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SEP Elements and Solutions of Certain Equations in a Ring with Involution

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Abstract. In this paper, we give a lot of new characterizations of SEP elements and partial isometries in rings with involution. Especially, we discuss these characterizations from the perspectives of the existence of solutions to certain equations, the general solutions of some equations.

1. Introduction

Let R be a ring with 1. $*$ is an anti-isomorphism of degree 2 in R, that is,

$$(1)(a^*)^* = a,\ (2)(a+b)^* = a^* + b^*,\ (3)(ab)^* = b^*a^*$$

for all $a, b \in R$. In this case, R is called a $*-$*ring*.

In the following research, we let R be a $*-$ring and $a \in R$.

a is called *group invertible* if there exists $b \in R$ such that

$$(1)\ aba = a,\ (2)\ bab = b,\ (3)\ ab = ba.$$

According to [2], b is uniquely determined by the above equations. We call it the *group inverse* of a and denote it by $a^\#$. The set of all group invertible elements of R is denoted by $R^\#$.

a is said to be *Moore-Penrose invertible* (or *MP-invertible*) if there exists $b \in R$ such that the following Penrose equations hold:

$$(1)\ aba = a,\ (2)\ bab = b,\ (3)\ (ab)^* = ab,\ (4)(ba)^* = ba.$$

There is at most one b such that the above conditions hold, see [7, 9, 11, 14, 15]. We call it the *Moore-Penrose inverse* (or *MP inverse*) of a and denote it by $a^\dagger$. The set of all MP-invertible elements of R is denoted by $R^\dagger$.

a is said to be EP [10] if $a \in R^\# \cap R^\dagger$ satisfying $a^\# = a^\dagger$. We denote the set of all EP elements of R by R^{EP}.

If $a = aa^*a$, then a is called a partial isometry. The set of all partial isometries of R is denoted by R^{PI}.

If $a \in R^{EP}$ and $a^* = a^\dagger$, we say the element a is a strongly EP element. We denote the set of all strongly EP elements of R by R^{SEP}.

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The research of *EP* elements in rings originates from *EP*-matrices. In [1], by using the representation of complex matrices and the matrix rank provided in [11], Baksalary, Styan and Trenkler investigated various classes of matrices such as partial isometries and EP matrices. In [18], Mosić and Djordjević studied the conditions involving $a^{\dagger}$, $a^{\#}$ and a^{*} to ensure that a is a partial isometry, giving several equivalent conditions under which an element in R is an EP element or a partial isometry based on ring theory only. More researches on EP elements and partial isometries have produced some meaningful findings, see [4, 16, 19–21, 23, 26].

In [5], using the generalized inverse of elements, common solutions of linear equations in a ring are discussed. Interesting research in this direction can be found in the literatures [6, 22, 25]. Recently, by means of the solution of constructed equations, a new kind of characterizations of generalized inverse elements are studied such as [24, 27].

Motivated by these results, this paper is intended to give a number of new characterizations of partial isometries and *SEP* elements [27] in rings with involution from some different angles. We characterize these elements by considering the existence of solutions to certain equations in a definite set, the general solutions of certain equations, and invertible elements in rings, which are all new approaches to study generalized inverses in rings.

2. Some characterizations of SEP elements

In [17, Theorem 1.5.3], It is proved that $a \in R^{SEP}$ if and only if $a \in R^{\#} \cap R^{+}$ and $aa^{*} = a^{+}a$. We can generalize this result as follows.

Lemma 2.1. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $aa^{*}a^{+} = a^{+}aa^{\#}$.*

Proof. " $\Longrightarrow$ " Assume that $a \in R^{SEP}$. Then $a \in R^{EP}$ and $aa^{*}a^{+} = a^{\#}$ by [17, Theorem 1.5.3]. Noting that $a \in R^{EP}$. Then $a^{+}aa^{\#} = a^{\#}$, this shows $aa^{*}a^{+} = a^{+}aa^{\#}$.

