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On Some Convex Combinations of Biholomorphic Mappings in Several Complex Variables

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Abstract. In this paper, our interest is devoted to study the convex combinations of the form $(1-\lambda)f+\lambda g$, where $\lambda \in (0,1)$, of biholomorphic mappings on the Euclidean unit ball $\mathbb{B}^n$ in the case of several complex variables. Starting from a result proved by S. Trimble [26] and then extended by P.N. Chichra and R. Singh [3, Theorem 2] which says that if f is starlike such that $\operatorname{Re}[f'(z)] > 0$, then $(1-\lambda)z+\lambda f(z)$ is also starlike, we are interested to extend this result to higher dimensions. In the first part of the paper, we construct starlike convex combinations using the identity mapping on $\mathbb{B}^n$ and some particular starlike mappings on $\mathbb{B}^n$. In the second part of the paper, we define the class $\mathcal{L}^*_\lambda(\mathbb{B}^n)$ and prove results involving convex combinations of normalized locally biholomorphic mappings and Loewner chains. Finally, we propose a conjecture that generalize the result proved by Chichra and Singh.

1. Introduction

Let $\mathbb{C}^n$ denote the space of n complex variables $z=(z_1,...,z_n)$ with the Euclidean inner product $\langle z,w\rangle = \sum_{j=1}^n z_j\overline{w}_j$ and the Euclidean norm $\|z\| = \sqrt{\langle z,z\rangle}$, for all $z,w\in\mathbb{C}^n$. Also, let $\mathbb{B}^n$ denote the Euclidean unit ball in $\mathbb{C}^n$. In the case of one complex variable, the unit disc $\mathbb{B}^1$ is denoted by U.

Let $H(\mathbb{B}^n)$ denote the set of all holomorphic mappings from $\mathbb{B}^n$ into $\mathbb{C}^n$. If $f\in H(\mathbb{B}^n)$, we say that f is normalized if $f(0)=0$ and $Df(0)=I_n$, where $Df(z)$ is the complex Jacobian matrix of f at z and I_n is the identity operator in $\mathbb{C}^n$. Let

$$S(\mathbb{B}^n) = \left\{ f\in H(\mathbb{B}^n) : f \textit{ is normalized and univalent} \right\}$$

be the set of all normalized biholomorphic mappings on $\mathbb{B}^n$.

A mapping $f\in S(\mathbb{B}^n)$ is called convex (starlike) if its image is a convex (respectively, starlike with respect to the origin) domain in $\mathbb{C}^n$. We denote by

$$K(\mathbb{B}^n) = \left\{ f\in S(\mathbb{B}^n) : f(\mathbb{B}^n) \textit{ is a convex domain in } \mathbb{C}^n \right\}$$

the class of normalized convex mappings on $\mathbb{B}^n$ and by

$$S^*(\mathbb{B}^n) = \left\{ f\in S(\mathbb{B}^n) : f(\mathbb{B}^n) \textit{ is a starlike domain with respect to zero in } \mathbb{C}^n \right\}.$$

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the class of normalized starlike mappings on $\mathbb{B}^n$. In the case of one complex variable, the sets $S(U)$, $K(U)$ and $S^*(U)$ are denoted by S, K and S^*.

If $f \in H(\mathbb{B}^n)$, we say that f is locally biholomorphic on $\mathbb{B}^n$ if $J_f(z) \neq 0$, for all $z \in \mathbb{B}^n$, where $J_f(z) = \det\big(Df(z)\big)$, for all $z \in \mathbb{B}^n$. We denote by

$$\mathcal{LS}_n(\mathbb{B}^n) = \Big\{f : \mathbb{B}^n \to \mathbb{C}^n : f \textit{ is normalized and locally biholomorphic on } \mathbb{B}^n\Big\}$$

the set of all normalized locally biholomorphic mappings on $\mathbb{B}^n$. If $n = 1$, then $\mathcal{LS}_1(\mathbb{B}^1)$ is denoted by $\mathcal{LS}$.

Another important class of normalized holomorphic functions on the unit disc U is the Carathéodory class (for details, one may consult [4, Chapter 2], [9, p. 27] or [22, Chapter 2]), denoted by

$$\mathcal{P} = \Big\{p \in H(U) : p(0) = 1 \textit{ and } \mathrm{Re}[p(\zeta)] > 0, \quad \zeta \in U\Big\}.$$

In the case of several complex variables, we use the family

$$\mathcal{M}(\mathbb{B}^n) = \Big\{h \in H(\mathbb{B}^n) : h(0) = 0, Dh(0) = I_n \textit{ and } \mathrm{Re}\langle h(z), z\rangle > 0, z \in \mathbb{B}^n \setminus \{0\}\Big\}$$

of normalized holomorphic mappings on the Euclidean unit ball. It is important to mention that the class $\mathcal{M}(\mathbb{B}^n)$ plays the role of the Carathéodory family in $\mathbb{C}^n$. This class will be very important in the section that contains remarks about Loewner chains and Herglotz vector fields. For more details, one may consult [6], [8], [9], [16], [20] and [23].

Next, we recall the notions of Loewner chain and Herglotz vector field on the Euclidean unit ball in $\mathbb{C}^n$. We will use these notions to prove that (under some particular assumptions) the convex combination of two Loewner chains is also a Loewner chain.

Definition 1.1. *(see e.g. [9, Definition 8.1.2] or [11, Definition 1.1]): A mapping $L = L(z,t) : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ is said to be a Loewner chain (normalized univalent subordination chain) if the following conditions hold:*

1. *$e^{-t}L(\cdot,t) \in S(\mathbb{B}^n)$, for all $t \in [0,\infty)$;*
2. *$L(\mathbb{B}^n, s) \subseteq L(\mathbb{B}^n, t)$, for all $0 \leq s \leq t < \infty$.*

Definition 1.2. *(see e.g. [9, Chapter 8]): Let $h : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ be a mapping. We say that h is a Herglotz vector field if the following conditions hold:*

1. *$h(\cdot,t) \in \mathcal{M}(\mathbb{B}^n)$, for all $t \in [0,\infty)$;*
2. *$h(z,\cdot)$ is measurable on $[0,\infty)$, for all $z \in \mathbb{B}^n$.*

The following theorem gives a sufficient condition for a mapping $L = L(z,t)$ to be a Loewner chain (see [6, Lemma 1.6], [11], [14, Lemma 2.3] or [20, Theorem 2.2]).

Theorem 1.3. *Let $L = L(z,t) : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ be a mapping which satisfies the following conditions:*

1. *$L(\cdot,t) \in H(\mathbb{B}^n)$, $L(0,t) = 0$ and $DL(0,t) = e^t I_n$, for all $t \in [0,\infty)$;*
2. *$L(z,\cdot)$ is locally Lipschitz continuous on the interval $[0,\infty)$ locally uniformly with respect to $z \in \mathbb{B}^n$.*

Assume that there exists a Herglotz vector field $h : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ such that

$$\frac{\partial L}{\partial t}(z,t) = DL(z,t)h(z,t), \quad a.e. \quad t \in [0,\infty), \quad z \in \mathbb{B}^n.$$

Moreover, assume that $\big\{e^{-t}L(\cdot,t)\big\}_{t\geq 0}$ is a normal family on $\mathbb{B}^n$. Then $L(z,t)$ is a Loewner chain.

The connection between the class S and the Loewner chains in the case of one complex variable is given by a result due to Pommerenke (see e.g. [9, Theorem 3.1.8]) which says that any function $f \in S$ can be embedded as the first element of a Loewner chain (i.e. for each $f \in S$, there exists a Loewner chain $L(\zeta, t)$ such that $L(\zeta, 0) = f(\zeta)$, for all $\zeta \in U$).

This result is no longer true for the class $S(\mathbb{B}^n)$, and by this reason I. Graham, H. Hamada and G. Kohr (see [6]) defined the class

$$S^0(\mathbb{B}^n) = \left\{ f \in S(\mathbb{B}^n) : \exists\, L(z,t) \text{ a Loewner chain s.t. } \{e^{-t}L(\cdot,t)\}_{t\geq 0} \text{ is a normal family on } \mathbb{B}^n \text{ and } f = L(\cdot, 0) \right\}$$

of normalized univalent mappings which have parametric representation on $\mathbb{B}^n$. It is clear that $S^0(\mathbb{B}^1) = S$ (see [22]), but $S^0(\mathbb{B}^n) \subsetneq S(\mathbb{B}^n)$, for $n \geq 2$ (see [6] and [9]). For details, one may consult also [6], [9, Chapter 8] and [11].

