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Fixed Point Results in M-Cone Metric Space Over Banach algebra with an Application

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Abstract. In this paper, we introduce the M-cone metric space over Banach algebra as a generalization of both M-metric space and cone metric space over Banach algebra and investigate some fixed point results in the new settings. Some examples are presented as illustrations. Finally, the results are supported by an application to examine the existence and uniqueness of solution for a Fredholm integral equation.

1. Introduction

In 1994, Matthews [8] introduced the notion of a partial metric space. In this space, the usual metric is replaced by a partial metric with a unique property that the self-distance of any point of space may not be zero. In 2007, the concept of cone metric spaces was introduced by Huang and Zhang [4] as a generalization of metric spaces. They also proved the Banach contraction principle in the setting of cone metric spaces over a normal cone. In order to generalize, Rezapour and Hamlbarani [10] omitted the assumption of normality of cone given in [4] and presented few examples to support the existence of non-normal cones, which shows that the results in the setting of cone metric spaces are appropriate only if the underlying cone is not necessarily normal. Afterwards, Liu and Xu [7] in 2013 introduced the notion of cone metric space over Banach algebra by replacing the Banach space E by Banach algebra A which clearly indicates that the existence of the fixed points of the mappings in cone metric spaces over Banach algebra are not equivalent to metric spaces. Moreover, they gave some examples to elucidate their results. Many authors have devoted their attention to generalizing cone metric spaces may be noted in (see [2], [3]). In 2014, Asadi et al. [1] introduced the concept of an M-metric space which is a generalization of a partial metric space and established some fixed point results for generalized contractions in the new setting.

In the present study, we introduce the structure of M-cone metric spaces over Banach algebra as a generalization of both M-metric space and cone metric space over Banach algebra. Also, we present the notion of generalized Lipschitz mapping in the framework of M-cone metric spaces over Banach algebra and investigate the existence of fixed point for such mappings. As an application, we examine the existence and uniqueness of solution for a Fredholm integral equation. Our results generalize and improve the main

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results in [1].

2. Preliminaries

First, we review some basic concepts about Banach algebra and cone metric spaces.

Let A always be a real Banach algebra i.e., A is a real Banach space in which an operation of multiplication is defined, subject to the following properties ($\forall \mu, \nu, \upsilon \in A, \alpha \in R$)

1. $(\mu\nu)\upsilon = \mu(\nu\upsilon)$,
2. $\mu(\nu + \upsilon) = \mu\nu + \mu\upsilon$ and $(\mu + \nu)\upsilon = \mu\upsilon + \nu\upsilon$,
3. $\alpha(\mu\nu) = (\alpha\mu)\nu = \mu(\alpha\nu)$,
4. $\| \mu\nu \| \leq \| \mu \| \| \nu \|$.

Throughout this paper, we shall assume that a Banach algebra has a unit (i.e., a multiplicative identity) e such that $e\mu = \mu e = \mu, \forall \mu \in A$. An element $\mu \in A$ is said to be invertible if $\exists$ an inverse element $\nu \in A$ such that $\mu\nu = \nu\mu = e$. The inverse of μ is denoted by μ^{-1}. For more details, we refer the reader to [11].

The following proposition is due to Rudin [11].

Proposition 2.1. Let A be Banach algebra with a unit e, and $\mu \in A$. If the spectral radius $\rho(\mu)$ of μ is less than 1, i.e.

$$\rho(\mu) = \lim_{n\to+\infty} \left\|\mu^n\right\|^{\frac{1}{n}} = \inf \left\|\mu^n\right\|^{\frac{1}{n}} < 1.$$

then $(e - \mu)$ is invertible. Actually,

$$(e - \mu)^{-1} = \sum_{i=0}^{+\infty} \mu^i.$$

Remark 2.2. From [11] we see that the spectral radius $\rho(\mu)$ of μ satisfies $\rho(\mu) \leq \left\|\mu\right\|$, $\forall\ \mu \in A$, where A is a Banach algebra with a unit e.

Remark 2.3. (See [12]). In Proposition 2.1, if the condition $'\rho(\mu) < 1'$ is replaced by $\left\|\mu\right\| \leq 1$, then the conclusion remains true.

Remark 2.4. (See [12]). If $\rho(\mu) < 1$ then $\left\|\mu^n\right\| \to 0\ \ (n \to +\infty)$.

Now let us recall the concepts of cone over Banach algebra

A subset P of A is called a cone if

1. P is non-empty closed and $\{\theta, e\} \subset P$;
2. $\alpha P + \beta P \subset P$ for all non-negative real numbers α, β;
3. $P^2 = PP \subset P$;
4. $P \cap (-P) = \{\theta\}$,

where θ denotes the null of the Banach algebra A. For a given cone $P \subset A$, we can define a partial ordering $\preceq$ with respect to P by $\mu \preceq \nu$ if and only if $\nu - \mu \in P$. $\mu \prec \nu$ will stand for $\mu \preceq \nu$ and $\mu \neq \nu$, while $\mu \ll \nu$ will stand for $\nu - \mu \in \operatorname{Int} P$, where $\operatorname{Int} P$ denotes the interior of P. If $\operatorname{Int} P \neq \emptyset$, then P is called a solid cone.

The cone P is called normal if there is a number $M > 1$ such that, $\forall \mu, \nu \in A$,

$$\theta \preceq \mu \preceq \nu$$

implies

$\| \mu \| \leq M \| \nu \|$.

The least positive number satisfying the above is called the normal constant of P [4].

In the following we always assume that A is Banach algebra with a unit e, P is a solid cone in A and $\leq$ is the partial ordering with respect to P.

We will require the following definitions and preliminary results to prove our results.

Definition 2.5. ([7]) Let X be a nonempty set. Suppose the mapping $d : X \times X \to A$ satisfies

(1) $\theta \prec d(\mu, \nu)\ \forall\ \mu, \nu \in X$ and $d(\mu, \nu) = \theta \Leftrightarrow \mu = \nu$,
(2) $d(\mu, \nu) = d(\nu, \mu)\ \forall\ \mu, \nu \in X$;
(3) $d(\mu, \nu) \leq d(\mu, \upsilon) + d(\upsilon, \nu)\ \forall\ \mu, \nu, \upsilon \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space over Banach algebra A.