" $\Longleftarrow$ " Assume that $aa^{*}a^{+} = a^{+}aa^{\#}$, then $a^{+}aa^{\#}(1 - aa^{+}) = aa^{*}a^{+}(1 - aa^{+}) = 0$, it follows $a^{+}aa^{\#} = a^{+}$, one obtains $aa^{*}a^{+} = a^{+}$. Hence $a \in R^{SEP}$ by [17, Theorem 1.5.3]. $\square$

The following lemma is inspired by [17, Lemma 1.3.4], which proof is routine.

Lemma 2.2. *Let $a \in R^{\#} \cap R^{+}$. Then*
1) $aa^{*}a^{+} \in R^{EP}$ with $(aa^{*}a^{+})^{+} = a(a^{\#})^{*}a^{+}$;
2) $a^{+}aa^{\#} \in R^{EP}$ with $(a^{+}aa^{\#})^{+} = a^{+}a^{2}$.

Lemma 2.1 and Lemma 2.2 imply the following theorem.

Theorem 2.3. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $a(a^{\#})^{*}a^{+} = a^{+}a^{2}$.*

Noting that $a \in R^{SEP}$ if and only if $a^{*} \in R^{SEP}$. Hence, stating a^{*} instead of a in Theorem 2.3, one obtains the following theorem.

Theorem 2.4. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $a^{*}a^{\#}(a^{+})^{*} = aa^{+}a^{*}$.*

Applying the involution on the equality of Theorem 2.4, one has

Corollary 2.5. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $a^{+}(a^{\#})^{*}a = a^{2}a^{+}$.*

It is well known that $a \in R^{SEP}$ if and only if $a^{+} \in R^{SEP}$. Also, we have $a^{+} \in R^{\#}$ with $(a^{+})^{\#} = (aa^{\#})^{*}a(aa^{\#})^{*}$. Hence, a^{+} instead of a in Corollary 2.5, one yields

Theorem 2.6. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $aa^{*}aa^{\#}a^{+} = a^{+}a^{+}a$.*

Noting that $a \in R^{EP}$ if and only if $a \in R^{\#} \cap R^{+}$ and $a^{+} = a^{+}a^{+}a$. Hence Theorem 2.6 leads to the following corollary.

Corollary 2.7. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $aa^{*}aa^{\#}a^{+} = a^{+}$.*

3. Consistency of certain equations

Corollary 2.7 inspires us to construct the following equation.

$$aa^*xa^+ = a^+. \tag{1}$$

Lemma 3.1. *Let $a \in R^\# \cap R^+$. Then $a \in R^{EP}$ if and only if Eq.(1) is consistent.*

Proof. " $\Longrightarrow$ " Assume that $a \in R^{EP}$. Then $aa^*(a^+)^*a^+a^+ = aa^+aa^+a^+ = aa^+a^+ = a^+$. Hence $x = (a^+)^*a^+$ is a solution, which implies Eq.(1) is consistent.

" $\Longleftarrow$ " Assume that Eq.(1) is consistent. Then $a^+ = aa^+a^+$, it follows that $a \in R^{EP}$. $\square$

Theorem 3.2. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(1) is consistent and the general solution is given by*

$$x = aa^\# + u - aa^+ua^+a, \quad where \quad u \in R. \tag{2}$$

Proof. " $\Longrightarrow$ " Assume that $a \in R^{SEP}$. Then $aa^*aa^\#a^+ = a^+$ by Corollary 2.7. It follows that $aa^*(aa^\# + u - aa^+ua^+a)a^+ = aa^*aa^\#a^+ = a^+$. Hence Eq.(2) is the solution of Eq.(1). Now let $x = x_0$ be any solution of Eq.(1). Then $aa^*x_0a^+ = a^+$. Choose $u = x_0 - aa^\#$. Then we have $aa^+ua^+a = aa^+(x_0 - aa^\#)a^+a = aa^+x_0a^+a - aa^\# = (a^+)^*(a^*x_0a^+)a - aa^\# = (a^+)^*a^+(aa^*x_0a^+)a - aa^\# = (a^+)^*a^+a^+a - aa^\#$. Noting that $a \in R^{SEP}$, then $(a^+)^* = a$ and $a^+a^+a = a^+ = a^\#$. Hence $aa^+ua^+a = aa^\# - aa^\# = 0$. One yields $x_0 = aa^\# + x_0 - aa^\# = aa^\# + u = aa^\# + u - aa^+ua^+a$. Therefore the general solution of Eq.(1) is given by Eq.(2).

