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A Note on Poincaré Constants

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Abstract. In this note, we give two comparison results of Poincaré constants:

(i) Let X and Y be Banach spaces. We give a relationship between the X-valued p-Poincaré constant and the Y-valued q-Poincaré constant introduced by Laat and Salle for all $1 \leq p, q < \infty$ when the unit sphere $S(X)$ of X is uniformly homeomorphic to the unit sphere $S(Y)$ of Y.

(ii) We provide an explicit relationship between the nonlinear spectral gap introduced by Mimura and the Poincaré constant introduced by Laat and Salle.

1. Introduction

Recently, the classical Mazur map has been extended to Banach-valued L_p spaces. These extension results have been successfully applied in comparing of Poinceará constants (see, e.g., [5, 12, 18, 22]).

In this note, continuing in this direction, we first give a comparison between the X-valued p-Poincaré constant and the Y-valued q-Poincaré constant introduced by Laat and Salle for all $1 \leq p, q < \infty$ when the unit sphere $S(X)$ of X is uniformly homeomorphic to the unit sphere $S(Y)$ of Y. Then we establish an explicit quantity relationship between Mimura's nonlinear spectral gap [18] (a variant of Poincaré constant) and Laat ans Salle's Poincaré constant [12]. Let's first recall some related materials.

The variants of Poincaré constants have been developed in a series of works, including [1, 5, 10, 13, 17–20, 22, 25], for several geometric and computer science applications, though many fundamental questions remain open. On the other hand, the explicit comparison of Poincaré constants, is a natural approach. See meta problem in [21, Question 1.1]. In this context, the sphere equivalence is often useful.

Let (M, d) and (M', d') be two metric spaces and let $f : M \to M'$ be any map. The modulus of continuity of f is the function $\omega_f : [0, \infty) \to [0, \infty)$ defined by

$$\omega_f(t) = \sup\{d'(f(x), f(y)) : x, y \in M \text{ and } d(x, y) \leq t\}$$

The map f is said to be *uniformly continuous* if $\lim_{t\to+0} \omega_f(t) = 0$, and a *uniform homeomorphism* if f is a bijection and f and f^{-1} are both uniformly continuous. The metric spaces M and M' is *uniform homeomorphic* provided there is a uniform homeomorphism between them. The map f is said to be *α-Hölder* if there exist an exponent $\alpha \in (0, 1]$ and a constant $c > 0$ such that $\omega_f(t) \leq ct^\alpha$ for all $t > 0$. We say that M and M' are

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Hölder equivalent or more precisely (α, β)-Hölder homeomorphic if there exists a uniform homeomorphism $f : M \to M'$ such that f is α-Hölder and f^{-1} is β-Hölder for some $\alpha, \beta \in (0, 1]$. In the present paper, we are mainly interested in the case when M and M' are the unit spheres or the unit balls of Banach spaces.

It is well known that the unit spheres of L_p-spaces $1 \le p < \infty$ are mutually uniformly (resp., Hölder) equivalent. This is achieved by the so-called *Mazur map* first used by Mazur in 1929 [16], which is an explicit uniform homeomorphism, with explicit modulus of continuity, between the unit spheres of different L_p-spaces, for $1 \le p < \infty$. More precisely, for given $1 \le p, q < \infty$ and a measure space (Ω, μ), the Mazur map from $L_p(\Omega, \mu, \mathbb{C})$ to $L_q(\Omega, \mu, \mathbb{C})$ is defined by

$$M_{p,q}(f) = |f|^{\frac{p}{q}} \operatorname{sign}(f).$$

Then $M_{p,q}$ is a uniform homeomorphism between the unit spheres such that $M_{p,q}^{-1} = M_{q,p}$ and $M_{p,q}$ is Lipschitz on the unit sphere if $p \ge q$, and $\frac{p}{q}$-Hölder if $p < q$. Thus the Mazur map provides a Hölder classification of unit spheres of L_p-spaces, $1 \le p < \infty$.

