based on these results, an upper bound for the error of approximation was also discussed. In [10], Tuncer et al. proved weighted Voronovskaya theorem for operators defined on unbounded intervals by using the estimation of remainder term in Taylor's formula. Furthermore, Grüss inequality and Grüss Voronovskaya-type theorem were also established.
The weighted approximation of modified Baskakov-Szász-Stancu operators were deeply examined by Bodur et al. [11]. They established Voronovskaya asymptotic formula, uniform convergence theorem in exponential weighted spaces, and obtained better rate of approximation. For more development on weighted approximation, we suggest [12], [13], [14], [15], [16], [17].

In 2019, Gupta [18] generated a parametric family of positive linear operators preserving linear functions. For the different values of parameters α,β and $\rho > 0$, this family contains discrete operators, Durrmeyer type operators and hybrid operators. For each $x \in [0,\infty)$, Gupta defined

$$\mathcal{A}_{n,\alpha}^{\beta,\rho}(g,x) = \sum_{k=1}^{\infty} p_{n,k}^{\alpha}(x) \int_0^{\infty} q_{n,k-1}^{\beta+1,\rho}(t)g(t)dt + p_{n,0}^{\alpha}(x)g(0), \tag{1}$$

where

$$p_{n,k}^{\alpha}(x) = \frac{(\alpha)_k}{k!}\frac{(\frac{nx}{\alpha})^k}{(1+\frac{nx}{\alpha})^{\alpha+k}},\ q_{n,k-1}^{\beta+1,\rho}(t) = \frac{n}{\beta.B(k\rho,\beta\rho+1)}\frac{(\frac{nt}{\beta})^{k\rho-1}}{(1+\frac{nt}{\beta})^{\beta\rho+k\rho+1}},$$

with the rising factorial $(\alpha)_k = \alpha(\alpha+1)...(\alpha+k-1)$ and $(\alpha)_0 = 1$.
He estimated the moments and performed error analysis of these operators for continuous and bounded functions with the aid of second order modulus of continuity. Further, a link between these operators and Miheşan operators is established as well.

This article is principally concerned with the approximation behavior of the operators $\mathcal{A}_{n,\alpha}^{\beta,\rho}(g,x)$ for some subspaces of $C[0,\infty)$. In section 2, we derive central moments and state an important Lemma based on certain assumptions on the parameters α and β. Section 3 contains the uniform convergence theorem and a Voronovskaya-type theorem for the operators (1), and an error of approximation for functions in Lipschitz-type space. Section 4 and 5 is devoted to weighted approximation theorems, i.e., a convergence theorem, quantitative Voronovskaya-type theorem and Grüss Voronovskaya-type theorem in polynomial weighted spaces.

2. Moments Estimation

Moments play a significant role in studying the approximation properties of positive linear operators. We calculate central moments for the operators (1) with the help of moments derived in [18].

Lemma 2.1. *For the operators $\mathcal{A}_{n,\alpha}^{\beta,\rho}(.,x)$, we have the following identities*

(i) $\mathcal{A}_{n,\alpha}^{\beta,\rho}(e_0,x) = 1$;

(ii) $\mathcal{A}_{n,\alpha}^{\beta,\rho}(e_1,x) = x$;

(iii) $\mathcal{A}_{n,\alpha}^{\beta,\rho}(e_2,x) = \dfrac{\beta}{\beta\rho-1}\left[\rho x^2\left(1+\dfrac{1}{\alpha}\right)+\dfrac{(\rho+1)x}{n}\right]$,

where $e_i(x) = x^i$, for $= 0, 1, 2$.

Lemma 2.2. *From Lemma 2.1, we get the following central moments for the operators (1).*

(i) $\mathcal{A}_{n,\alpha}^{\beta,\rho}\Big((t-x);x\Big) = 0$;

(ii) $\mathcal{A}_{n,\alpha}^{\beta,\rho}\Big((t-x)^2;x\Big) = \dfrac{x^2(\alpha+\beta\rho)}{\alpha(\beta\rho-1)}+\dfrac{x(\beta+\beta\rho)}{n(\beta\rho-1)}$;

(iii) $\mathcal{A}_{n,\alpha}^{\beta,\rho}\Big((t-x)^3;x\Big) = \dfrac{2x^3(\alpha+\beta\rho)(2\alpha+\beta\rho)}{\alpha^2(\beta\rho-1)(\beta\rho-2)}+\dfrac{3\beta x^2(\rho+1)(\beta\rho+2\alpha)}{n\alpha(\beta\rho-1)(\beta\rho-2)}+\dfrac{\beta^2 x(\rho+1)(\rho+2)}{n^2(\beta\rho-1)(\beta\rho-2)}$;

