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Some New Sequence Spaces in n–Normed Spaces Defined by a Museliak-Orlicz Function

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Abstract. In this paper, we introduce some new sequence spaces in n–normed spaces defined by Museliak-Orlicz function. Also we investigate some topological properties and inclusion relations between these spaces

1. Introduction

Let ω be the set of all sequences of real or complex numbers and $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of positive integers, set of real numbers and complex numbers, respectively. Also let ℓ_∞ and c be respectively the Banach spaces of bounded and convergent sequences $x = (x_k)$ with the usual norm $\|x\| = \sup |x_k|$. A sequence $x \in \ell_\infty$ is said to be almost convergent if all its Banach limits [1] coincide and the set of all almost convergent sequences is denoted by $\hat{c}$. Lorentz [10] proved that $x \in \hat{c}$ if and only if $\lim_n \frac{1}{n} \sum_{k=1}^{n} x_{k+m}$ exists uniformly in m.

Maddox [12, 13] defined x to be strongly almost convergent to a number L if

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+m} - L| = 0, \text{ uniformly in } m.$$

By $[\hat{c}]$ we denote the space of all strongly almost convergent sequences. It is easy to see that $c \subset [\hat{c}] \subset \hat{c} \subset \ell_\infty$.

In [3], Das and Sahoo defined the sequence spaces

$$(W) = \left\{ x : \frac{1}{n+1} \sum_{k=0}^{n} (t_{km}(x) - L) \to 0 \text{ as } n \to \infty, \text{ uniformly in } m, \text{ for some } L \right\}$$

and

$$[W] = \left\{ x : \frac{1}{n+1} \sum_{k=0}^{n} |t_{km}(x) - L| \to 0 \text{ as } n \to \infty, \text{ uniformly in } m, \text{ for some } L \right\}$$

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where $t_{km}(x) = \frac{(x_m+\cdots+x_{m+k})}{(k+1)}$.

An Orlicz function $M : [0, \infty) \to [0, \infty)$ is contiuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Lindenstrauss and Tzafriri [11] used the idea of an Orlicz function to define the following sequence space

$$\ell_M = \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

which is called an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}.$$

It is shown in [11] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p $(p \ge 1)$. An Orlicz function M satisfies the Δ_2-condition if and only if for any constant $L > 1$ there exists a constant $K(L)$ such that

$$M(Lu) \le K(L)M(u) \text{ for all values of } u \ge 0.$$

Subsequently Orlicz sequence spaces have been studied by Parashar and Chaudhry [19], R. Colak, M. Et and E. Malkowsky [2], Nuray and Gulcu [18], B. C. Tripathy and S. Mahanta [21]. Esi and M. Et [4], E. Savas [20], Bhardwaj and Singh [1], Mursaleen, Mushir A. Khan and Qamruddin [17] and many others.

Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers tending to ∞ and $\lambda_{n+1} \le \lambda_n + 1$, $\lambda_1 = 1$.

The generalized de Vallée-Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$.

A set of sequences $x = (x_k)$ which are strongly almost (v, λ)–summable was defined by Savas [20] as

$$[\hat{v}, \lambda] = \left\{ x = (x_k) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_{k+m} - L| = 0, \text{ for some } L, \text{ uniformly in } m \right\}.$$

Recently, R. Colak, M. Et and E. Malkowsky [2] defined the strongly almost (W, λ) summable sequences by using an Orlicz function as follows:

A sequence $x = (x_k)$ is said to be strongly (W, λ)–summable to L, if

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |t_{km}(x) - L| = 0, \text{ uniformly in } m.$$

In this case we write

$$[W, \lambda] = \left\{ x = (x_k) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |t_{km}(x) - L| = 0, \text{ for some } L, \text{ uniformly in } m \right\}$$

for the set of sequences $x = (x_k)$ which are strongly (W, λ)–summable to L; this is denoted by $x_k \to L[W, \lambda]$.