2. Remarks on convex combinations

An interesting fact about the class of normalized univalent functions on the unit disc U in $\mathbb{C}$ is that the class S is not convex. Namely, starting from two normalized univalent functions on the unit disc even the average of these functions does not necessarily belong to S. To show this, we present two examples in the case of one complex variable (see e.g. [4], [12] or [18]) and one example on the Euclidean unit ball in $\mathbb{C}^2$ (see e.g. [9] or [16]).

Example 2.1. *(see e.g. [4, Exercise 3, Chapter 2]): Let*

$$f(\zeta) = \frac{\zeta}{(1-\zeta)^2} \quad \text{and} \quad g(\zeta) = \frac{\zeta}{(1+\zeta)^2}, \quad \zeta \in U.$$

Then $h = (f+g)/2$ does not belong to S.

In Example 2.1, the functions f and g are not only normalized and univalent - they are even starlike on the unit disc U. However, the function h is not starlike on U (in fact, h is not even univalent on U). Hence, $f, g \in S^*$, but $h \notin S^*$ because $h \notin S$. Another important example was given by MacGregor in [18, Section 3]. He proved that the linear combination of two convex functions is not necessarily univalent on the unit disc.

Example 2.2. *(see [18, Section 3]): Let*

$$f(\zeta) = \frac{\zeta}{1-\zeta} \quad \text{and} \quad g(\zeta) = \frac{\zeta}{1+i\zeta}, \quad \zeta \in U.$$

Also let $h(\zeta) = tf(\zeta) + (1-t)g(\zeta)$, for all $0 < t < 1$. Then h is not univalent in U for each $t \in (0,1)$.

In the previous example, $f, g \in K$ are convex functions, but h is not univalent on U because there exists a point $z_0 = (a+i)/(a+1) \in \mathbb{C}$ with $|z_0| < 1$, where $a = \sqrt{(1-t)/t}$, $0 < t < 1$ such that $h'(z_0) = 0$ (for the complete proof, one may consult [18, Section 3]).

Next, we can extend the statement of Example 2.1 to the case of several complex variables. For $n = 2$, we obtain the following example (considered in [9, Problem 6.2.3] and [16, Problem 4.3.4]):

Example 2.3. *(see [9, Problem 6.2.3]): Let*

$$f(z) = \left(\frac{z_1}{(1-z_1)^2}, \frac{z_2}{(1-z_2)^2} \right) \quad \text{and} \quad g(z) = \left(\frac{z_1}{(1+z_1)^2}, \frac{z_2}{(1+z_2)^2} \right), \quad z = (z_1, z_2) \in \mathbb{B}^2.$$

Then $h = (f+g)/2$ is not starlike on the Euclidean unit ball $\mathbb{B}^2$. In fact h does not belong to $S(\mathbb{B}^2)$.

Remark 2.4. *We can also analyze the convex combination between the mapping* $f(z) = \frac{z}{(1-z_1)^2}$, *for* $z \in \mathbb{B}^n$ *and one of the mappings* $g(z) = \frac{z}{(1+z_1)^2}$ *or* $g(z) = \frac{z}{(1\pm z_2)^2}$, *for* $z = (z_1, ..., z_n) \in \mathbb{B}^n$. *Similar arguments like in the previous examples will show us if the mapping* $h = (f + g)/2$ *is starlike or not on the Euclidean unit ball* $\mathbb{B}^n$.

Although linear combinations of univalent functions are not always univalent (for more details about these results, one may consult [2], [12], [15] or [18]), there exist subclasses of the class S that satisfy this condition (see [3], [19] or [26] in the case $n = 1$). The goal of this paper is to extend in the case of several complex variables a result proved by P.N. Chichra and R. Singh in [3, Theorem 2] for the case of one complex variable. This result shows that the convex combination between a starlike function with positive real part of the derivative and the identity function is also starlike on the unit disc U, as it follows:

Theorem 2.5. *Let* $\lambda \in [0, 1]$. *If* $f \in S^*$ *and* $\mathrm{Re}[f'(\zeta)] > 0$, *for all* $\zeta \in U$, *then*

$$h(\zeta) = (1 - \lambda)\zeta + \lambda f(\zeta) \tag{1}$$

is starlike with respect to zero in U *and* $\mathrm{Re}[h'(\zeta)] > 0$, *for all* $\zeta \in U$.

In the case of several complex variables we start with some particular forms of the mapping f in order to construct convex combinations which are starlike on the Euclidean unit ball $\mathbb{B}^n$. Like in the case of result proved by Chichra and Singh, we also consider convex combinations between a starlike mapping and the identity map in $\mathbb{C}^n$. First, we prove a general result for normalized locally bihomomorphic mappings on $\mathbb{B}^n$ which satisfies some additional conditions in order to obtain the starlikeness of the convex combination. Then, in the final part, we propose a conjecture which generalize Theorem 2.5 in the case of several complex variables.

3. Preliminary results

Next we present some important results the will be used in the proofs of the main results from this paper. We recall, without proofs, the analytical characterization of starlikeness in $\mathbb{C}^n$ (proved by Matsuno, see [17] or [9, Theorem 6.2.2]; see also [5] and [25]), the connection between Loewner chains and starlike mappings (see [9, Corollary 8.2.3] or [21, Corollary 2]) and also some important criteria for univalence in $\mathbb{C}^n$.

Theorem 3.1. *Let* $f : \mathbb{B}^n \to \mathbb{C}^n$ *be a locally biholomorphic mapping such that* $f(0) = 0$. *Then* f *is starlike if and only if*

$$\mathrm{Re}\langle [Df(z)]^{-1} f(z), z\rangle > 0, \quad z \in \mathbb{B}^n \setminus \{0\}. \tag{2}$$

Using Theorem 3.1 we can prove that a locally biholomorphic mapping with $f(0) = 0$ is starlike on the Euclidean unit ball $\mathbb{B}^n$. Another important characterization of starlikeness was given by Pfaltzgraff and Suffridge (see [9, Corollary 8.2.3] or [21, Corollary 2]) in terms of Loewner chains.

Theorem 3.2. *Let* $f : \mathbb{B}^n \to \mathbb{C}^n$ *be a normalized locally biholomorphic mapping on* $\mathbb{B}^n$. *Then* f *is starlike on* $\mathbb{B}^n$ *if and only if* $L(z, t) = e^t f(z)$ *is a Loewner chain.*

We end this section with two important results that ensure the univalence of a normalized holmorphic mapping on the Euclidean unit ball $\mathbb{B}^n$. The first result, proved by Suffridge in [25, Theorem 7], is a version of the Noshiro-Warschawski's univalence criteria (see e.g. [4, Theorem 2.16] for the case $n = 1$) in the case of several complex variables.

Theorem 3.3. *(see [25, Theorem 7]): Let* $f : \mathbb{B}^n \to \mathbb{C}^n$ *be a normalized holomorphic mapping such that*

$$\mathrm{Re}\langle Df(z)(u), u\rangle > 0, \quad z \in \mathbb{B}^n, u \in \mathbb{C}^n, \|u\| = 1. \tag{3}$$

Then f *is univalent on* $\mathbb{B}^n$.

Another important criteria for univalence in $\mathbb{C}^n$ is presented in the following result proved by I. Graham, H. Hamada and G. Kohr in [7, Lemma 2.2] (see also [13] and [14]).

Theorem 3.4. *(see [7, Lemma 2.2]): Let $f : \mathbb{B}^n \to \mathbb{C}^n$ be a normalized holomorphic mapping such that*

$$\|Df(z) - I_n\| < 1, z \in \mathbb{B}^n. \tag{4}$$

Then $f \in S^0(\mathbb{B}^n)$. In particular, f is univalent on $\mathbb{B}^n$.

Taking into account that $S^0(\mathbb{B}^n) \subsetneq S(\mathbb{B}^n)$, for $n \geq 2$ (see [6]), it is clear that there is an important difference between Theorem 3.3 (which assures us the univalence of a mapping on $\mathbb{B}^n$) and Theorem 3.4 (which assures us that a mapping admits parametric representation on $\mathbb{B}^n$), i.e. there exist normalized holomorphic mappings on $\mathbb{B}^n$ that satisfy condition (3), but do not satisfy condition (4).