(iv)
$$\begin{aligned}\mathcal{A}_{n,\alpha}^{\beta,\rho}\Big((t-x)^4;x\Big) =& \frac{3(6\alpha^3+\alpha^2(12+\alpha)\beta\rho+2\alpha(4+\alpha)\beta^2\rho^2+(2+\alpha)\beta^3\rho^3)}{\alpha^3(\beta\rho-1)(\beta\rho-2)(\beta\rho-3)}x^4\\ &+\frac{6\beta(1+\rho)(6\alpha^2+\alpha(6+\alpha)\beta\rho+(2+\alpha)\beta^2\rho^2)}{n\alpha^2(\beta\rho-1)(\beta\rho-2)(\beta\rho-3)}x^3\\ &+\frac{\beta^2(\rho+1)(\beta\rho(7\rho+11)+3\alpha(8+\rho(4+\beta+\beta\rho)))}{n^2\alpha(\beta\rho-1)(\beta\rho-2)(\beta\rho-3)}x^2\\ &+\frac{\beta^3(\rho^3+6\rho^2+11\rho+6)}{n^3(\beta\rho-1)(\beta\rho-2)(\beta\rho-3)}x;\end{aligned}$$

(v)
$$\begin{aligned}\mathcal{A}_{n,\alpha}^{\beta,\rho}\Big((t-x)^6;x\Big) =& \frac{1}{A}\Big\{x^6\Big(n^5(\beta^5\rho^5(15\alpha^2+130\alpha+120)+\beta^4\rho^4(45\alpha^3+690\alpha^2+720\alpha)+\beta^3\rho^3(45\alpha^4\\ &+1420\alpha^3+1800\alpha^2)+\beta^2\rho^2(15\alpha^4+1290\alpha^4+2400\alpha^3)+\beta\rho(430\alpha^5+1800\alpha^4)\\ &+600\alpha^5)\Big)+x^5\Big(n^4\alpha\beta(\rho+1)(\beta^4\rho^4(45\alpha^2+390\alpha+360)+\beta^3\rho^3(1680\alpha^2+1800\alpha)\\ &+\beta^2\rho^2(45\alpha^4+2580\alpha^3+3600\alpha^2)+\beta^1\rho^1(1290\alpha^4+3600\alpha^3)+1800\alpha^4)\Big)\\ &+x^4\Big(n^3\alpha^2(\beta^5\rho^5(45\alpha^2-415\alpha+390)+\beta^5\rho^4(90\alpha^2+930\alpha+900)+\beta^5\rho^3(45\alpha^2\\ &+515\alpha+510)+\beta^4\rho^4(45\alpha^3+1305\alpha^2+1500\alpha)+\beta^4\rho^3(90\alpha^3+2970\alpha^2+3600\alpha)\\ &+\beta^4\rho^2(45\alpha^3+1665\alpha^2+2100\alpha)+\beta^3\rho^3(1160\alpha^3+2100\alpha^2)+\beta^3\rho^2(2580\alpha^3\\ &+5400\alpha^2)+\beta^3\rho(1420\alpha^3+3300\alpha^2)+\beta^2(1200\alpha^3\rho^2+3600\alpha^3\rho+2400\alpha^3))\Big)\\ &+x^3\Big(n^2\alpha^3(\beta^5\rho^5(15\alpha^2+180(\alpha+1))+\beta^5\rho^4(45\alpha^2+705\alpha+750)+\beta^5\rho^3(45\alpha^2\\ &+900\alpha+1020)+\beta^5\rho^2(-75\alpha^2+375\alpha+450)+\beta^4\rho^4(315\alpha^2+450\alpha)\end{aligned}$$

$$+ \beta^4\rho^3(1290\alpha^2 + 2100\alpha) + \beta^4\rho^2(1665\alpha^2 + 3150\alpha) + \beta^4\rho^1(690\alpha^2 + 1500\alpha)$$
$$+ 300\alpha^2\beta^3\rho^3 + 1800\alpha^2\beta^3\rho^2 + 3300\alpha^2\beta^3\rho + 1800\alpha^2\beta^3)))\Big)$$
$$+ x^2\Big(n\alpha^4\alpha(31\beta^5\rho^5(\alpha+1) + 225\beta^5\rho^4(\alpha+1) + 595\beta^5\rho^3(\alpha+1)$$
$$+ 675\beta^5\rho^2(\alpha+1) + 274\beta^5\rho(\alpha+1))\Big) + x\Big(\alpha^5(\beta^5(\rho^5 + 15\rho^4$$
$$+ 85\rho^3 + 225\rho^2 + 274\rho + 120))\Big)\Big\},$$

where $A = n^5\alpha^5(\beta\rho-1)(\beta\rho-2)(\beta\rho-3)(\beta\rho-4)(\beta\rho-5)$.

