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Hermitian Elements and its Generalizations in a Ring with Involution

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Abstract. We give some sufficient and necessary conditions for an element in a ring with involution to be a Hermitian by using certain equations admitting solutions in a definite set and the general solution representation.

1. Introduction

Let R be an associative ring with 1. An *involution* $* : a \mapsto a^*$ in R is an anti-isomorphism of degree 2 (see [15]), that is,

$$(1)(a^*)^* = a,\ \ (2)(a+b)^* = a^* + b^*,\ \ (3)(ab)^* = b^*a^*.$$

In this case, R is called a $*-$*ring*.

An element $a \in R$ is said to be *Moore−Penrose invertible* (or *MP−invertible*) [15] if there exists some $b \in R$ such that the following Penrose equations hold:

$$(1)\ aba = a,\ \ (2)\ bab = b,\ \ (3)\ ab = (ab)^*,\ \ (4)\ ba = (ba)^*.$$

There is at most one b such that the above conditions hold (see [3, 5, 8]). We call it the *Moore−Penrose inverse* (or *MP−inverse*) of a and denote it by $a^\dagger$. The set of all MP−invertible elements of R is denoted by $R^\dagger$.

An element $a \in R$ is said to be *group invertible* [4, 14] if there is some $b \in R$ satisfying the following conditions:

$$(1)aba = a,\ \ (2)bab = b,\ \ (3)ab = ba.$$

There is at most one b such that the above conditions hold. We call it the *group inverse* of a and denote it by $a^\#$. The set of all group invertible elements of R is denoted by $R^\#$.

An element $a \in R^\# \cap R^\dagger$ satisfying $a^\# = a^\dagger$ is said to be EP [6]. We denote the set of all EP elements of R by R^{EP}.

According to [10], an element $a \in R$ is called Hermitian if $a^* = a$. Cleraly, Hermitian elements are *EP*. We denote the set of all Hermitian elements of R by R^{Her}. An element $a \in R^\dagger$ satisfying $a^*a^\dagger = a^\dagger a^*$ is called to be star-dagger [10].

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In Section 2, some properties of Hermitian elements which are needed in this paper are given by referring to [10]. Motivated by these results, in Section 3, we intend to provide, by using certain equations admitting solutions in a definite set, further equivalent conditions for an element in a ring with involution to be a Hermitian element. In Section 4, we study the relationship between some constructed equations and the Hermitian elements. In Section 5, we introduce the concept of weakly Hermitian elements and some its properties. In Section 6, we discuss the relationship between the Hermitian elements and inverse representation.

2. Some properties of Hermitian elements

We give the following lemma at first.

Lemma 2.1. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Her}$ if and only if $a = a^{\dagger}a^3(a^{\dagger})^*$.*

Proof. " $\Longrightarrow$ " Assume that $a \in R^{Her}$. Then $a = a^*$ and $a^{\#} = a^{\dagger}$, it follows that $a^{\dagger}a^3(a^{\dagger})^* = a^2(a^{\dagger})^* = aa^*(a^{\dagger})^* = a$.

" $\Longleftarrow$ " From the assumption, we obtain $a^{\dagger}a^2 = a^{\dagger}a(a^{\dagger}a^3(a^{\dagger})^*) = a^{\dagger}a^3(a^{\dagger})^* = a$. Hence $a \in R^{EP}$, this gives $a = a^{\dagger}a^3(a^{\dagger})^* = a^2(a^{\dagger})^*$. Applying the involution on the last equality, one has $a^* = a^{\dagger}a^*a^* = a^{\#}a^*a^*$. Hence $a \in R^{Her}$ by [10, Theorem 1.4.2]. □

Observing the proof of Lemma 2.1, we have the following corollary.

Corollary 2.2. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Her}$ if and only if $a = a^{\dagger}a^3(a^{\#})^*$.*

The following lemma is inspired by [10, Lemma 1.3.4], which proof is routine.

Lemma 2.3. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then*
(1) $a^{\dagger}a^3(a^{\#})^* \in R^{\dagger}$ with $(a^{\dagger}a^3(a^{\#})^*)^{\dagger} = aa^{\dagger}a^*a^{\dagger}a^{\#}$;
(2) $a^{\dagger}a^3(a^{\#})^* \in R^{\#}$ with $(a^{\dagger}a^3(a^{\#})^*)^{\#} = a^*a^{\dagger}a^{\#}(aa^{\#})^*$;
(3) $a \in R^{EP}$ if and only if $a^{\dagger}a^3(a^{\#})^* \in R^{EP}$.

Lemma 2.1 and Lemma 2.3(1) leads to the following corollary.

Corollary 2.4. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Her}$ if and only if $a^{\dagger} = aa^{\dagger}a^*a^{\dagger}a^{\#}$.*

Also Lemma 2.1 and Lemma 2.3(2) imply the following corollary.

Corollary 2.5. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Her}$ if and only if $a^{\#} = a^*a^{\dagger}a^{\#}(aa^{\#})^*$.*

Proposition 2.6. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Her}$ if and only if $a^{\#}a^{\dagger} = a^*a^{\dagger}a^{\#}a^{\dagger}$.*

Proof. " $\Longrightarrow$ " Assume that $a \in R^{Her}$. Then $a^{\#} = a^*a^{\dagger}a^{\#}(aa^{\#})^*$ by Corollary 2.5. Post-multiplying the last equality by $a^{\dagger}$, one gets $a^{\#}a^{\dagger} = a^*a^{\dagger}a^{\#}a^{\dagger}$.

" $\Longleftarrow$ " Since $a^{\#}a^{\dagger} = a^*a^{\dagger}a^{\#}a^{\dagger}$, $a^{\#} = a^{\#}a^{\dagger}a = a^*a^{\dagger}a^{\#}a^{\dagger}a = a^*a^{\dagger}a^{\#}$. Hence $a \in R^{Her}$ by [10, Theorem 1.4.2]. □

The following lemma can be proved conventionally.