" $\Longleftarrow$ " From the assumption, one obtains $aa^*(aa^\# + u - aa^+ua^+a)a^+ = a^+$, this gives $aa^*aa^\#a^+ = a^+$. Hence $a \in R^{SEP}$ by Corollary 2.7. $\square$

Now, we construct equation as follows

$$a^+xa^+a^2a^+ = a^+. \tag{3}$$

Proposition 3.3. *Let $a \in R^\# \cap R^+$. Then the general solution of Eq.(3) is given by (2).*

Proof. First, we have $a^+(aa^\# + u - aa^+ua^+a)a^+a^2a^+ = a^+aa^\#a^+a^2a^+ + a^+ua^+a^2a^+ - a^+aa^+ua^+aa^+a^2a^+ = a^+$. Hence the formula(2) is the solution of Eq.(3). Next, let $x = x_0$ be any solution of Eq(3). Then $a^+x_0a^+a^2a^+ = a^+$. Noting that $aa^+x_0a^+a = a(a^+x_0a^+a^2a^+)a^\#a = aa^+a^\#a = a^\#a$. Hence $x_0 = aa^\# + x_0 - aa^+x_0a^+a$, this implies the general solution of Eq.(3) is given by Eq.(2). $\square$

The following corollary follows from Proposition 3.3 and Theorem 3.2.

Corollary 3.4. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(1) and Eq.(3) have the same solution.*

Eq.(3) induces us to construct the following equation.

$$a^+xa^+a^2a^+ = a^*. \tag{4}$$

Theorem 3.5. *Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if the general solution of Eq.(4) is given by (2).*

Proof. " $\Longrightarrow$ " Assume that $a \in R^{PI}$. Then $a^* = a^+$, it follows that Eq.(4) is the same as Eq.(3). Hence, by Proposition 3.3, the general solution of Eq.(4) is given by (2).

" $\Longleftarrow$ " From the assumption, one has

$$a^+(aa^\# + u - aa^+ua^+a)a^+a^2a^+ = a^*,$$

this gives $a^+ = a^*$. Hence $a \in R^{PI}$. $\square$

Similar to Proposition 3.3, we have the following proposition.

Proposition 3.6. *Let $a \in R^{\#} \cap R^{+}$. Then the general solution of Eq.(4) is given by*

$$x = aa^{*}aa^{\#} + u - aa^{+}ua^{+}a, \text{where } u \in R. \tag{5}$$

Corollary 3.7. *Let $a \in R^{\#} \cap R^{+}$. Then the general solution of Eq.(4) is given by*

$$x = u + aa^{+}(aa^{*}aa^{\#} - u)a^{+}a, \text{where } u \in R. \tag{6}$$

Proof. It is an immediate result of Proposition 3.6. □

Corollary 3.8. *Let $a \in R^{\#} \cap R^{+}$. Then the general solution of Eq.(4) is given by*

$$x = u + aa^{-}(aa^{*}aa^{\#} - aa^{+}ua^{+}a)a^{-}a, \text{for } u \in R \text{ and some inner inverse } a^{-} \text{ of } a. \tag{7}$$

Proof. It is an immediate result of Proposition 3.6. □

We don′t know whether we can change some inner inverse a^{-} to any inner inverse?

Theorem 3.9. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if Eq.(1) is consistent and the general solution is given by*

$$x = av + u - aa^{+}ua^{+}a, \text{ where } u \in R,\ v \in R^{-1} \text{ with } v^{-1} = a + 1 - a^{+}a. \tag{8}$$

Proof. " $\Longrightarrow$ " Since $a \in R^{SEP}$, the general solution of Eq.(1) is given by $x = aa^{\#} + u - aa^{+}ua^{+}a$ by Theorem 3.2. Choose $v = a^{\#} + 1 - a^{+}a$. Noting that $a \in R^{EP}$. Then v is invertible with $v^{-1} = a + 1 - a^{+}a$. Hence $x = av + u - aa^{+}ua^{+}a$, we are done.