Lemma 2.3. *If* $\alpha = \alpha_n \to \infty$ *and* $\beta = \beta_n \to \infty$, *as* $n \to \infty$, *and* $\lim_{n\to\infty} \frac{n}{\alpha_n} = a$, $\lim_{n\to\infty}\frac{n}{\beta_n} = b$; $a, b \in \mathbb{R}$, *then we have*

(i) $\lim_{n\to\infty} n\, \mathfrak{F}^{\beta,\rho,1}_{n,\alpha}(x) = 0;$

(ii) $\lim_{n\to\infty} n\, \mathfrak{F}^{\beta,\rho,2}_{n,\alpha}(x) = \frac{1}{2}\left[\left(a + \frac{b}{\rho}\right)x^2 + \left(1 + \frac{1}{\rho}\right)x\right];$

(iii) $\lim_{n\to\infty} n^2\, \mathfrak{F}^{\beta,\rho,4}_{n,\alpha}(x) = \left[\frac{3b^2}{\rho^2} + \frac{6ab}{\rho} + 3a^2\right]x^4 + \left[\frac{6a(\rho+1)}{\rho} + \frac{6b(\rho+1)}{\rho^2}\right]x^3 + \left[\frac{3(\rho^2+2\rho+1)}{\rho^2}\right]x^2;$

(iv) $$\lim_{n\to\infty} n^3\, \mathfrak{F}^{\beta,\rho,6}_{n,\alpha}(x) = \left[15a^3 + \frac{45a^2b}{\rho} + \frac{45ab^2}{\rho^2} + \frac{15b}{\rho^3}\right]x^6 + \left[\frac{45\rho^2(\rho+1)b^2}{\rho^5}\right]x^5$$
$$+ \left[45a + \frac{(90a+45b)}{\rho} + \frac{(45a+90b)}{\rho^2} + \frac{45b}{\rho^3}\right]x^4 + \left[15 + \frac{45}{\rho} + \frac{45}{\rho^2} - \frac{75}{\rho^3}\right]x^3,$$

where $\mathfrak{F}^{\beta,\rho,i}_{n,\alpha}(x) := \mathcal{A}^{\beta,\rho}_{n,\alpha}\big((t-x)^i; x\big), i = 1, 2, 4, 6.$

3. Direct Results

In this section, we prove a uniform convergence and a pointwise convergence theorem for functions in the space $C_\mu[0,\infty)$ which we define as:.
for $\mu > 0$, $C_\mu[0,\infty) := \big\{g \in C[0,\infty) : g(x) = O(x^\mu); x \geq 0\big\}$.

Theorem 3.1. *Suppose that* $g \in C_\mu[0,\infty)$. *Then* $\lim_{n\to\infty} \mathcal{A}^{\beta,\rho}_{n,\alpha}(g; x) = g(x)$, *uniformly in each closed and bounded subset of* $[0,\infty)$.

Proof. Since $\mathcal{A}^{\beta,\rho}_{n,\alpha}(e_0, x) = 1$ and $\mathcal{A}^{\beta,\rho}_{n,\alpha}(e_1, x) = x$, and by using Lemma 2.3 $\lim_{n\to\infty}\mathcal{A}^{\beta,\rho}_{n,\alpha}(e_2, x) = x^2$. Applying Korovokin's theorem, we immediately get the required proof. □

Theorem 3.2. *Let* $g \in C_\mu[0,\infty)$. *If* g''*exists at a point* $x \in [0,\infty)$, *then*
$\lim_{n\to\infty} n\left[\mathcal{A}^{\beta,\rho}_{n,\alpha}(g; x) - g(x)\right] = \frac{1}{2}\left[\left(a + \frac{b}{\rho}\right)x^2 + \left(1 + \frac{1}{\rho}\right)x\right]g''(x).$

Proof. By the virtue of Taylor's expansion, we get

$$g(t) = g(x) + g'(x)(t-x) + \frac{1}{2}g''(x)(t-x)^2 + \varphi(t,x)(t-x)^2, \tag{2}$$

and $\varphi(t,x) \to 0$, as $t \to x$.
Since the operators $\mathcal{A}^{\beta,\rho}_{n,\alpha}(g,x)$ preserve linear function, therefore may write

$$\mathcal{A}^{\beta,\rho}_{n,\alpha}(g; x) - g(x) = \mathcal{A}^{\beta,\rho}_{n,\alpha}((t-x); x)g'(x) + \frac{1}{2}\mathcal{A}^{\beta,\rho}_{n,\alpha}\big((t-x)^2; x\big)g''(x) + \mathcal{A}^{\beta,\rho}_{n,\alpha}\big(\varphi(t,x)(t-x)^2; x\big).$$

$$\lim_{n\to\infty} n\Big[\mathcal{A}^{\beta,\rho}_{n,\alpha}(g; x) - g(x)\Big] = \lim_{n\to\infty} n\, \mathcal{A}^{\beta,\rho}_{n,\alpha}\big((t-x); x\big)g'(x) + \lim_{n\to\infty}\frac{n}{2}\mathcal{A}^{\beta,\rho}_{n,\alpha}\big((t-x)^2; x\big)g''(x) + \lim_{n\to\infty} n\, \mathcal{A}^{\beta,\rho}_{n,\alpha}\big(\varphi(t,x)(t-x)^2; x\big).$$

