Filomat 36:16 (2022), 5449–5462
https://doi.org/10.2298/FIL2216449G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Chen-Ricci Inequalities in Slant Submersions for Complex Space Forms

Yılmaz Gündüzalp[a], Murat Polat[b]

[a]*Mathematics Education, Faculty of Education, Dicle University, 21280, Sur, Diyarbakır-Turkey*
[b]*Department of Mathematics, Faculty of Sci. Dicle University, 21280, Sur, Diyarbakır-Turkey*

Abstract. The goal of the present paper is to analyze sharp type inequalities including the scalar and Ricci curvatures of slant submersions in complex space forms.

1. Introduction

The intrinsic and extrinsic invariants that determined an inequality including the square mean curvature and the Ricci curvature of a submanifold in a real space form $R^n(c)$ devised by B.-Y. Chen in 1999 (see [5]). Also, for arbitrary submanifolds in an arbitrary Riemannian manifold, a generalization of this inequality was demonstrated in 2005 by B.-Y. Chen (see [6]). Subsequently, this inequality has been comprehensively examined for different ambient spaces by some authors who are achieved some results (see [3, 4, 6, 14, 16, 20, 21, 24, 29, 35]).

Given a $C^\infty-$submersion φ from a (semi)-Riemannian manifold (H_M, g_M) onto a (semi)-Riemannian manifold (H_N, g_N), according to the circumstances on the map $\varphi : (H_M, g_M) \rightarrow (H_N, g_N)$, we get the following: a slant submersion ([11, 28]), an almost Hermitian submersion ([22]), a (pseudo)-Riemannian submersion ([1], [7], [8],[12], [15],[18]), a quaternionic submersion ([13]), an anti-invariant submersion ([27]) ([19]), a Clairaut Submersion ([10]), conformal anti-invariant submersion([2]), a semi-invariant submersion ([17]), etc. As far as we know, Riemannian submersions were presented by B. O'Neill ([15]) and A. Gray ([8]) in 1960s, independently. Especially, by utilizing the notion of almost Hermitian submersions, B. Watson ([22]) presented some differential geometric features among fibers, base manifolds, and total manifolds. Subsequently, many results occur on this topic.

Watson in ([22]) studied Riemannian submersions between almost Hermitian manifolds under the name of holomorphic submersions. One of the most important consequences of this idea is that vertical and horizontal distributions are invariant under almost complex structure. He indicated that if the total manifold is a Kaehler manifold, then the base manifold is a Kaehler manifold. Slant submersions from almost Hermitian manifolds to Riemannian manifolds was introduced by Şahin in 2011 ([26]). In this article he showed that the geometry of slant submersions is quite different from holomorphic submersions.

2020 *Mathematics Subject Classification*. 2010; 53C15, 53B20, 53C50.
Keywords. Riemannian submersion, slant submersion, complex space form, Chen-Ricci inequality
Received: 29 September 2021; Revised: 24 December 2021; Accepted: 07 January 2022
Communicated by Mića Stanković
Corresponding author: Murat Polat
Email addresses: ygunduzalp@dicle.edu.tr (Yılmaz Gündüzalp), murat.polat@dicle.edu.tr (Murat Polat)

Let H_M be an almost Hermitian manifold with complex structure J of type $(1,1)$ and $H_{\overline{M}}$ a Riemannian manifold isometrically immersed in H_M. We can say that submanifolds of a Kaehler manifold are determined by the tangent bundle of the submanifold under the action of the complex structure of the ambient manifold. $T_xH_{\overline{M}}$ indicates the tangent space to $H_{\overline{M}}$ at the point x such that if $J(T_xH_{\overline{M}}) \subset T_xH_{\overline{M}}$, for every $x \in H_{\overline{M}}$, then A submanifold $H_{\overline{M}}$ is called holomorphic (complex). $T_xH^{\perp}_{\overline{M}}$ indicates the normal space to $H_{\overline{M}}$ at the any point x such that if $J(T_xH_{\overline{M}}) \subset T_xH^{\perp}_{\overline{M}}$ for every $x \in H_{\overline{M}}$, then $H_{\overline{M}}$ is called totally real. Slant submanifolds which a generalization of holomorphic and totally real submanifolds were introduced by Chen in ([25]). if for any $X \in T_xH_{\overline{M}}$ and $x \in H_{\overline{M}}$, the angle between $T_xH_{\overline{M}}$ and JX is a constant $\theta(X) \in \left[0, \frac{\pi}{2}\right]$,then the submanifold $H_{\overline{M}}$ is called slant ([25]) i.e, it does not depend on the choice of $X \in T_xH_{\overline{M}}$ and $x \in H_{\overline{M}}$. We can say that invariant and totally real immersions are slant immersions with $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant immersion which is neither invariant nor totally real is called a proper slant immersion.

