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Subclasses of Analytic Functions with Respect to Symmetric and Conjugate Points Connected with the q-Borel Distribution

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Abstract. In this article, by making use of a q-analogue of the familiar Borel distribution, we introduce two new subclasses:

$$\mathcal{S}^{\alpha,\lambda,q}_{\text{symmetric}}(b,A,B) \qquad \text{and} \qquad \mathcal{S}^{\alpha,\lambda,q}_{\text{conjugate}}(b,A,B)$$

of starlike and convex functions in the open unit disk Δ with respect to symmetric and conjugate points. We obtain some properties including the Taylor-Maclaurin coefficient estimates for functions in each of these subclasses and deduce various corollaries and consequences of the main results. We also indicate relevant connections of each of these subclasses $\mathcal{S}^{\alpha,\lambda,q}_{\text{symmetric}}(b,A,B)$ and $\mathcal{S}^{\alpha,\lambda,q}_{\text{conjugate}}(b,A,B)$ with the function classes which were investigated in several earlier works. Finally, in the concluding section, we choose to comment on the recent usages, especially in Geometric Function Theory of Complex Analysis, of the basic (or q-) calculus and also of its trivial and inconsequential (p,q)-variation involving an obviously redundant (or superfluous) parameter p.

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1. Introduction, Motivation and Definitions

In his survey-cum-expository review article, Srivastava [32] presented, and motivated the readers for further researches on the usage of the q-calculus in Geometric Function Theory, by means of a brief expository overview of the importance of classical q-analysis and the triviality and inconsequential nature of its so-called (p, q)-variant with an obviously redundant (or superfluous) forced-in parameter p (see, for details, [32, p. 340]; see also [33, Section 5 (pp. 1510–1512]). In the literature, one can find several families of such extensively- and widely-investigated linear convolution operators as (for example) the the Dziok-Srivastava, the Srivastava-Wright and the Srivastava-Attiya linear convolution operators (see also [29], [30] and [31]), together with their extended and generalized versions. The usages of the q-calculus and the fractional q-calculus in geometric function theory of complex analysis are believed to encourage and motivate significant further developments on these and other related topics (see also Srivastava and Karlsson [37, pp. 350–351]).

Our main objective in this article is apply a q-analogue of the familiar Borel distribution to introduce and investigate several properties of two new subclasses $\mathcal{S}^{\alpha,\lambda,q}_{\text{symmetric}}(b, A, B)$ and $\mathcal{S}^{\alpha,\lambda,q}_{\text{conjugate}}(b, A, B)$ of starlike and convex functions in the open unit disk Δ with respect to symmetric and conjugate points.

Let $\mathcal{A}$ denote the class of functions of the following normalized form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k\, z^k, \tag{1}$$

which are analytic in the open unit disk

$$\Delta := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

If the function $g \in \mathcal{A}$ is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k\, z^k \qquad (z \in \Delta), \tag{2}$$

then the Hadamard product (or convolution) of the functions f and g is defined by

$$(f * g)(z) := z + \sum_{k=2}^{\infty} a_k\, b_k\, z^k =: (g * f)(z) \qquad (z \in \Delta). \tag{3}$$

If f and F are analytic functions in Δ, we say that f is subordinate to F, written as $f \prec F$ or $f(z) \prec F(z)$ if there exists a Schwarz function w, which is analytic in Δ with $w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (\forall\ z \in \Delta)$, such that

$$f(z) = F\big(w(z)\big) \qquad (z \in \Delta).$$

Furthermore, if the function F is univalent in Δ, then we have the following equivalence (see [7] and [20]):

$$f(z) \prec F(z) \iff f(0) = F(0) \quad \text{and} \quad f(\Delta) \subset F(\Delta). \tag{4}$$

In the year 1959, Sakaguchi [27] introduced the class $\mathcal{S}^*_s$ of functions starlike with respect to symmetric points, which consists of functions $f \in \mathcal{A}$ satisfying the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0 \qquad (z \in \Delta).$$

Obviously, the above class $\mathcal{S}^*_s$ of univalent functions, which are starlike with respect to symmetric points, include the classes of convex functions and odd functions starlike with respect to the origin (see [27]).

Aouf *et al.* [3] introduced and studied the class $\mathcal{S}^*_{s,n}T(1,1)$ of functions n-starlike with respect to symmetric points, which consists of functions $f \in \mathcal{A}$ with $a_k \leqq 0$ for $k \geqq 2$, and satisfying the following inequality:

$$\Re\left(\frac{\mathcal{D}^{n+1}f(z)}{\mathcal{D}^n f(z) - \mathcal{D}^n f(-z)}\right) > 0 \qquad (z \in \Delta),$$

where $\mathcal{D}^n$ is the Sălăgean operator [28].

El-Ashwah and Thomas [8] introduced and studied the class $\mathcal{S}^*_c$ consisting of functions starlike with respect to conjugate points if it satisfies the following condition:

$$\Re\left(\frac{zf'(z)}{f(z) + \overline{f(\overline{z})}}\right) > 0 \qquad (z \in \Delta)$$

and Aouf *et al.* [3] introduced and studied the class $\mathcal{S}^*_{c,n}T(\alpha,\beta)$ of functions n-starlike with respect to conjugate points, which (for $\alpha = \beta = 1$) consists of functions $f \in \mathcal{A}$ with $a_k \leqq 0$ $(k \geqq 2)$ and satisfies the following inequality:

$$\Re\left(\frac{D^{n+1}f(z)}{D^n f(z) + D^n\overline{f(\overline{z})}}\right) > 0 \qquad (z \in \Delta).$$

Definition 1. Let Ω be the family of functions $w(z)$ which are analytic in Δ and satisfy the following conditions:

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \qquad (z \in \Delta).$$

Also, for arbitrary fixed numbers A and B such that $-1 \leqq B < A \leqq 1$, we denote by $\mathcal{P}[A,B]$ the family of functions normalized by

$$p(z) = 1 + b_1 z + b_2 z^2 + \cdots, \tag{5}$$

which are analytic in Δ, and are such that $p(z) \in \mathcal{P}[A,B]$ if and only if

$$p(z) \prec \frac{1+Az}{1+Bz} \quad \text{and} \quad p(z) = \frac{1+Aw(z)}{1+Bw(z)} \tag{6}$$

for some function $w(z) \in \Omega$ and for every $z \in \Delta$.