By using Lemma 2.3, we get

$$\lim_{n\to\infty} n\Big[\mathcal{A}_{n,\alpha}^{\beta,\rho}(g;x)-g(x)\Big]=\frac{g''(x)}{4}\left[\left(a+\frac{b}{\rho}\right)x^2+\left(1+\frac{1}{\rho}\right)x\right]+\lim_{n\to\infty} n\,\mathcal{A}_{n,\alpha}^{\beta,\rho}\Big(\varphi(t,x)(t-x)^2;x\Big). \tag{3}$$

On applying the Cauchy- Schwarz inequality in the last term of (3), we obtain

$$\lim_{n\to\infty} n\,\mathcal{A}_{n,\alpha}^{\beta,\rho}\Big(\varphi(t,x)(t-x)^2;x\Big)\le\lim_{n\to\infty}\left(\sqrt{\mathcal{A}_{n,\alpha}^{\beta,\rho}\Big(\varphi^2(t,x);x\Big)}\sqrt{n^2\mathcal{A}_{n,\alpha}^{\beta,\rho}\Big((t-x)^4;x\Big)}\right). \tag{4}$$

As $\varphi^2(x,x)=0$ and $\varphi^2(.,x)\in C_\mu[0,\infty)$, we have

$$\lim_{n\to\infty}\mathcal{A}_{n,\alpha}^{\beta,\rho}\Big(\varphi^2(t,x);x\Big)=\varphi^2(x,x)=0. \tag{5}$$

In view of the eq. (4) and (5) and Lemma 2.3, we get

$$\lim_{n\to\infty} n\,\mathcal{A}_{n,\alpha}^{\beta,\rho}\Big(\varphi(t,x)(t-x)^2;x\Big)=0. \tag{6}$$

Hence

$$\lim_{n\to\infty} n\left[\mathcal{A}_{n,\alpha}^{\beta,\rho}(g;x)-g(x)\right]=\frac{1}{2}\left[\left(a+\frac{b}{\rho}\right)x^2+\left(1+\frac{1}{\rho}\right)x\right]g''(x). \tag{7}$$

This completes the proof. □

Next, we establish direct estimate for the operators $\mathcal{A}_{n,\alpha}^{\beta,\rho}(g,x)$ via the Lipschitz type space considered in [19].
For the parameters $\varepsilon_1,\ \varepsilon_2>0$ and $r\in(0,1],\ N>0$,

$$Lip_N^{(\varepsilon_1,\varepsilon_2)}(r):=\left\{g\in C[0,\infty):|g(t)-g(x)|\le\frac{N.|t-x|^r}{(t+\varepsilon_1x^2+\varepsilon_2x)^{\frac{r}{2}}};x,t\in(0,\infty)\right\}.$$

Theorem 3.3. *Suppose $g\in Lip_N^{(\varepsilon_1,\varepsilon_2)}(r)$ and $0<r\le1$. Then, for all $x\ge0$,*
$\left|\mathcal{A}_{n,\alpha}^{\beta,\rho}(g,x)-g(x)\right|\le N\left(\frac{\mathfrak{F}_{n,\alpha}^{\beta,\rho,2}(x)}{(\varepsilon_1x^2+\varepsilon_2x)}\right)^{\frac{r}{2}}$*, where N is a positive real number.*

Proof. Using Hölder's inequality with $p=\frac{2}{r},q=\frac{2}{2-r}$, we obtain

$$\begin{aligned}\left|\mathcal{A}_{n,\alpha}^{\beta,\rho}(g,x)-g(x)\right|&=\sum_{k=1}^{\infty}p_{n,k}^{\alpha}(x)\int_0^{\infty}q_{n,k-1}^{\beta+1,\rho}(t)|g(t)-g(x)|dt+p_{n,0}^{\alpha}(x)|g(0)-g(x)|\\&\le\sum_{k=1}^{\infty}p_{n,k}^{\alpha}(x)\left(\int_0^{\infty}q_{n,k-1}^{\beta+1,\rho}(t)|g(t)-g(x)|^{\frac{2}{r}}dt\right)^{\frac{r}{2}}+p_{n,0}^{\alpha}(x)|g(0)-g(x)|\\&\le\left\{\sum_{k=1}^{\infty}p_{n,k}^{\alpha}(x)\int_0^{\infty}q_{n,k-1}^{\beta+1,\rho}(t)|g(t)-g(x)|^{\frac{2}{r}}dt+p_{n,0}^{\alpha}(x)|g(0)-g(x)|^{\frac{2}{r}}\right\}^{\frac{r}{2}}\left(\sum_{k=0}^{\infty}p_{n,k}^{\alpha}(x)\right)^{\frac{2-r}{2}}\\&=\left\{\sum_{k=1}^{\infty}p_{n,k}^{\alpha}(x)\int_0^{\infty}q_{n,k-1}^{\beta+1,\rho}(t)|g(t)-g(x)|^{\frac{2}{r}}dt+p_{n,0}^{\alpha}(x)|g(0)-g(x)|^{\frac{2}{r}}\right\}^{\frac{r}{2}}\end{aligned}$$