Let M be an Orlicz function and $p = (p_k)$ be any sequence of strictly positive real numbers. We define

$$[W, \lambda, M, p] = \left\{ x = (x_k) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|t_{km}(x) - L|}{\rho}\right)\right]^{p_k} = 0 \text{ uniformly in } m, \right.$$

for some L and for some $\rho > 0\}$,

$$[W,\lambda,M,p]_0 = \left\{x = (x_k) : \lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} \left[M\left(\frac{|t_{km}(x)|}{\rho}\right)\right]^{p_k} = 0 \text{ uniformly in } m, \text{ and for some } \rho > 0\right\},$$

$$[W,\lambda,M,p]_\infty = \left\{x = (x_k) : \sup_{m,n} \frac{1}{\lambda_n} \sum_{k\in I_n} \left[M\left(\frac{|t_{km}(x)|}{\rho}\right)\right]^{p_k} < \infty \text{ for some } \rho > 0\right\}.$$

The concept of 2–normed spaces was initially introduced by Gahler [5], that of n–normed spaces was introduced by Misiak [15], and this concept has been studied by many authors ([6–8]).

Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{R}$ of dimension d, where $d \geq n \geq 2$.
A real valued function $||\cdots||$ on X^n that satisfies the following four conditions:
(i) $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;
(ii) $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;
(iii) $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{R}$;
(iv) $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$;
is called an n–norm on X and the pair $(X, ||\cdots,||)$ is called an n–normed space over the field $\mathbb{R}$.

For example, we may take $X = \mathbb{R}^n$ being equipped with the n–norm $||x_1, x_2, \cdots, x_n||_E$ = the volume of the n–dimensional parallelepiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$.

Let $(X, ||.., .., ||)$ be an n–normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be a linearly independent set in X. Then the function $||.., .||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots x_{n-1}, a_i||; i = 1, 2, 3, \cdots, n\}$$

is called an $(n-1)$–norm on X with respect to $\{a_1, a_2, \cdots a_n\}$.

A sequence (x_k) in an n–normed space $(X, ||.., .., , ||)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, z_2, ..., z_{n-1} \in X.$$

A sequence (x_k) in an n–normed space $(X, ||.., ..., ||)$ is said to be Cauchy if

$$\lim_{k,p\to\infty} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, z_2, ..., z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n–norm. Any complete n–normed space is said to be an n–Banach space.

Let X be a linear metric space. A function $g : X \to \mathbb{R}$ is called paranorm if
(i) $g(x) \geq 0$, for all $x \in X$
(ii) $g(-x) = g(x)$, for all $x \in X$
(iii) $g(x + y) \leq g(x) + g(y)$, for all $x, y \in X$
(iv) If (α_n) is a sequence of scalars with $\alpha_n \to \alpha$ as $n \to \infty$ and (x_n) is a sequence of vectors with $g(x_n - x) \to 0$ as $n \to \infty$ then $g(\alpha_n x_n - \alpha x) \to 0$ as $n \to \infty$.

A sequence space X is said to be solid (or normal) if $(\alpha_k x_k) \in X$ whenever $(x_k) \in X$, for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$.

Lemma 1.1. ([9, p. 53]) *A sequence space X is normal implies that X is monotone.*

A sequence $\mathcal{M} = (M_k)$ of Orlicz function is called a Museliak–Orlicz function ([14, 16]). A sequence $\mathcal{N} = (N_k)$ with

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\}, \ k = 1, 2 \cdots$$

is called the complementary function of a Museliak–Orlicz function $\mathcal{M}$. For a given Museliak–Orlicz function $\mathcal{M}$, the Museliak–Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows:

$$t_{\mathcal{M}} = \{x \in \omega : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\},$$
$$h_{\mathcal{M}} = \{x \in \omega : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}^{(x)}} = \sum_{k=1}^{\infty} M_k(x_k), \; x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$\|x\| = \inf\{k > 0 : I_{\mathcal{M}}\left(\frac{x}{k}\right) \leq 1\}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf\left\{\frac{1}{k}\left(1 + I_{\mathcal{M}}(kx)\right) : k > 0\right\}.$$