In ([30]), the authors mainly studied Chen-Ricci inequality for Riemannian submersions from real space form. They also studied Chen-Ricci inequality for Riemannian maps from real space form and complex space forms by considering various cases such as anti-invariant, semi-invariant, Lagrangian etc. In this paper we consider Chen-Ricci inequality for slant Riemannian submersions from complex space forms. The rest of the paper is organized as follows: After giving some basic definitions and formulas in the second part, we research several inequalities including the Ricci and the scalar curvatures on vertical ($ker\varphi_*$) and horizontal ($(ker\varphi_*)^{\perp}$) distributions of slant submersions in complex space forms and then, we acquire Chen-Ricci inequalities on $ker\varphi_*$ and $(ker\varphi_*)^{\perp}$ of slant submersions in complex space forms.

2. Preliminaries

Let (H_M, g_M) be an almost Hermitian manifold. This implies ([23]) that H_M admits a tensor field J of type $(1,1)$ on H_M such that, $\forall X_1, X_2 \in \chi(H_M)$, we have

$$J^2 = -I, \quad g_M(JX_1, X_2) = -g_M(X_1, JX_2). \tag{1}$$

An almost Hermitian manifold H_M is called Kaehler manifold if

$$(\nabla^{LC}_{X_1})X_2 = 0, \quad \forall X_1, X_2 \in \chi(H_M), \tag{2}$$

here ∇^{LC} is the Levi-Civita connection on H_M. If $\{X_1, JX_1\}$ spans a plane section, the sectional curvature $F_{H_M}(X_1) = K_{H_M}(X_1 \wedge JX_1)$ of span$\{X_1, JX_1\}$ is called a sectional curvature. The Riemannian-Christoffel curvature tensor of a complex space form ([23]) $H_M(\alpha)$ of constant sectional curvature α satisfies

$$\begin{aligned} R_{H_M}(X_1, X_2, X_3, X_4) &= \frac{\alpha}{4}\{g_M(X_1, X_4)g_M(X_2, X_3) - g_M(X_1, X_3)g_M(X_2, X_4) \\ &+ g_M(JX_2, X_3)g_M(JX_1, X_4) - g_M(JX_1, X_3)g_M(JX_2, X_4) \\ &+ 2g_M(X_1, JX_2)g_M(JX_3, X_4)\} \end{aligned} \tag{3}$$

for all $X_1, X_2, X_3, X_4 \in \chi(H_M)$.

Let (H_M, g_M) and (H_N, g_N) be Riemannian manifolds. A Riemannian submersion is a smooth map $\varphi : H_M \to H_N$ which is onto and satisfies the following conditions:
(i) $\varphi_{*q} : T_qH_M \to T_{\varphi(q)}H_N$ is onto for all $q \in H_M$;
(ii) The fibres $\varphi_x^{-1}, x \in H_N$, are Riemannian submanifolds of H_M;
(iii) φ_{*q} preserves the length of the horizontal vectors.
The vectors tangent to the fibres are called vertical and those normal to the fibres are called horizontal. The tangent bundle of H_M splits as the Whitney sum of two distributions, the vertical one $ker\varphi_*$ and the orthogonal complementary distribution $(ker\varphi_*)^{\perp}$ called horizontal, and we denote by h and v the horizontal and vertical projections, respectively. A horizontal vector field X_1 on H_M is called as basic if X_1 is $\varphi-$related

to a vector field X_{1*} on H_N([15]). A Riemannian submersion $\varphi : H_M \to H_N$ specifies two $(1,2)$ tensor fields $\mathcal{T}$ and $\mathcal{A}$ on H_M, by the formulae([15]):

$$\mathcal{T}(E,G) = \mathcal{T}_E G = h\nabla^{LC}_E vG + v\nabla^{LC}_E hG \tag{4}$$

and

$$\mathcal{A}(E,G) = \mathcal{A}_E G = v\nabla^{LC}_E hG + h\nabla^{LC}_E vG \tag{5}$$

for all $E, G \in \chi(H_M)$.