Lemma 2.7. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then*
(1) $a^{\#}a^{\dagger} \in R^{EP}$ with $(a^{\#}a^{\dagger})^{\dagger} = a^3a^{\dagger}$;
(2) $a^*a^{\dagger}a^{\#}a^{\dagger} \in R^{\dagger}$ with $(a^*a^{\dagger}a^{\#}a^{\dagger})^{\dagger} = a^3(aa^{\#})^*(a^{\dagger})^*$;
(3) $a^*a^{\dagger}a^{\#}a^{\dagger} \in R^{\#}$ with $(a^*a^{\dagger}a^{\#}a^{\dagger})^{\#} = (aa^{\#})^*a^3(a^{\#})^*$;
(4) $a \in R^{EP}$ if and only if $a^*a^{\dagger}a^{\#}a^{\dagger} \in R^{EP}$.

Proposition 2.6 and Lemma 2.7(1), (2) infers the following corollary.

Corollary 2.8. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Her}$ if and only if $aa^{\dagger} = a(aa^{\#})^*(a^{\dagger})^*$.*

Corollary 2.9. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Her}$ if and only if $a^3a^{\dagger} = (aa^{\#})^*a^3(a^{\#})^*$.*

Proof. By Proposition 2.6, $a \in R^{Her}$ if and only if $a^{\#}a^{\dagger} = a^*a^{\dagger}a^{\#}a^{\dagger}$. According to Lemma 2.7(1) and (3), we have $a \in R^{Her}$ if and only if

$$a^3a^{\dagger} = (a^{\#}a^{\dagger})^{\dagger} = (a^{\#}a^{\dagger})^{\#} = (a^*a^{\dagger}a^{\#}a^{\dagger})^{\#} = (aa^{\#})^*a^3(a^{\#})^*.$$

□

Noting that $a^3a^{\dagger} = aa^{\dagger}a^3a^{\dagger} = aa^{\dagger}((aa^{\#})^*a^3(a^{\#})^*)a^*a^{\dagger}$. Hence Corollary 2.9 implies the following equation:

$$x = aa^{\dagger}xa^*a^{\dagger}. \tag{1}$$

Corollary 2.10. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Her}$ if and only if Eq.(1) has a solution $x = (aa^{\#})^*a^3(a^{\#})^*$.*

3. Solutions in χ_a of some constructed equations

Theorem 3.1. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Her}$ if and only if Eq.(1) has at least one solution in $\chi_a = \{a, a^{\#}, a^{\dagger}, a^*, (a^{\#})^*, (a^{\dagger})^*\}$.*

Proof. " $\Longrightarrow$ " Assume that $a \in R^{Her}$. Then $a = aa^*a^{\dagger}$ by [10, Theorem 1.4.2], this infers $x = a$ is a solution of Eq.(1).

" $\Longleftarrow$ " 1) If $x = a$ is a solution, then $a = aa^{\dagger}aa^*a^{\dagger}$, i.e. $a = aa^*a^{\dagger}$. Hence $a \in R^{Her}$ by [10, Theorem 1.4.2].

2) If $x = a^{\#}$, then $a^{\#} = aa^{\dagger}a^{\#}a^*a^{\dagger}$, e.g. $a^{\#} = a^{\#}a^*a^{\dagger}$. Pre-multiplying the equality by a^2, one has $a = aa^*a^{\dagger}$. Hence $a \in R^{Her}$ by [10, Theorem 1.4.2].

3) If $x = a^{\dagger}$, then $a^{\dagger} = aa^{\dagger}a^{\dagger}a^*a^{\dagger}$, it follows that $a^{\dagger}a^{\dagger} = a^{\dagger}a^{\dagger}a^*a^{\dagger}$. By [17, Lemma 2.11], we get $a^{\dagger} = a^{\dagger}a^*a^{\dagger}$, this gives $a^{\dagger} = aa^{\dagger}a^{\dagger}a^*a^{\dagger} = aa^{\dagger}a^{\dagger}$. Hence $a \in R^{EP}$. Now we conclude that $a^{\dagger} = a^{\dagger}a^*a^{\dagger} = a^{\#}a^*a^{\#}$. Then, by [10, Theorem 1.4.2], $a \in R^{Her}$.

4) If $x = a^*$, then $a^* = aa^{\dagger}a^*a^*a^{\dagger}$. Pre-multiplying the equality by $(aa^{\#})^*$, one has $a^* = a^*a^*a^{\dagger}$, this gives $a^* = aa^{\dagger}a^*$. Applying the involution on the last equality, on has $a = a^2a^{\dagger}$. Hence $a \in R^{EP}$. Now, from the equality $a^* = a^*a^*a^{\dagger}$, we obtain $a^* = a^*a^*a^{\#}$. Thus $a \in R^{Her}$ by [10, Theorem 1.4.2].

5) If $x = (a^{\dagger})^*$, then $(a^{\dagger})^* = aa^{\dagger}(a^{\dagger})^*a^*a^{\dagger}$, i.e. $(a^{\dagger})^* = aa^{\dagger}a^{\dagger}$. Pre-multiplying the equality by aa^*, we get $a = aa^*a^{\dagger}$. Hence $a \in R^{Her}$ by [10, Theorem 1.4.2].

6) If $x = (a^{\#})^*$, then $(a^{\#})^* = aa^{\dagger}(a^{\#})^*a^*a^{\dagger}$, i.e. $(a^{\#})^* = aa^{\dagger}a^{\dagger}$, this gives $aa^{\dagger}(a^{\#})^* = (a^{\#})^*$. Applying the involution on the equality, one has $a^{\#} = a^{\#}aa^{\dagger}$. Hence $a \in R^{EP}$ by [10, Theorem 1.2.1]. Now we obtain that $x = (a^{\dagger})^*$ is a solution. Hence $a \in R^{Her}$ by 5). □

Now we generalize Eq.(1) as follows.

$$x = aa^{\dagger}ya^*a^{\dagger}. \tag{2}$$

Lemma 3.2. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then the general solution of Eq.(2) is given by*

$$\begin{cases} x = aa^{\dagger}pa^*a^{\dagger} \\ y = p + v - aa^{\dagger}va^{\dagger}a \end{cases}, where \ \ p, v \in R. \tag{3}$$