A Museliak-Orlicz function $\mathcal{M} = (M_k)$ is said to satisfy the Δ_2–condition if there exist constants $a, K > 0$ and a sequence $c = (c_k)_{k=1}^{\infty} \in \ell'_+$ (the positive cone of ℓ') such that the inequality $M_k(2u) \leq KM_k(u) + c_k$ holds for all $k \in \mathbb{N}$ and $u \in \mathbb{R}_+$ whenever $M_k(u) \leq a$.

The following inequality will be used throughout the paper :
Let $p = (p_k)$ be a positive sequence of real numbers with $\inf_k p_k = h$, $\sup_k p_k = H$ and $K = \max\{1, 2^{H-1}\}$. Then for all $a_k, b_k \in \mathbb{C}$, for all $k \in \mathbb{N}$, we have

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{*}$$

and for $\lambda \in \mathbb{C}$, $|\lambda|^{p_k} \leq \max\{|\lambda|^h, |\lambda|^H\}$.

The main purpose of this paper is to introduce some new sequence spaces by using a Museliak–Orlicz function in n–normed spaces. Also we investigate some topological properties and inclusion relations between these spaces.

2. Some new sequence spaces

Let $\omega(n - X)$ denote X–valued sequence spaces defined in an n–normed space $(X, \|\cdots, \cdots\|)$. Clearly $w(n - X)$ is a linear space under addition and scalar multiplication.

Let $\mathcal{M} = (M_k)$ be a Museliak–Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers. We define for some $\rho > 0$

$$[W, \lambda, \mathcal{M}, p, \|\cdots\|]_0 = \left\{x \in \omega(n - X) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M_k\left(\left\|\frac{t_{km}(x)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k} = 0 \right.$$
$$\left. \text{uniformly in } , m \text{ for some } \rho > 0\right\},$$

$$[W, \lambda, \mathcal{M}, p, \|\cdots\|] = \left\{x \in \omega(n - X) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M_k\left(\left\|\frac{t_{km}(x) - L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k} = 0 \right.$$
$$\left. \text{for some } L, \text{ uniformly in } m, \text{ for some } \rho > 0\right\},$$

$$[W, \lambda, \mathcal{M}, p, \|\cdots\|]_\infty = \left\{x \in \omega(n - X) : \sup_{mn} \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M_k\left(\left\|\frac{t_{km}(x)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k} < \infty \right.$$

$$\text{for some } \rho > 0\Big\}.$$

Clearly the inclusions $[W,\lambda,\mathcal{M},p,\|\cdots\|]_0 \subset [W,\lambda,\mathcal{M},p,\|\cdots\|] \subset [W,\lambda,\mathcal{M},p,\|\cdots\|]_\infty$ hold.

If the sequence $x = (x_k)$ is convergent to the limit L in $[W,\lambda,\mathcal{M},p,\|\cdots\|]$, then we write

$$[W,\lambda,\mathcal{M},p,\|\cdots\|] - \lim x = L.$$

If we take $p_k = 1$ for all $k \in \mathbb{N}$, then the sequence spaces $[W,\lambda,\mathcal{M},p,\|\cdots\|]_0$, $[W,\lambda,\mathcal{M},p,\|\cdots\|]$, $[W,\lambda,\mathcal{M},p,\|\cdots\|]_\infty$ reduce to $[W,\lambda,\mathcal{M},\|\cdots\|]_0, [W,\lambda,\mathcal{M},\|\cdots\|], [W,\lambda,\mathcal{M},\|\cdots\|]_\infty$ as follows :