Lemma 2.1. *([15]) Let $\varphi : (H_M, g_M) \to (H_N, g_N)$ be a Riemannian submersion. Then, we have:*

$$\mathcal{A}_{X_1}X_2 = -\mathcal{A}_{X_2}X_1, \;\; X_1, X_2 \in \chi((ker\varphi_*)^\perp); \tag{6}$$

$$\mathcal{T}_{U_1}U_2 = \mathcal{T}_{U_2}U_1, \;\; U_1, U_2 \in \chi(ker\varphi_*); \tag{7}$$

$$g_M(\mathcal{T}_{U_1}X_2, X_3) = -g_M(\mathcal{T}_{U_1}X_3, X_2), \;\; U_1 \in \chi(ker\varphi_*), \;\; X_2, X_3 \in \chi(H_M); \tag{8}$$

$$g_M(\mathcal{A}_{X_1}X_2, X_3) = -g_M(\mathcal{A}_{X_1}X_3, X_2), \;\; X_1 \in \chi((ker\varphi_*)^\perp), \;\; X_2, X_3 \in \chi(H_M). \tag{9}$$

Let $R^{H_M}, R^{H_N}, R^{ker\varphi_*}$ and $R^{(ker\varphi_*)^\perp}$ stand for the Riemannian curvature tensors of Riemannian manifolds H_M, H_N, the vertical distribution $ker\varphi_*$ and the horizontal distribution $(ker\varphi_*)^\perp$, respectively.

Lemma 2.2. *([15]) Let $\varphi : (H_M, g_M) \to (H_N, g_N)$ be a Riemannian submersion. Then, we have:*

$$\begin{aligned} R^{H_M}(U_1, U_2, U_3, U_4) &= R^{ker\varphi_*}(U_1, U_2, U_3, U_4) + g_M(\mathcal{T}_{U_1}U_4, \mathcal{T}_{U_2}U_3) \\ &- g_M(\mathcal{T}_{U_2}U_4, \mathcal{T}_{U_1}U_3), \end{aligned} \tag{10}$$

$$\begin{aligned} R^{H_M}(X_1, X_2, X_3, X_4) &= R^{(ker\varphi_*)^\perp}(X_1, X_2, X_3, X_4) - 2g_M(\mathcal{A}_{X_1}X_2, \mathcal{A}_{X_3}X_4) \\ &+ g_M(\mathcal{A}_{X_2}X_3, \mathcal{A}_{X_1}X_4) - g_M(\mathcal{A}_{X_1}X_3, \mathcal{A}_{X_2}X_4), \end{aligned} \tag{11}$$

$$\begin{aligned} R^{H_M}(X_1, U_1, X_2, U_2) &= g_M((\nabla^{LC}\mathcal{T})(U_1, U_2), X_2) + g_M((\nabla^{LC}\mathcal{A})(X_1, X_2), U_2) \\ &- g_M(\mathcal{T}_{U_1}X_1, \mathcal{T}_{U_2}X_2) + g_M(\mathcal{A}_{X_2}U_2, \mathcal{A}_{X_1}U_1), \end{aligned} \tag{12}$$

for all $X_1, X_2, X_3, X_4 \in \chi((ker\varphi_*)^\perp)$ and $U_1, U_2, U_3, U_4 \in \chi(ker\varphi_*)$.

Further, the $\mathcal{H}$ mean curvature of every fibre of φ Riemannian submersion is defined

$$\mathcal{H} = \frac{1}{s}\mathcal{N}, \;\; \mathcal{N} = \sum_{p=1}^{s} \mathcal{T}_{E_p}E_p, \tag{13}$$

where $\{E_1, E_2, ..., E_s\}$ forms an orthonormal basis for the vertical distribution $ker\varphi_*$. Also, φ has totally geodesic fibres if $\mathcal{T} = 0$ on $ker\varphi_*$.