Proof. Clearly, (3) is the solution of Eq.(2). Now, let

$$\begin{cases} x = x_0 \\ y = y_0 \end{cases} \tag{4}$$

be a solution to Eq.(2). Then $x_0 = aa^{\dagger}y_0a^*a^{\dagger}$. Choose $p = x_0a(a^{\#})^*a^{\dagger}a$ and $v = y_0$.
We obtain

$$aa^{\dagger}pa^*a^{\dagger} = aa^{\dagger}(x_0a(a^{\#})^*a^{\dagger}a)a^*a^{\dagger} = aa^{\dagger}x_0aa^{\dagger} = aa^{\dagger}(aa^{\dagger}y_0a^*a^{\dagger})aa^{\dagger} = aa^{\dagger}y_0a^*a^{\dagger} = x_0,$$

$$aa^{\dagger}va^{\dagger}a = aa^{\dagger}y_0a^{\dagger}a = aa^{\dagger}y_0a^*(a^{\#})^*a^{\dagger}a = aa^{\dagger}y_0a^*a^{\dagger}a(a^{\#})^*a^{\dagger}a = x_0a(a^{\#})^*a^{\dagger}a = p.$$

Then, we deduce that

$$y_0 = p + v - aa^{\dagger}va^{\dagger}a.$$

Hence the general solution of Eq.(2) is given by (3). □

Theorem 3.3. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Her}$ if and only if the general solution of Eq.*(2) *is given by*

$$\begin{cases} x = a^*a^{\dagger}pa^*a^{\dagger} \\ y = p + v - aa^{\dagger}va^{\dagger}a \end{cases}, \textit{where } p, v \in R. \tag{5}$$

Proof. " $\Longrightarrow$ " Since $a \in R^{Her}$, $a^* = a$, it follows that the formula (3) is same as the formula (5). Hence, by Lemma 3.2, we know that the general solution of Eq.(2) is given by (5).

" $\Longleftarrow$ " From the assumption, we have $a^*a^{\dagger}pa^*a^{\dagger} = aa^{\dagger}(p + v - aa^{\dagger}va^{\dagger}a)a^*a^{\dagger}$, i.e. $a^*a^{\dagger}pa^*a^{\dagger} = aa^{\dagger}pa^*a^{\dagger}$ for all $p \in R$. Especially, choose $p = (a^{\#})^*$, we obtain $a^*a^{\dagger}a^{\dagger} = aa^{\dagger}a^{\dagger}$. By [17, Lemma 2.11], one has $a^*a^{\dagger} = aa^{\dagger}$, this gives $a^*a^{\dagger}a^{\#} = aa^{\dagger}a^{\#} = a^{\#}$. Hence $a \in R^{Her}$ by [10, Theorem 1.4.2]. □

Consider now the equation:

$$x = a^*a^{\dagger}ya^*a^{\dagger}. \tag{6}$$

The proof of the following lemma is routine.

Lemma 3.4. *Let $a \in R^{\#} \cap R^{\dagger}$. Then the general solution of Eq.*(6) *is given by* (5).

The following theorem follows from Theorem 3.3 and Lemma 3.4.

Theorem 3.5. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Her}$ if and only if Eq.*(2) *has the same solution as Eq.*(6).

Theorem 3.6. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Her}$ if and only if the general solution of Eq.*(2) *is given by*

$$\begin{cases} x = aa^{\dagger}pa^{\dagger}a \\ y = p + v - aa^{\dagger}va^{\dagger}a \end{cases}, \textit{where } p, v \in R. \tag{7}$$

Proof. " $\Longrightarrow$ " Assume that $a \in R^{Her}$. Then $a^*a^{\dagger} = aa^{\dagger} = a^{\dagger}a$ because a is *EP*, this implies the formula (3) is the same as the formula (7). Hence, by Lemma 3.2, we get the general solution of Eq.(1) is given by (7).

" $\Longleftarrow$ " From the assumption, we have $aa^{\dagger}pa^{\dagger}a = aa^{\dagger}(p + v - aa^{\dagger}va^{\dagger}a)a^*a^{\dagger}$, i.e. $aa^{\dagger}pa^{\dagger}a = aa^{\dagger}pa^*a^{\dagger}$ for all $p \in R$. Choose $p = a$, we have $a = aa^*a^{\dagger}$. Hence $a \in R^{Her}$ by [10, Theorem 1.4.2]. □

4. Consistency of equations

Consider now the equation:

$$aa^{\dagger}xa^*a^{\dagger} = (aa^{\#})^*a^3(a^{\#})^*. \tag{8}$$

Theorem 4.1. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Her}$ if and only if Eq.*(8) *is consistent, and the general solution is given by*

$$x = (aa^{\#})^*a^3(a^{\#})^* + u - aa^{\dagger}ua^{\dagger}a. \textit{ where } u \in R. \tag{9}$$

Proof. " $\Longrightarrow$ " Assume that $a \in R^{Her}$. Then, by Corollary 2.9, we know that $x = (aa^{\#})^* a^3 (a^{\#})^*$ is a solution of Eq.(8). Hence the formula (9) is also the solution. Now, let $x = x_0$ be any solution of Eq.(8). Then

$$aa^{\dagger} x_0 a^* a^{\dagger} = (aa^{\#})^* a^3 (a^{\#})^*.$$

Choose $u = x_0$. Then

$$(aa^{\#})^* a^3 (a^{\#})^* + u - aa^{\dagger} u a^{\dagger} a = (aa^{\#})^* a^3 (a^{\#})^* + u - aa^{\dagger} x_0 (a^* (a^{\#})^*) a^{\dagger} a =$$

$$(aa^{\#})^* a^3 (a^{\#})^* + x_0 - aa^{\dagger} x_0 a^* (a^{\dagger} a (a^{\#})^*) a^{\dagger} a = (aa^{\#})^* a^3 (a^{\#})^* + x_0 - (aa^{\#})^* a^3 (a^{\#})^* a (a^{\#})^* a^{\dagger} a.$$

Noting that $a \in R^{Her}$. Then

$$(aa^{\#})^* a^3 (a^{\#})^* a (a^{\#})^* a^{\dagger} a = (aa^{\#})^* a^3 (a^{\#})^* aa^{\#} a^{\dagger} a = (aa^{\#})^* a^3 (a^{\#})^* aa^{\#} =$$

$$(aa^{\#})^* a^3 (a^{\#})^* aa^{\dagger} = (aa^{\#})^* a^3 (a^{\#})^*.$$

Thus $(aa^{\#})^* a^3 (a^{\#})^* + u - aa^{\dagger} u a^{\dagger} a = x_0$. Hence the general solution of Eq.(8) is given by (9).