$$[W,\lambda,\mathcal{M},\|\cdots\|]_0 = \Big\{x \in \omega(n-X) : \lim_n \frac{1}{\lambda_n}\sum_{k\in I_n}\Big[M_k\Big(\Big\|\frac{t_{km}(x)}{\rho}, z_1,\cdots,z_{n-1}\Big\|\Big)\Big] = 0$$
$$\text{uniformly in } m, \text{ for some } \rho > 0\Big\},$$

$$[W,\lambda,\mathcal{M},\|\cdots\|] = \Big\{x \in \omega(n-X) : \lim_n \frac{1}{\lambda_n}\sum_{k\in I_n}\Big[M_k\Big(\Big\|\frac{t_{km}(x)-L}{\rho}, z_1,\cdots,z_{n-1}\Big\|\Big)\Big] = 0$$
$$\text{for some } L, \text{ uniformly in } m, \text{ for some } \rho > 0\Big\},$$

$$[W,\lambda,\mathcal{M},\|\cdots\|]_\infty = \Big\{x \in \omega(n-X) : \sup_{mn} \frac{1}{\lambda_n}\sum_{k\in I_n}\Big[M_k\Big(\Big\|\frac{t_{km}(x)}{\rho}, z_1,\cdots,z_{n-1}\Big\|\Big)\Big] < \infty$$
$$\text{for some } \rho > 0\Big\}.$$

Further if we take $\mathcal{M}(x) = x$ and $\rho = 1$ we get the following sequence spaces:

$$[W,\lambda,\|\cdots\|]_0 = \Big\{x \in \omega(n-X) : \lim_n \frac{1}{\lambda_n}\sum_{k\in I_n}[(\|t_{km}(x), z_1,\cdots,z_{n-1}\|)] = 0$$
$$\text{uniformly in } m\Big\},$$

$$[W,\lambda,\|\cdots\|] = \Big\{x \in \omega(n-X) : \lim_n \frac{1}{\lambda_n}\sum_{k\in I_n}[(\|t_{km}(x)-L, z_1,\cdots,z_{n-1}\|)] = 0$$
$$\text{for some } L, \text{ uniformly in } m\Big\},$$

$$[W,\lambda,\|\cdots\|]_\infty = \Big\{x \in \omega(n-X) : \sup_{mn} \frac{1}{\lambda_n}\sum_{k\in I_n}[(\|t_{km}(x), z_1,\cdots,z_{n-1}\|)] < \infty\Big\}.$$

3. Main Results

Theorem 3.1. *Let $\mathcal{M} = (M_k)$ be a Museliak–Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers. Then the sequence spaces $[W,\lambda,\mathcal{M},p,\|\cdots\|]_0$, $[W,\lambda,\mathcal{M},p,\|\cdots\|]$ and $[W,\lambda,\mathcal{M},p,\|\cdots\|]_\infty$ are linear spaces over the field $\mathbb{R}$.*

Proof. We consider only $[W,\lambda,\mathcal{M},p,\|\cdots\|]$. The other cases can be treated similarly.
Let $x, y \in [W,\lambda,\mathcal{M},p,\|\cdots\|]$ and α,β be scalars. Then there exist $L_1, L_2 \in X$ and positive real numbers ρ_1,ρ_2 such that for every $z_1,\cdots z_{n-1} \in X$

$$\begin{aligned}
&\frac{1}{\lambda_n}\sum_{k\in I_n}\Big[M_k\Big(\Big\|\frac{t_{km}(\alpha x+\beta y)-(\alpha L_1+\beta L_2)}{|\alpha|\rho_1+|\beta|\rho_2}, z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_k}\\
&\le \frac{1}{\lambda_n}\sum_{k\in I_n}\Big[M_k\Big(\frac{|\alpha|\rho_1}{|\alpha|\rho_1+|\beta|\rho_2}\Big\|\frac{t_{km}(x-L_1)}{\rho_1}, z_1,\cdots,z_{n-1}\Big\| + \frac{|\beta|\rho_2}{|\alpha|\rho_1+|\beta|\rho_2}\Big\|\frac{t_{km}(y-L_2)}{\rho_2}, z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_k}\\
&\le K\frac{1}{\lambda_n}\sum_{k\in I_n}\Big[M_k\Big(\Big\|\frac{t_{km}(x-L_1)}{\rho_1}, z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_k} + K\frac{1}{\lambda_n}\sum_{k\in I_n}\Big[M_k\Big(\Big\|\frac{t_{km}(y-L_2)}{\rho_2}, z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_k} \to 0
\end{aligned}$$

as $n \to \infty$ uniformly in m.