Definition 2.3. *([26]) Let φ be a Riemannian submersion from an almost Hermitian manifold (H_{M_1}, g_{M_1}, J) onto a Riemannian manifold (H_{M_2}, g_{M_2}). If the angle $\theta(X)$ between JX and the space $ker\varphi_{*q}$ is a constant for any non-zero vector $X \in ker\varphi_{*q}; q \in H_{M_1}$ i.e. it is independent of the point $q \in H_{M_1}$ and choice of the tangent vector X in $ker\varphi_{*q}$, then we say that φ is a slant submersion. In this case, the angle θ is called the slant angle of the slant submersion.*

Example 2.4. *Define a map* $\psi : \mathcal{R}^4 \to \mathcal{R}^2$ *by*

$$\psi(x_1, x_2, x_3, x_4) = (x_1 \cos a - x_3 \sin a, x_2 \sin b - x_4 \cos b),$$

here a and b are constant. Then the map ψ *is a slant submersion with the slant angle* θ *with* $\cos\theta = |\sin(a+b)|$, *where* J *is the canonical complex structure of* $\mathcal{R}^4$ *defined by* $J(a_1, a_2, a_3, a_4) = (-a_2, a_1, -a_4, a_3)$.

Let φ be a Riemannian submersion from an almost Hermitian manifold H_{M_1} with the structure (g_{M_1}, J) onto a Riemannian manifold (H_{M_2}, g_{M_2}). Then for $X \in \Gamma(ker\varphi_*)$,we can write

$$JX = \phi X + \omega X, \tag{14}$$

here $\phi X \in \Gamma(ker\varphi_*)$ and $\omega X \in \Gamma(ker\varphi_*)^\perp$. Also for $Z \in \Gamma(ker\varphi_*)^\perp$, we get

$$JZ = BZ + CZ, \tag{15}$$

where BZ and CZ are vertical and horizontal parts of JZ. Using (1), (14) and (15) we get

$$C^2X = -X - \omega BX \tag{16}$$

Example 2.5. *([26]) Every Hermitian submersion from an almost Hermitian manifold onto an almost Hermitian manifold is a slant submersion with* $\theta = 0$.

Example 2.6. *([26]) Every anti-invariant Riemannian submersion from an almost Hermitian manifold to a Riemannian manifold is a slant submersion with* $\theta = \frac{\pi}{2}$.

Theorem 2.7. *([26]) Let* φ *be a Riemannian submersion from an almost Hermitian manifold* (H_{M_1}, g_{M_1}, J) *onto a Riemannian manifold* (H_{M_2}, g_{M_2}). *Then* φ *is a proper slant submersion if and only if there exists a constant* $\lambda \in [-1, 0]$ *such that*

$$\phi^2 X = \lambda X \tag{17}$$

for $X \in \Gamma(ker\varphi_*)$. *If* φ *is a proper slant submersion, then* $\lambda = -\cos^2\theta$.

Using the above theorem, we have

$$g_{M_1}(\phi X, \phi Y) = \cos^2\theta g_{M_1}(X, Y) \tag{18}$$

$$g_{M_1}(\omega X, \omega Y) = \sin^2\theta g_{M_1}(X, Y) \tag{19}$$

3. Inequalities for slant submersions

Let's first give the following result:
Since φ is a slant submersion, and using (3) and (10) we get:

Lemma 3.1. $(H_M(\alpha), g_M)$ *and* (H_N, g_N) *denote a complex space form and a Riemannian manifold and let* $\varphi : (H_M(\alpha), g_M) \to (H_N, g_N)$ *be a slant submersion. Then, any for* $U_1, U_2, U_3, U_4 \in \chi(ker\varphi_*)$ *we obtain*