Theorem 1.1 (Mazur,1929). *The unit spheres of $L_p(\Omega, \mu, \mathbb{C})$ and $L_q(\Omega, \mu, \mathbb{C})$ are uniformly homeomorphic (resp., $(\min\{\frac{p}{q}, 1\}, \min\{\frac{q}{p}, 1\})$-Hölder homeomorphic) for every $1 \le p, q < \infty$ and every measure space (Ω, μ).*

Since then, the Mazur map has been generalized to more general situations and has been proven to be a strong tool in applications by people (see [3, 7, 8, 11, 21, 24, 26–29]). In recent years, the Mazur map has been extended to Banach-valued L_p spaces. Firstly, its qualitative version of Banach-valued L_p spaces was obtain in [6, Cheng-Dong].

Theorem 1.2. *[6] Let X, Y be Banach spaces. If their unit spheres $S(X)$ and $S(Y)$ are uniformly homeomorphic. Then the unit spheres of $L_p(\Omega, \mu, X)$ and $L_q(\Omega, \mu, Y)$ are uniformly homeomorphic for every measure space (Ω, μ) and for every $1 \le p, q < \infty$.*

Recently, Chen and Cheng give the quantitative version of Banach-valued L_p spaces of Mazur's result.

Theorem 1.3. *[4] Let X, Y be Banach spaces. If their unit spheres $S(X)$ and $S(Y)$ are (α, β)-Hölder homeomorphic. Then the unit spheres of $L_p(\Omega, \mu, X)$ and $L_q(\Omega, \mu, Y)$ are $(\min\{\frac{p}{q}, \alpha\}, \min\{\frac{q}{p}, \beta\})$-Hölder homeomorphic for every measure space (Ω, μ) and for every $1 \le p, q < \infty$.*

These results have been successfully applied in comparing of Poinceaŕe constants (see, e.g., [5, 12, 18, 22]).

For example, Mimura [18, Theorem 3.8] first established Theorem 1.2 in this case when $L_p(\Omega, \mu, X)$ and $L_q(\Omega, \mu, Y)$ are the sequence spaces $l_p(X)$ and $l_q(Y)$, respectively, and $X = Y$ is uniformly convex. He [18, Proposition 3.9] also proved Theorem 1.3 in this case when $L_p(\Omega, \mu, X)$ and $L_q(\Omega, \mu, Y)$ are the sequence spaces $l_p(X)$ and $l_q(Y)$, respectively, and $p = q$. These results were applied in generating an explicit comparison of Mimura's nonlinear spectral gap (see [18, Theorem 4.1]). Later, Cheng and Dong [6, Theorem 1.2] showed Theorem 1.2. Recently, Laat and Salle [12, Lemma 3.10] firstly proved Theorem 1.3 for different p, q but $X = Y$. They applied it to obtain a more general version of Banach-valued Matoušek's extrapolation phenomenon than the aforementioned Mimura's and Cheng's ones, in the sense that it does not rely on the maximal degree of the graph (see [12, Proposition 3.9]). This latter fact has a big advantage in applications. For example it is needed for Naor's result in [22](see details in Naor's remark [22, Remark 46]). More recently, Chen and Cheng [4, Theorem 1.2] completed the proof of Theorem 1.3.

This note consists of closely related three sections. In Section 2 we will apply Theorem 1.2 to give a relationship between the X-valued p-Poincaré constant and the Y-valued q-Poincaré constant introduced by Laat and Salle for all $1 \le p, q < \infty$. In Section 3 we give an explicit relation between the nonlinear spectral gap introduced by Mimura and the Poincaré constant introduced by Laat and Salle, As its direct application, a non-coarse embedding result of a sequence of expander graphs is obtained.

Our notation and terminology for Banach spaces are standard, as may be found for example in [14] and [15]. For a Banach space X, by $S(X)$ and B_X denote the unit sphere and unit ball of X, respectively.

2. Laat and Salle's extrapolation result for Poincaré inequalities

In this section, we aim to provide an extension of Laat and Salle's extrapolation result [12, Proposition 3.9].