Such an example, that shows us that $S^0(\mathbb{B}^n)$ is a proper subclass of $S(\mathbb{B}^n)$ for $n \geq 2$, is presented in [9, Example 8.3.21].

4. Univalence of convex combinations in $\mathbb{C}^n$

In view of the results presented in the previous section we can prove some criteria for univalence of a convex combination of normalized holomorphic mappings on the Euclidean unit ball $\mathbb{B}^n$. In fact, we can obtain a condition for a convex combination to be a mapping which has parametric representation on $\mathbb{B}^n$.

Lemma 4.1. *Let $f : \mathbb{B}^n \to \mathbb{C}^n$ be a normalized holomorphic mapping such that*

$$\mathrm{Re}\langle Df(z)(u), u\rangle > 0,$$

for all $z \in \mathbb{B}^n$, $u \in \mathbb{C}^n$ with $\|u\| = 1$. Also let

$$h(z) = (1-\lambda)z + \lambda f(z), \quad z \in \mathbb{B}^n, \quad \lambda \in [0, 1].$$

Then h is univalent on $\mathbb{B}^n$.

Proof. Clearly, h is a normalized holomorphic mapping. Moreover,

$$Dh(z) = (1-\lambda)I_n + \lambda Df(z), \quad z \in \mathbb{B}^n, \quad \lambda \in [0, 1]$$

and then

$$\mathrm{Re}\langle Dh(z)(u), u\rangle = \mathrm{Re}\Big\langle [(1-\lambda)I_n + \lambda Df(z)](u), u\Big\rangle$$

$$= \mathrm{Re}\langle (1-\lambda)I_n(u), u\rangle + \mathrm{Re}\langle \lambda Df(z)(u), u\rangle = (1-\lambda)\mathrm{Re}\langle u, u\rangle + \lambda \mathrm{Re}\langle Df(z)(u), u\rangle$$

$$= (1-\lambda)\|u\|^2 + \lambda \mathrm{Re}\langle Df(z)(u), u\rangle = (1-\lambda) + \lambda \mathrm{Re}\langle Df(z)(u), u\rangle > 0,$$

for all $z \in \mathbb{B}^n$, $u \in \mathbb{C}^n$ with $\|u\| = 1$. According to Theorem 3.3 we obtain that h is univalent on $\mathbb{B}^n$. Since h is also normalized, it means that $h \in S(\mathbb{B}^n)$. □

Remark 4.2. *Notice that, in view of the previous proof, we obtain that if $\mathrm{Re}\Big\langle Df(z)(u), u\Big\rangle > 0$, for all $z \in \mathbb{B}^n$ and $u \in \mathbb{C}^n$ with $\|u\| = 1$, then $\mathrm{Re}\Big\langle Dh(z)(u), u\Big\rangle > 0$, for all $z \in \mathbb{B}^n$ and $u \in \mathbb{C}^n$ with $\|u\| = 1$.*

Let $k \in \mathbb{N}^*$. We say that a mapping $P_k : \mathbb{C}^n \to \mathbb{C}^n$ is a homogenous polynomial of degree k if there exist $Q_k : \mathbb{C}^n \times ... \times \mathbb{C}^n \to \mathbb{C}^n$ an k-$\mathbb{C}$-linear operator such that $P_k(z) = Q_k(z^k)$. For details, one may consult [9] or [16]. Recall that we define the norm of the operator P_k by $\|P_k\| = \max\{\|P_k(z)\| : \|z\| = 1\}$.

Lemma 4.3. *Let $f : \mathbb{B}^n \to \mathbb{C}^n$ be a normalized holomorphic mapping such that*

$$\|Df(z) - I_n\| < 1,$$

for all $z \in \mathbb{B}^n$ and

$$h(z) = (1-\lambda)z + \lambda f(z), \quad z \in \mathbb{B}^n, \quad \lambda \in [0,1].$$

Then $h \in S^0(\mathbb{B}^n)$. In particular, h is univalent on $\mathbb{B}^n$.

Proof. Frist, it is clear that h is a normalized holomorphic mapping on $\mathbb{B}^n$. Moreover,

$$Dh(z) = (1-\lambda)I_n + \lambda Df(z), \quad z \in \mathbb{B}^n, \quad \lambda \in [0,1]$$

and then

$$\|Dh(z) - I_n\| = \|I_n - \lambda I_n + \lambda Df(z) - I_n\| = \|\lambda(Df(z) - I_n)\| = |\lambda| \cdot \|Df(z) - I_n\| \le \|Df(z) - I_n\| < 1,$$

for all $z \in \mathbb{B}^n$. In view of Theorem 3.4 (see [7, Lemma 2.2]) we obtain that $h \in S^0(\mathbb{B}^n)$, so h is also univalent on $\mathbb{B}^n$. □

Lemma 4.4. *Let $f : \mathbb{B}^n \to \mathbb{C}^n$ be a normalized holomorphic mapping such that $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$, for all $z \in \mathbb{B}^n$. Also let*

$$h(z) = (1-\lambda)z + \lambda f(z), \quad z \in \mathbb{B}^n, \quad \lambda \in (0,1).$$

If $\sum_{k=2}^{\infty} k\|A_k\| \le 1$, then $h \in S^0(\mathbb{B}^n)$.

Proof. Since $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$, for all $z \in \mathbb{B}^n$, we deduce that

$$Df(z) = I_n + \sum_{k=2}^{\infty} kA_k(z^{k-1}, \cdot), \quad z \in \mathbb{B}^n,$$

where we used the fact that $A_k(z^k)$ is a homogenous polynomial of degree k. On the other hand,

$$Dh(z) = (1-\lambda)I_n + \lambda Df(z), \quad z \in \mathbb{B}^n.$$

Hence,

$$\|Dh(z) - I_n\| = \|\lambda(Df(z) - I_n)\| = \Big\|\lambda \sum_{k=2}^{\infty} kA_k(z^{k-1}, \cdot)\Big\| = |\lambda| \cdot \Big\|\sum_{k=2}^{\infty} kA_k(z^{k-1}, \cdot)\Big\|$$

$$\le \sum_{k=2}^{\infty} k\|A_k(z^{k-1}, \cdot)\| \le \sum_{k=2}^{\infty} k\|A_k\| \cdot \|z\|^{k-1} \le \sum_{k=2}^{\infty} k\|A_k\| \cdot \|z\|.$$

Then we obtain

$$\|Dh(z) - I_n\| \le \|z\| \cdot \sum_{k=2}^{\infty} k\|A_k\| \le \|z\| < 1, \quad z \in \mathbb{B}^n.$$

Consequently,

$$\|Dh(z) - I_n\| < 1, \quad z \in \mathbb{B}^n$$

and in view of Theorem 3.4 we deduce that $h \in S^0(\mathbb{B}^n)$. □

5. Particular starlike mappings on the Euclidean unit ball in $\mathbb{C}^n$

In this section we present the first important result of this paper together with some examples to illustrate how this result can be applied in several particular cases. We begin this section with a well-known result related to starlike mappings on the Euclidean unit ball $\mathbb{B}^n$ (see [9]).

Lemma 5.1. *Let $f : \mathbb{B}^n \to \mathbb{C}^n$ be of the form $f(z) = \big(f_1(z_1), ..., f_n(z_n)\big)$, for all $z = (z_1, ..., z_n) \in \mathbb{B}^n$.*

1. *If $f_1, ..., f_n \in S^*$, then $f \in S^*(\mathbb{B}^n)$.*
2. *If, in addition, $\mathrm{Re}[f_j'(z_j)] > 0$, for all $j = \overline{1, n}$, then $\mathrm{Re}\langle Df(z)(u), u\rangle > 0$, for all $z \in \mathbb{B}^n$ and $u \in \mathbb{C}^n$ with $\|u\| = 1$.*

Proof. Indeed, if $f_1, ..., f_n \in S^*$, then $f \in S^*(\mathbb{B}^n)$ (see e.g. [9, Problem 6.2.5] or [16, Example 4.3.4]) and this completes the first part of the proof.