$$\begin{aligned} R^{ker\varphi_*}(U_1, U_2, U_3, U_4) \quad = \quad & \frac{\alpha}{4}\{g_M(U_1, U_4)g_M(U_2, U_3) \\ & -g_M(U_1, U_3)g_M(U_2, U_4) \\ & +g_M(\varphi U_1, U_4)g_M(\varphi U_2, U_3) \\ & -g_M(\varphi U_1, U_3)g_M(\varphi U_2, U_4) \\ & +2g_M(U_1, \varphi U_2)g_M(\varphi U_3, U_4)\} \\ & -g_M(\mathcal{T}_{U_1}U_4, \mathcal{T}_{U_2}U_3) + g_M(\mathcal{T}_{U_2}U_4, \mathcal{T}_{U_1}U_3), \end{aligned} \tag{20}$$

$$K^{ker\varphi_*}(U_1,U_2) = \frac{\alpha}{4}\{g_M^2(U_1,U_2) - \|U_1\|^2\|U_2\|^2 - 3g_M^2(\varphi U_1,U_2)\} - \|\mathcal{T}_{U_1}U_2\|^2 + g_M(\mathcal{T}_{U_2}U_2,\mathcal{T}_{U_1}U_1), \tag{21}$$

where $K^{ker\varphi_*}$ is a bi-sectional curvature of $ker\varphi_*$.

Let $\varphi : H_M(\alpha) \to H_N$ be an slant submersion. For every node $k \in H_M$, let $\{E_1,...,E_{2r},e_1,...,e_{2n}\}$ be an orthonormal basis of $T_kH_M(\alpha)$ such that $ker\varphi_* = span\{E_1,...,E_{2r}\}$, $(ker\varphi_*)^\perp = span\{\csc\theta\omega e_1,...,\csc\theta\omega e_{2n}\}$. Now, if we take $U_4 = U_1$ and $U_2 = U_3 = E_i$, $i = 1,2,...,2r$ in (20), and using (13) then we arrive at

$$Ric^{ker\varphi_*}(U_1) = \frac{\alpha}{4}\left\{(2r-1+3\cos^2\theta)g_M(U_1,U_1)\right\} -2rg_M(\mathcal{T}_{U_1}U_1,\mathcal{H}) + \sum_{i=1}^{2r} g_M(\mathcal{T}_{E_i}U_1,\mathcal{T}_{U_1}E_i). \tag{22}$$

From here, we get:

Theorem 3.2. *Let $\varphi : (H_M(\alpha),g_M) \to (H_N,g_N)$ be a slant submersion. Then, we have*

$$Ric^{ker\varphi_*}(U_1) \geq \frac{\alpha}{4}\left\{(2r-1+3\cos^2\theta)g_M(U_1,U_1)\right\} - 2rg_M(\mathcal{T}_{U_1}U_1,\mathcal{H}).$$

the equality status of the inequality satisfies if and only if every fibre is totally geodesic.

we can state the following result:

Corollary 3.3. *Let $\varphi : (H_M(\alpha),g_M) \to (H_N,g_N)$ be an anti-invariant Riemannian submersion with $\theta = \frac{\pi}{2}$. In this case,*

$$Ric^{ker\varphi_*}(U_1) \geq \frac{\alpha}{4}\{(2r-1)g_M(U_1,U_1)\} - 2rg_M(\mathcal{T}_{U_1}U_1,\mathcal{H})$$

the equality status of the inequality satisfies if and only if every fibre is totally geodesic.

Since the fibers are minimal in a Kaehler submersion ([7]), the following result can be written.

Corollary 3.4. *Let $\varphi : (H_M(\alpha),g_M) \to (H_N,J_N,g_N)$ be a Hermitian submersion with $\theta = 0$. In this case,*

$$Ric^{ker\varphi_*}(U_1) \geq \frac{\alpha}{2}\{(r+1)g_M(U_1,U_1)\}.$$

From (22), the following result is obtained.

Corollary 3.5. *Let $\varphi : (H_M(\alpha),g_M) \to (H_N,J_N,g_N)$ be a holomorphic Riemannian submersion. Then, the fibers are Einstein manifolds, i.e,*

$$Ric^{ker\varphi_*}(U_1) = \frac{\alpha}{2}(r+1)g_M(U_1,U_1).$$

Now, let's find the scalar curvature of the fiber from equation (22). Taking $U_1 = E_j$, $j = 1,...,2r$ and using (7), then we obtain

$$2scal^{ker\varphi_*} = \frac{\alpha}{4}\left\{2r(2r-1+3\cos^2\theta)\right\} - 4r^2\|\mathcal{H}\|^2 + \sum_{i,j=1}^{2r} g_M(\mathcal{T}_{E_i}E_j,\mathcal{T}_{E_i}E_j). \tag{23}$$

From here, the following theorem can be written.