2.1. *p*-Poincaré inequality

In this note, we let $G = (V, \omega)$ be a finite, undirected, connected graph with the vertex set V and the weight function $\omega\colon V \times V \to [0, \infty)$ such that $\omega(x, y) = \omega(y, x)$ for every $x, y \in V$. Unweighted graphs correspond to the case when ω takes values in $\{0, 1\}$, in which case ω is the indicator function of the edge set. The degree of a vertex x in a graph $G = (V, \omega)$ is defined as the number $d_\omega(x) = \sum_{y\in V} \omega(x, y)$.

Following [12], let $G = (V, \omega)$ be a finite connected graph. Equip $V \times V$ with the probability measure $\mathbb{P}(x, y) = \frac{\omega(x,y)}{\sum\limits_{x',y'\in V} \omega(x',y')}$ and V with the probability measure $\nu(x) = \frac{d_\omega(x)}{\sum_y d_\omega(y)}$. Note that ν is the stationary probability measure for the random walk on G with transition probability $p(x \to y) = \frac{\omega(x,y)}{d_\omega(x)}$. It is also the pushforward measure of $\mathbb{P}$ under both maps $(x, y) \mapsto x$ and $(x, y) \mapsto y$. Let $f : V \to X$. Its gradient $\nabla f : V \times V \to X$ is defined by $\nabla f(e) = f(x) - f(y)$ if $e = (x, y)$.

Consider a weighted, finite graph $G = (V, E)$, a number $1 \le p < \infty$ and a Banach space X. We denote by $\pi_{p,G}(X)$ the smallest real number π such that for all $f\colon V \to X$, the inequality

$$\inf_{x\in X} \|f - x\|_{L^p(V,\nu;X)} \le \pi \|\nabla f\|_{L^p(V\times V,\mathbb{P};X)} \tag{1}$$

holds. Following Laat and Salle [12, Definition 3.1], we call $\pi_{p,G}(X)$ the X-valued p-Poincaré constant of G. On a finite graph, the inequality (1) is always satisfied for some $\pi > 0$. For $p = 2$ and $X = L_2$, the constant $\pi_{2,G}(L_2) = \pi_{2,G}(\mathbb{R})$ can be expressed in terms of the first non-zero eigenvalue of the discrete Laplacian (see [12, Proposition 3.3 (iii)]).

Let $(Z_0, Z_1, \dots)$ be the random walk on G with Z_0 (and hence Z_n for all $n \ge 0$) distributed as ν. In this setting, the X-valued p-Poincaré constant of G is the smallest real number π such that for all $f\colon V \to X$, the following inequality holds:

$$\inf_{x\in X} \mathbb{E}[\|f(Z_0) - x\|^p] \le \pi^p \mathbb{E}[\|f(Z_0) - f(Z_1)\|^p].$$

2.2. Sphere equivalence and Laat and Salle's extrapolation result

Now we provide an extension of Laat and Salle's extrapolation result [12, Proposition 3.9]. Its proof is inspired by the original one in [12, Proposition 3.9].

Theorem 2.1. *Let X and Y be Banach spaces and assume that $\varphi : S(X) \to S(Y)$ is a uniform homeomorphism. Then, for every $1 \le p, q < \infty$ and for every finite connected graph G, we have the following inequality*

$$\frac{1}{\pi_{p,G}(X)} \ge \delta_1^{-1}\Big(\delta_2^{-1}(1)\frac{1}{\pi_{q,G}(Y)}\Big).$$

Here $\delta_1 = \omega_{M_{p,q}}$, $\delta_2 = \omega_{M_{q,p}}$ and $M_{p,q}$ is the Banach-valued Mazur map defined by $M_{p,q}(h) = \|h\|^{\frac{p}{q}}\varphi(\frac{h}{\|h\|})$ from $B_{L_p(X)}$ onto $B_{L_q(Y)}$.

