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# The Weakly Rothberger Property of Pixley–Roy Hyperspaces

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**Abstract.** Let PR($X$) denote the hyperspace of nonempty finite subsets of a topological space $X$ with Pixley–Roy topology. In this paper, by introducing closed-miss-finite networks and using principle ultrafilters, we proved that the following statements are equivalent for a space $X$:
(1) PR($X$) is weakly Rothberger;
(2) $X$ satisfies $\mathsf{S}_1(\Pi_{rcf}, \Pi_{wrcf})$;
(3) $X$ is separable and $X - \{x\}$ satisfies $\mathsf{S}_1(\Pi_{cf}, \Pi_{wcf})$ for each $x \in X$;
(4) $X$ is separable and each principal ultrafilter $\mathcal{F}[x]$ in PR($X$) is weakly Rothberger in PR($X$).
We also characterize the weakly Menger property and the weakly Hurewicz property of PR($X$).

## 1. Introduction

Throughout the paper all spaces are assumed to be infinite and $T_1$. $\mathbb{N}$ denotes the set of natural numbers. $\omega$ is the first infinite ordinal.

For a space $X$, let PR($X$) be the family of all nonempty finite subsets of $X$. For $A \in$ PR($X$) and an open set $U \subset X$, let

$$[A, U] = \{B \in \mathrm{PR}(X) : A \subset B \subset U\}.$$

The family $\{[A, U] : A \in \mathrm{PR}(X), U \text{ is open in } X\}$ is a base of PR($X$) for the *Pixley–Roy topology* [9] on PR($X$).

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of sets of an infinite set $X$.

$\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ denotes the selection principle: for each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$ there is a sequence $\{b_n : n \in \mathbb{N}\}$ such that $b_n \in A_n$ for each $n \in \mathbb{N}$ and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$\mathsf{S}_{\mathsf{fin}}(\mathcal{A}, \mathcal{B})$ denotes the selection principle: for each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$ there is a sequence $\{B_n : n \in \mathbb{N}\}$ such that $B_n$ is a finite subset of $A_n$ for each $n \in \mathbb{N}$ and $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{B}$.

We recall that an open cover $\mathcal{U}$ of a space $X$ is called an *ω-cover* of $X$ if every finite subset of $X$ is contained in a member of $\mathcal{U}$ and $X$ is not a member of $\mathcal{U}$. A family $\xi$ of subsets of a space $X$ is called a *π-network of* $X$ [3] if for each open set $U$ of $X$, there exists $M \in \xi$ such that $M \subset U$.

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For a space $X$, we write

- $\Omega$ : the collection of $\omega$-covers of $X$;
- $\mathcal{D}$ : the collection of dense subsets of $X$;
- $\Pi_\omega$ : the collection of $\pi$-networks of $X$ consisting of finite subsets of $X$;
- $\Pi_k$ : the collection of $\pi$-networks of $X$ consisting of compact subsets of $X$.

In the theory of selection principles, $\pi$-networks and $\omega$-covers play important roles. Some well-known weakly-versions of selection principles of Pixley–Roy space PR($X$) were established in terms of $\omega$-covers or $\pi$-networks of $X$. P. Daniels [2] introduced the weakly Rothberger property and the weakly Menger property and proved that for a metrizable space $X$, PR($X$) is weakly Rothberger (resp., weakly Menger) if and only if $X$ satisfies $\mathsf{S}_1(\Omega,\Omega)$ (resp., $\mathsf{S}_{\mathsf{fin}}(\Omega,\Omega)$). M. Sakai [11] and M. Bonanzinga, F. Cammaroto, B.A. Pansera, B. Tsaban [1] gave that for a countable space $X$, PR($X$) is weakly Rothberger (resp., weakly Menger) if and only if $X$ satisfies $\mathsf{S}_1(\Omega,\Omega)$ (resp., $\mathsf{S}_{\mathsf{fin}}(\Omega,\Omega)$) if and only if every finite power of $X$ satisfies $\mathsf{S}_1(\Pi_\omega,\Pi_\omega)$ (resp., $\mathsf{S}_{\mathsf{fin}}(\Pi_\omega,\Pi_\omega)$). M. Scheepers [12] obtained that for a subset $X$ of the real line, PR($X$) is weakly Rothberger (resp., weakly Menger) if and only if PR($X$) satisfies $\mathsf{S}_1(\mathcal{D},\mathcal{D})$ (resp., $\mathsf{S}_{\mathsf{fin}}(\mathcal{D},\mathcal{D})$) if and only if $X$ satisfies $\mathsf{S}_1(\Omega,\Omega)$ (resp., $\mathsf{S}_{\mathsf{fin}}(\Omega,\Omega)$).