Theorem 3.6. *Let $\varphi : (H_M(\alpha), g_M) \to (H_N, g_N)$ be a slant submersion. Then, we have*

$$2scal^{ker\varphi_*} \geq \frac{\alpha}{4}\left\{2r(2r-1+3\cos^2\theta)\right\} - 4r^2\|\mathcal{H}\|^2.$$

the equality status of the inequality satisfies if and only if every fibre is totally geodesic.

From (23), the following results can be written.

Corollary 3.7. *Let $\varphi : (H_M(\alpha), g_M) \to (H_N, g_N)$ be an anti-invariant Riemannian submersion with $\theta = \frac{\pi}{2}$. In this case,*

$$2scal^{ker\varphi_*} \geq \frac{\alpha}{2}\{r(2r-1)\} - 4r^2\|\mathcal{H}\|^2$$

the equality status of the inequality satisfies if and only if every fibre is totally geodesic.

Corollary 3.8. *Let $\varphi : (H_M(\alpha), g_M) \to (H_N, J_N, g_N)$ be a Hermitian submersion with $\theta = 0$. In this case,*

$$2scal^{ker\varphi_*} \geq \alpha r(r+1).$$

Since φ is a slant submersion, and using (3),(11), (15) We can write the following lemma:

Lemma 3.9. *Let $\varphi : (H_M(\alpha), g_M) \to (H_N, g_N)$ be a slant submersion. Then, for $X_1, X_2, X_3, X_4 \in \chi((ker\varphi_*)^\perp)$ we have*

$$\begin{aligned} R^{(ker\varphi_*)^\perp}(X_1,X_2,X_3,X_4) = {} & \frac{\alpha}{4}\{g_M(X_1,X_4)g_M(X_2,X_3) \\ & -g_M(X_1,X_3)g_M(X_2,X_4) \\ & +g_M(CX_1,X_4)g_M(CX_2,X_3) \\ & -g_M(CX_1,X_3)g_M(CX_2,X_4) \\ & +2g_M(X_1,CX_2)g_M(CX_3,X_4)\} \\ & +2g_M(\mathcal{A}_{X_1}X_2,\mathcal{A}_{X_3}X_4) \\ & -g_M(\mathcal{A}_{X_2}X_3,\mathcal{A}_{X_1}X_4) \\ & +g_M(\mathcal{A}_{X_1}X_3,\mathcal{A}_{X_2}X_4), \end{aligned} \tag{24}$$

$$\begin{aligned} K^{(ker\varphi_*)^\perp}(X_1,X_2) = {} & \frac{\alpha}{4}\{g_M^2(X_1,X_2) - \|X_1\|^2\|X_2\|^2 - 3g_M^2(CX_1,X_2)\} \\ & +3\|\mathcal{A}_{X_1}X_2\|^2, \end{aligned} \tag{25}$$

here $K^{(ker\varphi_*)^\perp}$ is a bi-sectional curvature of $(ker\varphi_*)^\perp$.

Since $\mathcal{A}$ =0 for Kaehler submersion ([7]), we have:

Corollary 3.10. *Let $\varphi : (H_M(\alpha), g_M) \to (H_N, g_N)$ be a Hermitian submersion with $\theta = 0$. In this case, we have*

i.

$$\begin{aligned} R^{(ker\varphi_*)^\perp}(X_1,X_2,X_3,X_4) = {} & \frac{\alpha}{4}\{g_M(X_1,X_4)g_M(X_2,X_3) - g_M(X_1,X_3)g_M(X_2,X_4) \\ & +g_M(CX_1,X_4)g_M(CX_2,X_3) - g_M(CX_1,X_3)g_M(CX_2,X_4) \\ & +2g_M(X_1,CX_2)g_M(CX_3,X_4)\} \end{aligned}$$

ii.