" $\Longleftarrow$ " From the assumption, we have $(a + 1 - a^{+}a)v = 1$, this gives $aa^{\#} = aa^{\#}(a + 1 - a^{+}a)v = av$. Hence $a^{+} = aa^{*}(av + u - aa^{+}ua^{+}a)a^{+} = aa^{*}ava^{+} = aa^{*}aa^{\#}a^{+}$. By Corollary 2.7, $a \in R^{SEP}$. □

Observing Theorem 3.9, we have $a = a^{2}v$. Hence we have the following corollary.

Corollary 3.10. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if Eq.(1) is consistent and the general solution is given by*

$$x = av + u - aa^{+}ua^{+}a, \text{ where } u \in R,\ v \in R^{-1} \text{ with } av^{-1} = a^{2}. \tag{9}$$

4. Univariate equation

If we multiply the equality in Theorem 2.3 on the left by $a^{\#}a^{\#}$, we have $a^{\#} = a^{\#}(a^{\#})^{*}a^{+}$, so we have the following Theorem.

Theorem 4.1. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $a^{\#}(a^{\#})^{*}a^{+} = a^{\#}$.*

Proof. " $\Longrightarrow$ " It is evident.

" $\Longleftarrow$ " Let $a^{\#} = a^{\#}(a^{\#})^{*}a^{+}$. Then $a^{\#}aa^{+} = a^{\#}(a^{\#})^{*}a^{+}aa^{+} = a^{\#}(a^{\#})^{*}a^{+} = a^{\#}$, this infers $a \in R^{EP}$. It follows $a^{+}a^{2} = a^{2}a^{+} = a^{2}a^{\#} = a^{2}a^{\#}(a^{\#})^{*}a^{+} = a(a^{\#})^{*}a^{+}$. Hence $a \in R^{SEP}$ by Theorem 2.3. □

Noting that $a^{\#} = a^{\#}a^{+}a$. Hence Theorem 4.1 inspires us to give the following equation.

$$x(a^{\#})^{*}a^{+} = xa^{+}a. \tag{10}$$

Theorem 4.2. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if Eq.(10) has at least one solution in $\chi_a=\{a, a^{\#}, a^{+}, a^{*}, (a^{+})^{*}, (a^{\#})^{*}\}$.*

Proof. " $\Longrightarrow$ " Assume that $a \in R^{SEP}$. Then, by Theorem 4.1, we obtain $a^{\#}(a^{\#})^*a^+ = a^{\#} = a^{\#}a^+a$. It follows that $x = a^{\#}$ is a solution.

" $\Longleftarrow$ " 1)If $x = a$ is a solution, then $a(a^{\#})^*a^+ = aa^+a = a$. Pre-multiplying the equality by $(a^{\#})^2$, one gets $a^{\#}(a^{\#})^*a^+ = a^{\#}$. Hence $a \in R^{SEP}$ by Theorem 4.1;
2)If $x = a^{\#}$, then $a^{\#}(a^{\#})^*a^+ = a^{\#}a^+a = a^{\#}$, which infers $a \in R^{SEP}$ by Theorem 4.1;
3)If $x = a^+$, then $a^+(a^{\#})^*a^+ = a^+a^+a$. Noting that $(a^{\#})^* = a^+a(a^{\#})^*$. Then we have $a^+a^+a(a^{\#})^*a^+ = a^+a^+a$. By [27, Lemma 3.11], one yields $(a^{\#})^*a^+ = a^+a$. Hence $a^{\#} = a^{\#}a^+a = a^{\#}(a^{\#})^*a^+$, which gives $a \in R^{SEP}$ by Theorem 4.1.
4)If $x = a^*$, then $a^*(a^{\#})^*a^+ = a^*a^+a$, e.g, $a^+ = a^*a^+a$. By [17, Theorem 1.5.3], $a \in R^{SEP}$.
5)If $x = (a^{\#})^*$, then $(a^{\#})^*(a^{\#})^*a^+ = (a^{\#})^*a^+a$. Pre-multiplying the equality by $(a^*)^2$, one gets $a^*(a^{\#})^*a^+ = a^*a^+a$. By 4), we have $a \in R^{SEP}$.
6)If $x = (a^+)^*$, then $(a^+)^*(a^{\#})^*a^+ = (a^+)^*a^+a = (a^+)^*$. Pre-multiplying the equality by aa^*, one obtains $a(a^{\#})^*a^+ = a$. Hence $a \in R^{SEP}$ by 1). □