We recall here that the function $\dfrac{1+Az}{1+Bz}$ maps Δ conformally onto a disk symmetrical with respect to the real axis, which is centered at the point

$$\frac{1-AB}{1-B^2} \qquad (B \neq \pm 1)$$

and with radius equal to

$$\frac{A-B}{1-B^2} \qquad (B \neq \pm 1).$$

In the year 1973, Janowski [16] introduced the following subclass of starlike functions:

$$\mathcal{S}^*(A,B) := \left\{f : f \in \mathcal{A} \quad \text{and} \quad \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \quad (-1 \leqq B < A \leqq 1;\ z \in \Delta)\right\}, \tag{7}$$

which has indeed been involved in several recent developments on the usages of the q-analysis in Geometric Function Theory of Complex Analysis (see, for example, [18] and [25]). On the other hand, Nasr and Aouf [22] introduced the subclass $\mathcal{S}(b)$ of starlike functions of complex order $b \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ as follows:

$$\mathcal{S}(b) := \left\{f : f \in \mathcal{A} \quad \text{and} \quad \Re\left(1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right) > 0 \quad (b \in \mathbb{C}^* := \mathbb{C} \setminus \{0\};\ z \in \Delta)\right\}.$$

Earlier in 1982, Goel and Mehrok [13] introduced a subclass $\mathcal{S}_s^*(A,B)$ of the above-mentioned Sakaguchi class $\mathcal{S}_s^*$ as follows:

$$\mathcal{S}_s^*(A,B) := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \frac{2zf'(z)}{f(z)-f(-z)} \prec \frac{1+Az}{1+Bz} \qquad (-1 \leqq B < A \leqq 1;\ z \in \Delta) \right\}.$$

More recently, for $-1 \leqq B < A \leqq 1$, $b \in \mathbb{C}^*$ and $z \in \Delta$, Aouf *et al.* [4] introduced a subclass $\mathcal{S}_s^*(b,A,B)$ of the Sakaguchi class $\mathcal{S}_s^*$ as follows:

$$\mathcal{S}_s^*(b,A,B) := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad 1 + \frac{1}{b}\left(\frac{2zf'(z)}{f(z)-f(-z)} - 1 \right) \prec \frac{1+Az}{1+Bz} \qquad (b \in \mathbb{C}^*;\ z \in \Delta) \right\}. \tag{8}$$

On the other hand, Arif *et al.* [5] introduced another subclass $C_s^*(b,A,B)$ of the Sakaguchi class $\mathcal{S}_s^\star$ as follows:

$$C_s^*(b,A,B) := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad 1 + \frac{1}{b}\left(\frac{2\,(zf'(z))'}{\big(f(z)-f(-z)\big)'} - 1 \right) \prec \frac{1+Az}{1+Bz} \qquad (-1 \leqq B < A \leqq 1;\ b \in \mathbb{C}^*;\ z \in \Delta) \right\}.$$

Such probability distributions as the Poisson, the Pascal, the Logarithmic, the Binomial, and other distributions have recently alleared in various context in the Geometric Function Theory of Complex Analysis mainly from a theoretical viewpoint (see [2, 9, 23, 24]). We recall that a discrete random variable x is said to have a Borel distribution if it takes on the values $1, 2, 3, \cdots$ with the following probabilities:

$$\frac{e^{-\lambda}}{1!}, \ \frac{2\lambda e^{-2\lambda}}{2!}, \ \frac{9\lambda^2 e^{-3\lambda}}{3!}, \ \cdots,$$

respectively, where λ is the parameter involved.

Recently, Wanas and Khuttar [44] introduced the Borel distribution (BD) whose probability mass function is given by

$$\text{Prob}\{x = \rho\} = \frac{(\rho\lambda)^{\rho-1}\ e^{-\lambda\rho}}{\rho!} \qquad (\rho = 1, 2, 3, \cdots).$$

Wanas and Khuttar [44] also introduced the following series $\mathcal{M}(\lambda; z)$ whose coefficients are probabilities of the Borel distribution (BD):

$$\begin{aligned} \mathcal{M}(\lambda; z) &:= z + \sum_{k=2}^{\infty} \frac{[\lambda\,(k-1)]^{k-2}\, e^{-\lambda(k-1)}}{(k-1)!}\, z^k \\ &= z + \sum_{k=2}^{\infty} \phi_k(\lambda)\, z^k \qquad (0 < \lambda \leqq 1), \end{aligned} \tag{9}$$

where, for convenience,

$$\phi_k(\lambda) := \frac{[\lambda\,(k-1)]^{k-2}\ e^{-\lambda(k-1)}}{(k-1)!}.$$

A linear operator $\mathcal{B}(\lambda; z)$ for functions $f : \mathcal{A} \to \mathcal{A}$ is now recalled as follows (see [11, 21, 35]):

$$\begin{aligned} \mathcal{B}(\lambda; z) f(z) &= \mathcal{M}(\lambda; z) * f(z) \\ &= z + \sum_{k=2}^{\infty} \frac{[\lambda\,(k-1)]^{k-2}\ e^{-\lambda\,(k-1)}}{(k-1)!}\, a_k\, z^k \qquad (0 < \lambda \leqq 1). \end{aligned} \tag{10}$$

Motivated essentially by the work of Srivastava [32], who made use of various operators of q-calculus and fractional q-calculus, we now recall some definitions and notations of the classical q-calulus. First of all, the q-Pochhammer symbol $(\lambda;q)_n$ is defined, for $\lambda, q \in \mathbb{C}$ and $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, by

$$(\lambda;q)_n := \begin{cases} 1 & (n=0) \\ (1-\lambda)(1-\lambda q)\cdots\left(1-\lambda q^{n-1}\right) & (n \in \mathbb{N}) \end{cases} \tag{11}$$

and

$$(\lambda;q)_\infty = \prod_{k=0}^{\infty}\left(1-\lambda q^k\right) \qquad \left(\left|q\right|<1\right).$$

In terms of the q-gamma function $\Gamma_q(z)$ defined by (see [12])

$$\Gamma_q(z) = (1-q)^{1-z}\ \frac{(q;q)_\infty}{(q^z;q)_\infty} \qquad \left(\left|q\right|<1;\ z \in \mathbb{C}\right),$$

it is easily seen from (11) that

$$\left(q^\lambda;q\right)_n = \frac{(1-q)^n\ \Gamma_q(\lambda+n)}{\Gamma_q(\lambda)} \qquad (n \in \mathbb{N}_0).$$

The q-gamma function $\Gamma_q(z)$ is known to satisfy the following recurrence relation:

$$\Gamma_q(z+1) = [z]_q\ \Gamma_q(z),$$

where $[\lambda]_q$ denotes the basic (or q-) number defined as follows:

$$[\lambda]_q := \begin{cases} \dfrac{1-q^\lambda}{1-q} & (\lambda \in \mathbb{C}) \\ 1+\sum\limits_{j=1}^{n-1} q^j & (n \in \mathbb{N}). \end{cases} \tag{12}$$

Using the definition (12), the q-factorial $[n]_q!$ is given by

$$[n]_q! := \begin{cases} 1 & (n=0) \\ \prod\limits_{j=1}^{n}[j]_q & (n \in \mathbb{N}). \end{cases}$$