Definition 2.6. ([2]) A partial cone metric on a nonempty set X is a function $p : X \times X \to A$ such that $\forall \mu, \nu, \upsilon \in X$, the following conditions hold:

(p_1) $\mu = \nu \Leftrightarrow p(\mu, \mu) = p(\mu, \nu) = p(\nu, \nu)$,

(p_2) $p(\mu, \mu) \leq p(\mu, \nu)$,

(p_3) $p(\mu, \nu) = p(\nu, \mu)$,

(p_4) $p(\mu, \nu) \leq p(\mu, \upsilon) + p(\upsilon, \nu) - p(\upsilon, \upsilon)$.

The pair (X, p) is called a partial cone metric space over Banach algebra. It is clear that, if $p(x, y) = \theta$, then from (p_1) and (p_2), $\mu = \nu$. But if $\mu = \nu, p(\mu, \nu)$ may not be θ.

Asadi et al. [1] gave a new generalization called M-metric space and defined it as follows:
Definition 2.7. ([1]) Let X be a non-empty set. A function $m : X \times X \to A$ is called an M-cone metric if the following conditions are satisfied.

$(m1)$ $m(\mu, \mu) = m(\nu, \nu) = m(\mu, \nu) \Leftrightarrow \mu = \nu$,
$(m2)$ $m_{\mu\nu} \leq m(\mu, \nu)$
$(m3)$ $m(\mu, \nu) = m(\nu, \mu)$
$(m4)$ $\Big(m(\mu, \nu) - m_{\mu\nu}\Big) \leq \Big(m(\mu, \upsilon) - m_{\mu\upsilon}\Big) + \Big(m(\upsilon, \nu) - m_{\upsilon\nu}\Big)$.

Then, the pair (X, m) is called an M-metric space.

3. M-cone metric space over Banach algebra

We now present the concept of M-cone metric spaces over Banach algebra with appropriate examples and study some of its properties needed in the sequel.

The following definitions, notations and lemmas are needed in the sequel:
For a non-empty set X and a function $m : X \times X \to R^+$. The following notation is useful in the sequel:

(1) $M_{\mu\nu} = \min\Big\{m(\mu, \mu), m(\nu, \nu)\Big\}$.

(2) $M_{\mu\nu} = \max\Big\{m(\mu, \mu), m(\nu, \nu)\Big\}$.

We present the definition of M-cone metric space over Banach algebra as follows:

Definition 3.1. Let X be a non-empty set. A function $m : X \times X \to A$ is called an M-cone metric if the following conditions are satisfied.

(m_1) $m(\mu,\mu) = m(\nu,\nu) = m(\mu,\nu) \Leftrightarrow \mu = \nu$,
(m_2) $m_{\mu\nu} \preceq m(\mu,\nu)$
(m_3) $m(\mu,\nu) = m(\nu,\mu)$
(m_4) $\Big(m(\mu,\nu) - m_{\mu\nu}\Big) \preceq \Big(m(\mu,\upsilon) - m_{\mu\upsilon}\Big) + \Big(m(\upsilon,\nu) - m_{\upsilon\nu}\Big)$.

Then, the pair (X, m) is called an M-cone metric space over Banach algebra.

According to the above definition the condition (p_1) in the Definition 2.6 changes to m_1 and p_2 is expressed for $p(\mu,\mu)$ where $p(\nu,\nu) = \theta$ may become $p(\nu,\nu) \neq \theta$. Thus, we improve that condition by replacing it by $\min\Big\{p(\mu,\mu), p(\nu,\nu)\Big\} \preceq p(\mu,\nu)$, and also we improve the condition (p_4) extending it to the form of m_4.

Thus, every partial cone metric space over Banach algebra is an M-cone metric space over Banach algebra. But converse may not be true as shown in the following examples:

Remark 3.2. For all $\mu, \nu \in X$

1. $\theta \preceq M_{\mu\nu} + m_{\mu\nu} = m(\mu,\mu) + m(\nu,\nu)$
2. $\theta \preceq M_{\mu\nu} - m_{\mu\nu} =\mid m(\mu,\mu) + m(\nu,\nu) \mid$
3. $M_{\mu\nu} - m_{\mu\nu} \preceq (M_{\mu\upsilon} - m_{\mu\upsilon}) + (M_{\upsilon\nu} - m_{\upsilon\nu})$.

In the following example we present an example of a M-cone space over Banach algebra which is not partial cone metric space over Banach algebra.

Example 3.3. Let $A = C^{'}_{R}[0,1]$ with norm defined by $\| \mu \| = \| \mu \|_\infty + \| \mu^{'} \|_\infty$ under usual multipli0cation, A is a real unit Banach algebra with unit $e = 1$. Consider a cone $P = \{\mu \in A : \mu \geq 0\}$ in A. Moreover, P is a non-normal cone ([12]). Let $M = [0, +\infty)$. Define $m : X \times X \to A$ by

$$m(\mu,\nu)(t) = \left(\frac{\mu+\nu}{2}\right)e^t$$

$\forall \mu, \nu \in X$. Then,

(i) $(m_1), (m_2), (m_3)$ are obvious.
(ii) Without loss of generality, assume $\mu \preceq \nu \in X$. Then,

$$m_{\mu\nu} = \mu,$$

and $m(\mu,\nu) - m_{\mu\nu} = \frac{\mu+\nu}{2} - \mu = \frac{\nu-\mu}{2}$.
Let $\upsilon \in X$ such that

Case 1: If $\mu \preceq \nu \preceq \upsilon$. Then,

$$\begin{aligned}\Big(m(\mu,\upsilon) - m_{\mu\upsilon}\Big) + \Big(m(\upsilon,\nu) - m_{\upsilon\nu}\Big) &= \frac{\upsilon-\mu}{2} + \frac{\upsilon-\nu}{2} \\ &\succeq \frac{\upsilon-\mu}{2} \\ &\succeq \frac{\nu-\mu}{2} \\ &= m(\mu,\nu) - m_{\mu\nu}.\end{aligned}$$

Case 2: If $\mu \preceq \upsilon \preceq \nu$. Then,

$$\begin{aligned}\Big(m(\mu,\upsilon) - m_{\mu\upsilon}\Big) + \Big(m(\upsilon,\nu) - m_{\upsilon\nu}\Big) &= \frac{\upsilon-\mu}{2} + \frac{\nu-\upsilon}{2}\\ &= \frac{\nu-\mu}{2} = m(\mu,\nu) - m_{\mu\nu}.\end{aligned}$$

Case 3: If $\upsilon \preceq \mu \preceq \nu$. Then,

$$\begin{aligned}\Big(m(\mu,\upsilon) - m_{\mu\upsilon}\Big) + \Big(m(\upsilon,\nu) - m_{\upsilon\nu}\Big) &= \frac{\mu-\upsilon}{2} + \frac{\nu-\upsilon}{2}\\ &\succeq \nu - \upsilon \succeq \nu - \mu = m(\mu,\nu) - m_{\mu\nu}.\end{aligned}$$

Clearly, if $\mu \prec \nu \in X$, then

$$\mu \prec \frac{\mu+\nu}{2} \prec \nu$$

i.e $m_{\mu\nu} \prec m(\mu,\nu) \prec m(\nu,\nu)$.