On the other hand, we find that $\omega$-covers or $\pi$-networks doesn't completely characterize the dual properties of these weak selection principles between a general space $X$ and its hyperspace PR($X$) ([7], Examples 2.9, 2.14 and Remark 2.15). We should introduce new networks or covers different from $\pi$-networks or $\omega$-covers of $X$ to be dual to selection principles in the hyperspace PR($X$).

For a general space $X$, the characterizations of the weakly Rothberger property, the weakly Menger property and the weakly Hurewicz property of PR($X$) are unknown. So the following natural questions arise.

**Question 1.1.** *For a space $X$, find the collections $\mathcal{A}$ and $\mathcal{B}$ of subsets of $X$ such that:*

*PR($X$) is weakly Rothberger if and only if $X$ satisfies $\mathsf{S}_1(\mathcal{A},\mathcal{B})$;*

*PR($X$) is weakly Menger if and only if $X$ satisfies $\mathsf{S}_{\mathsf{fin}}(\mathcal{A},\mathcal{B})$;*

*PR($X$) is weakly Hurewicz if and only if $X$ satisfies $\mathsf{S}_{\mathsf{fin}}(\mathcal{A},\mathcal{B})$.*

G.Di Maio, Lj.D.R. Kočinac and E. Meccariello [3] investigated the Rothberger property in $2^X$ under co-compact topology $F^+$ and co-finite topology $Z^+$. They proved that for a space $X$, $(2^X, F^+)$ (resp., $(2^X, Z^+)$) has the Rothberger property if and only if $X$ satisfies $\mathsf{S}_1(\Pi_k,\Pi_k)$ (resp., $\mathsf{S}_1(\Pi_\omega,\Pi_\omega)$).

In this paper, motivated by co-subset ideas of [3], we introduced a new kind of hit-and-miss networks: $rcf$-network, weakly $rcf$-network, $cf$-network and weakly $cf$-network. We obtain weakly selection principles of PR($X$). We prove that PR($X$) is weakly Rothberger (resp., weakly Menger and weakly Hurewicz) if and only if $X$ satisfies $\mathsf{S}_1(\Pi_{rcf},\Pi_{wrcf})$ (resp., $\mathsf{S}_{\mathsf{fin}}(\Pi_{rcf},\Pi_{wrcf})$ and $\mathsf{S}_{\mathsf{fin}}(\Pi_{rcf},\Pi^p_{wrcf})$) if and only if $X$ is separable and $X-\{x\}$ satisfies $\mathsf{S}_1(\Pi_{cf},\Pi_{wcf})$ (resp., $\mathsf{S}_{\mathsf{fin}}(\Pi_{cf},\Pi_{wcf})$ and $\mathsf{S}_{\mathsf{fin}}(\Pi_{cf},\Pi^p_{wcf})$) for each $x \in X$ if and only if $X$ is separable and each principal ultrafilter $\mathcal{F}[x]$ in PR($X$) is weakly Rothberger in PR($X$) (resp., weakly Menger and weakly Hurewicz).

## 2. Main results

**Definition 2.1.** ([1, 4, 8, 11]) A space $X$ is said to be weakly Rothberger if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of $X$, there exists $U_n \in \mathcal{U}_n$ such that $\bigcup_{n\in\mathbb{N}} U_n$ is dense in $X$.

**Definition 2.2.** ([1, 2, 4, 8]) A space $X$ is said to be weakly Menger if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of $X$, there exists a finite subset $\mathcal{V}_n \subset \mathcal{U}_n$ such that $\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n$ is dense in $X$.

**Definition 2.3.** ([4, 10]) A space $X$ is said to be weakly Hurewicz if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of $X$, there exists a finite subset $\mathcal{V}_n \subset \mathcal{U}_n$ such that for every nonempty open $U \subset X$, $U \cap (\bigcup \mathcal{V}_n) \neq \emptyset$ for all but finitely many $n \in \mathbb{N}$.

**Lemma 2.4.** *If PR(X) is weakly Menger, then X is separable.*

*Proof.* Let $\mathcal{U}_n = \{[\{x\}, X] : x \in X\}$. Then $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a sequence of open covers of PR($X$). There exists a finite subset $\mathcal{V}_n \subset \mathcal{U}_n$ for each $n \in \mathbb{N}$ such that $\mathrm{PR}(X) = \overline{\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n}$. Denote

$$\mathcal{V}_n = \{[\{x_{n,m}\}, X] : 1 \le m \le k_n\}.$$

We prove that $\{x_{n,m} : n \in \mathbb{N}, 1 \le m \le k_n\}$ is a countable dense subset of $X$. In fact, for each open subset $V$ of $X$, pick $y \in V$, then $[\{y\}, V]$ is an open subset of PR($X$). Thus $[\{y\}, V] \cap (\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n) \ne \emptyset$. There exists

$$x_{n_0,m_0} \in \{x_{n,m} : n \in \mathbb{N}, 1 \le m \le k_n\}$$

such that

$$[\{y\}, V] \cap [\{x_{n_0,m_0}\}, X] \ne \emptyset.$$

Thus $x_{n_0,m_0} \in V$. So $X$ is separable. □

In order to give characterizations of PR($X$) being weakly Rothberger, we define *rcf*-networks and weakly *rcf*-networks of $X$.