For $\lambda \in \mathbb{C}$, we shall also make use of the following notation for the basic (or q-) Pochhammer symbol defined above in (11):

$$\left(q^\lambda;q\right)_n := \begin{cases} 1 & (n=0) \\ (1-q^\lambda)(1-q^{\lambda+1})\cdots(1-q^{\lambda+n-1}) & (n \in \mathbb{N}) \end{cases}$$

and, for convenience, we write

$$[\lambda]_{q,n} := \begin{cases} 1 & (n=0) \\ \dfrac{(q^\lambda;q)_n}{(1-q)^n} = \prod\limits_{j=1}^{n}[\lambda+j-1]_q & (n \in \mathbb{N}) \end{cases} \tag{13}$$

in terms of the q-numbers $[\lambda]_q$ defined by (12). Clearly, from the definition (13), it is easy to see for the familiar Pocchammer symbol $(\lambda)_n$ that

$$\lim_{q\to 1-}\left\{[\lambda]_{q,n}\right\}=\lim_{q\to 1-}\left\{\frac{\left(q^{\lambda};q\right)_n}{(1-q)^n}\right\}=(\lambda)_n$$

and, for the classical (Euler's) gamma function $\Gamma(z)$, we have

$$\lim_{q\to 1-}\left\{\Gamma_q(z)\right\}=\Gamma(z).$$

For $0<q<1$ and the function $\mathcal{B}(\lambda;z)f(z)$ given by (10), when we apply the q-derivative operator D_q defined by (see [15]; see also [1] and [14])

$$D_q\Big(f(z)\Big):=\begin{cases}\dfrac{f(z)-f(qz)}{1-q} & (0<q<1)\\[2ex] f'(z) & (q\to 1-),\end{cases}$$

we get

$$\begin{aligned}D_q\Big(\mathcal{B}(\lambda;z)f(z)\Big):=&\ \frac{\mathcal{B}(\lambda;z)f(z)-\mathcal{B}(\lambda;z)f(qz)}{(1-q)z}\\ =&\ 1+\sum_{k=2}^{\infty}[k]_q\,\frac{[\lambda\,(k-1)]^{k-2}\,e^{-\lambda(k-1)}}{(k-1)!}\,a_k\,z^{k-1}\\ =&\ 1+\sum_{k=2}^{\infty}\Upsilon_k\,a_k\,z^{k-1}\qquad(0<\lambda\leqq 1;\ z\in\Delta),\end{aligned}$$

where the function $f(z)$ is given by (1).

Definition 2. For $\alpha>-1$ and $0<q<1$, the linear operator $\mathcal{B}_{\lambda}^{\alpha,q}$ for functions $f:\mathcal{A}\to\mathcal{A}$ is defined as follows:

$$\mathcal{B}_{\lambda}^{\alpha,q}f(z)*\mathcal{N}_{q,\alpha+1}(z)=z\,D_q\Big(\mathcal{B}(\lambda;z)f(z)\Big)\qquad(z\in\Delta),$$

where the function $\mathcal{N}_{q,\alpha+1}$ is given by

$$\mathcal{N}_{q,\alpha+1}(z):=z+\sum_{k=2}^{\infty}\frac{[\alpha+1]_{q,k-1}}{[k-1]_q!}\,z^k\qquad(z\in\Delta).$$

A simple computation shows that

$$\begin{aligned}\mathcal{B}_{\lambda}^{\alpha,q}f(z):=&\ z+\sum_{k=2}^{\infty}\frac{[k]_q!\,[\lambda\,(k-1)]^{k-2}\,e^{-\lambda(k-1)}}{[\alpha+1]_{q,k-1}\,(k-1)!}\,a_k\,z^k\\ =&\ z+\sum_{k=2}^{\infty}\Upsilon_k\,a_k\,z^k\qquad(0<\lambda\leqq 1;\ \alpha>-1;\ 0<q<1;\ z\in\Delta),\end{aligned}\tag{14}$$

where

$$\Upsilon_k=\frac{[k]_q!\,[\lambda\,(k-1)]^{k-2}\,e^{-\lambda(k-1)}}{[\alpha+1]_{q,k-1}\ (k-1)!}.\tag{15}$$

We also note that

$$\Upsilon_2 = \frac{[2]_q!\, e^{-\lambda}}{[\alpha+1]_{q,1}} \quad \text{and} \quad \Upsilon_3 = \frac{[3]_q!\, [\lambda]\, e^{-2\lambda}}{[\alpha+1]_{q,2}}. \tag{16}$$

From the definition (14), we can easily verify that each of the following relations holds true for all $f \in \mathcal{A}$:

$$[\alpha+1]_q\; \mathcal{B}_\lambda^{\alpha,q} f(z) = [\alpha]_q \mathcal{B}_\lambda^{\alpha+1,q} f(z) + q^\alpha\, z\, D_q \left(\mathcal{B}_\lambda^{\alpha+1,q} f(z)\right) \qquad (z \in \Delta) \tag{17}$$

and

$$\begin{aligned}\mathcal{R}_\lambda^\alpha f(z) := \lim_{q\to 1-} \left\{\mathcal{B}_\lambda^{\alpha,q} f(z)\right\} &= z + \sum_{k=2}^{\infty} \frac{k\,[\lambda\,(k-1)]^{k-2}\;\, e^{-\lambda(k-1)}}{(\alpha+1)_{k-1}}\, a_k\, z^k \\ &= z + \sum_{k=2}^{\infty} \Omega_k\, a_k\, z^k \qquad (z \in \Delta),\end{aligned} \tag{18}$$

where

$$\Omega_k = \frac{k\,[\lambda\,(k-1)]^{k-2}\;\, e^{-\lambda(k-1)}}{(\alpha+1)_{k-1}}. \tag{19}$$

The function class defined in (8) can now generalized by introducing the next class of functions which are defined with the aid of the operator $\mathcal{B}_\lambda^{\alpha,q}$.

Definition 3. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_{\text{symmetric}}^{\alpha,\lambda,q}(b, A, B)$ if and only if

$$1 + \frac{1}{b}\left(\frac{2z\left(\mathcal{B}_\lambda^{\alpha,q} f(z)\right)'}{\mathcal{B}_\lambda^{\alpha,q} f(z) - \mathcal{B}_\lambda^{\alpha,q} f(-z)} - 1\right) \prec \frac{1+Az}{1+Bz} \tag{20}$$

$(-1 \leqq A \leqq B \leqq 1;\; 0 < \lambda \leqq 1;\; \alpha > -1;\; 0 < q < 1;\; b \in \mathbb{C}^*)$.

Upon letting $q \to 1-$ in the class $\mathcal{S}_{\text{symmetric}}^{\alpha,\lambda,q}(b, A, B)$, we have

$$\lim_{q\to 1-} \left\{\mathcal{S}_{\text{symmetric}}^{\alpha,\lambda,q}(b, A, B)\right\} = \mathcal{G}_s^{\alpha,\lambda}(b, A, B),$$

where

$$\mathcal{G}_s^{\alpha,\lambda}(b, A, B) := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad 1 + \frac{1}{b}\left(\frac{2z\left(\mathcal{R}_\lambda^\alpha f(z)\right)'}{\mathcal{R}_\lambda^\alpha f(z) - \mathcal{R}_\lambda^\alpha f(-z)} - 1\right) \prec \frac{1+Az}{1+Bz} \right.$$
$$\left. (-1 \leqq A \leqq B \leqq 1;\; 0 < \lambda \leqq 1;\; \alpha > -1;\; b \in \mathbb{C}^*) \right\}.$$

Next, by using the operator $\mathcal{B}_\lambda^{\alpha,q}$, we define another function class given by Definition 4 below.