Then, m is an M-cone metric on X, but it is not a partial cone metric space over Banach algebra.

4. Topology on *M*-cone metric space over Banach algebra

In this section, we define topology on M-cone metric space over Banach algebra.

Definition 4.1. Let (X,m) be a M-cone metric space over Banach algebra. Then for $\mu \in X$ and $c \succ \theta$, the m-open ball with center μ and radius $c \succ \theta$ is

$$B_m(\mu,c) = \Big\{\nu \in X : m(\mu,\nu) + m(\nu,\nu) - m_{\mu\nu} - m(\mu,\mu) \ll c\Big\}.$$

Remark 4.2. We notice that

$$m(\mu,\mu) - m(\mu,\mu) + m(\mu,\mu) - m_{\mu\mu} = \theta,$$

i.e for every $c \succ \theta$, $\mu \in B_m(\mu,c)$. Additionally, if for some $\mu,\nu \in X$,

$$m_{\mu\nu} = m(\nu,\nu) \preceq m(\mu,\nu) \preceq m(\mu,\mu).$$

Then

$$\begin{aligned}m(\mu,\nu) + m(\nu,\nu) - m_{\mu\nu} - m(\mu,\mu) &= \Big(m(\mu,\nu) - m(\mu,\mu)\Big) + \Big(m(\nu,\nu) - m_{\mu\nu}\Big)\\ &= m(\mu,\nu) - m(\mu,\mu) \preceq \theta.\end{aligned}$$

Hence, every m-open ball centered at μ contains ν.

Lemma 4.3. Let (X,m) be a M-cone metric space over Banach algebra. The family of all m-open balls on X.

$$\mathfrak{B} = \Big\{B_m(\mu,c) : \mu \in M \text{ and } \theta \ll c\Big\}$$

forms a basis on X.

Proof. For every $\mu \in X$ and $\theta \ll c$, let $\nu \in B_m(\mu, c)$. Then,

$$m(\mu,\nu) + m(\nu,\nu) - m_{\mu\nu} - m(\mu,\mu) \prec c.$$

Take

$$\delta = c - m(\mu,\nu) - m(\nu,\nu) + m_{\mu\nu} + m(\mu,\mu) \succ \theta. \tag{4.1}$$

We claim that $B_m(\nu,\delta) \subseteq B_m(\mu,c)$.

If $\upsilon \in B_m(\nu,\delta)$, then

$$= m(\nu,\upsilon) + m(\upsilon,\upsilon) - m_{\nu\upsilon} - m(\nu,\nu) \prec \delta. \tag{4.2}$$

Hence, by definition of M-cone metric space over Banach algebra

$$\begin{aligned} m(\mu,\upsilon) + m(\upsilon,\upsilon) - m_{\mu\upsilon} - m(\mu,\mu) &= \Big(m(\mu,\upsilon) - m_{\mu\upsilon}\Big) + m(\upsilon,\upsilon) - m(\mu,\mu) \\ &\preceq \Big(m(\mu,\nu) - m_{\mu\nu} + m(\nu,\upsilon) - m_{\nu\upsilon}\Big) + m(\upsilon,\upsilon) - m(\mu,\mu). \end{aligned}$$

By adding and subtracting $m(\nu,\nu)$, we get

$$\Big(m(\mu,\nu) - m(\nu,\nu) - m_{\mu\nu} - m(\mu,\mu)\Big) + \Big(m(\nu,\upsilon) + m(\upsilon,\upsilon) - m_{\nu\upsilon} - m(\nu,\nu)\Big).$$

By (4.1) and (4.2), we get

$$\begin{aligned} m(\mu,\upsilon) + m(\upsilon,\upsilon) - m_{\mu\upsilon} - m(\mu,\mu) &\prec m(\mu,\nu) + m(\nu,\nu) - m_{\mu\nu} - m(\mu,\mu) + \delta \\ &= c \qquad \Big(\text{from (4.1)}\Big) \end{aligned}$$

$$\therefore\; B_m(\mu,\delta) \subseteq B_m(\mu,c)$$

and B_m is a basis on X.

Notation. Given an M-cone metric m on a set X. We denote τ_m, the topology generated by the m-open balls.

$$B_m(\mu,c) = \Big\{\nu \in X : m(\mu,\nu) + m(\nu,\nu) - m_{\mu,\nu} - m(\mu,\mu) \ll c\Big\}.$$

Lemma 4.4. Let (X,m) be a M-cone metric space over Banach algebra, then for each $\theta \ll c$, $c \in E$ $\exists\, \delta > 0$ such that $c - \mu \in \operatorname{Int} P$ (i.e $\mu \ll c$) whenever $\| \mu \| < \delta$, $\mu \in E$.

Proof. Since $\theta \ll c$, then $c \in \operatorname{Int} P$. Hence, find $\delta > 0$ such that $\Big\{\mu \in E : \| \mu - c \| < \delta\Big\} \subset \operatorname{Int} P$. Now, if $\| \mu \| < \delta$, then $\| (c-\mu) - c \| = \| -\mu \| = \| \mu \| < \delta$ and hence $(c-\mu) \in \operatorname{Int} P$ i.e $\mu \ll c$.

Lemma 4.5. Let (X,m) be a M-cone metric space over Banach algebra. Then for each $c_1 \gg \theta$ and $c_2 \gg \theta$, $c_1, c_2 \in E$, $\exists\, c \gg \theta$, $c \in E$ such that $c \ll c_1$ and $c \ll c_2$.

Proof. Since $c_2 \gg \theta$ then by Lemma 4.4, find $\delta > 0$ such that $\| \mu \| < \delta$ implies $\mu \ll c_2$ choose n_0 such that

$$\frac{1}{n_0} < \frac{\delta}{\| c_1 \|}.$$

Let $c = \frac{c_1}{n_0}$, then

$$\begin{aligned} \|c\| &= \left\|\frac{c_1}{n_0}\right\| \\ &= \frac{\|c_1\|}{n_0} < \delta, \end{aligned}$$

and hence $c \ll c_2$. But also it is clear that $c \gg \theta$ and $c \ll c_1$.