A pair $(C, F)$ of subsets of $X$ is called a *closed-miss-finite pair of* $X$, if $C$ is closed and $F$ is nonempty finite with $C \cap F = \emptyset$. A *closed-miss-finite family of* $X$ is a family of closed-miss-finite pairs of $X$.

Recall that a subset $U$ of $X$ is called a *co-finite subset of* $X$ [7] if $0 < |X - U| < \omega$. A family $\mathcal{U}$ consisting of co-finite subsets of $X$ is said to be a co-finite family of $X$. Let $Y \subsetneq X$. A subset $U$ of $Y$ is called a *co-finite subset of* $Y$ [7] if $0 \le |Y - U| < \omega$. A family $\mathcal{U}$ consisting of co-finite subsets of $Y$ is called a co-finite family of $Y$.

**Definition 2.5.** A closed-miss-finite family $\xi$ of $X$ is called a *regular closed-miss-finite network* (briefly, *rcf-network*) of $X$, if for each co-finite subset $U$ of $X$, there exists $(C, F) \in \xi$ such that $C \subset U$ and $F \cap U = \emptyset$.

**Definition 2.6.** A closed-miss-finite family $\xi$ of $X$ is called a *weakly rcf-network of* $X$, if for each co-finite subset $U$ of $X$ and $C \subset U$ closed in $X$, there exists $(C', F') \in \xi$ such that $C' \subset U$ and $F' \cap C = \emptyset$.

An *rcf*-network of $X$ is a weakly *rcf*-network of $X$. We write

- $\Pi_{rcf}$ : the collection of *rcf*-networks of $X$;
- $\Pi_{wrcf}$ : the collection of weakly *rcf*-networks of $X$.

**Theorem 2.7.** *For a space X, the following are equivalent:*
*(1)* PR($X$) *is weakly Rothberger;*
*(2) X satisfies* $\mathsf{S}_1(\Pi_{rcf}, \Pi_{wrcf})$*;*

*Proof.* (1)⇒(2) Let $\{\xi_n : n \in \mathbb{N}\}$ be a sequence of *rcf*-networks of $X$. For each $n \in \mathbb{N}$, let

$$\mathcal{U}_n = \{[F, X - C] : (C, F) \in \xi_n\}.$$

Then each $\mathcal{U}_n$ is an open cover of PR($X$). Indeed, let $A \in \mathrm{PR}(X)$, then $A^c = X - A$ is a co-finite subset of $X$. There exists $(C, F) \in \xi_n$ such that

$$C \subset A^c \text{ and } F \cap A^c = \emptyset.$$

Then $A \in [F, X - C]$. By (1), pick $[F_n, X - C_n] \in \mathcal{U}_n$ such that

$$\overline{\bigcup_{n\in\mathbb{N}} [F_n, X - C_n]} = \mathrm{PR}(X).$$

Then $(C_n, F_n) \in \xi_n$ and $\{(C_n, F_n) : n \in \mathbb{N}\}$ is a weakly *rcf*-network of $X$. In fact, let $U$ be a co-finite subset of $X$ and $C \subset U$ closed in $X$, then $[U^c, X - C]$ is an open subset of PR($X$). There is some $k \in \mathbb{N}$ such that

$$[U^c, X - C] \cap [F_k, X - C_k] \ne \emptyset.$$

Thus $C_k \subset U$ and $F_k \cap C = \emptyset$. So $X$ satisfies $\mathsf{S}_1(\Pi_{rcf}, \Pi_{wrcf})$.