Proof. Let $\pi_{q,G}(Y)$ be the Y-valued q-Poincaré constant for a finite connected graph G, and simply denoted by π_q. Let $f \in L_p(V, \nu, X)$. We next want to prove

$$\delta_1^{-1}\Big(\delta_2^{-1}(1)\frac{1}{\pi_q}\Big)\inf_{x\in X}\|f - x\|_p \le (\mathbb{E}\|f(Z_0) - f(Z_1)\|^p)^{\frac{1}{p}}, \tag{2}$$

where $(Z_0, Z_1, \cdots)$ be the random walk on G distributed as ν and $\delta_1 = \omega_{M_{p,q}}$, $\delta_2 = \omega_{M_{q,p}}$. Let $\varphi : S(X) \to S(Y)$ be the uniform homeomorphism. Let $M_{p,q} : L_p(V, \nu, X) \to L_q(V, \nu, Y)$ be the Banach valued Mazur map.

By Theorem 1.2 , we know that $M_{p,q} : L_p(V,\nu,X) \to L_q(V,\nu,Y)$ is also a uniform homeomorphism when restricted to their unit balls.

As in the proof of [12, Proposition 3.9], we may assume that $\inf_{x\in X}\|f-x\|_p = 1$, and replacing f by $f-x$ for a suitable x, we may assume that $\|f\|_p \le 1$. Let $g = M_{p,q}(f)$. Then $\|g\|_q \le 1$. For every $x \in B_X$ we regard x as a constant function on V then $x \in L_p(V,\nu,X)$ with $\|x\|_p \le 1$. Thus $y = M_{p,q}(x) = \|x\|^{\frac{p}{q}}\varphi(\frac{x}{\|x\|})$ is also a constant function in $L_q(V,\nu,Y)$ with $y \in B_Y$. Since $M_{q,p}$ is a uniform homeomorphism, it follows that

$$\|f-x\|_p = \|M_{q,p}(g) - M_{q,p}(y)\|_p \le \delta_2(\|g-y\|_q).$$

By taking the infimum over $x \in B_X$ and note that $M_{p,q}(B_X) = B_Y$ (regard as constant function), we obtain

$$1 = \inf_{x\in X,\|x\|\le 1}\|f-x\|_p \le \inf_{y\in Y,\|y\|\le 1}\delta_2(\|g-y\|_q).$$

This in particular implies, for every $y \in Y, \|y\| \le 1$,

$$\delta_2(\|g-y\|_q) \ge 1,$$

and so

$$\|g-y\|_q \ge \delta_2^{-1}(1).$$

Thus

$$\inf_{y\in Y,\|y\|\le 1}\|g-y\|_q \ge \delta_2^{-1}(1).$$

From the definition of π_q it follows that

$$\frac{\delta_2^{-1}(1)}{\pi_q} \le (\mathbb{E}\|g(Z_0)-g(Z_1)\|^q)^{\frac{1}{q}}. \tag{3}$$

Since $M_{p,q}$ is also a uniform homeomorphism we have

$$(\mathbb{E}\|g(Z_0)-g(Z_1)\|^q)^{\frac{1}{q}} \le \delta_1((\mathbb{E}\|f(Z_0)-f(Z_1)\|^p)^{\frac{1}{p}}).$$

This together (3) finally gives

$$\frac{\delta_2^{-1}(1)}{\pi_q} \le \delta_1((\mathbb{E}\|f(Z_0)-f(Z_1)\|^p)^{\frac{1}{p}}),$$

and so

$$\delta_1^{-1}(\frac{\delta_2^{-1}(1)}{\pi_q}) \le (\mathbb{E}\|f(Z_0)-f(Z_1)\|^p)^{\frac{1}{p}}.$$

This gives

$$\inf_{x\in X}\|f-x\|_p \le \frac{1}{\delta_1^{-1}(\frac{\delta_2^{-1}(1)}{\pi_q})}(\mathbb{E}\|f(Z_0)-f(Z_1)\|^p)^{\frac{1}{p}},$$

and so

$$\frac{1}{\pi_{p,G}(X)} \ge \delta_1^{-1}\Big(\delta_2^{-1}(1)\frac{1}{\pi_{q,G}(Y)}\Big). \tag{4}$$

This completes the proof. □

3. Comparing of Poincaré constants

In this section, we will give an explicit relationship between Mimura's nonlinear spectral gap and Laat and Salle's Poincaré constant for a regular graph.