Definition 4. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_{\text{conjugate}}^{\alpha,\lambda,q}(b, A, B)$ if and only if

$$1 + \frac{1}{b}\left(\frac{2z\left(\mathcal{B}_\lambda^{\alpha,q} f(z)\right)'}{\mathcal{B}_\lambda^{\alpha,q} f(z) + \mathcal{B}_\lambda^{\alpha,q} \overline{f(\overline{z})}} - 1\right) \prec \frac{1+Az}{1+Bz} \tag{21}$$

$(-1 \leqq A \leqq B \leqq 1;\; 0 < \lambda \leqq 1;\; \alpha > -1;\; 0 < q < 1;\; b \in \mathbb{C}^*)$.

If we let $q \to 1-$ in the function class $\mathcal{S}^{\alpha,\lambda,q}_{\text{conjugate}}(b, A, B)$, we find that

$$\lim_{q\to 1-}\left\{\mathcal{S}^{\alpha,\lambda,q}_{\text{conjugate}}(b, A, B)\right\} = \mathcal{G}^{\alpha,\lambda}_{c}(b, A, B),$$

where

$$\mathcal{G}^{\alpha,\lambda}_{c}(b, A, B) := \left\{ f : f \in \mathbb{A} \quad \text{and} \quad 1 + \frac{1}{b}\left(\frac{2z\left(\mathcal{R}^{\alpha}_{\lambda} f(z)\right)'}{\mathcal{R}^{\alpha}_{\lambda} f(z) + \mathcal{R}^{\alpha}_{\lambda}\overline{f(\overline{z})}} - 1\right) \prec \frac{1 + Az}{1 + Bz} \right.$$
$$\left. (-1 \leqq A \leqq B \leqq 1;\ 0 < \lambda \leqq 1;\ \alpha > -1;\ b \in \mathbb{C}^{*}) \right\}.$$

The following lemmas will be needed to prove our results.

Lemma 1. (see [13, Lemma 2]) *If*

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in \mathcal{P}[A, B],$$

then

$$\left|p_n\right| \leqq A - B \qquad (n \in \mathbb{N};\ -1 \leqq A \leqq B \leqq 1).$$

Lemma 2. (see [13, Lemma 3]) *If the function N is analytic and the function M is starlike functions in Δ with $N(0) = M(0) = 0$, then the following condition*:

$$\frac{\left|\dfrac{N'(z)}{M'(z)} - 1\right|}{\left|A - B\left(\dfrac{N'(z)}{M'(z)}\right)\right|} < 1 \qquad (z \in \Delta;\ -1 \leqq A \leqq B \leqq 1)$$

implies that

$$\frac{\left|\dfrac{N(z)}{M(z)} - 1\right|}{\left|A - B\left(\dfrac{N(z)}{M(z)}\right)\right|} < 1 \qquad (z \in \Delta;\ -1 \leqq A \leqq B \leqq 1).$$

2. Properties of the Subclass $\mathcal{S}^{\alpha,\lambda,q}_{\text{symmetric}}(b, A, B)$

Unless otherwise mentioned, we shall assume in the remainder of this paper that

$$-1 \leqq B \leqq A \leqq 1,\ \ 0 < \lambda \leqq 1,\ \ \alpha > -1,\ \ 0 < q < 1 \quad \text{and} \quad b \in \mathbb{C}^{*}$$

and also that the complex powers are understood as principal values. Throughout this work, we assume that an empty sum is 0 and an empty product is 1.

One of our main results in this section is stated as Theorem 1 below.

Theorem 1. *Let $f(z) \in \mathcal{S}^{\alpha,\lambda,q}_{\text{symmetric}}(\gamma, A, B)$. Then the following condition*:

$$1 + \frac{1}{b}\left(\frac{z\left(\mathcal{B}^{\alpha,q}_{\lambda}\psi(z)\right)'}{\mathcal{B}^{\alpha,q}_{\lambda}\psi(z)} - 1\right) \prec \frac{1 + Az}{1 + Bz} \tag{22}$$

is satisfied for the odd function ψ given by

$$\psi(z) := \frac{1}{2}[f(z) - f(-z)]. \tag{23}$$

Proof. If $f \in \mathcal{S}^{\alpha,\lambda,q}_{\text{symmetric}}(b, A, B)$, then there exists a function $h \in \mathcal{P}[A, B]$ such that

$$h(z) = 1 + \frac{1}{b}\left(\frac{2z\left(\mathcal{B}^{\alpha,q}_{\lambda} f(z)\right)'}{\mathcal{B}^{\alpha,q}_{\lambda} f(z) - \mathcal{B}^{\alpha,q}_{\lambda} f(-z)} - 1\right), \tag{24}$$

that is, that

$$b[h(z) - 1] = \frac{2z\left(\mathcal{B}^{\alpha,q}_{\lambda} f(z)\right)'}{\mathcal{B}^{\alpha,q}_{\lambda} f(z) - \mathcal{B}^{\alpha,q}_{\lambda} f(-z)} - 1 \tag{25}$$

and

$$b[h(-z) - 1] = \frac{-2z\left(\mathcal{B}^{\alpha,q}_{\lambda} f(-z)\right)'}{\mathcal{B}^{\alpha,q}_{\lambda} f(z) - \mathcal{B}^{\alpha,q}_{\lambda} f(-z)} - 1, \tag{26}$$

which, together, imply that

$$\frac{1}{2}[h(z) + h(-z)] = 1 + \frac{1}{b}\left(\frac{z\left(\mathcal{B}^{\alpha,q}_{\lambda} \psi(z)\right)'}{\mathcal{B}^{\alpha,q}_{\lambda} \psi(z)} - 1\right). \tag{27}$$

On the other hand,

$$h(z) \prec \frac{1 + Az}{1 + Bz}$$

and $\frac{1+Az}{1+Bz}$ is univalent. Thus, by (4), we have

$$\frac{1}{2}[h(z) + h(-z)] \prec \frac{1 + Az}{1 + Bz}.$$

This leads us to the assertion (22) of Theorem 1. The proof of Theorem 1 is thus completed. □

In the limit case when $q \to 1^-$ in Theorem 1, we obtain the following corollary.

Corollary 1. *Let $f(z) \in \mathcal{G}^{\alpha,\lambda}_{s}(b, A, B)$. Then the following condition*:

$$1 + \frac{1}{b}\left(\frac{z\left(\mathcal{R}^{\alpha}_{\lambda} \psi(z)\right)'}{\mathcal{R}^{\alpha}_{\lambda} \psi(z)} - 1\right) \prec \frac{1 + Az}{1 + Bz}.$$

is satisfied for the odd function ψ given by (23).