$$K^{(ker\varphi_*)^\perp}(X_1,X_2) = \frac{\alpha}{4}\{g_M^2(X_1,X_2) - \|X_1\|^2\|X_2\|^2 - 3g_M^2(CX_1,X_2)\}.$$

Now, if we take $X_4 = X_1$ and $X_2 = X_3 = \csc\theta\omega e_j$, $j = 1, 2, ..., 2n$ in (24), and using (6),(16) then we get

$$Ric^{(ker\varphi_*)^\perp}(X_1) = \frac{\alpha}{4}\{(2n+2)g_M(X_1, X_1) + 3g_M(\omega BX_1, X_1)\} + 3\sum_{j=1}^{2n} g_M(\mathcal{A}_{X_1}\csc\theta\omega e_j, \mathcal{A}_{\csc\theta\omega e_j}X_1). \tag{26}$$

From here, the following theorem can be written.

Theorem 3.11. *Let $\varphi : (H_M(\alpha), g_M) \to (H_N, g_N)$ be a slant submersion. Then,*

$$Ric^{(ker\varphi_*)^\perp}(X_1) \leq \frac{\alpha}{4}\{(2n+2)g_M(X_1, X_1) + 3g_M(\omega BX_1, X_1)\}$$

the equality status of the inequality satisfies if and only if horizontal distribution is integrable.

From (26), the following result can be written.

Corollary 3.12. *Let φ be a holomorphic Riemannian submersion from a Kaehler manifold $(H_M(\alpha), g_M)$ onto a Kaehler manifold (H_N, J_N, g_N) with $\theta = 0$. Then,*

$$Ric^{(ker\varphi_*)^\perp}(X_1) = \frac{\alpha}{4}\{(2n+2)g_M(X_1, X_1) + 3g_M(\omega BX_1, X_1)\}.$$

Taking $X_1 = \csc\theta\omega e_i$, $i = 1, 2, ..., 2n$ in (26), then we obtain:

$$2scal^{(ker\varphi_*)^\perp} = \frac{\alpha}{4}\{2n(2n+2) + 6n + 3tr(\omega B)\} - 3\sum_{i,j=1}^{2n} g_M(\mathcal{A}_{\csc\theta\omega e_i}\csc\theta\omega e_j, \mathcal{A}_{\csc\theta\omega e_i}\csc\theta\omega e_j). \tag{27}$$

Then, we write

$$2scal^{(ker\varphi_*)^\perp} \leq \frac{\alpha}{4}\{2n(2n+2) + 6n + 3tr(\omega B)\}. \tag{28}$$

Thus, we can give:

Theorem 3.13. *Let $\varphi : (H_M(\alpha), g_M) \to (H_N, g_N)$ be a slant submersion. Then,*

$$2scal^{(ker\varphi_*)^\perp} \leq \frac{\alpha}{4}\{2n(2n+2) + 6n + 3tr(\omega B)\}$$

The equality status of (28) satisfies if and only if $(ker\varphi_*)^\perp$ horizontal distribution is integrable.

From (27), Since $\mathcal{A}$ =0 for Kaehler submersion, the following result can be written.

Corollary 3.14. *Let φ be a holomorphic Riemannian submersion from a Kaehler manifold $(H_M(\alpha), g_M)$ onto a Kaehler manifold (H_N, J_N, g_N) with $\theta = 0$. Then,*

$$2scal^{(ker\varphi_*)^\perp} = \frac{\alpha}{4}\{2n(2n+2) + 6n + 3tr(\omega B)\}$$

4. Chen-Ricci inequalities for slant submersions

Let $(H_M(\alpha), g_M)$ be a complex space form, (H_N, g_N) a Riemannian manifold and $\varphi : H_M(\alpha) \to H_N$ be a slant submersion. For every node $p \in H_M$, let $\{E_1, ..., E_{2r}, \csc\theta\omega e_1, ..., \csc\theta\omega e_{2n}\}$ be an orthonormal basis of $T_pH_M(\alpha)$ such that $ker\varphi_* = span\{E_1, ..., E_{2r}\}$ and $(ker\varphi_*)^\perp = span\{\csc\theta\omega e_1, ..., \csc\theta\omega e_{2n}\}$. Let's state $\mathcal{T}^t_{ij}$ by