Noting that $a^{\#} = aa^+a^{\#}$. Hence Theorem 4.1 induces the following equation.

$$x(a^{\#})^*a^+ = aa^+x. \tag{11}$$

Similar to Theorem 4.2, we have the following theorem.

Theorem 4.3. *Let $a \in R^{\#} \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(11) has at least one solution in χ_a.*

It is known that $a \in R^{SEP}$ if and only if $a^* \in R^{SEP}$. Hence a^* instead of a in Eq.(11), we have

$$xa^{\#}(a^+)^* = a^+ax. \tag{12}$$

Also Theorem 4.3 implies the following theorem.

Theorem 4.4. *Let $a \in R^{\#} \cap R^+$. Then $a \in R^{SEP}$ if and only if equation (12) has at least one solution in χ_a.*

5. Bivariate equations

We can generalize Eq.(12) as follows:

$$xa^{\#}(a^+)^* = a^+ay. \tag{13}$$

Theorem 5.1. *Let $a \in R^{\#} \cap R^+$. Then the general solution of Eq.(13) is given by*

$$\begin{cases} x = a^+ap + u - uaa^+ \\ y = pa^{\#}(a^+)^* + v - a^+av \end{cases}, \text{where } u, v, p \in R. \tag{14}$$

Proof. Noting that

$$(a^+ap + u - uaa^+)a^{\#}(a^+)^* = a^+apa^{\#}(a^+)^* = a^+a(pa^{\#}(a^+)^* + v - a^+av).$$

Then the formula (14) is the solution of Eq.(13).

Now, let

$$\begin{cases} x = x_0 \\ y = y_0 \end{cases}$$

be any solution of Eq.(13). Then we have $x_0a^{\#}(a^+)^* = a^+ay_0$, it follows that

$$x_0aa^+ = x_0a^{\#}a^2a^+ = x_0a^{\#}aa^+a^2a^+ = x_0a^{\#}(a^+)^*a^*a^2a^+ = a^+ay_0a^*a^2a^+.$$

Choose $p = a^+ay_0a^*a^2a^+$. Then $x_0aa^+ = a^+ap$, which gives $x_0 = a^+ap + x_0 - x_0aa^+$.
Since

$$\begin{aligned} pa^{\#}(a^+)^* &= a^+ay_0a^*a^2a^+a^{\#}(a^+)^* = a^+ay_0a^*aa^{\#}(a^+)^* \\ &= a^+ay_0a^*(a^+)^* = a^+ay_0a^+a = x_0a^{\#}(a^+)^*a^+a = x_0a^{\#}(a^+)^* = a^+ay_0 \end{aligned}$$

one gets $y_0 = pa^{\#}(a^+)^* + y_0 - a^+ay_0$. Hence the general solution of Eq.(13) is given by (14). □

Theorem 5.2. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{PI}$ if and only if the general solution of Eq.(13) is given by*

$$\begin{cases} x = a^*ap + u - uaa^+ \\ y = pa^{\#}(a^+)^* + v - a^+av \end{cases}, \text{where } u, v, p \in R. \tag{15}$$

Proof. $''\Longrightarrow''$ Assume that $a \in R^{PI}$. Then $a^+ = a^*$, it follows that the formula (15) is the same as the formula (14). Hence, by Theorem 5.1, the general solution of Eq.(13) is given by Eq.(15).