Theorem 2. *A function $f \in \mathcal{S}^{\alpha,\lambda,q}_{\text{symmetric}}(b, A, B)$ if and only if there exists a function $p \in \mathcal{P}[A, B]$ such that*

$$\left(\mathcal{B}^{\alpha,q}_{\lambda} f(z)\right)' = \left(b[h(z) - 1] + 1\right) \exp\left(\frac{b}{2}\int_0^z \frac{h(t) + h(-t) - 2}{t}\, dt\right). \tag{28}$$

Proof. In our proof of Theorem 1, we have (27), which implies that

$$\frac{\left(\mathcal{B}^{\alpha,q}_{\lambda} \psi(z)\right)'}{\mathcal{B}^{\alpha,q}_{\lambda} \psi(z)} = \frac{1}{z} + \frac{b}{2}\left(\frac{h(z) + h(-z) - 2}{z}\right),$$

which, upon integration, yields

$$\mathcal{B}_{\lambda}^{\alpha,q}\psi(z) = z\exp\left(\frac{b}{2}\int_0^z \frac{h(t)+h(-t)-2}{t}\,dt\right). \tag{29}$$

Since $f \in \mathcal{S}_{\text{symmetric}}^{\alpha,\lambda,q}(b,A,B)$, we find from (24) that

$$z\left(\mathcal{B}_{\lambda}^{\alpha,q}f(z)\right)' = \left(b[h(z)-1]+1\right)\mathcal{B}_{\lambda}^{\alpha,q}\psi(z)$$

Using (29) and this last equation, we get (28). This evidently completes the proof of Theorem 2. □

Taking $q \to 1-$ in Theorem 2, we obtain the following corollary.

Corollary 2. *A function $f \in \mathcal{G}_s^{\alpha,\lambda}(b,A,B)$ if and only if there exists a function $p \in \mathcal{P}[A,B]$ such that*

$$\left(\mathcal{R}_{\lambda}^{\alpha}f(z)\right)' = \left(b[h(z)-1]+1\right)\exp\left(\frac{b}{2}\int_0^z \frac{h(t)+h(-t)-2}{t}\,dt\right).$$

Theorem 3. *Let $f(z) \in \mathcal{S}_{\text{symmetric}}^{\alpha,\lambda,q}(b,A,B)$. Then, for all $n \geqq 1$,*

$$|a_{2n}| \leqq \frac{|b|\,(A-B)}{2^n\; n!\; |\Upsilon_{2n}|}\prod_{k=1}^{n-1}\left(|b|\,(A-B)+2k\right) \tag{30}$$

and

$$|a_{2n+1}| \leqq \frac{|b|\,(A-B)}{2^n\; n!\; |\Upsilon_{2n+1}|}\prod_{k=1}^{n-1}\left(|b|\,(A-B)+2k\right), \tag{31}$$

where Υ_k $(\forall\, k \geqq 2)$ are given by (15).

Proof. Since $f \in \mathcal{S}_{\text{symmetric}}^{\alpha,\lambda,q}(b,A,B)$, Definition 3 yields

$$1+\frac{1}{b}\left(\frac{2z\left(\mathcal{B}_{\lambda}^{\alpha,q}f(z)\right)'}{\mathcal{B}_{\lambda}^{\alpha,q}f(z)-\mathcal{B}_{\lambda}^{\alpha,q}f(-z)}-1\right) = \frac{1+Aw(z)}{1+Bw(z)}, \tag{32}$$

which can be simplified to the following form:

$$2z\left(\mathcal{B}_{\lambda}^{\alpha,q}f(z)\right)' = \left[\mathcal{B}_{\lambda}^{\alpha,q}f(z)-\mathcal{B}_{\lambda}^{\alpha,q}f(-z)\right]\left(1+b\sum_{k=1}^{\infty}c_k z^k\right), \tag{33}$$

where we have assumed that

$$h(z) = 1+\sum_{k=1}^{\infty}c_k z^k = \frac{1+Aw(z)}{1+Bw(z)} \tag{34}$$

Thus, in view of (33), it follows from (20) that

$$\begin{aligned} z + 2\Upsilon_2 a_2 z^2 + 3\Upsilon_3 a_3 z^3 + 4\Upsilon_4 a_4 z^4 + \cdots + 2n\Upsilon_{2n}a_{2n}z^{2n} \\ + (2n+1)\,\Upsilon_{2n+1}a_{2n+1}z^{2n+1} + \cdots \\ = \left(z + \Upsilon_3 a_3 z^3 + \Upsilon_5 a_5 z^5 + \cdots + \Upsilon_{2n-1}a_{2n-1}z^{2n-1} + \Upsilon_{2n+1}a_{2n+1}z^{2n+1} + \cdots\right) \\ \cdot\left(1 + bc_1 z + bc_2 z^2 + \cdots\right) \end{aligned}$$

Equating the coefficients of like powers of z in this last equation, we obtain

$$2\Upsilon_2 a_2 = \gamma c_1 \quad \text{and} \quad 2\Upsilon_3 a_3 = bc_2, \tag{35}$$

$$4\Upsilon_4 a_4 = \gamma c_3 + \gamma c_1 \Upsilon_3 a_3 \quad \text{and} \quad 4\Upsilon_5 a_5 = bc_4 + bc_2 \Upsilon_3 a_3, \tag{36}$$

$$2n\Upsilon_{2n} a_{2n} = bc_{2n-1} + bc_{2n-3}\Upsilon_3 a_3 + bc_{2n-5}\Upsilon_5 a_5 + \cdots + bc_1 \Upsilon_{2n-1} a_{2n-1} \tag{37}$$

and

$$2n\Upsilon_{2n+1} a_{2n+1} = bc_{2n} + bc_{2n-2}\Upsilon_3 a_3 + bc_{2n-4}\Upsilon_5 a_5 + \cdots + bc_2 \Upsilon_{2n-1} a_{2n-1}. \tag{38}$$

We prove (30) and (31) by using mathematical induction, together with Lemma 1, (35) and (36). We thus find that

$$|a_2| \leqq \frac{|b|}{2\,|\Upsilon_2|}\,(A-B) \quad \text{and} \quad |a_3| \leqq \frac{|b|}{2\,|\Upsilon_3|}\,(A-B)\,,$$

$$|a_4| \leqq \frac{|b|\,(A-B)}{8\,|\Upsilon_4|}\,[2+|b|\,(A-B)]$$

and

$$|a_5| \leqq \frac{|b|\,(A-B)}{8\,|\Upsilon_5|}\,[2+|b|\,(A-B)]\,.$$

It follows that (30) and (31) hold true for $n = 1, 2$. Equation (37) in conjuction with Lemma 1 yields

$$|a_{2n}| \leqq \frac{|b|\,(A-B)}{2n\,|\Upsilon_{2n}|}\left(1+\sum_{r=1}^{n-1}|\Upsilon_{2r+1}|\,|a_{2r+1}|\right)$$

Next, we assume that (30) and (31) hold true for $3, 4, \cdots, n-1$. Indeed, the above inequality yields

$$|a_{2n}| \leqq \frac{|b|\,(A-B)}{2n\,|\Upsilon_{2n}|}\left(1+\sum_{r=1}^{n-1}\frac{|b|\,(A-B)}{2^r r!}\prod_{k=1}^{r-1}[|b|\,(A-B)+2k]\right). \tag{39}$$

To complete the proof of Theorem 3, it is sufficient to show that

$$\begin{aligned}\frac{|b|\,(A-B)}{2m\,|\Upsilon_{2m}|}&\left(1+\sum_{r=1}^{m-1}\frac{|b|\,(A-B)}{2^r r!}\prod_{k=1}^{r-1}(|b|\,(A-B)+2k)\right)\\ &= \frac{|b|\,(A-B)}{2^m m!\,|\Upsilon_{2m}|}\prod_{k=1}^{m-1}[|b|\,(A-B)+2k] \qquad (m = 3, 4, \cdots, n).\end{aligned} \tag{40}$$