For the second part of the lemma, we have

$$Df(z)(u) = \begin{pmatrix} f_1'(z_1) & 0 & 0 & ... & 0 \\ 0 & f_2'(z_2) & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & f_n'(z_n) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ ... \\ u_n \end{pmatrix} = \big(u_1 f_1'(z_1), ..., u_n f_n'(z_n)\big)$$

and

$$\mathrm{Re}\langle Df(z)(u), u\rangle = |u_1|^2 \cdot \mathrm{Re}[f_1'(z_1)] + ... + |u_n|^2 \cdot \mathrm{Re}[f_n'(z_n)] > 0,$$

for all $z \in \mathbb{B}^n$ and $u \in \mathbb{C}^n$ with $\|u\| = 1$ and this completes the proof. □

According to Lemma 5.1 we can obtain a first version of Theorem 2.5 in the case of several complex variables. However, in this case we have a particular form of the mapping f (it has on each component a starlike function of one variable). The following result is very simple and its proof is immediate.

Proposition 5.2. *Let $0 \le \lambda \le 1$ and let $f_1, ..., f_n \in S^*$ be such that $\mathrm{Re}[f_j'(z_j)] > 0$, for all $j = \overline{1, n}$. Also, let $f(z) = \big(f_1(z_1), ..., f_n(z_n)\big)$, for all $z \in \mathbb{B}^n$. Then*

$$h(z) = (1 - \lambda)z + \lambda f(z) \tag{5}$$

is starlike, for all $z \in \mathbb{B}^n$ and $\lambda \in [0, 1]$. Moreover, $\mathrm{Re}\langle Dh(z)(u), u\rangle > 0$, for all $z \in \mathbb{B}^n$ and $u \in \mathbb{C}^n$ with $\|u\| = 1$. In particular, h is univalent on $\mathbb{B}^n$.

Proof. In view of Lemma 5.1 we have that $f \in S^*(\mathbb{B}^n)$ and $\mathrm{Re}\langle Df(z)(u), u\rangle > 0$, for all $z \in \mathbb{B}^n$ and $u \in \mathbb{C}^n$ with $\|u\| = 1$. On the other hand,

$$h(z) = (1 - \lambda)z + \lambda f(z) = (1 - \lambda)\big(z_1, ..., z_n\big) + \lambda\big(f_1(z_1), ..., f_n(z_n)\big)$$
$$= \big((1 - \lambda)z_1 + \lambda f_1(z_1), ..., (1 - \lambda)z_n + \lambda f_n(z_n)\big).$$

If we denote $h_j(z_j) = (1 - \lambda)z_j + \lambda f_j(z_j)$, for all $j = \overline{1, n}$ and $z_j \in U$, then according to Theorem 2.5 we obtain that $h_j \in S^*$, for all $j = \overline{1, n}$ and $\lambda \in [0, 1]$. Hence,

$$h(z) = \big(h_1(z_1), ..., h_n(z_n)\big)$$

is a starlike mapping on $\mathbb{B}^n$, for all $\lambda \in [0, 1]$. Moreover,

$$Dh(z)(u) = \begin{pmatrix} h_1'(z_1) & 0 & 0 & ... & 0 \\ 0 & h_2'(z_2) & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & h_n'(z_n) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ ... \\ u_n \end{pmatrix} = \big(u_1 h_1'(z_1), ..., u_n h_n'(z_n)\big)$$

and

$$\mathrm{Re}\langle Dh(z)(u), u\rangle = |u_1|^2 \cdot \mathrm{Re}[h_1'(z_1)] + ... + |u_n|^2 \cdot \mathrm{Re}[h_n'(z_n)] > 0,$$

for all $z \in \mathbb{B}^n$ and $u \in \mathbb{C}^n$ with $\|u\| = 1$ and this completes the proof. □

It is clear that the mapping f used in the previous result has a very particular form (has on each component a starlike function of one complex variable). However, we can obtain similar results for an arbitrary starlike mapping on the Euclidean unit ball.

In the following examples (considered also in [5], [10, Example 3.5], [13, Example 3.4], [24] or [25, Examples 3 and 7]) we use arbitrary starlike mappings to construct starlike mappings (as convex combinations on $\mathbb{B}^n$). On the other hand, we can obtain starlike mappings h of the form (5) on $\mathbb{B}^n$ using starlike mappings f that do not satisfy condition $\mathrm{Re}\langle Df(z)(u), u\rangle > 0$, for all $z \in \mathbb{B}^n$ and $u \in \mathbb{C}^n$ with $\|u\| = 1$.

Example 5.3. *Let $n = 2$ and $f : \mathbb{B}^2 \to \mathbb{C}^2$ be given by*

$$f(z) = \left(z_1 + az_2^2, z_2\right), \quad z = (z_1, z_2) \in \mathbb{B}^2 \tag{6}$$

with $|a| \leq 3\sqrt{3}/2$. According to [24, Example 5], we know that $f \in S^*(\mathbb{B}^2)$. Moreover,

$$h(z) = (1-\lambda)z + \lambda f(z) = (1-\lambda)\left(z_1, z_2\right) + \lambda\left(z_1 + az_2^2, z_2\right) = \left((1-\lambda)z_1 + \lambda z_1 + \lambda a z_2^2, (1-\lambda)z_2 + \lambda z_2\right),$$

so

$$h(z) = \left(z_1 + \lambda a z_2^2, z_2\right), \quad z = (z_1, z_2) \in \mathbb{B}^2.$$

Then $h \in S^*(\mathbb{B}^2)$ if and only if $|\lambda a| \leq 3\sqrt{3}/2$. In particular, this is true for $\lambda \in [0, 1]$ and $|a| \leq 3\sqrt{3}/2$. Hence, $h \in S^*(\mathbb{B}^2)$.

Example 5.4. *Let $n = 2$ and $f : \mathbb{B}^2 \to \mathbb{C}^2$ be given by*

$$f(z) = \left(z_1 + az_1z_2, z_2\right), \quad z = (z_1, z_2) \in \mathbb{B}^2 \tag{7}$$

with $|a| \leq 1$. According to [24, Example 6], we know that $f \in S^*(\mathbb{B}^2)$. Moreover,

$$h(z) = (1-\lambda)z + \lambda f(z) = (1-\lambda)\left(z_1, z_2\right) + \lambda\left(z_1 + az_1z_2, z_2\right) = \left((1-\lambda)z_1 + \lambda z_1 + \lambda a z_1 z_2, (1-\lambda)z_2 + \lambda z_2\right),$$

so

$$h(z) = \left(z_1 + \lambda a z_1 z_2, z_2\right), \quad z = (z_1, z_2) \in \mathbb{B}^2.$$

Then $h \in S^*(\mathbb{B}^2)$ if and only if $|\lambda a| \leq 1$. In particular, this is true for $\lambda \in [0, 1]$ and $|a| \leq 1$. Hence, $h \in S^*(\mathbb{B}^2)$.

It is important to mention here that the results contained in Examples 5.3 and 5.4 can be directly verified. However, we can obtain starlike mappings h as convex combinations of two starlike mappings such that at least one of them does not satisfy the condition $\mathrm{Re}\langle Df(z)(u), u\rangle > 0$, for all $z \in \mathbb{B}^n$ and $u \in \mathbb{C}^n$ with $\|u\| = 1$.

Remark 5.5. *Notice that in Examples 5.3 and 5.4 we can consider also a general case: a complex parameter $\lambda \in \mathbb{C}$ with the property that $|\lambda| \leq 1$.*

In the following example (considered also in [9] and [24]) we use a convex mapping f on $\mathbb{B}^n$ to construct a starlike univalent mapping h on the Euclidean unit ball $\mathbb{B}^n$. The condition $|a| \leq 1/2$ also ensures the univalence of the mapping h on $\mathbb{B}^n$.

Example 5.6. *Let $n = 2$ and $f : \mathbb{B}^2 \to \mathbb{C}^2$ be given by*

$$f(z) = \left(z_1 + az_2^2, z_2\right), \quad z = (z_1, z_2) \in \mathbb{B}^2 \tag{8}$$

with $|a| \leq 1/2$. Then $h(z) = (1-\lambda)z + \lambda f(z)$ is starlike on $\mathbb{B}^2$, for all $z \in \mathbb{B}^2$ and $\lambda \in [0, 1]$.