Therefore, $(\alpha x + \beta y) \in [W, \lambda, \mathcal{M}, p, \|\cdots\|]$. This proves that $[W, \lambda, \mathcal{M}, p, \|\cdots\|]$ is a linear space. □

Theorem 3.2. *Let $\mathcal{M} = (M_k)$ be a Museliak–Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers. Then the sequence space $[W, \lambda, \mathcal{M}, p, \|\cdots\|]_\infty$ is a paranormed space for some $\rho > 0$ and $h > 0$ with respect to the paranorm defined by*

$$g(x) = \inf\left\{\rho^{p_n/H} : \left(\sup_{mn} \frac{1}{\lambda_n} \sum_{k\in I_n} \left[M_k\left(\left\|\frac{t_{km}(x)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H} < \infty\right\}.$$

Proof. Clearly $g(-x) = g(x)$ and $g(\theta) = 0$ where $\theta = (0, 0, \cdots, 0)$ is the zero sequence.
Let $x, y \in [W, \lambda, \mathcal{M}, p, \|\cdots\|]_\infty$. Also let

$$A(x) = \left\{\rho > 0 : \frac{1}{\lambda_n} \sum_{k\in I_n} \left[M_k\left\|\left(\frac{t_{km}(x)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k} < \infty\right\},$$

and

$$A(y) = \left\{\rho > 0 : \frac{1}{\lambda_n} \sum_{k\in I_n} \left[M_k\left(\left\|\frac{t_{km}(y)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k} < \infty\right\}.$$

Let $\rho_1 \in A(x)$ and $\rho_2 \in A(y)$. By using Minkowski's inequality for $p = (p_k)$ with $p_k > 1$ for all k, we have

$$\begin{aligned}
&\left(\frac{1}{\lambda_n} \sum_{k\in I_n} \left[M_k\left(\left\|\frac{t_{km}(x+y)}{\rho_1+\rho_2}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H} \\
&\quad\le \left(\frac{1}{\lambda_n} \sum_{k\in I_n} \left[M_k\left(\frac{\rho_1}{\rho_1+\rho_2}\left\|\frac{t_{km}(x)}{\rho_1}, z_1, \cdots, z_{n-1}\right\| + \frac{\rho_2}{\rho_1+\rho_2}\left\|\frac{t_{km}(y)}{\rho_2}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H} \\
&\quad\le \left(\frac{1}{\lambda_n} \sum_{k\in I_n} \left[\frac{\rho_1}{\rho_1+\rho_2} M_k\left(\left\|\frac{t_{km}(x)}{\rho_1}, z_1, \cdots, z_{n-1}\right\|\right) + \frac{\rho_2}{\rho_1+\rho_2} M_k\left(\left\|\frac{t_{km}(y)}{\rho_2}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H} \\
&\quad\le \left(\frac{1}{\lambda_n} \sum_{k\in I_n} \left[M_k\left(\left\|\frac{t_{km}(x)}{\rho_1}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H} + \left(\frac{1}{\lambda_n} \sum_{k\in I_n} \left[M_k\left(\left\|\frac{t_{km}(y)}{\rho_2}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H} < \infty.
\end{aligned}$$

Thus

$$\begin{aligned}
g(x+y) &= \inf\{(\rho_1+\rho_2)^{p_n/H} : \rho_1 \in A(x) \text{ and } \rho_2 \in A(y)\} \\
&\le \inf\{\rho_1^{p_n/H} : \rho_1 \in A(x)\} + \inf\{\rho_2^{p_n/H} : \rho_2 \in A(y)\} = g(x) + g(y).
\end{aligned}$$

For the case $0 < p = (p_k) < 1$ with $0 < p_k < 1$ for all k, we have $g(x+y) \le g(x) + g(y)$ from (*).