It easy to see that (40) is valid for $m = 3$. We now suppose that (40) is true for $4, \cdots, m-1$. Then it follows from (39) that

$$
\begin{aligned}
&\frac{|b|(A-B)}{2m|\Upsilon_{2m}|}\left(1+\sum_{r=1}^{m-1}\frac{|b|(A-B)}{2^r r!}\prod_{k=1}^{r-1}[|b|(A-B)+2k)\right]\\
&=\frac{|b|(A-B)}{2m|\Upsilon_{2m}|}\left(1+\sum_{r=1}^{m-2}\frac{|b|(A-B)}{2^r r!}\prod_{k=1}^{r-1}[|b|(A-B)+2k]\right.\\
&\qquad\left.+\frac{|b|)(A-B)}{2^{m-1}(m-1)!}\prod_{k=1}^{m-2}[|b|(A-B)+2k]\right)\\
&=\frac{(m-1)|\Upsilon_{2m-2}|}{m|\Upsilon_{2m}|}\left(\frac{\left|\gamma\right|(A-B)}{2^{m-1}(m-1)!|\Upsilon_{2m-2}|}\prod_{k=1}^{m-2}(\left|\gamma\right|(A-B)+2k)\right.\\
&\qquad\left.+\frac{\left|\gamma\right|(A-B)}{2m|\Upsilon_{2m}|}\frac{\left|\gamma\right|(A-B)}{2^{m-1}(m-1)!}\prod_{k=1}^{m-2}[\left|\gamma\right|(A-B)+2k]\right)\\
&=\frac{(m-1)|\Upsilon_{2m-2}|}{m|\Upsilon_{2m}|}\left(\frac{|b|(A-B)}{2^{m-1}(m-1)!|\Upsilon_{2m-2}|}\prod_{k=1}^{m-2}(|b|(A-B)+2k)\right.\\
&\qquad\left.+\frac{|b|(A-B)}{2m|\Upsilon_{2m}|}\frac{|b|(A-B)}{2^{m-1}(m-1)!}\prod_{k=1}^{m-2}[|b|(A-B)+2k]\right)\\
&=\frac{(m-1)|b|(A-B)}{2^{m-1}m!|\Upsilon_{2m}|}\prod_{k=1}^{m-2}[|b|(A-B)+2k]\\
&\qquad+\frac{|b|(A-B)}{2|\Upsilon_{2m}|}\frac{|b|(A-B)}{2^{m-1}m!}\prod_{k=1}^{m-2}[|b|(A-B)+2k]\\
&=\frac{|b|(A-B)}{2^{m-1}m!|\Upsilon_{2m}|}\prod_{k=1}^{m-2}(|b|(A-B)+2k)\left((m-1)+\frac{|b|(A-B)}{2}\right)\\
&=\frac{|b|(A-B)}{2^{m-1}m!|\Upsilon_{2m}|}\prod_{k=1}^{m-2}(|b|(A-B)+2k)\left(\frac{|b|(A-B)+2(m-1)}{2}\right)\\
&=\frac{|b|(A-B)}{2^m m!\ |\Upsilon_{2m}|}\prod_{k=1}^{m-1}[|b|(A-B)+2k],
\end{aligned}
$$

that is, (40) holds true for $m = n$. From (39) and (40), we obtain (30). Similary, we can prove (31). This completes the proof of Theorem 3. □

Letting $q \to 1-$ in Theorem 3, we obtain the following corollary.

Corollary 3. *Let $f(z) \in \mathcal{G}_s^{\alpha,\lambda}(\gamma, A, B)$. Then, for all $n \geqq 1$,*

$$|a_{2n}| \leqq \frac{|b|(A-B)}{2^n n!\,|\Omega_{2n}|}\prod_{k=1}^{n-1}[|b|(A-B)+2k]$$

and

$$|a_{2n+1}| \leqq \frac{|b|(A-B)}{2^n n!\,|\Omega_{2n+1}|}\prod_{k=1}^{n-1}[|b|(A-B)+2k].$$

where Ω_k ($\forall\ k \geqq 2$) are given by (19).

Theorem 4. *If the function* $f \in \mathcal{S}^{\alpha,\lambda,q}_{\text{symmetric}}(b,A,B)$*, then* $F \in \mathcal{S}^{\alpha,\lambda,q}_{\text{symmetric}}(b,A,B)$*, where*

$$F(z) = \frac{2}{z}\int_0^z f(t)\,dt. \tag{41}$$

Proof. It is easy to see from (41) that

$$\begin{aligned}1 &+ \frac{1}{b}\left(\frac{2z\left(\mathcal{B}^{\alpha,q}_{\lambda}F(z)\right)'}{\mathcal{B}^{\alpha,q}_{\lambda}F(z) - \mathcal{B}^{\alpha,q}_{\lambda}F(-z)} - 1\right)\\ &= \frac{2z\mathcal{B}^{\alpha,q}_{\lambda}f(z) + (b-3)\int_0^z \mathcal{B}^{\alpha,q}_{\lambda}f(t)\,dt + (b-1)\int_0^z \mathcal{B}^{\alpha,q}_{\lambda}f(-t)\,dt}{b\left(\int_0^z \mathcal{B}^{\alpha,q}_{\lambda}f(t)\,dt + \int_0^z \mathcal{B}^{\alpha,q}_{\lambda}f(-t)\,dt\right)}.\end{aligned} \tag{42}$$

If, for convenience, we denote by N and M the numerator and the denominator of the right-hand side of (42), we get

$$\begin{aligned}\frac{zM'(z)}{M(z)} &= \frac{z\mathcal{B}^{\alpha,q}_{\lambda}f(z) - z\mathcal{B}^{\alpha,q}_{\lambda}f(-z)}{\int_0^z \mathcal{B}^{\alpha,q}_{\lambda}f(t)\,dt + \int_0^z \mathcal{B}^{\alpha,q}_{\lambda}f(-t)\,dt}\\ &= \frac{1}{2}\left(\frac{2zG'(z)}{G(z)-G(-z)} + \frac{2(-z)G'(-z)}{G(-z)-G(z)}\right),\end{aligned} \tag{43}$$

where

$$G(z) = \int_0^z \mathcal{B}^{\alpha,q}_{\lambda}f(t)\,dt.$$

Since $f \in \mathcal{S}^{\alpha,\lambda,q}_{\text{symmetric}}(b,A,B)$, it follows that

$$1 + \frac{1}{b}\left(\frac{2zG''(z)}{G'(z)-G'(-z)} - 1\right) \prec \frac{1+Az}{1+Bz}$$

and that

$$G(z) \in \mathcal{C}^*_s(b,A,B) \subset \mathcal{S}^*_{\text{symmetric}}(b,A,B) \subset \mathcal{S}^*_s.$$