" $\Longleftarrow$ " From the assumption, we have

$$aa^{\dagger}((aa^{\#})^* a^3 (a^{\#})^* + u - aa^{\dagger} u a^{\dagger} a) a^* a^{\dagger} = (aa^{\#})^* a^3 (a^{\#})^*.$$

By a simple calculation, one obtains $a^3 a^{\dagger} = (aa^{\#})^* a^3 (a^{\#})^*$. Hence $a \in R^{Her}$ by Corollary 2.9. □

Assume that $a \in R^{EP}$. Then

$$(aa^{\#})^* a^3 (a^{\#})^* = a^{\dagger} a (aa^{\#})^* a^3 (a^{\#})^* aa^{\dagger} = aa^{\dagger}((aa^{\#})^* a^3 (a^{\#})^* a (a^{\#})^*) a^* a^{\dagger}.$$

Hence we have the following theorem.

Theorem 4.2. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{EP}$ if and only if Eq.*(8) *is consistent.*
In this case, the general solution is given by

$$x = (aa^{\#})^* a^3 (a^{\#})^* a (a^{\#})^* + u - aa^{\dagger} u a^{\dagger} a. \text{ where } u \in R. \tag{10}$$

Now we construct the following equation.

$$a^* a^{\dagger} a^{\#} a^{\dagger} x a^* a^{\dagger} a^3 (a^{\#})^* = (aa^{\#})^* a^3 (a^{\#})^*. \tag{11}$$

The following theorem points out that which equation'*s* general solution is given by (10).

Theorem 4.3. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then the general solution of Eq.*(11) *is given by* (10)*.*

Theorem 4.2 and Theorem 4.3 induce the following corollary.

Corollary 4.4. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{EP}$ if and only if Eq.*(8) *has the same solution as Eq.*(11)*.*

Which equation'*s* general solution is given by the formula (9) So we construct the following equation.

$$(aa^{\#})^* x (aa^{\#})^* = (aa^{\#})^* a^3 (a^{\#})^*. \tag{12}$$

Theorem 4.5. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then the general solution of Eq.*(12) *is given by* (9)*.*

Hence Theorem 4.1 and Theorem 4.5 imply the following corollary.

Corollary 4.6. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Her}$ if and only if Eq.*(8) *has the same solution as Eq.*(12)*.*

5. Weakly Hermitian elements

Suppose that $a \in R^{\#} \cap R^{\dagger}$. If $(a^{\dagger})^{*} = a^{\#}$, then a is called weakly Hermitian element. Clearly, Henritian elements are weakly Hermitian. While the following example illustrates the converse is not true in general.

Example 5.1. *Suppose that $R = M_3(Z_2)$, where the involution is the transpose of matrix. Suppose that $a = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in R$. Then, by [2], we have $a^{\#} = a$ and $a^{\dagger} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = a^{*}$. Hence a is weakly Hermitian, while it is not Hermitian.*

We denote the set of all weakly Hermitian elements of R by R^{Wher}.

Observing Corollary 2.9, we can construct the following equation.

$$xa^{\dagger} = (aa^{\#})^{*}x(a^{\#})^{*}. \tag{13}$$

Theorem 5.2. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Wher}$ if and only if Eq.(13) has at least one solution in χ_a.*

Proof. $'' \Longrightarrow ''$ Assume that $a \in R^{Wher}$. Then $(a^{\dagger})^{*} = a^{\#}$. Hence $x = a^{*}$ is a solution.

$'' \Longleftarrow ''$ 1) If $x = a$ is a solution, then $aa^{\dagger} = (aa^{\#})^{*}a(a^{\#})^{*}$, it follows that

$$a^{\dagger} = a^{\dagger}aa^{\dagger} = a^{\dagger}(aa^{\#})^{*}a(a^{\#})^{*} = (a^{\#})^{*}.$$

Hence $a \in R^{Wher}$.

2) If $x = a^{\#}$, then $a^{\#}a^{\dagger} = (aa^{\#})^{*}a^{\#}(a^{\#})^{*}$. Pre-multiplying the equality by $a^{\dagger}a$, one has

$$a^{\#}a^{\dagger} = a^{\dagger}aa^{\#}a^{\dagger},$$

it follows that

$$a^{\#} = a^{\#}a^{\dagger}a = a^{\dagger}aa^{\#}a^{\dagger}a = a^{\dagger}aa^{\#}.$$

Hence $a \in R^{EP}$ by [10, Theorem 1.2.1], this gives

$$a^{\#}a^{\dagger} = (aa^{\#})^{*}a^{\#}(a^{\#})^{*} = (aa^{\dagger})^{*}a^{\#}(a^{\#})^{*} = a^{\#}(a^{\#})^{*}.$$

Now we have

$$a^{\dagger} = a^{\dagger}a^{2}a^{\#}a^{\dagger} = a^{\dagger}a^{2}a^{\#}(a^{\#})^{*} = (a^{\#})^{*}.$$

Hence $a \in R^{Wher}$.

3) If $x = a^{\dagger}$, then $a^{\dagger}a^{\dagger} = (aa^{\#})^{*}a^{\dagger}(a^{\#})^{*}$, i.e.

$$a^{\dagger}a^{\dagger} = a^{\dagger}(a^{\#})^{*} = a^{\dagger}a^{\dagger}a(a^{\#})^{*}.$$

By [17, Lemma 2.11], one yields $a^{\dagger} = a^{\dagger}a(a^{\#})^{*} = (a^{\#})^{*}$. Hence $a \in R^{Wher}$.

4) If $x = a^{*}$, then $a^{*}a^{\dagger} = (aa^{\#})^{*}a^{*}(a^{\#})^{*}$, i.e.

$$a^{*}a^{\dagger} = (aa^{\#})^{*}.$$

Pre-multiplying the last equality by $(a^{\#})^{*}$, one has $a^{\dagger} = (a^{\#})^{*}$. Hence $a \in R^{Wher}$.