Theorem 4.6. Every M-cone metric space over Banach algebra (X, m) is a topological space.

Proof. For $c \gg \theta$, let $B_m(\mu, c) = \left\{\nu \in X : m(\mu,\nu)+m(\nu,\nu)-m_{\mu,\nu}-m(\mu,\mu) \ll c\right\}$ and $\mathfrak{B} = \left\{B_m(\mu, c) : \mu \in X, c \gg \theta\right\}$ then $\tau_m = \left\{U \subset \mu : \forall\ \mu \in U,\ \exists\ B \in \mathfrak{B}, \mu \in B \subset U\right\}$ is a topology on X indeed.

(τ_{m_1}): $\emptyset,\ X \in \tau_m$
(τ_{m_2}): Let $U, V \in \tau_m$ and let $\mu \in U \cap V$ then $\mu \in U$ and $\mu \in V$ find $c_1 \gg \theta$, $c_2 \gg \theta$ such that $\mu \in B_m(\mu, c_1) \subset U$ and $\mu \in B_m(\mu, c_2) \subset V$. By Lemma 4.5, find $c \gg \theta$ such that $c \ll c_1$ and $c \ll c_2$. Then clearly, $\mu \in B_m(\mu, c) \subset B_m(\mu, c_1) \cap B_m(\mu, c_2) \subset U \cap V$. Hence $U \cap V \in \tau_m$.
(τ_{m_3}): Let $U_i \in \tau_m$ for each $i \in I$ and let $\mu \in U_{i\in I}$, then $\exists\ i_0 \in I$ such that $\mu \in U_{i_0}$. Hence, find $c \gg \theta$ such that $\mu \in B_m(\mu, c) \subset U_i \subset U_{i\in I}$ that is $U_{i\in I} \in \tau_m$.

Lemma 4.7. Let (X, m) be a M-cone metric space over Banach algebra A and let P be a solid cone in a Banach algebra A where $k \in P$ is an arbitrarily vector, then (X, m) is a T_0-space.

Proof. Let (X, m) be an M-cone metric space over Banach algebra with two distinct elements $\mu, \nu \in X$. Without the loss of generality, we can consider two cases:

Case 1. If $m(\mu, \mu) = m(\nu, \nu)$ then by (m_1) and (m_2) and since $\mu \neq \nu$, we have

$$m_{\mu,\nu} = m(\mu, \mu) = m(\nu, \nu) \prec m(\mu, \nu).$$

Hence,

$$m(\mu, \nu) + m(\nu, \nu) - m_{\mu,\nu} - m(\mu, \mu) = m(\mu, \nu) - m(\mu, \mu) \succ \theta.$$

Therefore, if $c = m(\mu, \nu) - m(\mu, \mu)$ then $\nu \notin B_m(\mu, c)$.

Case 2. If $m(\mu, \mu) \prec m(\nu, \nu)$, then by (m_1)

$$\begin{aligned} & m_{\mu,\nu} \preceq m(\mu, \nu) \\ \text{or} \quad & m(\mu, \nu) - m_{\mu,\nu} \succeq \theta. \end{aligned}$$

Hence,

$$m(\mu, \nu) + m(\nu, \nu) - m_{\mu,\nu} - m(\mu, \mu) \succeq m(\nu, \nu) - m(\mu, \mu) \succ \theta.$$

Therefore, if $c = m(\nu, \nu) - m(\mu, \mu)$ then $y \notin B_m(\mu, c)$.
Consequently, we find that M-cone metric space over Banach algebra (X, m) is T_0 space.

Now, we define convergent and θ-Cauchy sequence in M-cone metric space over Banach algebra.

Definition 4.8. Let (X, m) be a M-cone metric space over Banach algebra and $\{\mu_n\}$ be a sequence in X. If for every $c \in \operatorname{Int} P$ there is a positive integer n_0 such that $m(\mu_n, \mu) \ll c + m_{\mu_n,\mu}$, $\forall\ n > n_0$, then $\{\mu_n\}$ is said to be convergent and converges to x.

Definition 4.9. Let (X, m) be M-cone metric space over Banach algebra. A sequence $\{\mu_n\}$ in (μ, m) is called a θ-Cauchy sequence if for every $c \gg \theta$, there is $n_0 \in N$ such that $m(\mu_n, \mu_m) - m_{\mu_n,\mu_m} \ll c$ or $M(\mu_n, \mu_m) - m_{\mu_n,\mu_m} \ll c$, $\forall n, m \geqslant n_0$.

Definition 4.10. Let (X, m) be M-cone metric space over Banach algebra. Then (X, m) is said to be θ-complete if every θ-Cauchy sequence $\{\mu_n\}$ in X converges to a point $\mu \in X$, i.e.

$$\lim_{n\to+\infty} M_{a,\mu_n} = \lim_{n\to+\infty} m_{a,\mu_n} = m(a, a) = \theta.$$

Lemma 4.11. Let (X, m) be a M-cone space over Banach algebra and $\{\mu_n\}$, $\{\nu_n\}$ be sequences in X. Assume that $\mu_n \to \mu \in X$ and $\nu_n \to \nu \in X$ as $n \to +\infty$. Then

$$\lim_{n\to+\infty} \left(m(\mu_n, \nu_n) - m_{\mu_n,\nu_n}\right) = m(\mu, \nu) - m_{\mu\nu}.$$

Proof. We have

$$\begin{aligned}\left|\left(m(\mu_n, \nu_n) - m_{\mu_n,\nu_n}\right) - \left(m(\mu, \nu) - m_{\mu,\nu}\right)\right| &= \Big|\left(m(\mu_n, \mu) - m_{\mu_n,\mu}\right) + \left(m(\mu, \nu_n) - m_{\mu,\nu_n}\right) \\ &\quad -\left(m(\mu, \nu) - m_{\mu,\nu}\right)\Big| \\ &= \left|m(\mu_n, \mu) - m_{\mu_n,\mu} + m(\nu_n, \nu) - m_{\nu_n,\nu}\right| \\ &= \left|m(\mu_n, \mu) - m_{\mu_n,\mu} + m(\nu_n, \nu) - m_{\nu_n,\nu}\right| \\ &\le \left|m(\mu_n, \mu) - m_{\mu_n,\mu}\right| + \left|m(\nu_n, \nu) - m_{\nu_n,\nu}\right|\end{aligned}$$

From Lemma 4.11, we deduce the following lemma.