Finally we prove that the scalar multiplication is continuous. Whenever $\alpha \to 0$ and x is fixed imply $g(\alpha x) \to 0$. Also, whenever $x \to \theta$ and α is any number imply $g(\alpha x) \to 0$. By using the definition g, we get that

$$g(\alpha x) = \inf\left\{\rho^{p_n/H} : \left(\frac{1}{\lambda_n} \sum_{k\in I_n} \left[M_k\left(\left\|\frac{t_{km}(\alpha x)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H} < \infty\right\}.$$

Then,

$$g(\alpha x) \leq \inf\left\{(\alpha\sigma)^{p_n/H} : \left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M_k\left(\left\|\frac{t_{km}(x)}{\sigma}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H} < \infty\right\},$$

where $\sigma = \rho/\alpha$. Since $|\alpha|^{p_k} \leq \max\{|\alpha|^h, |\alpha|^H\}$, therefore $|\alpha|^{p_k/H} \leq (\max\{|\alpha|^h, |\alpha|^H\})^{1/H}$. Then

$$\begin{aligned} g(\alpha x) &\leq \left(\max\left\{|\alpha|^h, |\alpha|^H\right\}\right)^{1/H} \times \\ &\quad \inf\left\{\sigma^{p_n/H} : \left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M_k\left(\left\|\frac{t_{km}(x)}{\sigma}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H} < \infty\right\}. \\ &= (\max\{|\alpha|^h, |\alpha|^H\})^{1/H} g(x), \quad h > 0. \end{aligned}$$

This completes the proof. □

Theorem 3.3. *Let $\mathcal{M} = (M_k)$, $\mathcal{M}' = (M'_k)$ and $\mathcal{M}'' = (M''_k)$ be Museliak–Orlicz functions. Then the following statements hold:*
(i) Let $0 < h \leq p_k \leq 1$ for all k. Then

$$[W, \lambda, \mathcal{M}, p, \|\cdots\|]_0 \subseteq [W, \lambda, \mathcal{M}, \|\cdots\|]_0, \quad [W, \lambda, \mathcal{M}, p, \|\cdots\|] \subseteq [W, \lambda, \mathcal{M}, \|\cdots\|].$$

(ii) Let $1 < p_k \leq H < \infty$ for all k. Then

$$[W, \lambda, \mathcal{M}, \|\cdots\|]_0 \subseteq [W, \lambda, \mathcal{M}, p, \|\cdots\|]_0, \quad [W, \lambda, \mathcal{M}, \|\cdots\|] \subseteq [W, \lambda, \mathcal{M}, p, \|\cdots\|].$$

(iii) Finally

$$[W, \lambda, \mathcal{M}', p, \|\cdots\|]_0 \cap [W, \lambda, \mathcal{M}'', p, \|\cdots\|]_0 \subseteq [W, \lambda, \mathcal{M}' + \mathcal{M}'', p, \|\cdots\|]_0.$$

Proof. (i) We shall prove the result for the space $[W, \lambda, \mathcal{M}, p, \|\cdots\|]_0$. The other cases can be treated similarly. Let $x \in [W, \lambda, \mathcal{M}, p, \|\cdots\|]_0$, $0 < h \leq p_k \leq 1$ for all k. Then

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M_k\left(\left\|\frac{t_{km}(x)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right] \leq \frac{1}{\lambda_n}\sum_{k\in I_n}\left[M_k\left(\left\|\frac{t_{km}(x)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k} \to 0 \text{ as } n \to \infty,$$
$$\text{uniformly in } m.$$

Hence $[W, \lambda, \mathcal{M}, p, \|\cdots\|]_0 \subseteq [W, \lambda, \mathcal{M}, \|\cdots\|]_0$.