$''\Longleftarrow''$ From the assumption, we get $(a^*ap + u - uaa^+)a^{\#}(a^+)^* = a^+a(pa^{\#}(a^+)^* + v - a^+av)$, e.g. $a^*apa^{\#}(a^+)^* = a^+apa^{\#}(a^+)^*$ for all $p \in R$. Especially, choose $p = 1$, one yields $a^+a = a^+(a^+)^*$, it follows that $a = aa^+a = aa^+(a^+)^* = (a^+)^*$. Hence $a \in R^{PI}$. □

Theorem 5.3. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if the general solution of Eq.(13) is given by*

$$\begin{cases} x = a^{\#}(a^+)^*p + u - uaa^+ \\ y = pa^{\#}(a^+)^* + v - a^+av \end{cases}, \text{where } u, v, p \in R. \tag{16}$$

Proof. $''\Longrightarrow''$ Assume that $a \in R^{SEP}$. Then $a^{\#} = a^*$ and $a = (a^+)^*$. Hence, by Theorem 5.1, we are done.

$''\Longleftarrow''$ From the assumption, we have

$$(a^{\#}(a^+)^*p + u - uaa^+)a^{\#}(a^+)^* = a^+a(pa^{\#}(a^+)^* + v - a^+av),$$

this gives $a^{\#}(a^+)^*pa^{\#}(a^+)^* = a^+apa^{\#}(a^+)^*$ for all $p \in R$. Choose $p = a^*a$, we get $a^{\#}(a^+)^* = a^+a$. Pre-multiplying the last equality by a^+a, one has $a^+(a^+)^* = a^+a$. Hence $a \in R^{PI}$ by Theorem 5.2. It follows that $a^+a = a^{\#}(a^+)^* = a^{\#}a$, which infers $a \in R^{SEP}$. □

Which equation whose general solution is given by the formula (16)? To do this, we construct the following equation.

$$xa^{\#}(a^+)^* = a^{\#}(a^+)^*y. \tag{17}$$

Evidently, we have the following theorem.

Theorem 5.4. *Let $a \in R^{\#} \cap R^{+}$. Then the general solution of Eq.(17) is given by Eq.(16).*

The following theorem follows from Theorem 5.3 and Theorem 5.4.

Theorem 5.5. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if Eq.(13) and Eq.(17) have the same solution.*

6. Constructing (b, c)-inverse to characterize *SEP* elements

Let $a, b, c \in R$. If there exists $y \in R$ such that $y \in bRy \cap yRc$; $yab = b$; $cay = c$, then a is called $(b, c)-$invertible [8], and y is called $(b, c)-$inverse of a.

It is well known that if y exists, it is unique and is denoted by $a^{\|(b,c)}$. Many studies on $(b, c)-$inverses appears in [3, 12, 13].

Evidently, $a^{\|(b,c)}$ exists if and only if $b \in bRcab$ and $c \in cabRc$.

Theorem 4.1 implies the following theorem.

Theorem 6.1. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $\left(a^{\#}\right)^{*\|\left(a^+,a^{\#}\right)} = a^{\#}$.*

Proof. $''\Longrightarrow''$ If $a \in R^{SEP}$, then, by Theorem 4.1, $a^{\#} = a^{\#}(a^{\#})^*a^+$ and $a^{\#} = a^+$. By a routine verification, we have $\left(a^{\#}\right)^{*\|\left(a^+,a^{\#}\right)} = a^{\#}$.

$''\Longleftarrow''$ If $\left(a^{\#}\right)^{*\|\left(a^+,a^{\#}\right)} = a^{\#}$, then $a^{\#}(a^{\#})^*a^+ = a^+$, it follows that $a^{\#}aa^+ = a^{\#}aa^{\#}(a^{\#})^*a^+ = a^{\#}(a^{\#})^*a^+ = a^+$. Hence $a \in R^{EP}$, which implies $a^{\#} = a^+ = a^{\#}(a^{\#})^*a^+$. By Theorem 4.1, we have $a \in R^{SEP}$. □

Corollary 6.2. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $\left(a^{\#}\right)^{*\|\left(a^{+},a^{\#}\right)} = a^{+}$.*

Proof. $'' \Longrightarrow ''$ If $a \in R^{SEP}$, then, by Theorem 6.1, $\left(a^{\#}\right)^{*\|\left(a^{+},a^{\#}\right)} = a^{\#}$. Noting that $a^{\#} = a^{+}$. Then $\left(a^{\#}\right)^{*\|\left(a^{+},a^{\#}\right)} = a^{+}$.