5) If $x = (a^{\dagger})^{*}$, then $(a^{\dagger})^{*}a^{\dagger} = (aa^{\#})^{*}(a^{\dagger})^{*}(a^{\#})^{*}$, i.e.

$$(a^{\dagger})^{*}a^{\dagger} = (a^{\#}a^{\#})^{*}.$$

Pre-multiplying the last equality by a^{*}, one has $a^{\dagger} = (a^{\#})^{*}$. Hence $a \in R^{Wher}$.

6) If $x = (a^{\#})^{*}$, then $(a^{\#})^{*}a^{\dagger} = (aa^{\#})^{*}(a^{\#})^{*}(a^{\#})^{*}$, i.e.

$$(a^{\#})^{*}a^{\dagger} = (a^{\#}a^{\#})^{*}.$$

Pre-multiplying the last equality by a^{*}, one has $a^{\dagger} = (a^{\#})^{*}$. Hence $a \in R^{Wher}$. $\square$

Assume that $a \in R^{Wher}$, then $(aa^{\#})^* = a^{\dagger}a^*$. Hence Eq.(13) can be transformed as follows.

$$xa^{\dagger} = a^{\dagger}a^*x(a^{\#})^*. \tag{14}$$

Theorem 5.3. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then a is star-dagger if and only if Eq.*(14) *has at least one solution in χ_a.*

Proof. " $\Longrightarrow$ " Assume that a is star-dagger. Then $x = a^*$ is a solution.

" $\Longleftarrow$ " 1) If $x = a$ is a solution, then $aa^{\dagger} = a^{\dagger}a^*a(a^{\#})^*$. Post-multiplying the equality by $a^*a^{\dagger}$, one has

$$aa^{\dagger}a^*a^{\dagger} = a^{\dagger}a^*.$$

Pre-multiplying the last equality by $a^{\dagger}$, one gets

$$a^{\dagger}a^*a^{\dagger} = a^{\dagger}a^{\dagger}a^*.$$

By [17, Lemma 2.11], one yields $a^*a^{\dagger} = a^{\dagger}a^*$. Hence a is star-dagger.

2) If $x = a^{\#}$, then $a^{\#}a^{\dagger} = a^{\dagger}a^*a^{\#}(a^{\#})^*$. Pre-multiplying the equality by $a^{\dagger}a$, one has

$$a^{\#}a^{\dagger} = a^{\dagger}aa^{\#}a^{\dagger},$$

it follows that $a \in R^{EP}$ by the proof of 2) in Theorem 5.2. This gives

$$a^{\dagger}a^{\dagger} = a^{\#}a^{\dagger} = a^{\dagger}a^*a^{\#}(a^{\#})^*.$$

Hence $a^{\dagger} = a^*a^{\#}(a^{\#})^* = a^*a^{\dagger}(a^{\#})^*$, this infers $a^{\dagger}a^* = a^*a^{\#}(a^{\#})^*a^* = a^*a^{\dagger}$. Hence a is star-dagger.

3) If $x = a^{\dagger}$, then $a^{\dagger}a^{\dagger} = a^{\dagger}a^*a^{\dagger}(a^{\#})^*$, it follows that $a^{\dagger} = a^*a^{\dagger}(a^{\#})^*$. Hence a is star-dagger by 2).

4) If $x = a^*$, then $a^*a^{\dagger} = a^{\dagger}a^*a^*(a^{\#})^*$, i.e. $a^*a^{\dagger} = a^{\dagger}a^*$. Hence a is star-dagger.

5) If $x = (a^{\dagger})^*$, then $(a^{\dagger})^*a^{\dagger} = a^{\dagger}a^*(a^{\dagger})^*(a^{\#})^*$, i.e. $(a^{\dagger})^*a^{\dagger} = a^{\dagger}(a^{\#})^*$. Multiplying the equality on the left by a^* and on the right by a^*, one has $a^{\dagger}a^* = a^*a^{\dagger}$. Hence a is star-dagger.

6) If $x = (a^{\#})^*$, then $(a^{\#})^*a^{\dagger} = a^{\dagger}a^*(a^{\#})^*(a^{\#})^*$, i.e.

$$(a^{\#})^*a^{\dagger} = a^{\dagger}(a^{\#})^*.$$

Multiplying the equality on the left by a^* and on the right by a^*, one has $a^{\dagger}a^* = a^*a^{\dagger}$. Hence a is star-dagger. $\square$

We can generalize Eq.(14) as follows.

$$xa^{\dagger} = a^{\dagger}a^*y(a^{\#})^*. \tag{15}$$

Theorem 5.4. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then the general solution of Eq.*(15) *is given by*

$$\begin{cases} x = a^{\dagger}a^*p + u - ua^{\dagger}a \\ y = pa^{\dagger}a^* + v - aa^{\dagger}va^{\dagger}a \end{cases}, \text{where } p, u, v \in R. \tag{16}$$

Proof. First, clearly, the formula (16) is the solution of Eq.(15). Next, let

$$\begin{cases} x = x_0 \\ y = y_0 \end{cases} \tag{17}$$

be any solution of Eq.(15). Then $x_0a^{\dagger} = a^{\dagger}a^*y_0(a^{\#})^*$. Choose

$$p = y_0(a^{\#})^*a, \ \ u = x_0 \ \ \text{and} \ \ v = y_0 - pa^{\dagger}a^*.$$

Then, we have

$$ua^{\dagger}a = x_0a^{\dagger}a = (a^{\dagger}a^*y_0(a^{\#})^*)a = a^{\dagger}a^*p,$$
$$aa^{\dagger}va^{\dagger}a = aa^{\dagger}(y_0 - pa^{\dagger}a^*)a^{\dagger}a = aa^{\dagger}y_0a^{\dagger}a - aa^{\dagger}pa^{\dagger}a^*a^{\dagger}a =$$
$$aa^{\dagger}y_0a^{\dagger}a - aa^{\dagger}y_0(a^{\#})^*aa^{\dagger}a^*a^{\dagger}a = aa^{\dagger}y_0a^{\dagger}a - aa^{\dagger}y_0a^{\dagger}a = 0.$$

Hence $x_0 = a^{\dagger}a^*p + u - ua^{\dagger}a$ and $y_0 = pa^{\dagger}a^* + v - aa^{\dagger}va^{\dagger}a$. Therefore the general solution of Eq.(16) is given by (15). $\square$

Theorem 5.5. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then a is star-dagger if and only if the general solution of Eq.*(15) *is given by*

$$\begin{cases} x = a^*a^\dagger p + u - ua^\dagger a \\ y = pa^\dagger a^* + v - aa^\dagger va^\dagger a \end{cases}, \text{where } p, u, v \in R. \tag{18}$$

Proof. $'' \Longrightarrow ''$ Assume that a is star-dagger. Then $a^\dagger a^* = a^* a^\dagger$, it follows that the formula (16) is the same as the formula (18). Hence, by Theorem 5.4, we are done.