Proof. According to [24, Example 7] we know that $f \in K(\mathbb{B}^2)$. Moreover,

$$h(z) = \left(z_1 + \lambda a z_2^2, z_2\right), \quad z = (z_1, z_2) \in \mathbb{B}^2.$$

Then $h \in S^*(\mathbb{B}^2)$ because $|\lambda a| \leq \frac{1}{2} < \frac{3\sqrt{3}}{2}$. In addition, $Df(z) = \begin{pmatrix} 1 & 2az_2 \\ 0 & 1 \end{pmatrix}$ and

$$\mathrm{Re}\langle Df(z)(u), u\rangle = \mathrm{Re}\langle (u_1 + 2az_2u_2, u_2), (u_1, u_2)\rangle = |u_1|^2 + \mathrm{Re}(2az_2u_2\overline{u}_1) + |u_2|^2 = 1 + 2\mathrm{Re}(az_2u_2\overline{u}_1)$$

$$\geq 1 - 2|a||z_2||u_1||u_2| > 0,$$

for all $z \in \mathbb{B}^2$, $u \in \mathbb{C}^2$ with $\|u\| = 1$ and $|a| \leq 1/2$. On the other hand,

$$Dh(z) = D\Big((1-\lambda)z + \lambda f(z)\Big) = (1-\lambda)I_2 + \lambda Df(z) \tag{9}$$

and

$$Dh(z)(u) = (1-\lambda)u + \lambda Df(z)(u), \quad u \in \mathbb{C}^2, \|u\| = 1.$$

Then

$$\langle Dh(z)(u), u\rangle = \langle (1-\lambda)u + \lambda Df(z)(u), u\rangle = \langle (1-\lambda)u, u\rangle + \langle \lambda Df(z)(u), u\rangle = (1-\lambda)\langle u, u\rangle + \lambda\langle Df(z)(u), u\rangle$$

$$= (1-\lambda)\|u\|^2 + \lambda\langle Df(z)(u), u\rangle = (1-\lambda) + \lambda\langle Df(z)(u), u\rangle$$

and

$$\mathrm{Re}\langle Dh(z)(u), u\rangle = (1-\lambda) + \lambda\mathrm{Re}\langle Df(z)(u), u\rangle > 0, \tag{10}$$

for all $z \in \mathbb{B}^2$, $u \in \mathbb{C}^2$ with $\|u\| = 1$ and $\lambda \in [0, 1]$. Hence, h is a starlike univalent mapping on the Euclidean unit ball $\mathbb{B}^2$. □

6. A general result on convex combinations of locally biholomorphic mappings in several complex variables

Next, we present another suggestive result which can be seen as a second version of Theorem 2.5 in the case of several complex variables. We can consider the following result as a generalization of the theorem proved by Chichra and Singh (see [3, Theorem 2]).

Theorem 6.1. *Let $0 < \lambda < 1$ and $\mu = \lambda/(1-\lambda)$. Also let $f : \mathbb{B}^n \to \mathbb{C}^n$ be a normalized locally biholomorphic mapping such that*

$$\left\|Df(z) - I_n\right\| < \frac{1}{\lambda} \tag{11}$$

and

$$\mathrm{Re}\Big\langle \Big(I_n + \mu Df(z)\Big)^{-1}\Big(z + \mu f(z)\Big), z\Big\rangle > 0, \tag{12}$$

for all $z \in \mathbb{B}^n \setminus \{0\}$. Consider $h : \mathbb{B}^n \to \mathbb{C}^n$ be given by

$$h(z) = (1-\lambda)z + \lambda f(z), \quad z \in \mathbb{B}^n. \tag{13}$$

Then $h \in S^*(\mathbb{B}^n)$, for all $\lambda \in (0, 1)$.

Proof. In order to prove that $h \in S^*(\mathbb{B}^n)$, for all $\lambda \in (0,1)$, it is enough to show that h is locally biholomorphic on $\mathbb{B}^n$, $h(0) = 0$ and

$$\mathrm{Re}\langle [Dh(z)]^{-1} h(z), z\rangle > 0, \quad z \in \mathbb{B}^n \setminus \{0\}.$$

Since f is normalized on $\mathbb{B}^n$ it follows that

$$h(0) = \lambda f(0) = 0 \quad \text{and} \quad Dh(0) = (1-\lambda)I_n + \lambda Df(0) = I_n.$$

Moreover,

$$\left\|Dh(z) - I_n\right\| = \left\|(1-\lambda)I_n + \lambda Df(z) - I_n\right\| = \left\|\lambda Df(z) - \lambda I_n\right\| = |\lambda| \cdot \left\|Df(z) - I_n\right\| = \lambda \cdot \left\|Df(z) - I_n\right\| < \frac{\lambda}{\lambda} = 1,$$

for all $\lambda \in (0,1)$, in view of relation (11). According to Theorem 3.4 the above inequality assures us that h is univalent on $\mathbb{B}^n$. Obviously, in particular, h is locally biholomorphic on the Euclidean unit ball $\mathbb{B}^n$. Moreover, the relation

$$\left\|Dh(z) - I_n\right\| < 1$$

implies that $Dh(z)$ is invertible, i.e. there exists the inverse operator $[Dh(z)]^{-1}$. Finally, the inequality

$$\mathrm{Re}\langle [Dh(z)]^{-1} h(z), z\rangle > 0, \quad z \in \mathbb{B}^n \setminus \{0\}$$

is equivalent to

$$\mathrm{Re}\left\langle \left((1-\lambda)I_2 + \lambda Df(z)\right)^{-1}\left((1-\lambda)z + \lambda f(z)\right), z\right\rangle = \mathrm{Re}\left\langle \frac{1}{1-\lambda}\left(I_2 + \mu Df(z)\right)^{-1}(1-\lambda)\left(z + \mu f(z)\right), z\right\rangle$$

$$= \mathrm{Re}\left\langle \left(I_2 + \mu Df(z)\right)^{-1}\left(z + \mu f(z)\right), z\right\rangle > 0, \quad z \in \mathbb{B}^n \setminus \{0\}.$$

Hence, in view of relation (12), we obtain that

$$\mathrm{Re}\langle [Dh(z)]^{-1} h(z), z\rangle > 0,$$

for all $z \in \mathbb{B}^n \setminus \{0\}$. According to Theorem 3.1 we conclude that $h \in S^*(\mathbb{B}^n)$, for all $\lambda \in (0,1)$. □

Remark 6.2. *It is clear that*

- *if* $\lambda = 0$, *then* $h(z) = z$;
- *if* $\lambda = 1$, *then* $h(z) = f(z)$,

for all $z \in \mathbb{B}^n$ *and then* $h \in S^*(\mathbb{B}^n)$.

7. The class $\mathcal{L}^*_\lambda(\mathbb{B}^n)$

Taking into account the main result from the previous section, we can define a class of normalized locally biholomorphic mappings on the Euclidean unit ball that satisfies conditions from Theorem 6.1. Hence, the convex combination between the identity map and a function from this class will be a starlike mapping on the Euclidean unit ball $\mathbb{B}^n$.

Definition 7.1. *Let us consider* $\lambda \in (0,1)$ *and* $\mu = \lambda/(1-\lambda)$. *We say that* $f \in \mathcal{L}^*_\lambda(\mathbb{B}^n)$ *if* f *is normalized locally biholomorphic on* $\mathbb{B}^n$ *and* f *satisfies*

(a_1) $\left\|Df(z) - I_n\right\| < \frac{1}{\lambda}$;

(a_2) $\mathrm{Re}\left\langle \left(I_n + \mu Df(z)\right)^{-1}\left(z + \mu f(z)\right), z\right\rangle > 0$, *for all* $z \in \mathbb{B}^n \setminus \{0\}$.

In view of the above definition, we denote the class

$$\mathcal{L}_\lambda^*(\mathbb{B}^n) = \Big\{ f \in \mathcal{LS}_n(\mathbb{B}^n) : f \text{ satisfies } (a_1) \text{ and } (a_2) \Big\}$$

Remark 7.2. *Note that $\mathcal{L}_\lambda^*(\mathbb{B}^n) \neq \emptyset$ as the identity mapping $\phi : \mathbb{B}^n \to \mathbb{C}^n$ given by $\phi(z) = z$, for all $z \in \mathbb{B}^n$ belongs to $\mathcal{L}_\lambda^*(\mathbb{B}^n)$.*

Next, we present an example (considered also in [5], [10, Example 3.5] or [25, Examples 3 and 7]) of a mapping f that has been used in Example 5.6 and which satisfies also the conditions (a_1) and (a_2) from Definition 7.1. This means that $f \in \mathcal{L}_\lambda^*(\mathbb{B}^2)$ and then we can construct a starlike mapping h which is a convex combination between the identity mapping and the mapping f.