Let's recall some related materials. Following [19], a sequence of metric spaces $\{(X_n, d_{X_n})\}$ is said to embed coarsely (with the same moduli) into a metric space (Y, d_Y) if there exist two non-decreasing functions $\rho_1, \rho_2 : [0, \infty) \to [0, \infty)$ such that $\lim_{t\to\infty} \rho_1(t) = \infty$, and there exist mappings $f_n : X_n \to Y$, such that for all $n \in \mathbb{N}$ and $x, y \in X_n$ we have

$$\rho_1\Big(d_{X_n}(x, y)\Big) \le d_Y(f_n(x), f_n(y)) \le \rho_2\Big(d_{X_n}(x, y)\Big).$$

As said in [19], since coarse embeddability is a weak requirement, it is quite difficult to prove coarse non-embeddability. The nonlinear spectral gaps have been the well-known obstacle for coarse embeddability, as pioneered by Gromov [10].

We now turn to Mimura's nonlinear spectral gap [18], which is an variant of Poincaré constant. Consider a finite connected graph $G = (V, E)$, a number $1 \le p < \infty$ and a Banach space X. We denote by $\lambda_1(G; X, p)$ the largest real number λ such that for all $f : V \to X$, the inequality

$$\|f - m(f)\|^p_{l_p(V,X)} \le \frac{1}{\lambda}\|\nabla^E f\|^p_{l_p(V,X)}$$

holds. Here $m(f) = \frac{1}{|V|}\sum_{x\in V} f(x)$ is the mean value of f. By $\|h\|_{l_p(V,X)}$ we denote the p-norm $\Big(\sum_{x\in V}\|h(x)\|^p\Big)^{\frac{1}{p}}$ of $h : V \to X$ and $\|\nabla^E h\|_{l_p(V,X)}$ is the semi-norm defined by

$$\|\nabla^E h\|_{l_p(V,X)} = \Big(\sum_{(x,y)\in E}\|h(y) - h(x)\|^p\Big)^{\frac{1}{p}}.$$

Following Mimura [18, Definition 1.1], we call $\lambda_1(G; X, p)$ the (X, p)-spectral gap of G. In particular, for $p = 2$ and $X = (\mathbb{R}, |\cdot|)$, where we denote by $|\cdot|$ the Euclidean distance on $\mathbb{R}$, we have that $\lambda_1(G; \mathbb{R}, 2)$ is the just classical spectral gap of G. Following [18], a sequence of finite connected graphs $\{G_n = (V_n, E_n)\}_{n=1}^{\infty}$ is called (X, p)-anders if they have uniformly bounded degree, $\lim_{n\to\infty}|V_n| = \infty$ and $\inf_{n\in\mathbb{N}} \lambda_1(G_n; X, p) > 0$. It is classical that for a fixed $d \in \mathbb{N}$ a sequence of d-regular graphs is a sequence of expander graphs if and only if they are $(\mathbb{R}, 2)$-anders.

Gromov [9] observed that every sequence of expander graphs $\{G_n\}_{n\in\mathbb{N}}$ (here every G_n is considered as a metric space equipped with the shorted-path metric $d_{G_n}(x, y)$ between $x, y \in V_n$.) does not admit coarse embeddings into a Hilbert space. More generally, being (X, p)-anders for some fixed p and a Banach space X implies poor coarse embeddability into X, which means that if expander graphs $\{G_n\}_{n\in\mathbb{N}}$ are (X, p)-anders then $\{G_n\}_{n\in\mathbb{N}}$ do not coarsely embed into X.

On the other hand, Matoušek proved in [17] that being $(\mathbb{R}, p)$-anders does not depend on p (his strategy is often called Matoušek extrapolation [18, 19]). This was greatly generalized to Banach spaces setting by Mimura in [18], and finally Cheng proved in [5, Theorem 4.9] that for any fixed Banach space X the property of being an (X, p)-anders does not depend on $p \in [1, \infty)$. A key fact used there is that the property of being an (X, p)-anders is stable under sphere equivalence of Banach spaces (see [18, Theorem A] and [5, Theorem 4.9]).