Lemma 4.12. Let (X, m) be an M-cone metric space over Banach algebra and $\{\mu_n\}$ be a sequences in X. Assume that $\mu_n \to \mu \in X$ as $n \to +\infty$. Then

$$\lim_{n\to+\infty} \left(m(\mu_n, \nu) - m_{\mu_n,\nu}\right) = m(\mu, \nu) - m_{\mu,\nu}$$

for all $\nu \in X$.

Lemma 4.13. Let (X, m) be an M-cone metric space over Banach algebra and $\{\mu_n\}$ be a sequences in X. Assume that $\mu_n \to \mu \in X$ and $\mu_n \to \nu \in X$ as $n \to +\infty$. Then, $m(\mu, \nu) = m_{\mu,\nu}$. Furthermore, if $m(\mu, \mu) = m(\nu, \nu)$, then $\mu = \nu$.

Proof. By Lemma 4.12, we have

$$\theta = \lim_{n\to+\infty} \left(m(\mu_n, \mu_n) - m_{\mu_n,\mu_n}\right) = m(\mu, \nu) - m_{\mu,\nu}.$$

Lemma 4.14. Let $\{\mu_n\}$ be a sequence in an M-cone metric space over Banach algebra (X, m). Then

(i) $\lim_{n\to+\infty} m(\mu_n, \mu_n) = \theta$.
(ii) $\lim_{n,m\to+\infty} m_{\mu_n,\mu_m} = \theta$.
(iii) $\lim_{n,m\to+\infty} M_{\mu_n,\mu_m} = \theta$.

Proof. The proof of Lemma 4.14 is quite straight forward. (i) follows trivially from Definition 1. (ii) and (iii) follow trivially from (i).

5. Generalized Lipschitz mapping

In this section, we introduce the concept of generalized Lipschitz map on M-cone metric space over Banach algebra.

Definition 5.1. Let (X, m) be a M-cone metric space over Banach algebra. A mapping $T : X \to X$ is called a generalized Lipschitz mapping if $\exists$ a vector $k \in P$ with $\rho(k) < 1$ and $\forall \mu, \nu \in X$, one has

$$m(T\mu, T\nu) \preceq k\, m(\mu, \nu).$$

Example 5.2. Let A be a Banach algebra and P be a cone as in Example 3.3, and let $X = R^+$. Define a map $m : X \times X \to A$ by

$$m(\mu, \nu)(t) = \left(\frac{\mu + \nu}{2}\right)e^t,$$

for all $\mu, \nu \in X$. Then, (X, m) is a M-cone metric space over Banach algebra A. Take $T : X \to X$ by $T(\mu) = \frac{\mu}{\mu+1}$, where $\mu \in [3, +\infty)$ all $\mu, \nu \in X$, we have when $\mu \neq \nu$

$$\begin{aligned}
m(T\mu, T\nu)(t) &= \frac{1}{2}\left(\frac{\mu}{\mu + 1} + \frac{\nu}{\nu + 1}\right)e^t \\
&\leq \frac{1}{2}\left(\frac{\mu}{4} + \frac{\nu}{4}\right)e^t \\
&= \frac{1}{4}\left(\frac{\mu + \nu}{2}\right)e^t \\
&= km(\mu, \nu)(t).
\end{aligned}$$

Hence, T is a generalized Lipschitz mapping on μ, where $k = \frac{1}{4}$.

Now, we discuss some facts on c-sequence theory.

Definition 5.3 ([6]). Let P be a solid cone in a Banach space E. A sequence $\{u_n\} \subset P$ is said to be a c-sequence if for each $c \gg \theta$ $\exists$ a natural number N such that $u_n \ll c$ $\forall n > N$.

Lemma 5.4 ([12]). Let P be a solid cone in a Banach algebra A. Suppose that $k \in P$ be an arbitrary vector and $\{u_n\}$ is a c-sequence in P. Then $\{ku_n\}$ is a c-sequence.

Lemma 5.5 ([11]). Let A be a Banach algebra with a unit $e, k \in A$, then $\lim_{n\to+\infty} \| k^n \|^{\frac{1}{n}}$ exists and the spectral radius $\rho(k)$ satisfies

$$\rho(k) = \lim_{n\to+\infty} \| k^n \|^{\frac{1}{n}} = \inf \| k^n \|^{\frac{1}{n}} .$$

If $\rho(k) < |\lambda|$, then $(\lambda e - k)$ is invertible in A. Moreover,

$$(\lambda e - k)^{-1} = \sum_{i=0}^{+\infty} \frac{k^i}{\lambda^{i+1}}$$

where λ is a complex constant.

Lemma 5.6 ([11]). Let A be a Banach algebra with a unit $e, a, b \in A$. If a commutes with b, then

$$\rho(a + b) \leq \rho(a) + \rho(b), \quad \rho(ab) \leq \rho(a)\, \rho(b).$$

Lemma 5.7 ([5]). Let A be Banach algebra with a unit e and P be a solid cone in A. Let $a, k, l \in P$ hold $l \preceq k$ and $a \preceq la$. If $\rho(k) < 1$, then $a = \theta$.

Lemma 5.8 ([5]). If E is a real Banach space with a solid cone P and $\{u_n\} \subset P$ be a sequence with $\| u_n \| \to 0$ $(n \to +\infty)$, then $\{u_n\}$ is a c-sequence.

Lemma 5.9 ([5]). If E is a real Banach space with a solid cone P

(1) If $a, b, c \in E$ and $a \preccurlyeq b \preccurlyeq c$, then $a \ll c$.
(2) If $a \in P$ and $a \ll c$ for each $c \gg \theta$, then $a = \theta$.

Lemma 5.10 ([5]). Let A be a Banach algebra with a unit e and $k \in A$. If λ is a complex constant and $\rho(k) < |\lambda|$, then

$$\rho\Big(\big(\lambda e - k\big)^{-1}\Big) \leq \frac{1}{|\lambda| - \rho(k)}.$$

6. Fixed point results

Now we shall prove some fixed point theorem for generalized Lipschitz maps in setting of M-cone metric space over Banach algebra.