$'' \Longleftarrow ''$ Assume that $\left(a^{\#}\right)^{*\|\left(a^{+},a^{\#}\right)} = a^{+}$. Then $a^{\#} = a^{\#}(a^{\#})^{*}a^{+}$. Hence $a \in R^{SEP}$ by Theorem 4.1. $\square$

Lemma 6.3. *Let $a, b, c \in R$. Then $a^{\|(b,c)}$ exists if and only if*

$$\begin{cases} b = bxcab \\ c = cabxc \end{cases}$$

has at least one solution.

Proof. $'' \Longrightarrow ''$ Let $a^{\|(b,c)} = y$. Then $y \in bRy \cap yRc$ and $yab = b$, $c = cay$. Set $y = bsy = ytc$. Then $b = yab = bsyab = bsytcab = b(syt)cab$, $c = caytc = cabsytc = cab(syt)c$. Hence

$$\begin{cases} b = bxcab \\ c = cabxc \end{cases}$$

has a solution whcih is $x = syt$.

$'' \Longleftarrow ''$ Let $x = x_0$ be a solution. Then

$$\begin{cases} b = bx_0cab \\ c = cabx_0c \end{cases}$$

Choose $y = bx_0c$, then $b = yab$, $c = cay$. For $y = bx_0c = bx_0cay \in bRy$, $y = bx_0c = yabx_0c \in yRc$. Hence $a^{\|(b,c)} = y$. $\square$

Theorem 6.4. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if*

$$\begin{cases} a^{+} = a^{+}xa^{\#}(a^{\#})^{*}a^{+} \\ a^{\#} = a^{\#}(a^{\#})^{*}a^{+}xa^{\#} \\ a^{+} = a^{+}xa^{\#} \end{cases} \tag{18}$$

has a solution.

Proof. $'' \Longrightarrow ''$ Assume that $a \in R^{SEP}$, then by Theorem 6.1, we have $\left(a^{\#}\right)^{*\|\left(a^{+},a^{\#}\right)} = a^{\#}$. By Lemma 6.3, we get

$$\begin{cases} a^{+} = a^{+}xa^{\#}(a^{\#})^{*}a^{+} \\ a^{\#} = a^{\#}(a^{\#})^{*}a^{+}xa^{\#} \end{cases}$$

has a solution.
By Theorem 4.1, we know $a^{\#} = a^{\#}(a^{\#})^{*}a^{+}$, then

$$\begin{cases} a^{+} = a^{+}xa^{+} \\ a^{\#} = a^{\#}xa^{\#} \end{cases}$$

has a solution. Hence Eq.(18) has a solution.

$'' \Longleftarrow ''$ If Eq.(18) has a solution $x = x_0$, then we have

$$\begin{cases} a^{\#} = a^{\#}(a^{\#})^{*}a^{+}x_0a^{\#} \\ a^{+} = a^{+}x_0a^{\#} \end{cases}$$

Then $a^{\#} = a^{\#}(a^{\#})^{*}a^{+}$. Hence $a \in R^{SEP}$ by Theorem 4.1. $\square$

7. Constructing invertible elements to characterize SEP elements

Theorem 7.1. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $a^{\#} = u(a^{\#})^{*}a^{+}$, where u is invertible with $u^{-1} = a+1-a^{+}a$.*

Proof. " $\Longrightarrow$ " If $a \in R^{SEP}$, then $a^{\#} = a^{\#}(a^{\#})^{*}a^{+}$ by Theorem 4.1. Choose $u = a^{\#} + 1 - a^{+}a$. Then u is invertible and $u^{-1} = a + 1 - a^{+}a$. Also, $u(a^{\#})^{*}a^{+} = (a^{\#} + 1 - a^{+}a)(a^{\#})^{*}a^{+} = a^{\#}(a^{\#})^{*}a^{+} = a^{\#}$.