We see from (43) that $M(z)$ is a starlike function. In addition, we have

$$\frac{N'(z)}{M'(z)} = 1 + \frac{1}{b}\left(\frac{2z\left(\mathcal{B}^{\alpha,q}_{\lambda}f(z)\right)'}{\mathcal{B}^{\alpha,q}_{\lambda}f(z) - \mathcal{B}^{\alpha,q}_{\lambda}f(-z)} - 1\right),$$

so that

$$\frac{N'(z)}{M'(z)} = \frac{1+Aw(z)}{1+Bw(z)}$$

It follows that

$$\left|\frac{N'(z)}{M'(z)} - 1\right| < \left|A - B\left(\frac{N'(z)}{M'(z)}\right)\right|.$$

Now, by applying Lemma 2, we have

$$\left|\frac{N(z)}{M(z)} - 1\right| < \left|A - B\left(\frac{N(z)}{M(z)}\right)\right|,$$

which implies that $F \in \mathcal{S}^{\alpha,\lambda,q}_{\text{symmetric}}(b,A,B)$, just as asserted by Theorem 4. □

Taking $q \to 1^-$ in Theorem 4, we obtain the following corollary:

Corollary 4. *If the function* $f \in \mathcal{G}^{\alpha,\lambda}_s(b,A,B)$*, then* F *given by* (41) *belongs to the class* $\mathcal{G}^{\alpha,\lambda}_s(b,A,B)$.

3. Coefficient Bounds for the Subclass $\mathcal{S}^{\alpha,\lambda,q}_{\text{conjugate}}(b, A, B)$

Theorem 5. *Let the function $f \in \mathcal{S}^{\alpha,\lambda,q}_{\text{conjugate}}(b, A, B)$. Then, for all $n \geqq 1$,*

$$|a_{2n}| \leqq \frac{|b|\,(A-B)}{(2n-1)!\,|\Upsilon_{2n}|} \prod_{k=1}^{2n-2} [|b|\,(A-B)+k] \tag{44}$$

and

$$|a_{2n+1}| \leqq \frac{|b|\,(A-B)}{2n!\,|\Upsilon_{2n+1}|} \prod_{k=1}^{2n-1} [|b|\,(A-B)+k], \tag{45}$$

where Υ_k $(\forall\, k \geqq 2)$ are given by (15).

Proof. Since $f \in \mathcal{S}^{\alpha,\lambda,q}_{\text{conjugate}}(b, A, B)$, Definition 4 yields

$$1 + \frac{1}{b}\left(\frac{2z\left(\mathcal{B}^{\alpha,q}_{\lambda} f(z)\right)'}{\mathcal{B}^{\alpha,q}_{\lambda} f(z) + \mathcal{B}^{\alpha,q}_{\lambda}\overline{f(\overline{z})}} - 1\right) = \frac{1 + Aw(z)}{1 + Bw(z)}, \tag{46}$$

which, upon setting

$$h(z) = 1 + \sum_{k=1}^{\infty} c_k z^k = \frac{1 + Aw(z)}{1 + Bw(z)}, \tag{47}$$

can be written in the following form:

$$2z\left(\mathcal{B}^{\alpha,q}_{\lambda} f(z)\right)' = \left(\mathcal{B}^{\alpha,q}_{\lambda} f(z) + \mathcal{B}^{\alpha,q}_{\lambda}\overline{f(\overline{z})}\right)\left(1 + b\sum_{k=1}^{\infty} c_k z^k\right).$$

It follows from (21) that

$$\begin{aligned} z + 2a\Upsilon_2 z^2 &+ 3\Upsilon_3 a_3 z^3 + 4\Upsilon_4 a_4 z^4 + \cdots + 2n\Upsilon_{2n} a_{2n} z^{2n} \\ &+ (2n+1)\Upsilon_{2n+1} a_{2n+1} z^{2n+1} + \cdots \\ &= (z + \Upsilon_2 a_2 z^2 + \Upsilon_3 a_3 z^3 + \Upsilon_4 a_4 z^4 + \cdots + \Upsilon_{2n} a_{2n} z^{2n} + \Upsilon_{2n+1} a_{2n+1} z^{2n+1} + \cdots) \\ &\quad \cdot \left(1 + bc_1 z + bc_2 z^2 + \cdots\right), \end{aligned}$$

which, upon equating the coefficients of like powers of z, yields

$$\Upsilon_2 a_2 = bc_1 \quad \text{and} \quad 2\Upsilon_3 a_3 = bc_2 + bc_1\Upsilon_2 a_2, \tag{48}$$

$$3\Upsilon_4 a_4 = bc_3 + bc_2\Upsilon_2 a_2 + bc_1\Upsilon_3 a_3 \quad \text{and} \quad 4\Upsilon_5 a_5 = bc_4 + bc_3\Upsilon_2 a_2 + bc_2\Upsilon_3 a_3 + bc_1\Upsilon_4 a_4, \tag{49}$$

$$(2n-1)\Upsilon_{2n} a_{2n} = bc_{2n-1} + bc_{2n-2}\Upsilon_2 a_2 + \cdots + bc_2\Upsilon_{2n-2} a_{2n-2} + bc_1\Upsilon_{2n-1} a_{2n-1} \tag{50}$$

and

$$2n\Upsilon_{2n+1} a_{2n+1} = bc_{2n} + bc_{2n-1}\Upsilon_2 a_2 + \cdots + bc_2\Upsilon_{2n-1} a_{2n-1} + bc_1\Upsilon_{2n} a_{2n}. \tag{51}$$

We now apply Lemma 1, together with (48) and (49). We thus obtain

$$|a_2| \leqq \frac{|b|}{|\Upsilon_2|}\,(A-B) \quad \text{and} \quad |a_3| \leqq \frac{|b|\,(A-B)}{2\,|\Upsilon_3|}\Big(1 + |b|\,(A-B)\Big),$$

$$|a_4| \leqq \frac{|b|(A-B)}{2.3\,|\Upsilon_4|}\,(1+|b|(A-B))\,[2+|b|(A-B)]$$

and

$$|a_5| \leqq \frac{|b|(A-B)}{2.3.4\,|\Upsilon_5|}\,(1+|b|(A-B))\,(2+|b|(A-B))\,[3+|b|(A-B)]\,.$$

It follows that (44) and (45) hold true for $n = 1, 2$. Equation (50) in conjuction with Lemma 1 yeilds

$$|a_{2n}| \leqq \frac{|b|(A-B)}{(2n-1)\,|\Upsilon_{2n}|}\left(1+\sum_{r=1}^{n-1}|\Upsilon_{2r}|\,|a_{2r}|+\sum_{r=1}^{n-1}|\Upsilon_{2r+1}|\,|a_{2r+1}|\right)$$

Next, we assume that (44) and (45) hold true for $3, 4, \cdots, n-1$. Thus the above inequality leads us to the following inequality:

$$\begin{aligned}|a_{2n}| \leqq \frac{|b|(A-B)}{(2n-1)\,|\Upsilon_{2n}|}\Bigg(1&+\sum_{r=1}^{n-1}\frac{|b|(A-B)}{(2r-1)!}\prod_{j=1}^{2r-2}[j+|b|(A-B)]\\&+\sum_{r=1}^{n-1}\frac{|b|(A-B)}{(2r)!}\prod_{i=1}^{2r-1}\big(i+|b|(A-B)\big)\Bigg).\end{aligned} \tag{52}$$