$$\leq N\left(\sum_{k=1}^{\infty} p_{n,k}^{\alpha}(x)\int_0^{\infty} q_{n,k-1}^{\beta+1,\rho}(t)\frac{(t-x)^2}{(t+\varepsilon_1 x^2+\varepsilon_2 x)}dt + p_{n,0}^{\alpha}(x)\frac{x^2}{(\varepsilon_1 x^2+\varepsilon_2 x)}\right)^{\frac{r}{2}}$$
$$\leq \frac{N}{(\varepsilon_1 x^2+\varepsilon_2 x)^{\frac{r}{2}}}\left(\sum_{k=1}^{\infty} p_{n,k}^{\alpha}(x)\int_0^{\infty} q_{n,k-1}^{\beta+1,\rho}(t)(t-x)^2 dt + x^2 p_{n,0}^{\alpha}(x)\right)^{\frac{r}{2}}$$
$$= \frac{N}{(\varepsilon_1 x^2+\varepsilon_2 x)^{\frac{r}{2}}}(\mathcal{A}_{n,\alpha}^{\beta,\rho}((t-x)^2);x))^{\frac{r}{2}} = \frac{N}{(\varepsilon_1 x^2+\varepsilon_2 x)^{\frac{r}{2}}}\left(\mathfrak{V}_{n,\alpha}^{\beta,\rho,2}(x)\right)^{\frac{r}{2}}.$$

which is our required result. □

4. Weighted Approximation

To prove weighted convergence theorems, we consider the following spaces with weight function $\gamma(x) = 1 + x^2$.

(i) $K_\gamma[0,\infty) := \left\{g : [0,\infty) \to \mathbb{R} : |g(x)| \leq M_g\gamma(x), M_g \in \mathbb{R}^+, M_g \text{ depends on } g\right\}$.

(ii) $C_\gamma[0,\infty)$:= The space of all continuous functions in $K_\gamma[0,\infty)$ endowed with the norm $\|g\|_\gamma := \sup_{x\in[0,\infty)} \frac{|g(x)|}{\gamma(x)}$.

(iii) $C_\gamma^0[0,\infty) := \left\{g \in C_\gamma[0,\infty) : \lim_{x\to\infty} \frac{|g(x)|}{\gamma(x)} \text{ exists and finite}\right\}$.

Theorem 4.1. *Suppose that $g \in C_\gamma^0[0,\infty)$ and $r > 0$, then*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{|\mathcal{A}_{n,\alpha}^{\beta,\rho}(g;x) - g(x)|}{(1+x^2)^{1+r}} = 0.$$

Proof. For any arbitrary and fixed real number $x_0 > 0$, we can write

$$\sup_{x\in[0,\infty)} \frac{\left|\mathcal{A}_{n,\alpha}^{\beta,\rho}(g;x) - g(x)\right|}{(1+x^2)^{1+r}} \leq \sup_{x\leq x_0} \frac{\left|\mathcal{A}_{n,\alpha}^{\beta,\rho}(g;x) - g(x)\right|}{(1+x^2)^{1+r}} + \sup_{x>x_0} \frac{\left|\mathcal{A}_{n,\alpha}^{\beta,\rho}(g;x) - g(x)\right|}{(1+x^2)^{1+r}}$$
$$\leq \sup_{x\leq x_0}\left|\mathcal{A}_{n,\alpha}^{\beta,\rho}(g;x) - g(x)\right| + \sup_{x>x_0} \frac{\left|\mathcal{A}_{n,\alpha}^{\beta,\rho}(g;x)\right|}{(1+x^2)^{1+r}} + \sup_{x>x_0} \frac{|g(x)|}{(1+x^2)^{1+r}}.$$

Since $|g(x)| \leq \|g\|_\gamma(1+x^2)$ for all $x \geq 0$, therefore

$$\sup_{x\in[0,\infty)} \frac{\left|\mathcal{A}_{n,\alpha}^{\beta,\rho}(g;x) - g(x)\right|}{(1+x^2)^{1+r}} \leq \|\mathcal{A}_{n,\alpha}^{\beta,\rho}(g;x) - g(x)\|_{C_{[0,x_0]}} + \|g\|_\gamma \sup_{x>x_0} \frac{\left|\mathcal{A}_{n,\alpha}^{\beta,\rho}(1+t^2;x)\right|}{(1+x^2)^{1+r}} + \sup_{x\geq x_0} \frac{\|g\|_\gamma}{(1+x^2)^r}$$
$$= Z_1 + Z_2 + Z_3, (say). \tag{8}$$

On applying the basic convergence theorem, for any given $\epsilon > 0$, $\exists\, k_1 \in \mathbb{N}$, such that

$$Z_1 = \|\mathcal{A}_{n,\alpha}^{\beta,\rho}(g;.) - g\|_{C_{[0,x_0]}} < \frac{\epsilon}{3}, \quad \text{for all } n \geq k_1. \tag{9}$$