(ii) Let $1 < p_k \leq H < \infty$ for all k and $x \in [W, \lambda, \mathcal{M}, \|\cdots\|_0]$. Then for each $0 < \epsilon < 1$ there exists a positive integer m_0 such that

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M_k\left(\left\|\frac{t_{km}(x)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right] < \epsilon < 1 \text{ for all } m > m_0.$$

This implies that

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M_k\left(\left\|\frac{t_{km}(x)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k} \leq \frac{1}{\lambda_n}\sum_{k\in I_n}\left[M_k\left(\left\|\frac{t_{km}(x)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right].$$

Therefore $x \in [W, \lambda, \mathcal{M}, p, \|\cdots\|]_0$, for each $\rho > 0$. Hence $[W, \lambda, \mathcal{M}, \|\cdots\|]_0 \subseteq [W, \lambda, \mathcal{M}, p, \|\cdots\|]_0$.

(iii) Suppose that $x \in [W, \lambda, \mathcal{M}', p, \|\cdots\|]_0 \cap [W, \lambda, \mathcal{M}'', p, \|\cdots\|]_0$. Then

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[(M'_k + M''_k)\left(\left\|\frac{t_{km}(x)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k}$$

$$= \frac{1}{\lambda_n}\sum_{k\in I_n}\left[M_k'\left(\left\|\frac{t_{km}(x)}{\rho},z_1,\cdots,z_{n-1}\right\|\right)+M_k''\left(\left\|\frac{t_{km}(x)}{\rho},z_1,\cdots,z_{n-1}\right\|\right)\right]^{p_k}$$

$$\leq K\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M_k'\left(\left\|\frac{t_{km}(x)}{\rho},z_1,\cdots,z_{n-1}\right\|\right)\right]^{p_k}+K\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M_k''\left(\left\|\frac{t_{km}(x)}{\rho},z_1,\cdots,z_{n-1}\right\|\right)\right]^{p_k}\to 0$$

$$\text{as } n\to\infty \text{ uniformly in } m.$$

Thus $x \in [W, \lambda, \mathcal{M}' + \mathcal{M}'', p, \|\cdots\|]_0$. This completes the proof of the theorem. □

Theorem 3.4. *The sequence spaces* $[W, \lambda, \mathcal{M}, p, \|\cdots\|]_0$ *and* $[W, \lambda, \mathcal{M}, p, \|\cdots\|]_\infty$ *are solid.*

Proof. We give the proof for $[W, \lambda, \mathcal{M}, p, \|\cdots\|]_0$.
Let $x \in [W, \lambda, \mathcal{M}, p, \|\cdots\|]_0$ and $\alpha = (\alpha_k)$ be any sequence of scalars such that $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M_k\left(\left\|\frac{t_{km}(\alpha x)}{\rho},z_1,\cdots,z_{n-1}\right\|\right)\right]^{p_k}\leq\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M_k\left(\left\|\frac{t_{km}(x)}{\rho},z_1,\cdots,z_{n-1}\right\|\right)\right]^{p_k}\to 0$$

$$\text{as } n\to\infty, \text{ uniformly in } m.$$

Hence $\alpha x \in [W, \lambda, \mathcal{M}, p, \|\cdots\|]_0$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$, whenever $x \in [W, \lambda, \mathcal{M}, p, \|\cdots\|]_0$. Hence the space $[W, \lambda, \mathcal{M}, p, \|\cdots\|]_0$ is a solid sequence space. □

Corollary 3.5. *The sequence spaces* $[W, \lambda, \mathcal{M}, p, \|\cdots\|]_0$ *and* $[W, \lambda, \mathcal{M}, p, \|\cdots\|]_\infty$ *are monotone.*

Proof. The proof follows from Lemma 1.1. □

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