Example 7.3. *Let $n = 2$ and $f : \mathbb{B}^2 \to \mathbb{C}^2$ be given by*

$$f(z) = \big(z_1 + az_2^2, z_2\big), \quad z = (z_1, z_2) \in \mathbb{B}^2, \tag{14}$$

where $|a| \leq 1/2$. Then $f \in \mathcal{L}_\lambda^*(\mathbb{B}^2)$. In particular, $h \in S^*(\mathbb{B}^2)$, where $h(z) = (1-\lambda)z + \lambda f(z)$, for all $z \in \mathbb{B}^2$ and $\lambda \in (0,1)$.

Proof. Let $0 < \lambda < 1$ and $\mu = \lambda/(1-\lambda)$. Then f is normalized locally biholomorphic on $\mathbb{B}^2$ and

$$Df(z) = \begin{pmatrix} 1 & 2az_2 \\ 0 & 1 \end{pmatrix}.$$

If we denote

$$A = Df(z) - I_2 = \begin{pmatrix} 1 & 2az_2 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2az_2 \\ 0 & 0 \end{pmatrix},$$

then

$$\big\|Df(z) - I_2\big\| = \big\|A\big\| = \max\Big\{\|A(w)\| : \|w\| = 1\Big\}.$$

Let us consider $w \in \mathbb{C}^n$ such that $\|w\| = 1$. It follows that

$$\|A(w)\| = |2az_2w_2| = 2 \cdot |a| \cdot |z_2| \cdot |w_2|$$

and $\|A\| < \frac{1}{\lambda}$ if and only if $|a| \leq \frac{1}{2\lambda}$. Since $\lambda \in (0,1)$ and $|a| \leq 1/2$, we obtain that condition (a_1) from Definition 7.1 is satisfied. On the other hand,

$$B = I_2 + \mu Df(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \mu & 2a\mu z_2 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} 1+\mu & 2a\mu z_2 \\ 0 & 1+\mu \end{pmatrix}$$

and

$$B^{-1} = \frac{1}{(1+\mu)^2} \begin{pmatrix} 1+\mu & -2a\mu z_2 \\ 0 & 1+\mu \end{pmatrix} = \begin{pmatrix} \frac{1}{1+\mu} & -\frac{2a\mu z_2}{(1+\mu)^2} \\ 0 & \frac{1}{1+\mu} \end{pmatrix}. \tag{15}$$

Let us denote

$$C(z) = z + \mu f(z) = (z_1, z_2) + \big(\mu z_1 + 2a\mu z_2^2, \mu z_2\big) = \big((1+\mu)z_1 + 2a\mu z_2^2, (1+\mu)z_2\big), \quad z = (z_1, z_2) \in \mathbb{B}^2.$$

Then

$$\begin{aligned} \operatorname{Re}\big\langle B^{-1}C(z), z\big\rangle &= \operatorname{Re}\Big\langle\Big(\frac{(1+\mu)z_1 + 2a\mu z_2^2}{1+\mu} - \frac{2a\mu z_2^2(1+\mu)}{(1+\mu)^2}, \frac{(1+\mu)z_2}{1+\mu}\Big), (z_1, z_2)\Big\rangle \\ &= \operatorname{Re}\Big\langle\Big(z_1 + \frac{2a\mu z_2^2}{1+\mu} - \frac{2a\mu z_2^2}{1+\mu}, z_2\Big), (z_1, z_2)\Big\rangle = \operatorname{Re}\langle(z_1, z_2), (z_1, z_2)\rangle = \|z\|^2 > 0, \end{aligned}$$

for all $z \in \mathbb{B}^2 \setminus \{0\}$. Hence, condition (a_2) from Definition 7.1 is also satisfied and then we conclude that $f \in \mathcal{L}_\lambda^*(\mathbb{B}^2)$. In particular, in view of Theorem 6.1 we obtain that $h \in S^*(\mathbb{B}^2)$, where h is given by relation (13). □

Remark 7.4. *In the second part of this section, let us to refer to the case $n = 1$. Consider $\lambda \in (0, 1)$, $\mu = \lambda/(1 - \lambda)$ and $f : U \to \mathbb{C}$ a normalized locally univalent function on the unit disc U in $\mathbb{C}$. Then*

1. *Condition (a_1) can be written in one of the following form*

$$|f'(\zeta) - 1| < \frac{1}{\lambda} \Leftrightarrow f'(\zeta) \in U_\lambda(1; 1/\lambda), \quad \zeta \in U, \tag{16}$$

where $U_\lambda(1; 1/\lambda)$ is the disc with center $w_\lambda = 1$ and radius $r_\lambda = 1/\lambda$. The smallest disc U_λ can be constructed for $\lambda \to 1$ (in this case, we obtain the disc U_1 of center $w_1 = 1$ and radius $r_1 = 1$). On the other hand, for $\lambda \to 0$, it is clear that $\mathrm{Re}[f'(\zeta)] > 0$ for $\zeta \in U$ implies condition (16), but the converse result is not necessarily true.

2. *Condition (a_2) can be written in one of the following form*

$$\mathrm{Re}\Big\langle \big(1 + \mu f'(\zeta)\big)^{-1}\big(\zeta + \mu f(\zeta)\big), \zeta \Big\rangle > 0$$

or

$$\mathrm{Re}\left[\frac{\overline{\zeta}\big(\zeta + \mu f(\zeta)\big)}{1 + \mu f'(\zeta)}\right] > 0 \quad \Leftrightarrow \quad |\zeta|^2 \mathrm{Re}\left[\frac{\zeta + \mu f(\zeta)}{\zeta + \mu \zeta f'(\zeta)}\right] > 0, \tag{17}$$

for all $\zeta \in U$ with $\zeta \neq 0$. Clearly, if $f \in S^*$ and $\mathrm{Re}[f'(\zeta)] > 0$ for $\zeta \in U$, then the above condition is satisfied (for details, one may consult [1] or [3]). But again, the converse result, is not necessarily true.

Hence, we conclude that if $\lambda \in (0, 1)$ is sufficiently small and $f \in S^*$ is a function with the property that $\mathrm{Re}[f'(\zeta)] > 0$ for $\zeta \in U$, then (a_1) and (a_2) take place, but the converse implication is not necessarily true.

3. *In view of previous remarks we can define the class*

$$\mathcal{L}^*_\lambda(\mathbb{B}^1) = \mathcal{L}^*_\lambda(U) = \left\{ f \in H(U) : f(0) = 0, f'(0) = 1, |f'(\zeta) - 1| < \frac{1}{\lambda} \text{ and } \mathrm{Re}\left[\frac{\zeta + \mu f(\zeta)}{\zeta + \mu \zeta f'(\zeta)}\right] > 0, \zeta \in U \setminus \{0\} \right\}$$

for the case of one complex variable, where $\lambda \in (0, 1)$ and $\mu = \lambda/(1 - \lambda)$.

Remaining in the case $n = 1$ we obtain the following result:

Proposition 7.5. *Let $\lambda \in (0, 1)$ and $f \in \mathcal{L}^*_\lambda(U)$. Consider the function $h : U \to \mathbb{C}$ be given by*

$$h(\zeta) = (1 - \lambda)\zeta + \lambda f(\zeta), \quad \zeta \in U.$$

Then $h \in S^*$.

Proof. Since $f \in \mathcal{L}^*_\lambda(U)$ we deduce that $h \in H(U)$, $h(0) = 0$ and $h'(0) = 1$ (in fact, $h \in S$). Moreover,

$$\mathrm{Re}\left[\frac{\zeta h'(\zeta)}{h(\zeta)}\right] = \mathrm{Re}\left[\frac{\zeta + \mu \zeta f'(\zeta)}{\zeta + \mu f(\zeta)}\right] > 0,$$

for all $\zeta \in U \setminus \{0\}$, where $\mu = \lambda/(1 - \lambda)$. Hence, in view of the analytical characterization of starlikeness in $\mathbb{C}$ (see [9, Theorem 2.2.2]) we obtain that $h \in S^*$. □

Question 7.6. *What is the connection between the starlikeness of the mapping f and conditions (a_1) and (a_2) for $n \geq 2$? For sure, there will be no equivalence between the conditions, but the question would be whether the implication from the case $n = 1$ is true.*

8. Remarks on Loewner chains

Another interesting approach to the class $\mathcal{L}^*_\lambda$ is that in terms of Loewner chains. We can prove that starting from a function $f \in \mathcal{L}^*_\lambda(\mathbb{B}^n)$ we can easily construct an associated Loewner chain according to Theorem 6.1 and the characterization of starlikeness with Loewner chains given by Theorem 3.2. In particular, we can obtain a Loewner chain that is the convex combination of another two Loewner chains.