We first need the following result.

Lemma 3.1. *Let $G = (V, E)$ be a finite connected graph with weight $\omega(x, y)$ and let X be a Banach space. Set $\upsilon(x) = \frac{\sum_{y\in V}\omega(x,y)}{\sum_{x',y'\in V}\omega(x',y')}$ and let $f : V \to X$. Assume that $m(f) = \sum_{x\in V} f(x)\upsilon(x)$ is the mean value of f with respect to the probability measure υ on V. Then for every $1 \le p < \infty$ we have*

$$\inf_{x\in X}\|f - x\|_{L_p(V,\upsilon,X)} \le \|f - m(f)\|_{L_p(V,\upsilon,X)} \le 2\inf_{x\in X}\|f - x\|_{L_p(V,\upsilon,X)}. \tag{5}$$

Proof. The first inequality in (5) is obvious. We turn to prove the second inequality. Note that, for a given $1 \le p < \infty$, the function $g(x) = \|x\|^p$ is a convex function on X. This gives

$$\begin{aligned}\int_V \|f(x) - m(f)\|^p dv &= \int_V \|\sum_{y\in V}(f(x) - f(y))v(y)\|^p dv \\ &\le \int_V \Big[\sum_{y\in V} \|f(x) - f(y)\|^p v(y)\Big] dv \\ &= \sum_{x,y\in V} \|f(x) - f(y)\|^p v(y)v(x).\end{aligned}$$

Choose $x_0 \in X$ such that

$$\|f - x_0\|_{L_p(V,v,X)} = \inf_{x\in X} \|f - x\|_{L_p(V,v,X)}.$$

By the triangle inequality we have

$$\begin{aligned}&\Big(\sum_{x,y\in V} \|f(x) - f(y)\|^p v(x)v(y)\Big)^{\frac{1}{p}} \\ =\;& \Big(\sum_{x,y\in V} \|[f(x) - f(y)][v(x)v(y)]^{\frac{1}{p}}\|^p\Big)^{\frac{1}{p}} \\ \le\;& \Big(\sum_{x,y\in V} (\|[f(x) - x_0][v(x)v(y)]^{\frac{1}{p}}\| + \|[f(y) - x_0][v(x)v(y)]^{\frac{1}{p}}\|)^p\Big)^{\frac{1}{p}} \\ \le\;& \Big(\sum_{x,y\in V} (\|[f(x) - x_0][v(x)v(y)]^{\frac{1}{p}}\|^p\Big)^{\frac{1}{p}} + \Big(\sum_{x,y\in V} (\|[f(y) - x_0][v(x)v(y)]^{\frac{1}{p}}\|^p\Big)^{\frac{1}{p}} \\ =\;& 2\Big(\sum_{x\in V} \|f(x) - x_0\|^p v(x)\Big)^{\frac{1}{p}} \\ =\;& 2\|f - x_0\|_{L_p(V,v,X)}.\end{aligned}$$

Thus

$$\|f - m(f)\|_{L_p(V,v,X)} \le 2 \inf_{x\in X} \|f - x\|_{L_p(V,v,X)}.$$

Which completes the proof. □

Theorem 3.2. *Let $G = (V, E)$ be a finite connected, d-regular, unweighted graphs. Then for every $1 \le p < \infty$ and every Banach space X, we have*

$$\frac{1}{2}\Big(\frac{1}{\lambda_1(G; p, X)}\Big)^{\frac{1}{p}} \le \pi_{p,G}(X) \le \Big(\frac{1}{\lambda_1(G; p, X)}\Big)^{\frac{1}{p}}.$$

Proof. Note first that in this case we have

$$m(f) = \frac{1}{|V|}\sum_{x\in V} f(x) = \sum_{x\in V} f(x)v(x),$$

where $v(x) = \frac{1}{|V|}$ for every $x \in V$. Thus for every $f : V \to X$, we have

$$\|f - m(f)\|^p_{l_p(V,X)} = \sum_{x\in V} \|f(x) - m(f)\|^p = |V| \int_V \|f(x) - m(f)\|^p dv.$$