As $\lim_{n\to\infty} \sup_{x>x_0} \frac{\mathcal{A}_{n,\alpha}^{\beta,\rho}(1+t^2;x)}{(1+x^2)} = 1$, therefore there exists $k_2 \in \mathbb{N}$ such that

$$\sup_{x>x_0} \frac{\mathcal{A}_{n,\alpha}^{\beta,\rho}(1+t^2;x)}{(1+x^2)} \leq \frac{(1+x_0^2)^r}{\|g\|_\gamma}\frac{\epsilon}{3} + 1, \text{ for all } n \geq k_2.$$

Hence,

$$Z_2 = \|g\|_\gamma \sup_{x>x_0} \frac{\left|\mathcal{A}_{n,\alpha}^{\beta,\rho}(1+t^2;x)\right|}{(1+x^2)^{1+r}} \leq \frac{\|g\|_\gamma}{(1+x_0^2)^r} \sup_{x>x_0} \frac{\mathcal{A}_{n,\alpha}^{\beta,\rho}(1+t^2;x)}{(1+x^2)}$$

$$\leq \frac{\|g\|_\gamma}{(1+x_0^2)^r} + \frac{\epsilon}{3}, \text{ for all } n \geq k_2. \tag{10}$$

Let us choose x_0 to be so large that

$$\frac{\|g\|_\gamma}{(1+x_0^2)^r} < \frac{\epsilon}{6},$$

then

$$Z_2 \leq \frac{\epsilon}{6} + \frac{\epsilon}{3} = \frac{\epsilon}{2} \text{ and } Z_3 = \sup_{x>x_0} \frac{\|g\|_\gamma}{(1+x^2)^r} \leq \frac{\|g\|_\gamma}{(1+x_0^2)^r} < \frac{\epsilon}{6}. \tag{11}$$

Let $k_0 = \max\{k_1, k_2\}$, then by combining eq. (9) and (11), we obtain

$$\sup_{x\in[0,\infty)} \frac{\left|\mathcal{A}_{n,\alpha}^{\beta,\rho}(g;x) - g(x)\right|}{(1+x^2)^{1+r}} < \epsilon, \text{ for all } n \geq k_0.$$

Hence, the proof is completed. □

Theorem 4.2. *Suppose the function $g \in C_\gamma[0,\infty)$, then*

$$\lim_{n\to\infty} \left\|\mathcal{A}_{n,\alpha}^{\beta,\rho}(g) - g\right\|_\gamma = 0. \tag{12}$$

Proof. To prove eq.(12) by Korovkin type theorem [20], it is sufficient to show the following:

$$\lim_{n\to\infty} \left\|\mathcal{A}_{n,\alpha}^{\beta,\rho}(t^\nu;x) - e_\nu\right\|_\gamma = 0, \quad \nu = 0,1,2. \tag{13}$$

Since $\mathcal{A}_{n,\alpha}^{\beta,\rho}(1;x) = 1$ and $\mathcal{A}_{n,\alpha}^{\beta,\rho}(t;x) = x$, so eq.(13), holds true for $\nu = 0,1$.

Finally, we obtain

$$\left\|\mathcal{A}_{n,\alpha}^{\beta,\rho}(t^2;x) - x^2\right\|_\gamma = \sup_{x\geq 0} \frac{1}{1+x^2} \left|\frac{\beta}{\beta\rho - 1}\left(\rho x^2(1+\frac{1}{\alpha}) + \frac{(\rho+1)x}{n}\right) - x^2\right|$$

$$\leq \sup_{x\geq 0} \frac{x^2}{1+x^2} \left|\frac{\beta\rho}{\beta\rho-1}\left(1+\frac{1}{\alpha}\right) - 1\right| + \sup_{x\geq 0} \frac{x}{1+x^2} \left|\frac{\beta(\rho+1)}{n(\beta\rho-1)}\right|,$$

which implies that $\lim_{n\to\infty} \left\|\mathcal{A}_{n,\alpha}^{\beta,\rho}(t^2;x) - x^2\right\|_\gamma = 0$. □

5. Voronovskaya-Type Approximation Theorem

Yüksel and Ispir [21] analyzed the approximation properties of Srivastava-Gupta operators in weighted spaces of continuous and unbounded functions working on the interval $[0,\infty)$. To obtain the rate of convergence of these operators, they defined the weighted modulus of continuity $\Omega_2(g;\eta)$ as follows :
for every function g $\in C_\gamma[0,\infty)$ and $\eta > 0$,

$$\Omega_2(g;\eta) = \sup_{0\leq h<\eta, x\in[0,\infty)} \frac{\left|g(x+h) - g(x)\right|}{(1+h^2)(1+x^2)}. \tag{14}$$

Interestingly, the weighted modulus of continuity $\Omega_2(g;\eta)$ has common properties with the classical modulus of continuity. In the following Lemma, we mention some of them.