Proposition 8.1. *Let $\lambda \in (0,1)$. If $f \in \mathcal{L}^*_\lambda(\mathbb{B}^n)$, then*

$$H(z,t) = (1-\lambda)e^t z + \lambda e^t f(z) \tag{18}$$

is a Loewner chain, for all $z \in \mathbb{B}^n$ and $t \in [0,\infty)$.

Proof. Let $f \in \mathcal{L}^*_\lambda(\mathbb{B}^n)$ and $h : \mathbb{B}^n \to \mathbb{C}^n$ be given by

$$h(z) = (1-\lambda)z + \lambda f(z), \quad z \in \mathbb{B}^n, \quad \lambda \in (0,1).$$

Then $h(0) = 0$, $Dh(0) = I_n$ and h is locally biholomorphic on $\mathbb{B}^n$. Moreover,

$$H(z,t) = (1-\lambda)e^t z + \lambda e^t f(z) = e^t h(z), \quad z \in \mathbb{B}^n, \quad t \in [0,\infty).$$

According to Theorem 6.1 we know that $h \in S^*(\mathbb{B}^n)$. Since h is normalized locally biholomorphic on $\mathbb{B}^n$, it follows in view of Theorem 3.2 that $H(z,t)$ is a Loewner chain, for all $z \in \mathbb{B}^n$ and $t \in [0,\infty)$. □

Remark 8.2. *In view of Proposition 7.5 and Theorem 3.2 we obtain the previous result also in the case of one complex variable.*

In the following remark we replace the mapping $f \in \mathcal{L}^*_\lambda(\mathbb{B}^n)$ with a starlike mapping on the Euclidean unit ball. However in order to obtain a Loewner chain (which is also a convex combination of the identity mapping and a starlike mapping on $\mathbb{B}^n$) we still need the assumption (a_2) from Definition 7.1. According to this remark we deduce that in our context, for $n \geq 2$ this condition is very important.

Remark 8.3. *Let $\lambda \in (0,1)$ and $f \in S^*(\mathbb{B}^n)$ be such that $\|Df(z) - I_n\| < \frac{1}{\lambda}$, for all $z \in \mathbb{B}^n$. Also consider the mapping $H = H(z,t) : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ be given by*

$$H(z,t) = (1-\lambda)e^t z + \lambda e^t f(z), \quad z \in \mathbb{B}^n, \quad t \in [0,\infty). \tag{19}$$

In this case, $H = H(z,t)$ is the convex combination of two Loewner chains

$$L_1(z,t) = e^t z$$

and

$$L_2(z,t) = e^t f(z), \quad z \in \mathbb{B}^n, \quad t \in [0,\infty).$$

Moreover, $H(\cdot,t)$ is holomorphic on $\mathbb{B}^n$, $H(0,t) = 0$ and $DH(0,t) = e^t I_n$, for all $t \in [0,\infty)$. On the other hand, $H(z,\cdot)$ is locally Lipschitz continuous on $[0,\infty)$ locally uniformly with respect to $z \in \mathbb{B}^n$.

According to relation (19) we have that

$$e^{-t}H(z,t) = e^{-t}\Big[(1-\lambda)e^t z + \lambda e^t f(z)\Big] = (1-\lambda)z + \lambda f(z),$$

for all $z \in \mathbb{B}^n$ and $t \in [0,\infty)$. Let us denote $h : \mathbb{B}^n \to \mathbb{C}^n$ given by

$$h(z) = (1-\lambda)z + \lambda f(z) = e^{-t}H(z,t),$$

for all $z \in \mathbb{B}^n$, $t \in [0,\infty)$ and $\lambda \in (0,1)$. Since $f \in S^*(\mathbb{B}^n)$, we deduce that $\left\{e^{-t}H(\cdot,t)\right\}_{t\geq 0}$ is a normal family on $\mathbb{B}^n$ (see [4, Chapter 1] and [16]).

In order to prove that $H = H(z,t)$ is a Loewner chain (according to Theorem 1.3) we have to construct a Herglotz vector field $P = P(z,t) : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ such that

$$\frac{\partial H}{\partial t}(z,t) = DH(z,t)P(z,t), \quad a.e. \quad t \in [0,\infty), \quad z \in \mathbb{B}^n. \tag{20}$$

Using the assumption $\|Df(z) - I_n\| < \frac{1}{\lambda}$, for all $z \in \mathbb{B}^n$, we deduce that

$$\|Dh(z) - I_n\| = \|(1-\lambda)I_n + \lambda Df(z) - I_n\| = \|\lambda Df(z) - \lambda I_n\| = |\lambda| \cdot \|Df(z) - I_n\| < \frac{\lambda}{\lambda} = 1,$$

for all $z \in \mathbb{B}^n$. Then the operator $I_n + \left[Dh(z) - I_n\right] = Dh(z)$ is invertible on $\mathbb{B}^n$ and we can consider the inverse operator $[Dh(z)]^{-1}$ on the Euclidean unit ball $\mathbb{B}^n$. In view of this remark and relation (20) we obtain

$$P(z,t) = \left[DH(z,t)\right]^{-1}\frac{\partial H}{\partial t}(z,t) = e^{-t}\left[(1-\lambda)I_n + \lambda Df(z)\right]^{-1}H(z,t)$$

or equivalently,

$$P(z,t) = \left[(1-\lambda)I_n + \lambda Df(z)\right]^{-1}\left[(1-\lambda)z + \lambda f(z)\right]$$

and hence

$$P(z,t) = \left[Dh(z)\right]^{-1}h(z), \tag{21}$$

for all $z \in \mathbb{B}^n$ and $t \in [0,\infty)$. Clearly, $P(z,\cdot)$ is measurable on $[0,\infty)$, for all $z \in \mathbb{B}^n$ because is constant with respect to t and then it remains to prove that $P(\cdot,t) \in \mathcal{M}(\mathbb{B}^n)$.

For simplicity let us consider $n = 2$. Since $Dh(z) = (1-\lambda)I_2 + \lambda Df(z)$ is invertible, it follows that $P(\cdot,t)$ is holomorphic on $\mathbb{B}^2$, $P(0,t) = 0$ and $DP(0,t) = I_2$, for all $t \in [0,\infty)$. Indeed,

$$P(0,t) = \left[(1-\lambda)I_2 + \lambda Df(0)\right]^{-1}\left[\lambda f(0)\right] = I_2(0) = 0,$$

for all $t \in [0,\infty)$. On the other hand, if we denote $h(z) = \big(h_1(z), h_2(z)\big)$, then

$$Dh(z) = \begin{pmatrix} \frac{\partial h_1}{\partial z_1} & \frac{\partial h_1}{\partial z_2} \\ \frac{\partial h_2}{\partial z_1} & \frac{\partial h_2}{\partial z_2} \end{pmatrix} \qquad \textit{and} \qquad \left[Dh(z)\right]^{-1} = \frac{1}{J_h(z)}\begin{pmatrix} \frac{\partial h_2}{\partial z_2} & -\frac{\partial h_1}{\partial z_2} \\ -\frac{\partial h_2}{\partial z_1} & \frac{\partial h_1}{\partial z_1} \end{pmatrix},$$

where $J_h(z) = \det\big(Dh(z)\big)$, for all $z \in \mathbb{B}^2$. Taking into account the previous relations we obtain the mapping

$$P(z,t) = \frac{1}{J_h(z)}\Big(h_1(z)\frac{\partial h_2}{\partial z_2}(z) - h_2(z)\frac{\partial h_1}{\partial z_2}(z), h_2(z)\frac{\partial h_1}{\partial z_1}(z) - h_1(z)\frac{\partial h_2}{\partial z_1}(z)\Big),$$

for all $z \in \mathbb{B}^2$ and $t \in [0,\infty)$. Now it is clear that $P(0,t) = 0$, for all $t \in [0,\infty)$. Moreover, after some computations we deduce that