(2)⇒(1) Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of PR($X$). Without loss of generality, suppose that each $\mathcal{U}_n$ is a family of basic open sets. For each $n \in \mathbb{N}$, put

$$\xi_n = \{(X - V, A) : [A, V] \in \mathcal{U}_n\}.$$

Let $U$ be a co-finite subset of $X$, there is some $[A, V] \in \mathcal{U}_n$ such that $U^c \in [A, V]$. Thus $X - V \subset U$ and $A \cap U = \emptyset$. So $\{\xi_n : n \in \mathbb{N}\}$ is a sequence of $rcf$-networks of $X$. There exists $(X - V_n, A_n) \in \xi_n$ such that $\{(X - V_n, A_n) : n \in \mathbb{N}\}$ is a weakly $rcf$-network of $X$. It is clear that $[A_n, V_n] \in \mathcal{U}_n$ for $n \in \mathbb{N}$ and $\overline{\bigcup_{n\in\mathbb{N}}[A_n, V_n]} = \mathrm{PR}(X)$. □

A space $X$ is said to be *weakly Lindelöf* [3] if for every open cover $\mathcal{U}$ of $X$, there exists a set $\{U_n : n \in \mathbb{N}\} \subset \mathcal{U}$ such that $\bigcup_{n\in\mathbb{N}} U_n$ is dense in $X$.

Using *rcf*-networks and weakly *rcf*-networks, we can obtain a characterization of the weakly Lindelöf property of PR($X$).

**Theorem 2.8.** *For a space X, the following are equivalent:*

*(1)* PR($X$) *is weakly Lindelöf;*

*(2) For each rcf-network $\xi$ of X, there exists a set $\{(C_n, F_n) : n \in \mathbb{N}\} \subset \xi$ such that $\{(C_n, F_n) : n \in \mathbb{N}\}$ is a weakly rcf-network of X.*

Next, we define *cf*-networks and weakly *cf*-networks of a subset $Y$ of $X$.

Let $Y$ be a subset of $X$. A *closed-miss-finite family $\xi$ of $Y$* denotes the following family.

$$\xi = \{(C, F) : C,\ F \subset Y,\ C \text{ is closed in } X \text{ and } F \text{ is finite with } C \cap F = \emptyset\}.$$

**Definition 2.9.** Let $Y \subsetneq X$. A closed-miss-finite family $\xi$ of $Y$ is called a *closed-miss-finite network* (briefly, *$cf$-network*) of $Y$, if for each co-finite subset $U$ of $Y$, there exists $(C, F) \in \xi$ such that $C \subset U$ and $F \cap U = \emptyset$.

**Definition 2.10.** Let $Y \subsetneq X$. A closed-miss-finite family $\xi$ of $Y$ is called a *weakly $cf$-network of $Y$*, if for each co-finite subset $U$ of $Y$ and $C \subset U$ closed in $X$, there exists $(C', F') \in \xi$ such that $C' \subset U$ and $F' \cap C = \emptyset$.

A $cf$-network of $Y$ is a weakly $cf$-network of $Y$. We write

- $\Pi_{cf}$ : the collection of *cf*-networks of each $Y \subsetneq X$;
- $\Pi_{wcf}$ : the collection of weakly *cf*-networks of each $Y \subsetneq X$.

**Theorem 2.11.** *For a space X, the following are equivalent:*

*(1) X satisfies $\mathsf{S}_1(\Pi_{rcf}, \Pi_{wrcf})$;*

*(2) X is separable and $X - \{x\}$ satisfies $\mathsf{S}_1(\Pi_{cf}, \Pi_{wcf})$ for each $x \in X$;*

*Proof.* (1)⇒(2) By Lemma 2.4, $X$ is separable since the weakly Menger property is weaker than the weakly Rothberger property. Let $\{\xi_n : n \in \mathbb{N}\}$ be a sequence of $cf$-networks of $X - \{x\}$. For each $(C, A) \in \xi_n$, put

$$\zeta^{(n)}_{(C,A)} = \begin{cases} \{(C, A)\}, & \text{if } A \neq \emptyset; \\ \{(C, F) : F \in [X - C]^{<\omega} \backslash \{\emptyset\}\}, & \text{if } A = \emptyset. \end{cases}$$

For each $n \in \mathbb{N}$, let

$$\zeta_n = \bigcup_{(C,A)\in\xi_n} \zeta^{(n)}_{(C,A)}.$$

Then each $\zeta_n$ is an $rcf$-network of $X$. In fact, let $U$ be a co-finite subset of $X$, then $U \cap (X - \{x\})$ is a co-finite subset of $X - \{x\}$. There exists $(C, A) \in \xi_n$ such that

$$C \subset U \cap (X - \{x\}) \text{ and } A \cap (U \cap (X - \{x\})) = \emptyset.$$

**Case 1.** If $A \neq \emptyset$, then $(C, A) \in \zeta_n$. Since $A \subset X - \{x\}$, then

$$C \subset U \text{ and } A \cap U = (A \cap (X - \{x\})) \cap U = A \cap (U \cap (X - \{x\})) = \emptyset.$$