" $\Longleftarrow$ " Assume that $a^{\#} = u(a^{\#})^{*}a^{+}$, and $u^{-1} = a + 1 - a^{+}a$. Then $a^{\#}aa^{+} = u(a^{\#})^{*}a^{+}aa^{+} = u(a^{\#})^{*}a^{+} = a^{\#}$, this gives $a \in R^{EP}$. Hence $a^{+} = a^{\#}$. Since $(a + 1 - a^{+}a)u = u^{-1}u = 1$, $a^{\#} = a^{\#}(a + 1 - a^{+}a)u = a^{\#}au$, which gives $a^{\#} = a^{\#}aa^{\#} = a^{\#}au(a^{\#})^{*}a^{+} = a^{\#}(a^{\#})^{*}a^{+}$. Hence $a \in R^{SEP}$ by Theorem 4.1. $\square$

Theorem 7.2. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $a^{\#} = u(a^{\#})^{*}v$, where u, v are invertible with $u^{-1} = a + 1 - a^{+}a$, $v^{-1} = a + 1 - aa^{+}$ and $av = ua$.*

Proof. " $\Longrightarrow$ " Assume that $a \in R^{SEP}$. Then, by Theorem 7.1, we have $a^{\#} = u(a^{\#})^{*}a^{+}$, where $u^{-1} = a + 1 - a^{+}a$. Choose $v = a^{+} + 1 - aa^{+}$. Then $v = a^{\#} + 1 - aa^{\#}$ because $a \in R^{EP}$. Clearly, $v^{-1} = a + 1 - aa^{\#} = a + 1 - aa^{+}$ and $u(a^{\#})^{*}v = u(a^{\#})^{*}(a^{+} + 1 - aa^{+}) = u(a^{\#})^{*}a^{+} = a^{\#}$. Noting that $u^{-1} = v^{-1}$. Then $av = ua$.

" $\Longleftarrow$ " Assume that $a^{\#} = u(a^{\#})^{*}v$, where $u^{-1} = a+1-a^{+}a$, $v^{-1} = a+1-aa^{+}$ and $av = ua$. Then $u^{-1}a = av^{-1}$, it follows that $a^{+}a^{2} = a^{2}a^{+}$, $aa^{+} = a^{\#}a^{2}a^{+} = a^{\#}a^{+}a^{2} = a^{\#}a$. Hence $a \in R^{EP}$. Noting that $v(a + 1 - aa^{+}) = vv^{-1} = 1$, $a^{+} = v(a + 1 - aa^{+})a^{+} = vaa^{+}$, then $a^{\#} = a^{\#}aa^{+} = u(a^{\#})^{*}vaa^{+} = u(a^{\#})^{*}a^{+}$. Hence $a \in R^{SEP}$ by Theorem 7.1. $\square$

Theorem 7.3. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $a^{\#} = a^{\#}ua^{+}$, where u is invertible with $u^{-1} = a^{*}+1-a^{+}a$.*

Proof. " $\Longrightarrow$ " If $a \in R^{SEP}$, then $a^{\#} = a^{\#}(a^{\#})^{*}a^{+}$ and $a^{\#} = a^{+}$. Choose $u = (a^{\#})^{*} + 1 - a^{+}a$, then $u(a^{*} + 1 - a^{+}a) = 1 = (a^{*} + 1 - a^{+}a)u$. It infers u is invertible with $u^{-1} = a^{*} + 1 - a^{+}a$. Also, we have $a^{\#}ua^{+} = a^{\#}((a^{\#})^{*} + 1 - a^{+}a)a^{+} = a^{\#}(a^{\#})^{*}a^{+} = a^{\#}$.

" $\Longleftarrow$ " By the assumption, we have $u(a^{*} + 1 - a^{+}a) = 1$, this gives $(a^{\#})^{*} = u(a^{*} + 1 - a^{+}a)(a^{\#})^{*} = u(aa^{\#})^{*}$. Since $a^{+} = (aa^{\#})^{*}a^{+}$, $a^{\#} = a^{\#}ua^{+} = a^{\#}u(aa^{\#})^{*}a^{+} = a^{\#}(a^{\#})^{*}a^{+}$. Hence $a \in R^{SEP}$ by Theorem 4.1. $\square$

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