$$DP(z,t) = \frac{1}{J_h(z)}\begin{pmatrix} p_{11}(z) & p_{12}(z) \\ p_{21}(z) & p_{22}(z) \end{pmatrix},$$

where

$$p_{11}(z) = \frac{\partial h_1}{\partial z_1}(z)\frac{\partial h_2}{\partial z_2}(z) + h_1(z)\frac{\partial^2 h_2}{\partial z_2 \partial z_1}(z) - \frac{\partial h_2}{\partial z_1}(z)\frac{\partial h_1}{\partial z_2}(z) - h_2(z)\frac{\partial^2 h_1}{\partial z_2 \partial z_1}(z)$$

$$p_{12}(z) = \frac{\partial h_1}{\partial z_2}(z)\frac{\partial h_2}{\partial z_2}(z) + h_1(z)\frac{\partial^2 h_2}{\partial z_2^2}(z) - \frac{\partial h_2}{\partial z_2}(z)\frac{\partial h_1}{\partial z_2}(z) - h_2(z)\frac{\partial^2 h_1}{\partial z_2^2}(z)$$

$$p_{21}(z) = \frac{\partial h_2}{\partial z_1}(z)\frac{\partial h_1}{\partial z_1}(z) + h_2(z)\frac{\partial^2 h_1}{\partial z_1^2}(z) - \frac{\partial h_1}{\partial z_1}(z)\frac{\partial h_2}{\partial z_1}(z) - h_1(z)\frac{\partial^2 h_2}{\partial z_1^2}(z)$$

$$p_{22}(z) = \frac{\partial h_2}{\partial z_2}(z)\frac{\partial h_1}{\partial z_1}(z) + h_2(z)\frac{\partial^2 h_1}{\partial z_1 \partial z_2}(z) - \frac{\partial h_1}{\partial z_2}(z)\frac{\partial h_2}{\partial z_1}(z) - h_1(z)\frac{\partial^2 h_2}{\partial z_1 \partial z_2}(z)$$

and then

$$DP(0,t) = \frac{1}{J_h(0)}\begin{pmatrix} \frac{\partial h_1}{\partial z_1}(0)\frac{\partial h_2}{\partial z_2}(0) - \frac{\partial h_2}{\partial z_1}(0)\frac{\partial h_1}{\partial z_2}(0) & 0 \\ 0 & \frac{\partial h_1}{\partial z_1}(0)\frac{\partial h_2}{\partial z_2}(0) - \frac{\partial h_2}{\partial z_1}(0)\frac{\partial h_1}{\partial z_2}(0) \end{pmatrix}$$

$$= \frac{\frac{\partial h_1}{\partial z_1}(0)\frac{\partial h_2}{\partial z_2}(0) - \frac{\partial h_2}{\partial z_1}(0)\frac{\partial h_1}{\partial z_2}(0)}{J_h(0)} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{J_h(0)}{J_h(0)} \cdot I_2 = I_2,$$

for all $t \in [0,\infty)$ since f is normalized and $h_1(0) = h_2(0) = 0$. In order to complete the proof it remains to show that

$$\operatorname{Re}\Big\langle P(z,t), z\Big\rangle > 0, \quad z \in \mathbb{B}^n \setminus \{0\}, \quad t \in [0,\infty).$$

Indeed, we have

$$\operatorname{Re}\Big\langle P(z,t), z\Big\rangle = \operatorname{Re}\Big\langle \Big[Dh(z)\Big]^{-1} h(z), z\Big\rangle = \operatorname{Re}\Big\langle \Big[(1-\lambda)I_n + \lambda Df(z)\Big]^{-1}\Big[(1-\lambda)z + \lambda f(z)\Big], z\Big\rangle > 0, \tag{22}$$

for all $z \in \mathbb{B}^n \setminus \{0\}$ and $t \in [0,\infty)$. But relation (22) is the same as condition (a_2) from Definition 7.1. Concluding the above arguments, we obtain the following result:

Proposition 8.4. *Let $\lambda \in (0,1)$ and $f \in S^*(\mathbb{B}^n)$ be such that*

$$\|Df(z) - I_n\| < \frac{1}{\lambda} \tag{23}$$

and

$$\operatorname{Re}\Big\langle \Big[(1-\lambda)I_n + \lambda Df(z)\Big]^{-1}\Big[(1-\lambda)z + \lambda f(z)\Big], z\Big\rangle > 0, \tag{24}$$

for all $z \in \mathbb{B}^n \setminus \{0\}$. Then $H = H(z,t)$ given by (19) is a Loewner chain. In particular, the Loewner chain $H = H(z,t)$ is the convex combination of two Loewner chains.

Remark 8.5. *Let $n = 1$, $\lambda \in (0,1)$ and $f \in S^*$ be such that $\big|f'(\zeta) - 1\big| < 1$, for all $\zeta \in U$. In view of relation (17) we obtain that condition (24) is*

$$\operatorname{Re}\left[\frac{\zeta + \mu f(\zeta)}{\zeta + \mu\zeta f'(\zeta)}\right] > 0, \quad \zeta \in U \setminus \{0\}, \quad \mu = \lambda/(1-\lambda). \tag{25}$$

If we denote $h(\zeta) = (1-\lambda)\zeta + \lambda f(\zeta)$, for all $\zeta \in U$, then h is holomorphic and normalized on U. In fact, in view of the assumption $|f'(\zeta) - 1| < 1$, for all $\zeta \in U$, we obtain that $h \in S$. Next, let us define $p : U \times [0,\infty) \to \mathbb{C}$ given by

$$p(\zeta,t) = \frac{h(\zeta)}{\zeta h'(\zeta)}, \quad \zeta \in U, \quad t \in [0,\infty).$$

According to previous remarks, we deduce that $p(\cdot,t) \in \mathcal{P}$, for all $t \in [0,\infty)$ and $p(\zeta,\cdot)$ is measurable on the interval $[0,\infty)$, for all $\zeta \in U$, where

$$\mathcal{P} = \Big\{p \in H(U) : p(0) = 1 \text{ and } \operatorname{Re}[p(\zeta)] > 0, \quad \zeta \in U\Big\}$$

is the Carathéodory class in the case of one complex variable (for details, one may consult [4, Chapter 2], [9, p. 27] or [22, Chapter 2]).

If we consider $H = H(\zeta, t) : U \times [0, \infty) \to \mathbb{C}$ *given by* $H(\zeta, t) = e^t h(\zeta)$*, then* $H(\cdot, t) \in H(U)$, $H(0, t) = 0$ *and* $H'(0, t) = e^t$*, for all* $t \in [0, \infty)$*. Moreover,*

$$\frac{\partial H}{\partial t}(\zeta, t) = e^t h(\zeta) = \zeta e^t h'(\zeta) p(\zeta, t) = zH'(\zeta, t) p(\zeta, t) \quad a.e. \quad t \in [0, \infty), \quad \zeta \in U.$$

Hence, taking into account the $n = 1$ *version of Theorem 1.3 (see [9, Theorem 3.1.13]), we conclude that* $H = H(\zeta, t)$ *is a Loewner chain in* $\mathbb{C}$.

9. Conjecture related to Chichra-Singh's result in several complex variables

In this last section we propose a conjecture (for the case of several complex variables) which generalize Theorem 2.5 proved by Chichra and Singh in [3] (in the case of one complex variable).

Conjecture 9.1. *Let* $\lambda \in [0, 1]$*. If* $f \in S^*(\mathbb{B}^n)$ *and* $\mathrm{Re}\left\langle Df(z)(u), u \right\rangle > 0$*, for all* $z \in \mathbb{B}^n$ *and* $u \in \mathbb{C}^n$ *with* $\|u\| = 1$*, then*

$$h(z) = (1 - \lambda)z + \lambda f(z)$$

is a starlike mapping on $\mathbb{B}^n$*. Moreover,* $\mathrm{Re}\left\langle Dh(z)(u), u \right\rangle > 0$*, for all* $z \in \mathbb{B}^n$ *and* $u \in \mathbb{C}^n$ *with* $\|u\| = 1$*. In particular, h is univalent on* $\mathbb{B}^n$*.*

Remark 9.2. *In the case of one complex variable, the statement of Conjecture 9.1 is true, as it reduces to Theorem 2.5 obtained by Chichra and Singh in [3].*

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