$'' \Longleftarrow ''$ From the assumption, we have

$$(a^*a^\dagger p + u - ua^\dagger a)a^\dagger = a^\dagger a^*(pa^\dagger a^* + v - aa^\dagger va^\dagger a)(a^{\#})^*,$$

i.e.

$$a^*a^\dagger pa^\dagger = a^\dagger a^* pa^\dagger a^*(a^{\#})^* \text{ for all } p \in R.$$

Choose $p = a$. One yields $a^\dagger a^* = a^* a^\dagger$. Hence a is star-dagger. □

Corollary 5.6. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Her}$ if and only if the general solution of Eq.*(15) *is given by*

$$\begin{cases} x = aa^\dagger p + u - ua^\dagger a \\ y = pa^\dagger a^* + v - aa^\dagger va^\dagger a \end{cases}, \text{where } p, u, v \in R. \tag{19}$$

Proof. $'' \Longrightarrow ''$ Assume that $a \in R^{Her}$. Then a is star-dagger and $a = a^*$. We get the formula (19) is the same as the formula (18). Hence, by Theorem 5.5, we are done.

$'' \Longleftarrow ''$ From the assumption, we have

$$(aa^\dagger p + u - ua^\dagger a)a^\dagger = a^\dagger a^*(pa^\dagger a^* + v - aa^\dagger va^\dagger a)(a^{\#})^*,$$

i.e.

$$aa^\dagger pa^\dagger = a^\dagger a^* pa^\dagger a^*(a^{\#})^* \text{ for all } p \in R.$$

Choose $p = a$. One yields $aa^\dagger = a^\dagger a^*$ and $a = aa^\dagger a = a^\dagger a^* a$. Hence $a \in R^{Her}$ by [10, Theorem 1.4.2]. □

6. Hermitian elements and inverse representation

If $a \in R^{\#}$, then $a + 1 - aa^{\#}$ is invertible and $(a + 1 - aa^{\#})^{-1} = a^{\#} + 1 - aa^{\#}$. Hence Lemma 2.7 and Proposition 2.6 give the following theorem.

Theorem 6.1. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then*

(1) $a \in R^{Her}$ if and only if $((aa^{\#})^* a^3 (a^{\#})^* + 1 - (aa^{\#})^*)^{-1} = a^{\#}a^\dagger + 1 - (aa^{\#})^*$;

(2) $a \in R^{Her}$ if and only if $(a^3 a^\dagger + 1 - aa^\dagger)^{-1} = a^* a^\dagger a^{\#} a^\dagger + 1 - aa^\dagger$;

(3) $a \in R^{EP}$ if and only if $(a^* a^\dagger a^{\#} a^\dagger + 1 - (aa^{\#})^*)^{-1} = a^3 (aa^{\#})^* (a^\dagger)^* + 1 - (aa^{\#})^*$.

It is well known that for any $a, b \in R$, $1 - ab$ is invertible, we conclude that $1 - ba$ is invertible and $(1 - ba)^{-1} = 1 + b(1 - ab)^{-1}a$. Hence we have the following theorem.

Theorem 6.2. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Her}$ if and only if $1 - (aa^{\#})^* + a^3(a^{\#})^*$ is invertible and $(1 - (aa^{\#})^* + a^3(a^{\#})^*)^{-1} = 1 - (aa^{\#})^* + a^\dagger a^\dagger$;*

Proof. $'' \Longrightarrow ''$ Assume that $a \in R^{Her}$. Then $(1 - (aa^{\#})^*(1 - a^3(a^{\#})^*))^{-1} = 1 - (aa^{\#})^* + a^{\#}a^\dagger$ by Theorem 6.1, it follows that

$$(1 - (aa^{\#})^*(1 - a^3(a^{\#})^*))^{-1} = 1 + (1 - a^3(a^{\#})^*)(1 - (aa^{\#})^*(1 - a^3(a^{\#})^*))^{-1}(aa^{\#})^* =$$

$$1 + (1 - a^3(a^{\#})^*)(1 - (aa^{\#})^* + a^{\#}a^\dagger)(aa^{\#})^* = 1 + (1 - a^3(a^{\#})^*)a^{\#}a^\dagger = 1 - a^3(a^{\#})^* a^{\#} a^\dagger + a^{\#} a^\dagger.$$

i.e.

$$(1-(aa^{\#})^{*}+a^{3}(a^{\#})^{*})^{-1}=1-a^{3}(a^{\#})^{*}a^{\#}a^{\dagger}+a^{\#}a^{\dagger}.$$

Assume that $a \in R^{Her}$, then $(a^{\#})^{*}=a^{\#}=a^{\dagger}$, it follows that

$$a^{3}(a^{\#})^{*}a^{\#}a^{\dagger}=aa^{\dagger}=(aa^{\dagger})^{*}=(aa^{\#})^{*}$$

and

$$a^{\#}a^{\dagger}=a^{\dagger}a^{\dagger}.$$

Hence

$$(1-(aa^{\#})^{*}+a^{3}(a^{\#})^{*})^{-1}=1-(aa^{\#})^{*}+a^{\dagger}a^{\dagger}.$$

$'' \Longleftarrow ''$ From the assumption, we have

$$1=(1-(aa^{\#})^{*}+a^{3}(a^{\#})^{*})(1-(aa^{\#})^{*}+a^{\dagger}a^{\dagger})=1-(aa^{\#})^{*}+a^{3}(a^{\#})^{*}a^{\dagger}a^{\dagger}.$$