**Case 2.** If $A = \emptyset$, take

$$F = X - U \in [X - C]^{<\omega} \backslash \{\emptyset\}.$$

Then $(C, F) \in \zeta^{(n)}_{(C,A)} \subset \zeta_n$ such that $C \subset U$ and $F \cap U = \emptyset$. By (1), there exists $(C_n, F_n) \in \zeta_n$ for $n \in \mathbb{N}$ such that $\{(C_n, F_n) : n \in \mathbb{N}\}$ is a weakly $rcf$-network of $X$. One readily check that each $(C_n, A_n) \in \xi_n$. We show that $\{(C_n, A_n) : n \in \mathbb{N}\}$ is a weakly $cf$-network of $X - \{x\}$. Let $U$ be a co-finite subset of $X - \{x\}$ and $C \subset U$ closed in $X$, then $U$ is a co-finite subset of $X$. There exists some $(C_k, F_k) \in \{(C_n, F_n) : n \in \mathbb{N}\}$ such that

$$C_k \subset U \text{ and } F_k \cap C = \emptyset.$$

Then $C_k \subset U$ and $A_k \cap C \subset F_k \cap C = \emptyset$.

(2)⇒(1) Let $\{\xi_n : n \in \mathbb{N}\}$ be a sequence of $rcf$-networks of $X$ and rearrange $\{\xi_n : n \in \mathbb{N}\}$ as $\{\xi_{n,m} : n, m \in \mathbb{N}\}$. Suppose that $\{x_m : m \in \mathbb{N}\}$ is a countable dense subset of $X$. For each $m \in \mathbb{N}$, let

$$\zeta_{n,m} = \{(C, A \cap (X - \{x_m\})) : (C, A) \in \xi_{n,m} \text{ and } C \subset X - \{x_m\}\}.$$

Then $\{\zeta_{n,m} : n \in \mathbb{N}\}$ is a sequence of $cf$-networks of $X - \{x_m\}$. In fact, for each co-finite $U$ of $X - \{x_m\}$, then $U$ is a co-finite subset of $X$. There exists $(C, A) \in \xi_{n,m}$ such that

$$C \subset U \subset X - \{x_m\} \text{ and } A \cap U = \emptyset.$$

Thus $(C, A \cap (X - \{x_m\})) \in \zeta_{n,m}$ such that

$$C \subset U \text{ and } [A \cap (X - \{x_m\})] \cap U = \emptyset.$$

By (2), for each $n \in \mathbb{N}$, there exists

$$(C_{n,m}, A_{n,m} \cap (X - \{x_m\})) \in \zeta_{n,m}$$

such that $\{(C_{n,m}, A_{n,m} \cap (X - \{x_m\})) : n \in \mathbb{N}\}$ is a weakly $cf$-network of $X - \{x_m\}$. One readily check that $(C_{n,m}, A_{n,m}) \in \xi_{n,m}$ for each $m, n \in \mathbb{N}$. We show that $\{(C_{n,m}, A_{n,m}) : m, n \in \mathbb{N}\}$ is a weakly $rcf$-network of $X$. Indeed, let $U$ be a co-finite subset of $X$ and $C \subset U$ closed in $X$.

**Case 1.** $U - C \neq \emptyset$. Take $x_m \in (X - C) \cap U$, then

$$C \subset U - \{x_m\} \subset X - \{x_m\}.$$

Since $U - \{x_m\}$ is a co-finite subset of $X - \{x_m\}$, there exists

$$(C_{n,m}, A_{n,m} \cap (X - \{x_m\})) \in \zeta_{n,m}$$

such that

$$C_{n,m} \subset U - \{x_m\} \text{ and } [A_{n,m} \cap (X - \{x_m\})] \cap C = \emptyset.$$

Hence $C_{n,m} \subset U$ and $A_{n,m} \cap C = \emptyset$ since $x_m \notin C$.

**Case 2.** $C = U$. Take $x_m \in X - C$, then

$$C \subset U \subset X - \{x_m\}.$$

Since $U$ is a co-finite subset of $X - \{x_m\}$, there exists

$$(C_{n,m}, A_{n,m} \cap (X - \{x_m\}) \in \zeta_{n,m}$$

such that

$$C_{n,m} \subset U \text{ and } [A_{n,m} \cap (X - \{x_m\})] \cap C = \emptyset.$$

Hence $C_{n,m} \subset U$ and $A_{n,m} \cap C = \emptyset$. So $X$ satisfies $\mathsf{S}_1(\Pi_{rcf}, \Pi_{wrcf})$. □

Finally, for a subset $Y$ of $X$, we define the weakly Rothberger property, the weakly Menger property and the weakly Hurewicz property of $Y$.