Theorem 6.1. Let (X, m) be a M-cone metric space over Banach algebra A and $T : X \to X$ be a mapping satisfying the following conditions $\forall\ \mu, \nu \in X$

$$m(T\mu, T\nu) \preceq k\Big[m(\mu, T\mu) + m(\nu, T\nu)\Big] \tag{6.1}$$

where $\rho(k) < 1$. Then T admits a unique fixed point.

Proof. Let $\mu_0 \in X$ and define a sequence $\{\mu_n\}$ in X such that $\mu_n = T\mu_{n-1}$ $\forall n \in N$.

From (6.1) and (m_4), we get

$$\begin{aligned} m(\mu_{n+1}, \mu_n) &= m(T\mu_n, T\mu_{n-1}) \\ &\preceq k\Big[m(\mu_n, T\mu_n) + m(\mu_{n-1}, T\mu_{n-1})\Big] \\ &= k\Big[m(\mu_n, \mu_{n+1}) + m(\mu_{n-1}, \mu_n)\Big] \end{aligned}$$

$$So,\ \ (e-k)m(\mu_{n+1}, \mu_n) \preceq km(\mu_{n-1}, \mu_n).$$

Since $\rho(k) < 1$. By Lemma 5.5, $(e - k)$ is invertible.

So,

$$m(\mu_{n+1}, \mu_n) \preceq k(e-k)^{-1} m(\mu_{n-1}, \mu_n).$$

Put $h = k(e - k)^{-1}$.

Hence, $m(\mu_{n+1}, \mu_n) \preceq h\, m(\mu_n, \mu_{n-1})$.

By Lemma 5.6 and Lemma 5.10, we have

$$\begin{aligned} \rho(h) &= \rho\Big[k(e-k)^{-1}\Big] \\ &\leq \rho(k) \cdot \rho\Big((e-k)^{-1}\Big) \\ &\leq \frac{\rho(k)}{1 - \rho(k)} < 1. \end{aligned}$$

Thus, $(e-h)$ is invertible. Continuing in the same argument, we will get

$$m(\mu_n, \mu_{n+1}) \preceq h^n m(\mu_0, \mu_1).$$

Using Lemma 4.14, we get (i), (ii), (iii) of Lemma 4.14 hold.
Moreover, $\forall n, m \in N; n > m$, we have

$$\begin{aligned} m(\mu_n, \mu_m) &\preceq m(\mu_n, \mu_{n+1}) - m_{\mu_n,\mu_{n+1}} + m(\mu_{n+1}, \mu_m) - m_{\mu_{n+1},\mu_n} \\ &\preceq m(\mu_n, \mu_{n+1}) + m(\mu_{n+1}, \mu_m) \\ &\preceq m(\mu_n, \mu_{n+1}) + m(\mu_{n+1}, \mu_{n+2}) - m_{\mu_{n+1},\mu_{n+2}} + m(\mu_{n+2}, \mu_m) - m_{\mu_{n+2},\mu_m} \\ &\preceq m(\mu_n, \mu_{n+1}) + m(\mu_{n+1}, \mu_{n+2}) + m(\mu_{n+2}, \mu_m) \\ &\preceq m(\mu_n, \mu_{n+1}) + m(\mu_{n+1}, \mu_{n+2}) + m(\mu_{n+2}, \mu_{n+1}) + \cdots + m(\mu_{m-1}, \mu_m) \\ &\preceq h^n m(\mu_0, \mu_1) + h^{n+1} m(\mu_0, \mu_1) + \cdots + h^{m-1} m(\mu_0, \mu_1) \\ &= h^n \Big[e + h + h^2 + \cdots + h^{m-n-1}\Big] m(\mu_0, \mu_1) \\ &= (e-h)^{-1} h^n m(\mu_0, \mu_1). \end{aligned}$$

In view of Remark 2.4, $\left\|h^m m(\mu_0, \mu_1)\right\| \leq \|h^n\| \left\|m(\mu_0, \mu_1)\right\| \to 0$ as $n \to +\infty$, by Lemma 5.8, we have $\{h^n m(\mu_0, \mu_1)\}$ is a c-sequence. Using Lemma 5.4 and Lemma 5.9, $\{\mu_n\}$ is a θ-Cauchy sequence in X. By the θ-completeness of X, there is $\mu \in X$ so that

$$\lim_{n \to +\infty} m(\mu_n, \mu) = \lim_{n,m \to +\infty} m(\mu_n, \mu_m) = m(\mu, \mu) = \theta. \tag{6.2}$$

As $\mu_n \to \mu$ as $n \to +\infty$ for some μ. So, $m(\mu_n, \mu) - m_{\mu_n,\mu} \to \theta$ as $n \to +\infty$,

$$m_{\mu_n,\mu} = \min\Big\{m(\mu_n, \mu_n), m(\mu, \mu)\Big\} \to \theta, \tag{6.3}$$

$$m_{\mu_n,T\mu} = \min\Big\{m(\mu_n, \mu_n), m(T\mu, T\mu)\Big\} = \theta. \tag{6.4}$$

Next, we will show that $m(\mu, T\mu) = \theta$.
From (m_4), we get

$$m(\mu, T\mu) - m_{\mu,T\mu} \preceq m(\mu, \mu_n) - m_{\mu,\mu_n} + m(\mu_n, T\mu) - m_{\mu_n,T\mu}$$

$\forall n \in N$.

So,

$$\begin{aligned} m(\mu, T\mu) - m_{\mu,T\mu} &\preceq m(\mu, \mu_n) + m(\mu_n, T\mu) \\ &= m(\mu, \mu_n) + m(T\mu_{n-1}, T\mu) \\ &\preceq m(\mu, \mu_n) + k\Big[m(\mu_{n-1}, T\mu_{n-1}) + m(\mu, T\mu)\Big] \\ &= m(\mu, \mu_n) + k\, m(\mu_{n-1}, \mu_n) + k\, m(\mu, T\mu) \\ (e-k) m(\mu, T\mu) - m_{\mu,T\mu} &\preceq m(\mu, \mu_n) + k\, m(\mu_{n-1}, \mu_n) \end{aligned}$$

$\Rightarrow$

$$(e-k) m(\mu, T\mu) \preceq \theta.$$

The multiplication by

$$(e-k)^{-1} = \sum_{i=0}^{+\infty} k^i \geq 0$$

yields that $m(\mu, T\mu) \preceq \theta$ thus,

$$m(\mu, T\mu) = \theta. \tag{6.5}$$

By condition (6.1), we have

$$\begin{aligned} m(T\mu, T\mu) &\preceq k\Big[m(\mu, T\mu) + m(\mu, T\mu)\Big] \\ &= 2k\, m(\mu, T\mu). \end{aligned}$$