In order to complete the proof of Theorem 5, it is sufficient to show that

$$\begin{aligned}&\frac{\left|\gamma\right|(A-B)}{(2m-1)\,|\Upsilon_{2m}|}\Bigg(1+\sum_{r=1}^{m-1}\frac{\left|\gamma\right|(A-B)}{(2r-1)!}\prod_{i=1}^{2r-2}\big(i+\left|\gamma\right|(A-B)\big)\\&\qquad+\sum_{r=1}^{m-1}\frac{\left|\gamma\right|(A-B)}{(2r)!}\prod_{j=1}^{2r-1}[j+\left|\gamma\right|(A-B)]\Bigg)\\&\quad=\frac{|b|(A-B)}{(2m-1)!\,|\Upsilon_{2m}|}\prod_{i=1}^{2m-2}\big(i+(A-B)\,|b|\big).\end{aligned} \tag{53}$$

It easy to see that (53) is valid for $m = 3$. We now suppose that (53) is true for $4, \cdots, m-1$. Then it follows from (52) that

$$\begin{aligned}
&\frac{|b|(A-B)}{(2m-1)|\Upsilon_{2m}|}\Bigg(1+\sum_{r=1}^{m-1}\frac{|b|(A-B)}{(2r-1)!}\prod_{j=1}^{2r-2}[|b|(A-B)+j] \\
&\qquad +\sum_{r=1}^{m-1}\frac{|b|(A-B)}{2r!}\prod_{j=1}^{2r-1}[|b|(A-B)+j] \\
&=\frac{(2m-3)|\Upsilon_{2m-2}|}{(2m-1)|\Upsilon_{2m}|}\Bigg[\frac{|b|(A-B)}{(2m-3)|\Upsilon_{2m-2}|}\Bigg(1+\sum_{r=1}^{m-2}\frac{|b|(A-B)}{(2r-1)!}\prod_{j=1}^{2r-2}[|b|(A-B)+j] \\
&\qquad +\sum_{r=1}^{m-2}\frac{|b|(A-B)}{(2r)!}\prod_{j=1}^{2r-1}[|b|(A-B)+j]\Bigg)\Bigg] \\
&\qquad +\frac{|b|(A-B)}{(2m-1)|\Upsilon_{2m}|}\left(\frac{|b|(A-B)}{(2m-3)!}\prod_{i=1}^{2m-4}(|b|(A-B)+i)\right) \\
&\qquad +\frac{|b|(A-B)}{(2m-1)|\Upsilon_{2m}|}\left(\frac{|b|(A-B)}{(2m-2)!}\prod_{j=1}^{2m-3}[|b|(A-B)+j]\right) \\
&=\frac{(2m-3)|\Upsilon_{2m-2}|}{(2m-1)|\Upsilon_{2m}|}\left(\frac{|b|(A-B)}{(2m-3)!\,|\Upsilon_{2m-2}|}\prod_{j=1}^{2m-4}[|b|(A-B)+j]\right) \\
&\qquad +\frac{|b|(A-B)}{(2m-1)|\Upsilon_{2m}|}\left(\frac{|b|(A-B)}{(2m-3)!}\prod_{i=1}^{2m-4}(|b|(A-B)+i)\right) \\
&\qquad +\frac{|b|(A-B)}{(2m-1)|\Upsilon_{2m}|}\left(\frac{|b|(A-B)}{(2m-2)!}\prod_{j=1}^{2m-3}[|b|(A-B)+j]\right) \\
&=\frac{1}{(2m-1)|\Upsilon_{2m}|}\frac{|b|(A-B)}{(2m-3)!}\prod_{j=1}^{2m-4}[|b|(A-B)+j]\Big)\Big(|b|(A-B)+2m-3\Big) \\
&\qquad +\frac{|b|(A-B)}{|\Upsilon_{2m}|}\left(\frac{|b|(A-B)}{(2m-1)!}\prod_{j=1}^{2m-3}[|b|(A-B)+j]\right) \\
&=\left(\frac{|b|(A-B)}{(2m-1)!\;|\Upsilon_{2m}|}\prod_{j=1}^{2m-3}[|b|(A-B)+j]\right)\Big(|b|(A-B)+2m-2\Big) \\
&=\frac{|b|(A-B)}{(2m-1)!\,|\Upsilon_{2m}|}\prod_{j=1}^{2m-2}[|b|(A-B)+j],
\end{aligned}$$

that is, (53) holds true for $m=n$. Hence, from (52) and (53), we obtain (44). Similary, we can prove (45). The proof of Theorem 5 is thus completed. □

4. Concluding Remarks and Observations

In the present investigation, we have successfully made use of a q-analogue of the familiar Borel distribution in order to introduce and study two new subclasses $\mathcal{S}^{\alpha,\lambda,q}_{\text{symmetric}}(b,A,B)$ and $\mathcal{S}^{\alpha,\lambda,q}_{\text{conjugate}}(b,A,B)$ of starlike and convex functions in the open unit disk Δ with respect to symmetric and conjugate points.

For functions in each of these subclasses $\mathcal{S}^{\alpha,\lambda,q}_{\text{symmetric}}(b,A,B)$ and $\mathcal{S}^{\alpha,\lambda,q}_{\text{conjugate}}(b,A,B)$, we have derived several properties including the Taylor-Maclaurin coefficient estimates. We have also indicated relevant connections of each of these subclasses $\mathcal{S}^{\alpha,\lambda,q}_{\text{symmetric}}(b,A,B)$ and $\mathcal{S}^{\alpha,\lambda,q}_{\text{conjugate}}(b,A,B)$ with the function classes which were invesigated in several earlier works.

In his survey-cum-expository review article, Srivastava [32] demonstrated how the theories of the basic (or q-) calculus and the fractional q-calculus have significantly encouraged and motivated further developments in Geometric Function Theory of Complex Analysis. As a matter of fact, the subject of the basic or quantum (or q-) analysis has found widespread applications which are based upon the extensive study of q-series and q-polynomials and, especially, q-hypergeometric functions and q-hypergeometric polynomials (see, for details, [37, pp. 350–351]). Therefore, with a view to aiding and motivating the interested reader for further researches on the subject, we choose to cite several recent developments (see, for example, [6], [17], [19], [26], [34], [36], [38], [39], [40], [42] and [43]) on various usages of the basic or quantum (or q-) calculus in Geometric Function Theory of Complex Analysis.

In concluding this investigation, we choose to reiterate an important observation, which was presented in the above-mentioned review-cum-expository review article by Srivastava [32, p. 340], as well as in the recently-published review article by Srivastava [33, Section 5 (pp. 1510–1512)], who pointed out the fact that the results for the above-mentioned known or new q-analogues can easily (and possibly trivially) be translated into the corresponding results for the so-called (p,q)-analogues (with $0 < |q| < p \leqq 1$) by applying some obvious parametric and argument variations, the additional parameter p being superfluous of redundant.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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