**Definition 2.12.** A subset $Y$ of a space $X$ is said to be *weakly Rothberger* in $X$ if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of covers of $Y$ sets open in $X$, there exists $U_n \in \mathcal{U}_n$ such that $Y \subset \overline{\bigcup_{n\in\mathbb{N}} U_n}$.

**Definition 2.13.** A subset $Y$ of a space $X$ is said to be *weakly Menger* in $X$ if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of covers of $Y$ sets open in $X$, there exists a finite subset $\mathcal{V}_n \subset \mathcal{U}_n$ such that $Y \subset \overline{\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n}$.

**Definition 2.14.** A subset $Y$ of $X$ is said to be *weakly Hurewicz* in $X$ if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of covers of $Y$ sets open in $X$, there are finite subfamilies $\mathcal{V}_n \subset \mathcal{U}_n$ such that for every nonempty open $U$ of $X$ with $U \cap Y \neq \emptyset$, $U \cap (\bigcup \mathcal{V}_n) \neq \emptyset$ for all but finitely many $n \in \mathbb{N}$.

Let $\mathcal{R}$ be a family of sets. By a *filter in* $\mathcal{R}$ [5] we means a subfamily $\mathcal{F} \subset \mathcal{R}$ satisfying the following conditions:
(1) $\emptyset \notin \mathcal{F}$;
(2) If $A_1, A_2 \in \mathcal{F}$, then $A_1 \cap A_2 \in \mathcal{F}$;
(3) If $A \in \mathcal{F}$ and $A \subset A_1$, then $A_1 \in \mathcal{F}$.

A filter $\mathcal{F}$ is called an *ultrafilter in* $\mathcal{R}$, if every filter $\mathcal{F}'$ in $\mathcal{R}$ that contains $\mathcal{F}$ we have $\mathcal{F}' = \mathcal{F}$. Let $\mathcal{F}$ be an ultrafilter. If $\bigcap \mathcal{F} \neq \emptyset$, then $\mathcal{F}$ is called a *principal ultrafilter*.

For $x \in X$, Let

$$\mathcal{F}[x] = \{A \in \mathrm{PR}(X) : x \in A\}.$$

Then $\mathcal{F}[x]$ is a principal ultrafilter since $\bigcap \mathcal{F}[x] = \{x\}$.

**Theorem 2.15.** *For a space X, the following are equivalent:*
*(1) X is separable and* $X - \{x\}$ *satisfies* $\mathsf{S}_1(\Pi_{cf}, \Pi_{wcf})$ *for each* $x \in X$*;*
*(2) X is separable and each principal ultrafilter* $\mathcal{F}[x]$ *in* PR(X) *is weakly Rothberger in* PR(X)*.*

*Proof.* (1)⇒(2) Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of base open covers of $\mathcal{F}[x]$ in PR(X). For each $n \in \mathbb{N}$, put

$$\xi_n = \{(X - V, A \cap (X - \{x\})) : [A, V] \in \mathcal{U}_n\}.$$

Then $\{\xi_n : n \in \mathbb{N}\}$ is a sequence of $cf$-networks of $X - \{x\}$. In fact, let $U$ be a co-finite subset of $X - \{x\}$, then $U^c \in \mathcal{F}[x]$. There exists $[A, V] \in \mathcal{U}_n$ such that $U^c \in [A, V]$. Then

$$X - V \subset U \text{ and } (A \cap (X - \{x\})) \cap U = A \cap U = \emptyset.$$

By (1), there exists $(X - V_n, A_n \cap (X - \{x\})) \in \xi_n$ for each $n \in \mathbb{N}$ such that $\{(X - V_n, A_n \cap (X - \{x\})) : n \in \mathbb{N}\}$ is a weakly $cf$-network of $X - \{x\}$. Then each $[A_n, V_n] \in \mathcal{U}_n$. We show that $\mathcal{F}[x] \subset \overline{\bigcup_{n\in\mathbb{N}}[A_n, V_n]}$. Indeed, let $B \in \mathcal{F}[x]$ and $[A, V]$ a neighbourhood of $B$ in PR(X), then $X - V \subset X - B \subset X - \{x\}$. There exists $(X - V_k, A_k \cap (X - \{x\}))$ such that

$$X - V_k \subset X - B \text{ and } (A_k \cap (X - \{x\})) \cap (X - V) = A_k \cap (X - V) = \emptyset.$$

Thus

$$A \subset B \subset V_k \text{ and } A_k \subset V.$$

So $A \cup A_k \subset V \cap V_k$. It implies that $[A, V] \cap [A_k, V_k] \neq \emptyset$.