This together with the inequality (5) and the definition of $\pi_{p,G}(X)$ (in short π_p) gives

$$\begin{aligned}
\|f-m(f)\|_{l_p(V,X)}^p &= |V|\Big\|f-m(f)\Big\|_{L_p(V,\nu,X)}^p \\
&\leq |V|\Big(2\inf_{x\in X}\|f-x\|_{L_p(V,\nu,X)}\Big)^p \\
&\leq |V|\Big(2\pi_p\|\nabla f\|_{L_p(V\times V,\mathbb{P},X)}\Big)^p \\
&= |V|2^p\pi_p^p\sum_{(x,y)\in V\times V}\|f(x)-f(y)\|^p\mathbb{P}(x,y) \\
&= |V|2^p\pi_p^p(\frac{1}{|V|}\sum_{(x,y)\in E}\|f(x)-f(y)\|^p) \\
&= 2^p\pi_p^p\|\nabla^E f\|_p^p.
\end{aligned}$$

Thus

$$\frac{1}{\lambda_1(G;p,X)}\leq 2^p\pi_p^p. \tag{6}$$

On the other hand, for every $f:V\to X$, by again the inequality (5) and the definition of $\lambda_1(G;p,X)$ we have

$$\begin{aligned}
\inf_{x\in X}\|f-x\|_{L_p(V,\nu,X)}^p\leq\|f-m(f)\|_{L_p(V,\nu,X)}^p &= \frac{1}{|V|}\|f-m(f)\|_{l_p(V,X)}^p \\
&\leq \frac{1}{|V|}(\frac{1}{\lambda_1(G;p,X)}\|\nabla^E f\|_p^p) \\
&= \frac{1}{|V|}\frac{1}{\lambda_1(G;p,X)}\Big(|V|\Big\|\nabla f\Big\|_{L_p(V\times V,\mathbb{P},X)}^p\Big) \\
&= \frac{1}{\lambda_1(G;p,X)}\|\nabla f\|_{L_p(V\times V,\mathbb{P},X)}^p.
\end{aligned}$$

This gives $\pi_{p,G}^p(X)\leq\frac{1}{\lambda_1(G;p;X)}$. Combine this with the inequality (6) we obtain

$$\frac{1}{2^p\lambda_1(G;p,X)}\leq\pi_{p,G}^p(X)\leq\frac{1}{\lambda_1(G;p,X)}.$$

Thus the proof of theorem 3.2 is complete. □

Remark 3.3. *Here we need to point out that some (implicit) relationship between the variants of Poincaré constants may be well-known to experts. As in Laat and Salle's paper [12], they wrote "Let us point out that our p-Poincaré constant differs (by a factor or power) from the p-Poincaré constants in [2] and [23], neither does it exactly coincide with the conventions of [19]", and for them it was enough about the relationship between these variants of Poincaré constants.*

Theorem 3.2 particularly implies the following non-coarse embedding result of expander graphs.

Corollary 3.4. *Let X be a Banach space. Assume that a sequence of expander graphs $\{G_n\}_{n\in\mathbb{N}}$ is (X,p)-anders in the sense of Laat and Salle, this means that $\{G_n\}_{n\in\mathbb{N}}$ is d-regular for some $d\in\mathbb{N}$, $\lim_{n\to\infty}|V_n|=\infty$ and $\inf_{n\in\mathbb{N}}\pi_{p,G_n}(X)>0$ for some pair (X,p) $(1\leq p<\infty)$, then the sequence of expander graphs does not coarsely embed into the Banach space X.*

Proof. By the assumption and Theorem 3.2, we have that the sequence of expander graphs $\{G_n\}_{n\in\mathbb{N}}$ is also (X,p)-anders in the sense of Mimura. Thus the sequence of expander graphs does not coarsely embed into X from the aforementioned fact. □

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