Hence

$$(aa^{\#})^{*}=a^{3}(a^{\#})^{*}a^{\dagger}a^{\dagger}.$$

Pre-multiplying the equality by $aa^{\dagger}$, we obtain that $(aa^{\#})^{*}=aa^{\dagger}$, it follows that $a \in R^{EP}$.
Hence $aa^{\dagger}=a^{3}(a^{\#})^{*}a^{\dagger}a^{\dagger}$, pre-multiplying the last equality by $a^{\dagger}a^{\dagger}$, one has $a^{\dagger}a^{\dagger}=a(a^{\#})^{*}a^{\dagger}a^{\dagger}$, this gives $a^{\dagger}=a(a^{\#})^{*}a^{\dagger}$ by [17, Lemma 2.1].
So

$$a^{\dagger}a^{\dagger}=a^{\dagger}a(a^{\#})^{*}a^{\dagger}=(a^{\#})^{*}a^{\dagger}.$$

Hence

$$a^{*}a^{\#}a^{\#}=a^{*}a^{\dagger}a^{\dagger}=a^{*}(a^{\#})^{*}a^{\dagger}=a^{\dagger}=a^{\#}.$$

By [10, Theorem 1.4.2], $a \in R^{Her}$. □

Corollary 6.3. *Suppose that $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{Her}$ if and only if $(1-a^{\dagger}a+a^{\dagger}a^{\dagger})^{-1}=1-a^{\dagger}a+a^{\dagger}a^{3}(a^{\#})^{*}a$.*

Proof. $'' \Longrightarrow ''$ Assume that $a \in R^{Her}$. Then

$$(1-(aa^{\#})^{*}aa^{\dagger}+a^{\dagger}a^{\dagger})^{-1}=1-(aa^{\#})^{*}+a^{3}(a^{\#})^{*}$$

by Theorem 6.2, it follows that

$$(1-((aa^{\#})^{*}a-a^{\dagger})a^{\dagger})^{-1}=1-(aa^{\#})^{*}+a^{3}(a^{\#})^{*}.$$

Hence

$$\begin{aligned}(1-a^{\dagger}((aa^{\#})^{*}a-a^{\dagger}))^{-1}&=1+a^{\dagger}(1-(aa^{\#})^{*}+a^{3}(a^{\#})^{*})((aa^{\#})^{*}a-a^{\dagger})=\\ &1+a^{\dagger}a^{3}(a^{\#})^{*}((aa^{\#})^{*}a-a^{\dagger})=1-a^{\dagger}a^{3}(a^{\#})^{*}a^{\dagger}+a^{\dagger}a^{3}(a^{\#})^{*}a.\end{aligned}$$

Since $a \in R^{Her}$, then

$$a^{\dagger}a^{3}(a^{\#})^{*}a^{\dagger}=a^{\dagger}a^{3}a^{\#}a^{\#}=a^{\dagger}a.$$

Thus

$$(1-a^{\dagger}a+a^{\dagger}a^{\dagger})^{-1}=(1-a^{\dagger}((aa^{\#})^{*}a-a^{\dagger}))^{-1}=1-a^{\dagger}a+a^{\dagger}a^{3}(a^{\#})^{*}a.$$

$'' \Longleftarrow ''$ From the assumption, we have

$$1=(1-a^{\dagger}a+a^{\dagger}a^{\dagger})(1-a^{\dagger}a+a^{\dagger}a^{3}(a^{\#})^{*}a)=1-a^{\dagger}a+a^{\dagger}a^{\dagger}-a^{\dagger}a^{\dagger}a^{\dagger}a+a^{\dagger}a^{\dagger}a^{\dagger}a^{3}(a^{\#})^{*}a,$$

it follows that

$$a^{\dagger}a^{\dagger}=a^{\dagger}a+a^{\dagger}a^{\dagger}a^{\dagger}a-a^{\dagger}a^{\dagger}a^{\dagger}a^{3}(a^{\#})^{*}a.$$

Post-multiplying the equality by $a^\dagger a$, one has

$$a^\dagger a^\dagger = a^\dagger a^\dagger a^\dagger a.$$

By [17, Lemma 2.11], $a^\dagger = a^\dagger a^\dagger a$, this infers $a \in R^{EP}$. Hence

$$a^\dagger a^\dagger = a^\dagger a + a^\dagger a^\dagger - (a^\#)^* a,$$

one gets $a^\dagger a = (a^\#)^* a$ and $a^\# = a^\dagger = a^\dagger a a^\dagger = (a^\#)^* a a^\dagger = (a^\#)^*$.Thus $a \in R^{Her}$. □

Lemma 6.4. *Suppose that $a \in R^\# \cap R^\dagger$. Then*
(1) $a^\dagger a^\dagger \in R^\#$ with $(a^\dagger a^\dagger)^\# = (aa^\#)^* a(aa^\#)^* a(aa^\#)^*$;
(2) $a^\dagger a^\dagger \in R^\dagger$ with $(a^\dagger a^\dagger)^\dagger = a(aa^\#)^* a$;
(3) $a^\dagger a^3 (a^\#)^* a \in R^{EP}$ with $(a^\dagger a^3(a^\#)^* a)^\dagger = a^\dagger a^* a^\dagger a^\#$.

Proof. 1) Noting that $a^\dagger (aa^\#)^* = a^\dagger = (aa^\#)^* a^\dagger$ and $a^\dagger a(aa^\#)^* = (aa^\#)^* = (aa^\#)^* aa^\dagger$. Then

$$(a^\dagger a^\dagger)((aa^\#)^* a(aa^\#)^* a(aa^\#)^*) = a^\dagger a^\dagger a(aa^\#)^* a(aa^\#)^* = a^\dagger (aa^\#)^* a(aa^\#)^* = (aa^\#)^*,$$

$$((aa^\#)^* a(aa^\#)^* a(aa^\#)^*)(a^\dagger a^\dagger) = (aa^\#)^* a(aa^\#)^* aa^\dagger a^\dagger = (aa^\#)^* a(aa^\#)^* a^\dagger = (aa^\#)^*,$$

$$(a^\dagger a^\dagger)((aa^\#)^* a(aa^\#)^* a(aa^\#)^*)(a^\dagger a^\dagger) = (aa^\#)^* (a^\dagger a^\dagger) = a^\dagger a^\dagger,$$

and $((aa^\#)^* a(aa^\#)^* a(aa^\#)^*)(a^\dagger a^\dagger)((aa^\#)^* a(aa^\#)^* a(aa^\#)^*) = (aa^\#)^* a(aa^\#)^* a(aa^\#)^*$.
Hence $a^\dagger a^\dagger \in R^\#$ and $(a^\dagger a^\dagger)^\# = (aa^\#)^* a(aa^\#)^* a(aa^\#)^*$.