Lemma 5.1. *[21] If $g \in C_\gamma[0,\infty)$, then*

(i) $\Omega_2(g;\eta)$ *is an increasing function of* η.

(ii) $\lim\limits_{n\to\infty} \Omega_2(g;\eta) = 0$

(iii) *For given* $\lambda > 0$, $\Omega_2(g;\lambda\eta) \le 2(1+\lambda)(1+\eta^2)\Omega_2(g;\eta)$.

Remark 5.2. *For every* $g \in C^0_\gamma[0,\infty)$ *and* $\lambda = \dfrac{|t-x|}{\eta}$, *by eq.*(14) *and Lemma* 5.1(*iii*), *we can write*

$$\left.\begin{aligned} \left|g(t)-g(x)\right| &\le \left(1+(t-x)^2\right)(1+x^2)\Omega_2\left(g;|t-x|\right)\\ &\le 2\left(1+\frac{|t-x|}{\eta}\right)(1+\eta^2)\Omega_2(g;\eta)(1+(t-x)^2)(1+x^2) \end{aligned}\right\}. \tag{15}$$

The next theorem determines the rate of approximation of the operators defined in (1) for functions belonging in the weighted space $C^0_\gamma[0,\infty)$ by using $\Omega_2(.;\eta)$.

Theorem 5.3. *Let* $g \in C^0_\gamma[0,\infty)$ *with the condition that* $g'(x), g''(x) \in C^0_\gamma[0,\infty)$.*Then, for sufficiently large* n *and each* $x \ge 0$,

$$\left|n\left\{\mathcal{A}^{\beta,\rho}_{n,\alpha}(g,x) - g(x) - g'(x)\mathcal{A}^{\beta,\rho}_{n,\alpha}\left((t-x);x\right) - \frac{g''(x)}{2!}\mathcal{A}^{\beta,\rho}_{n,\alpha}\left((t-x)^2;x\right)\right\}\right| = O(1)\Omega_2\left(g'';\sqrt{1/n}\right).$$

Proof. Using Taylor's formula for the function g, we have

$$\begin{aligned} g(t) &= g(x) + g'(x)(t-x) + \frac{g''(\nu)}{2!}(t-x)^2\\ &= g(x) + g'(x)(t-x) + \frac{g''(x)}{2!}(t-x)^2 + k_2(t,x), \end{aligned} \tag{16}$$

$$\text{where } k_2(t,x) = \frac{g''(\nu)-g''(x)}{2!}(t-x)^2,\ \nu \in (x,t). \tag{17}$$

By remark 5.2, we can write

$$\begin{aligned} \left|g''(\nu)-g''(x)\right| &\le \left(1+(\nu-x)^2\right)(1+x^2)\Omega\left(g'';|\nu-x|\right)\\ &\le \left(1+(t-x)^2\right)(1+x^2)\Omega\left(g'';|\nu-x|\right)\\ &\le 2\left(1+(t-x)^2\right)(1+x^2)\left(1+\frac{|t-x|}{\eta}\right)(1+\eta^2)\Omega(g'';\eta), \end{aligned} \tag{18}$$

but

$$\left(1+\frac{|t-x|}{\eta}\right)\left(1+(t-x)^2\right) \le \begin{cases} 2(1+\eta^2), & \text{if } |t-x| < \eta,\\ \dfrac{2(t-x)^4}{\eta^4}(1+\eta^2), & \text{if } |t-x| \ge \eta, \end{cases}$$

i.e.,

$$\left(1+\frac{|t-x|}{\eta}\right)\left(1+(t-x)^2\right) \le 2\left(1+\frac{(t-x^4)}{\eta^4}\right)(1+\eta^2). \tag{19}$$

Combining eq.(17) and (19), and choosing $0 < \eta < 1$, we obtain

$$\left|k_2(t,x)\right| \le 2(1+\eta^2)^2(1+x^2)\Omega_2(g'';\eta)\left(1+\frac{(t-x)^4}{\eta^4}\right)(t-x)^2. \tag{20}$$

Operating $\mathcal{A}_{n,\alpha}^{\beta,\rho}$ and Lemma 2.2 on both sides of (16), we get

$$\left|\mathcal{A}_{n,\alpha}^{\beta,\rho}(g,x) - g(x) - g'(x)\mathcal{A}_{n,\alpha}^{\beta,\rho}((t-x);x) - \frac{g''(x)}{2!}\mathcal{A}_{n,\alpha}^{\beta,\rho}\Big((t-x)^2;x\Big)\right| \leq \mathcal{A}_{n,\alpha}^{\beta,\rho}\Big(|k_2(t,x)|;x\Big). \tag{21}$$