From (6.5), we get

$$m(T\mu, T\mu) = \theta. \tag{6.6}$$

From (6.2), (6.5) and (6.6), we obtain,

$$m(\mu, \mu) = m(T\mu, T\mu) = m(\mu, T\mu).$$

Using (m_1), we get $T\mu = \mu$. Finally, we will show that T has a unique fixed point. Assume that ν is an another fixed point of T from (6.1), we get

$$\begin{aligned} m(\mu, \nu) &= m(T\mu, T\nu) \\ &\preceq k\Big[m(\mu, T\nu) + m(\nu, T\nu)\Big] \\ &= k\Big[m(\mu, \mu) + m(\nu, \nu)\Big]. \end{aligned}$$

From (6.2),

$$m(\mu, \nu) = \theta.$$

$\Rightarrow$

$$\mu = \nu.$$

Therefore, μ is a unique fixed point of T. This finishes the proof.

Theorem 6.2. Let (X, m) be a M-cone metric space over Banach algebra A and let $T : X \to X$ be a mapping satisfying the following conditions $\forall \mu, \nu \in X$

$$m(T\mu, T\nu) \preceq k\Big[m(\mu, T\nu) + m(T\mu, \nu)\Big] \tag{6.7}$$

where $\rho(k) < \dfrac{1}{s+1}$. Then, T admits a unique fixed point.

Proof. Let $\mu_0 \in X$ and define $\mu_n = T\mu_{n-1}$ $\forall n \in N$.

From (6.7), and (m_4), we get

$$\begin{aligned} m(\mu_{n+1}, \mu_n) &= k\Big[m(\mu_n, T\mu_{n-1}) + m(T\mu_n, \mu_{n-1})\Big] \\ &\preceq k\Big[m(\mu_n, \mu_n) + m(\mu_{n+1}, \mu_{n-1})\Big] \\ &\preceq k\Big[m(\mu_n, \mu_n) + m(\mu_{n+1}, \mu_n) - m_{\mu_{n+1}, \mu_n} + m(\mu_n, \mu_{n-1}) - m_{\mu_n, \mu_{n-1}} + m_{\mu_{n+1}, \mu_{n-1}}\Big] \\ &= k\Big[m(\mu_n, \mu_n) + m(\mu_{n+1}, \mu_n) - m(\mu_{n+1}, \mu_{n+1}) + m(\mu_n, \mu_{n-1}) - m(\mu_n, \mu_n) + m(\mu_{n+1}, \mu_{n-1})\Big] \\ &= k\Big[m(\mu_{n+1}, \mu_n) + m(\mu_n, \mu_{n-1})\Big] \end{aligned}$$

$$(e - k)m(\mu_{n+1}, \mu_n) \preceq km(\mu_n, \mu_{n-1}).$$

Since, $\rho(k) < 1$. By Lemma 5.5, $(e-k)$ is invertible.

So,

$$m(\mu_{n+1}, \mu_n) \preceq k(e-k)^{-1} m(\mu_n, \mu_{n-1}).$$

It is evident that

$$m(\mu_{n+1}, \mu_n) \preceq h\, m(\mu_n, \mu_{n-1}).$$

Put $h = k(e-k)^{-1}$.

Hence,

$$\begin{aligned}
\rho(h) &= \rho\Big(k(e-k)^{-1}\Big) \\
&\leq \rho(k) \cdot \rho\Big((e-k)^{-1}\Big) \\
&\leq \frac{\rho(k)}{1-\rho(k)} < 1.
\end{aligned}$$

Thus, $(e-h)$ is invertible. By using Lemma 4.14, we get (i), (ii), (iii) of Lemma 4.14 hold. By the mimic of the proof of Theorem 6.1, we can show that $\{\mu_n\}$ is a θ-Cauchy sequence. Since, (X, m) is a θ-complete M-cone metric space over a Banach algebra, there is $\mu \in X$ so that

$$\lim_{n\to+\infty} m(\mu_n, \mu) = \lim_{n,m\to+\infty} m(\mu_n, \mu_m) = m(\mu, \mu) = \theta. \tag{6.8}$$

As $\mu_n \to \mu$ as $n \to +\infty$ for some μ. So, $m(\mu_n, \mu) - m_{\mu_n,\mu} \to \theta$ as $n \to +\infty$ and $M_{\mu_n,\mu} - m_{\mu_n,\mu} \to \theta$ as $n \to +\infty$. From (ii) of Lemma 4.14, we get $m(\mu_n, \mu_n) \to \theta$ as $n \to +\infty$ and so

$$\begin{aligned}
m_{\mu_n,\mu} &= \min\Big\{m(\mu_n, \mu_n), m(\mu, \mu)\Big\} \to \theta \ \text{ as } \ n \to +\infty \\
\text{and} \ \ m_{\mu_n,T\mu} &= \min\Big\{m(\mu_n, \mu_n), m(T\mu, T\mu)\Big\} \to \theta \ \text{ as } \ n \to +\infty.
\end{aligned}$$

Next, we will show that $m(\mu, T\mu) = \theta$.

From (m_4), we get

$$\begin{aligned}
m(\mu, T\mu) &= m_{\mu,T\mu} - m_{\mu,T\mu} \\
&\preceq m(\mu, \mu_n) - m_{\mu,\mu_n} + m(\mu_n, T\mu) - m_{\mu_n,T\mu}
\end{aligned}$$

$\forall n \in N$.