2) Clearly,

$$(a^\dagger a^\dagger)(a(aa^\#)^* a) = a^\dagger (aa^\#)^* a = a^\dagger a,$$

$$(a(aa^\#)^* a)(a^\dagger a^\dagger) = a(aa^\#)^* a^\dagger = aa^\dagger,$$

$$(a^\dagger a^\dagger)(a(aa^\#)^* a)(a^\dagger a^\dagger) = a^\dagger a(a^\dagger a^\dagger) = a^\dagger a^\dagger,$$

and $(a(aa^\#)^* a)(a^\dagger a^\dagger)(a(aa^\#)^* a) = aa^\dagger (a(aa^\#)^* a) = a(aa^\#)^* a$.
Thus $a^\dagger a^\dagger \in R^\dagger$ and $(a^\dagger a^\dagger)^\dagger = a(aa^\#)^* a$.

3)

$$(a^\dagger a^3(a^\#)^* a)(a^\dagger a^* a^\dagger a^\#) = a^\dagger a^3(a^\#)^* a^* a^\dagger a^\# = a^\dagger a^3 a^\dagger a^\# = a^\dagger a,$$

$$(a^\dagger a^* a^\dagger a^\#)(a^\dagger a^3(a^\#)^* a) = a^\dagger a^* a^\dagger a(a^\#)^* a = a^\dagger a^* (a^\#)^* a = a^\dagger a,$$

$$(a^\dagger a^3(a^\#)^* a)(a^\dagger a^* a^\dagger a^\#)(a^\dagger a^3(a^\#)^* a) = a^\dagger a(a^\dagger a^3(a^\#)^* a) = a^\dagger a^3(a^\#)^* a,$$

$$(a^\dagger a^* a^\dagger a^\#)(a^\dagger a^3(a^\#)^* a)(a^\dagger a^* a^\dagger a^\#) = a^\dagger a(a^\dagger a^* a^\dagger a^\#) = a^\dagger a^* a^\dagger a^\#.$$

Hence $a^\dagger a^3(a^\#)^* a \in R^{EP}$ with $(a^\dagger a^3(a^\#)^* a)^\dagger = a^\dagger a^* a^\dagger a^\# = (a^\dagger a^3(a^\#)^* a)^\#$. □

Corollary 6.5. *Suppose that $a \in R^\# \cap R^\dagger$. Then $a \in R^{Her}$ if and only if $a^\dagger a^\dagger = a^\dagger a^* a^\dagger a^\#$.*

Proof. Since $a^\dagger a^3(a^\#)^* a \in R^{EP}$ by Lemma 6.4,

$$(a^\dagger a^3(a^\#)^* a + 1 - a^\dagger a)^{-1} = a^\dagger a^* a^\dagger a^\# + 1 - a^\dagger a.$$

By Corollary 6.3, we have $a \in R^{Her}$ if and only if $(a^\dagger a^3(a^\#)^* a + 1 - a^\dagger a)^{-1} = a^\dagger a^\dagger + 1 - a^\dagger a$. It follows that $a \in R^{Her}$ if and only if

$$a^\dagger a^* a^\dagger a^\# + 1 - a^\dagger a = a^\dagger a^\dagger + 1 - a^\dagger a.$$

This implies $a \in R^{Her}$ if and only if $a^\dagger a^\dagger = a^\dagger a^* a^\dagger a^\#$. □

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References

[1] O. M. Baksalary, G. P. H. Styan, G. Trenkler, On a matrix decomposition of Hartwig and Spindelböck, Linear Algebra Appl. 430 (10) (2009) 2798-2812.
[2] W. Chen, On EP elements, normal elements and partial isometries in rings with involution, Electron. J. Linear Algebra 23 (1) (2012) 553-561.
[3] D. Cvetkovíc, D. S. Djordjevíc, J. J. Koliha, Moore-Penrose inverse in rings with involution, Linear Algebra Appl. 426 (2007) 371-381.
[4] M. P. Drazin, Pseudoinverse in associative rings and semigroups, Amer. Math. Monthly 65 (1958) 506-514.
[5] R. E. Harte, M. Mbekhta, On generalized inverses in C^*--algebras, Studia Math. 103 (1992) 71-77.
[6] R. E. Hartwig, Block generalized inverses, Arch. Rational Mech. Anal. 61 (3) (1976) 197-251.
[7] R. E. Hartwig, K. Spindelböck, Matrices for which A^* and $A^\dagger$ commute, Linear Multilinear Algebra 14 (1984) 241-256.
[8] J. J. Koliha, The Drazin and Moore-Penrose inverse in C^*--algebras, Math. Proc. Royal Irish Acad. 99A (1999) 17-27.
[9] J. J. Koliha, P. Patrício, Elements of rings with equal spectral idempotents, J. Aust. Math. Soc. 72 (1) (2002) 137-152.
[10] D. Mosić, Generalized inverses, Faculty of Sciences and Mathematics, University of Niš, 2018.
[11] D. Mosić, D. S. Djordjević, Further results on partial isometries and EP elements in rings with involution, Math. Comput. Modelling 54 (1) (2011) 460-465.
[12] D. Mosić, D. S. Djordjević, Partial isometries and EP elements in rings with involution, Electron. J. Linear Algebra 18 (2009) 761-772.
[13] D. Mosić, D. S. Djordjević, Further results on partial isometries and EP elements in rings with involution, Math. Comput. Modelling 54 (1) (2011) 460-465.
[14] D. Mosić, D. S. Djordjević, J. J. Koliha, EP elements in rings, Linear Algebra Appl. 431 (2009) 527-535.
[15] R. Penrose, A generalized inverse for matrices, Proc. Cambridge Philos. Soc. 51 (1955) 406-413.
[16] Z. C. Xu, R. J. Tang, J. C. Wei, Strongly EP elements in a ring with involution, Filomat 34 (6) (2020) 2101-2107.
[17] D. D. Zhao, J. C. Wei. Strongly EP elements in rings with involution, J. Algebra Appl., (2022) 2250088 (10 pages), DOI: 10.1142/S0219498822500888.