(2)⇒(1) Suppose that $\{\xi_n : n \in \mathbb{N}\}$ is a sequence of $cf$-networks of $X - \{x\}$. For each $n \in \mathbb{N}$, let

$$\mathcal{U}_n = \{[F \cup \{x\}, X - C] : (C, F) \in \xi_n\}.$$

Then $\mathcal{U}_n$ is an open cover of $\mathcal{F}[x]$ in PR($X$). In fact, let $B = B_1 \cup \{x\} \in \mathcal{F}[x]$, where $B_1 \in [X - \{x\}]^{<\omega}$, then $(X - \{x\}) - B_1$ is a co-finite subset of $X - \{x\}$. There exists $(C, F) \in \xi_n$ such that

$$C \subset (X - \{x\}) - B_1 \text{ and } F \cap ((X - \{x\}) - B_1) = \emptyset.$$

Since $(X - \{x\}) - B_1 = X - B$, we have

$$C \subset X - B$$

and

$$(F \cup \{x\}) \cap (X - B) = (F \cup \{x\}) \cap ((X - \{x\}) - B_1) = F \cap ((X - \{x\}) - B_1) = \emptyset.$$

So $B \in [F \cup \{x\}, X - C]$. By (2), there exists $[F_n \cup \{x\}, X - C_n] \in \mathcal{U}_n$ such that

$$\mathcal{F}[x] \subset \overline{\bigcup_{n\in\mathbb{N}} [F_n \cup \{x\}, X - C_n]}.$$

Then each $(C_n, F_n) \in \xi_n$ and $\{(C_n, F_n) : n \in \mathbb{N}\}$ is a weakly $cf$-network of $X - \{x\}$. Indeed, for a co-finite subset $U$ of $X - \{x\}$ and $C \subset U$ closed in $X$, $[U^c, X - C]$ is a neighbourhood of $U^c \in \mathcal{F}[x]$. There exists $[F_k \cup \{x\}, X - C_k]$ such that

$$[U^c, X - C] \cap [F_k \cup \{x\}, X - C_k] \neq \emptyset.$$

Then $U^c \subset X - C_k$ and $F_k \cup \{x\} \subset X - C$. So $C_k \subset U$ and $F_k \cap C = \emptyset$. □

**Corollary 2.16.** *For a space X, the following are equivalent:*
*(1)* PR($X$) *is weakly Rothberger;*
*(2) X satisfies* $S_1(\Pi_{rcf}, \Pi_{wrcf})$*;*
*(3) X is separable and* $X - \{x\}$ *satisfies* $S_1(\Pi_{cf}, \Pi_{wcf})$ *for each* $x \in X$*;*
*(4) X is separable and each principal ultrafilter* $\mathcal{F}[x]$ *in* PR(X) *is weakly Rothberger in* PR($X$).

Using the methods of Theorems 2.7, 2.11 and 2.15, we can obtain the following.

**Theorem 2.17.** *For a space X, the following are equivalent:*
*(1)* PR($X$) *is weakly Menger;*
*(2) X satisfies* $S_{fin}(\Pi_{rcf}, \Pi_{wrcf})$*;*
*(3) X is separable and* $X - \{x\}$ *satisfies* $S_{fin}(\Pi_{cf}, \Pi_{wcf})$ *for each* $x \in X$*;*
*(4) X is separable and each principal ultrafilter* $\mathcal{F}[x]$ *in* PR(X) *is weakly Menger in* PR(X).

Motivated by groupability ideas of [6], we introduced $p$-$rcf$-networks and weakly $p$-$cf$-networks to characterize the weakly Hurewicz property.

**Definition 2.18.** A partitioned closed-miss-finite family $\xi = \bigcup_{n\in\mathbb{N}} \xi_n$ of $X$ is said to be a *weakly p-rcf-network* of $X$, if for each co-finite subset $U$ of $X$ and closed set $C \subset U$, there exists $(C_n, F_n) \in \xi_n$ such that $C_n \subset U$ and $F_n \cap C = \emptyset$ for all but finitely many $n \in \mathbb{N}$.

Denote $\Pi^p_{wrcf}$ the collection of weakly $p$-$rcf$-networks of $X$.

**Theorem 2.19.** *For a space X, the following are equivalent:*
*(1)* PR($X$) *is weakly Hurewicz;*
*(2) X satisfies* $S_{fin}(\Pi_{rcf}, \Pi^p_{wrcf})$.