So,

$$\begin{aligned}
m(\mu, T\mu) - m_{\mu,T\mu} &\preceq m(\mu, \mu_n) + m(\mu_n, T\mu) \\
&= m(\mu, \mu_n) + m(T\mu_{n-1}, T\mu) \\
&\preceq m(\mu, \mu_n) + k\Big[m(\mu_{n-1}, T\mu) + m(T\mu_{n-1}, \mu)\Big] \\
&= m(\mu, \mu_n) + k\, m(\mu_{n-1}, T\mu) + k\, m(\mu_n, \mu) \\
&\preceq m(\mu, \mu_n) + k\Big[m(\mu_{n-1}, \mu) - m_{\mu_{n-1},\mu} + m(\mu, T\mu) - m_{\mu,T\mu}\Big] + k\, m(\mu_n, \mu)
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
&(e-k) m(\mu, T\mu) \preceq \theta \\
\text{Thus,} \qquad & m(\mu, T\mu) = \theta.
\end{aligned}$$

By condition (6.7), we have

$$\begin{aligned} m(T\mu, T\mu) &\preceq k\Big[m(\mu, T\mu) + m(T\mu, \mu)\Big] \\ &= 2k\, m(\mu, T\mu). \end{aligned}$$

From (6.8), we get

$$m(T\mu, T\mu) = \theta. \tag{6.9}$$

From (6.7), (6.8) and (6.9), we obtain,

$$m(\mu, \mu) = m(T\mu, T\mu) = m(\mu, T\mu).$$

Using (m_1), we get $T\mu = \mu$. Finally, we will show that T has a unique fixed point. Assume that ν is another fixed point of T. From (6.7), we get

$$\begin{aligned} m(\mu, \nu) &= m(T\mu, T\nu) \\ &\preceq k\Big[m(\mu, T\nu) + m(\nu, T\mu)\Big] \\ &= k\Big[m(\mu, \nu) + m(\nu, \mu)\Big] \\ &= 2k\, m(\mu, \nu) \end{aligned}$$

implies

$$(e - 2k)m(\mu, \nu) \preceq \theta.$$

Since $\rho(k) < 1$, we conclude that

$$m(\mu, \nu) = \theta$$

$\Rightarrow$

$$\mu = \nu.$$

Therefore, T has a unique fixed point. This completes the proof.

Corollary 6.3. Let (X, m) be a M-cone metric space over a Banach algebra A and let $T : X \to X$ be a mapping satisfying the following conditions $\forall \mu, \nu \in X$.

$$m(T\mu, T\nu) \preceq k\, m(\mu, \nu)$$

where $\rho(k) < 1$. Then, T possesses a unique fixed point.

Now, we give the following examples in the support of our main results.

Example 6.4. Let $A = C_1^R[0, 1]$ and define a norm on A by $\| \mu \| = \| \mu \|_\infty + \| \mu' \|_\infty$ $\forall \mu \in A$. Let the multiplication in A be the point wise multiplication. Then A is a real Banach algebra with unit $e = 1$. Set $P = \Big\{\mu \in A : \mu \geq 0\Big\}$ which is normal in A. Moreover, P is not normal (see[12]). Let $X = [0, +\infty)$. Define a mapping $m : X \times X \to A$ by

$$m(\mu, \nu)(t) = \left(\frac{\mu + \nu}{2}\right)e^t,$$

$\forall \mu, \nu \in X$. Then, (X, m) is a θ-complete M-cone metric space over Banach algebra A. Now define a mapping $T : X \to X$ by $T\mu = \log\left(1 + \frac{\mu}{2}\right)$. Since $\log(1 + u) \leq u$, for each $u \in [0, +\infty)$ $\forall \mu \in X$. Observe that, $\forall \mu, \nu \in X$, we obtain

$$\begin{aligned} m(T\mu, T\nu)(t) &= \frac{1}{2}\left[\log\left(1 + \frac{\mu}{2}\right) + \log\left(1 + \frac{\nu}{2}\right)\right]e^t \\ &\leq \left[\frac{\mu}{2} + \frac{\mu}{2}\right]e^t \\ &= \frac{1}{2}m(\mu, \nu)(t). \end{aligned}$$

where $k = \frac{1}{2}$. Thus, all the conditions of Corollary 6.3, holds and T has a unique fixed point $\mu = \theta$.

7. Application to the Existence of a Solution of Integral Equations

An application of the theorem stated in the previous part will be presented in this section.

Consider $C\Big([0, T], R\Big)$, the class of continuous functions on $[0, T]$, $T > 0$. Let $A = C[0, T]$ be equipped with the norm $\| \mu \| = \| \mu \|_\infty + \| \mu' \|_\infty$. Take the usual multiplication, then A is a Banach algebra with the unit $e = 1$. Let m be the M-cone metric given as

$$m(\mu, \nu)(t) = \sup_{t \in [a,b]} \left(\frac{\mu + \nu}{2}\right)e^t, \tag{7.1}$$

$\forall \mu, \nu \in C\Big([0, T], R\Big)$. Note that $\Big(C\Big([0, T], R\Big), m\Big)$ is a θ-complete M-cone metric space over Banach algebra $\Big(C[0, T], R\Big)$.

Theorem 7.1. Assume that $\forall \mu, \nu \in C\Big([0, T], R\Big)$

$$\left|K\Big(t, s, \mu(t)\Big) + K\Big(t, s, \nu(t)\Big)\right| \leq \lambda \left|\mu(t) + \nu(t)\right|, \tag{7.2}$$

$\forall t, s \in [0, T]$, where $\lambda \in [0, 1)$. Then, the integral equation

$$\mu(t) = \int_0^T K\Big(t, s, \mu(t)\Big)ds \tag{7.3}$$

where $t \in [0, T]$, admits a unique solution in $C\Big([0, T], R\Big)$.

Proof. Define $T : X \to X$ by

$$T\mu(t) = \int_0^T K\Big(t, s, \mu(t)\Big)ds \tag{7.4}$$

$\forall t, s \in [0, T]$.

We have,

$$
\begin{aligned}
m(T\mu, T\nu)(t) &= \left|\frac{T\mu(t) + T\nu(t)}{2}\right| \\
&= \left|\int_0^T \left(\frac{K\big(t, s, \mu(t)\big) + K\big(t, s, \nu(t)\big)}{2}\right) ds\right| e^t \\
&\leq \left(\int_0^T \left|\frac{K\big(t, s, \mu(t)\big) + K\big(t, s, \nu(t)\big)}{2}\right| ds\right) e^t \\
&\leq \left(\lambda \int_0^T \left|\frac{\mu(t) + \nu(t)}{2}\right| ds\right) e^t \\
&\leq \left(\lambda \int_0^T \left(\frac{|\mu(t)| + |\nu(t)|}{2}\right) ds\right) e^t \\
&\leq \left(\lambda \sup_{t\in[0,b]} \left(\frac{|\mu(t)| + |\nu(t)|}{2}\right) \int_0^T ds\right) e^t \\
&\leq \lambda\, m(\mu, \nu)(t).
\end{aligned}
$$

Thus, condition (6.1) is satisfied. Therefore, all conditions of Theorem 6.1 are satisfied. Hence, T has a unique fixed point, which means that the Fredholm integral equation (7.3) has a unique solution. This completes the proof.

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