Applying Lemma 2.3 and using Eq.(20), we get

$$\begin{aligned}\mathcal{A}_{n,\alpha}^{\beta,\rho}\Big(|k_2(t,x)|;x\Big) &\leq 2(1+\eta^2)^2(1+x^2)\Omega_2(g'';\eta)\mathcal{A}_{n,\alpha}^{\beta,\rho}\left(\left((t-x)^2) + \frac{(t-x)^6}{\eta^4}\right);x\right)\\ &= 2(1+\eta^2)^2(1+x^2)\Omega_2(g'';\eta)\left(\mathcal{A}_{n,\alpha}^{\beta,\rho}\Big((t-x)^2;x\Big) + \frac{1}{\eta^4}\mathcal{A}_{n,\alpha}^{\beta,\rho}\Big((t-x)^6;x\Big)\right)\\ &= 2(1+\eta^2)^2(1+x^2)\Omega_2(g'';\eta)\Big(O(1/n) + \frac{1}{\eta^4}O\left(1/n^3\right)\Big).\end{aligned}$$

If we choose $\eta = \sqrt{1/n}$, then

$$n.\mathcal{A}_{n,\alpha}^{\beta,\rho}\Big(|k_2(t,x)|;x\Big) = O(1)\Omega_2(g''; \sqrt{1/n}). \tag{22}$$

Hence, from (21) and (22), we obtain the required result. □

6. Grüss Voronovskaya-Type Theorem

Theorem 6.1. *Suppose that g, h and $gh \in C_\gamma^0[0,\infty)$ such that g', h', $(gh)'$, g'', h'' and $(gh)'' \in C_\gamma^0[0,\infty)$. Then, for each $x \in [0,\infty)$, we have*

$$\lim_{n\to\infty} n\Big\{\mathcal{A}_{n,\alpha}^{\beta,\rho}((gh);x) - \mathcal{A}_{n,\alpha}^{\beta,\rho}(g;x)\mathcal{A}_{n,\alpha}^{\beta,\rho}(h;x)\Big\} = \frac{g'(x)\,h'(x)}{2}\left\{\left(a+\frac{b}{\rho}\right)x^2 + \left(1+\frac{1}{\rho}\right)x\right\}.$$

Proof. Since

$$(gh)(x) = g(x)h(x),\ (gh)'(x) = g'(x)h(x) + g(x)\,h'(x) \text{ and } (gh)''(x) = g''(x)h(x) + 2g'(x)\,h'(x) + g(x)\,h''(x),$$

therefore, we may write
$\mathcal{A}_{n,\alpha}^{\beta,\rho}\Big((gh);x\Big) - \mathcal{A}_{n,\alpha}^{\beta,\rho}(g;x)\mathcal{A}_{n,\alpha}^{\beta,\rho}(h;x)$

$$\begin{aligned}&= \left\{\mathcal{A}_{n,\alpha}^{\beta,\rho}((gh);x) - g(x)h(x) - (gh)'(x)\mathcal{A}_{n,\alpha}^{\beta,\rho}\Big((t-x);x\Big) - \frac{(gh)''(x)}{2!}\mathcal{A}_{n,\alpha}^{\beta,\rho}\Big((t-x)^2;x\Big)\right\}\\ &- h(y)\left\{\mathcal{A}_{n,\alpha}^{\beta,\rho}(g,x) - g(x) - g'(x)\mathcal{A}_{n,\alpha}^{\beta,\rho}((t-x);x) - \frac{g''(x)}{2!}\mathcal{A}_{n,\alpha}^{\beta,\rho}\Big((t-x)^2;x\Big)\right\}\\ &- \mathcal{A}_{n,\alpha}^{\beta,\rho}(g,x)\left\{\mathcal{A}_{n,\alpha}^{\beta,\rho}(h,x) - h(x) - h'(x)\mathcal{A}_{n,\alpha}^{\beta,\rho}\Big((t-x);x\Big) - \frac{h''(x)}{2!}\mathcal{A}_{n,\alpha}^{h,\rho}\Big((t-x)^2;x\Big)\right\}\\ &+ \frac{1}{2!}\mathcal{A}_{n,\alpha}^{\beta,\rho}\Big((t-x)^2;x\Big)\Big\{g(x)\,h''(x) + 2g'(x)\,h'(x) - h''(x)\mathcal{A}_{n,\alpha}^{\beta,\rho}(g,x)\Big\}\\ &+ \mathcal{A}_{n,\alpha}^{\beta,\rho}\Big((t-x);x\Big)\Big\{g(x)\,h'(x) - h'(x)\mathcal{A}_{n,\alpha}^{\beta,\rho}(g,x)\Big\}.\end{aligned}$$

Now, by using lemma 2.3 and Theorem 3.1, we obtain

$$\lim_{n\to\infty} n\Big\{\mathcal{A}_{n,\alpha}^{\beta,\rho}\Big((gh);x\Big) - \mathcal{A}_{n,\alpha}^{\beta,\rho}(g;x)\mathcal{A}_{n,\alpha}^{\beta,\rho}(h;x)\Big\} = \frac{g'(x)\,h'(x)}{2}\left\{\left(a+\frac{b}{\rho}\right)x^2 + \left(1+\frac{1}{\rho}\right)x\right\},$$

which proves our result. □

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Conflict of interest The authors declare that they have no conflict of interest.

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