*Proof.* (1)⇒(2) Let $\{\xi_n : n \in \mathbb{N}\}$ be a sequence of $rcf$-networks of $X$. For each $n \in \mathbb{N}$,

$$\mathcal{U}_n = \{[F, X - C] : (C, F) \in \xi_n\}$$

is an open cover of PR($X$). By (1), pick a finite subset $\mathcal{V}_n \subset \mathcal{U}_n$ for each $n \in \mathbb{N}$ such that for every nonempty open $W \subset$ PR($X$), $W \cap \bigcup \mathcal{V}_n \neq \emptyset$ for all but finitely many $n \in \mathbb{N}$. Let

$$\zeta_n = \{(C, F) : [F, X - C] \in \mathcal{V}_n\}.$$

Then $\zeta_n \subset \xi_n$ is finite. We show that $\bigcup_{n\in\mathbb{N}} \zeta_n$ is a weakly $p$-$rcf$-network of $X$. Let $U$ be a co-finite subset of $X$ and $C \subset U$ closed in $X$, then $[U^c, X - C]$ is an open subset of PR($X$). There exists $[F_n, X - C_n] \in \mathcal{V}_n$ such that

$$[U^c, X - C] \cap [F_n, X - C_n] \neq \emptyset$$

for all but finitely many $n \in \mathbb{N}$. So $(C_n, F_n) \in \zeta_n$ such that $C_n \subset U$ and $F_n \cap C = \emptyset$ for all but finitely many $n \in \mathbb{N}$.

(2)⇒(1) Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of PR($X$). For each $n \in \mathbb{N}$, suppose that $\mathcal{U}_n$ is a family of base open sets of PR($X$). Let

$$\xi_n = \{(X - V, A) : [A, V] \in \mathcal{U}_n\}.$$

Then $\{\xi_n : n \in \mathbb{N}\}$ is a sequence of $rcf$-networks of $X$. For each $n \in \mathbb{N}$, there exists a finite subset $\zeta_n \subset \xi_n$ such that $\bigcup_{n\in\mathbb{N}} \zeta_n$ is a weakly $p$-$rcf$-network of $X$. Let

$$\mathcal{V}_n = \{[A, V] : (X - V, A) \in \zeta_n\}.$$

Then $\mathcal{V}_n \subset \mathcal{U}_n$ is finite. For every nonempty open $[B, U]$ of PR($X$), $X - B$ is a co-finite subset of $X$ and $X - U \subset X - B$ is closed in $X$. There exists $(X - V_n, A_n) \in \zeta_n$ such that

$$X - V_n \subset X - B \text{ and } A_n \cap (X - U) = \emptyset$$

for all but finitely many $n \in \mathbb{N}$. Thus $[B, U] \cap [A_n, V_n] \neq \emptyset$, i.e., $[B, U] \cap \bigcup \mathcal{V}_n \neq \emptyset$ for all but finitely many $n \in \mathbb{N}$. So PR($X$) is weakly Hurewicz. □

**Definition 2.20.** Let $Y \subsetneq X$. A partitioned closed-miss-finite family $\xi = \bigcup_{n\in\mathbb{N}} \xi_n$ of $Y$ is called a *weakly $p$-$cf$-network* of $Y$, if for each co-finite subset $U$ of $Y$ and $C \subset U$ closed in $X$, there exists $(C_n, F_n) \in \xi_n$ such that $C_n \subset U$ and $F_n \cap C = \emptyset$ for all but finitely many $n \in \mathbb{N}$.

We write $\Pi^p_{wcf}$ the collection of weakly $p$-$cf$-networks of each subset $Y \subsetneq X$.

From Theorem 2.19, in a similar way of Theorems 2.7, 2.11 and 2.15, one can prove the following.

**Theorem 2.21.** *For a space $X$, the following are equivalent:*
*(1)* PR($X$) *is weakly Hurewicz;*
*(2) $X$ satisfies $\mathsf{S}_{fin}(\Pi_{rcf}, \Pi^p_{wrcf})$;*
*(3) $X$ is separable and $X - \{x\}$ satisfies $\mathsf{S}_{fin}(\Pi_{cf}, \Pi^p_{wcf})$ for each $x \in X$;*
*(4) $X$ is separable and each principal ultrafilter $\mathcal{F}[x]$ in* PR($X$) *is weakly Hurewicz in* PR($X$).

In [4], G. Di Maio and Lj.D.R. Kočinac introduced the definitions of the quasi-Rothberger property, the quasi-Menger property and the quasi-Hurewicz property. They pointed that a space $X$ is quasi-Rothberger (resp., quasi-Menger and quasi-Hurewicz) if and only if every closed subspace of $X$ is weakly Rothberger (resp., weakly Menger and weakly Hurewicz). So we ask:

**Question 2.22.** *For a space $X$, find the collections $\mathcal{A}$ and $\mathcal{B}$ of subsets of $X$ such that:*

*PR($X$) is quasi-Rothberger if and only if $X$ satisfies $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$;*
*PR($X$) is quasi-Menger if and only if $X$ satisfies $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$;*
*PR($X$) is quasi-Hurewicz if and only if $X$ satisfies $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$.*

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