$$\mathcal{T}^t_{ij} = g_M(\mathcal{T}_{E_i}E_j, \csc\theta\omega e_t) \tag{29}$$

where $1 \le i, j \le 2r$ and $1 \le t \le 2n$. Similarly, let's state $\mathcal{A}^\alpha_{ij}$ by

$$\mathcal{A}^\alpha_{ij} = g_M(\mathcal{A}_{\csc\theta\omega ei}e_j, E_\alpha) \tag{30}$$

in which $1 \le i, j \le 2n$ and $1 \le \alpha \le 2r$ and we employee

$$\delta(\mathcal{N}) = \sum_{i=1}^{2n}\sum_{k=1}^{2r}((\nabla^{LC}\mathcal{T})_{E_k}E_k, \csc\theta\omega e_i). \tag{31}$$

Now, from *(19)*, we get

$$2scal^{ker\varphi_*} = \frac{\alpha}{4}\left\{2r(2r-1+3\cos^2\theta)\right\} - 4r^2\|\mathcal{H}\|^2 + \sum_{i,j=1}^{2r} g_M(\mathcal{T}_{E_i}E_j, \mathcal{T}_{E_i}E_j).$$

Using *(7)* and *(29)*, we arrive at

$$2scal^{ker\varphi_*} = \frac{\alpha}{4}\left\{2r(2r-1+3\cos^2\theta)\right\} - 4r^2\|\mathcal{H}\|^2 + \sum_{t=1}^{2n}\sum_{i,j=1}^{2r}(\mathcal{T}^t_{ij})^2. \tag{32}$$

From ([9]), we know that

$$\begin{aligned}\sum_{t=1}^{2n}\sum_{i,j=1}^{2r}(\mathcal{T}^t_{ij})^2 &= \frac{1}{2}4r^2\|\mathcal{H}\|^2 + \frac{1}{2}\sum_{t=1}^{2n}\left[\mathcal{T}^t_{11} - \mathcal{T}^t_{22} - ... - \mathcal{T}^t_{rr}\right]^2 \\ &\quad +2\sum_{t=1}^{2n}\sum_{j=2}^{2r}(\mathcal{T}^t_{1j})^2 - 2\sum_{t=1}^{2n}\sum_{2\le i<j\le 2r}\left[\mathcal{T}^t_{ii}\mathcal{T}^t_{jj} - \left(\mathcal{T}^t_{ij}\right)^2\right].\end{aligned} \tag{33}$$

If we put *(33)* in *(32)*, we obtain

$$\begin{aligned}2scal^{ker\varphi_*} &= \frac{\alpha}{4}\left\{2r(2r-1+3\cos^2\theta)\right\} - 2r^2\|\mathcal{H}\|^2 \\ &\quad +\frac{1}{2}\sum_{t=1}^{2n}\left[\mathcal{T}^t_{11} - \mathcal{T}^t_{22} - ... - \mathcal{T}^t_{rr}\right]^2 + 2\sum_{t=1}^{2n}\sum_{j=2}^{2r}(\mathcal{T}^t_{1j})^2 \\ &\quad -2\sum_{t=1}^{2n}\sum_{2\le i<j\le 2r}\left[\mathcal{T}^t_{ii}\mathcal{T}^t_{jj} - \left(\mathcal{T}^t_{ij}\right)^2\right].\end{aligned} \tag{34}$$

From here, we have

$$2scal^{ker\varphi_*} \ge \frac{\alpha}{4}\left\{2r(2r-1+3\cos^2\theta)\right\} - 2r^2\|\mathcal{H}\|^2 - 2\sum_{t=1}^{2n}\sum_{2\le i<j\le 2r}\left[\mathcal{T}^t_{ii}\mathcal{T}^t_{jj} - \left(\mathcal{T}^t_{ij}\right)^2\right]. \tag{35}$$

On the other hand, from *(10)*, taking $U_1 = U_4 = E_i, U_2 = U_3 = E_j$ and using (29), we have

$$2\sum_{2\le i<j\le 2r} R^{H_M}(E_i, E_j, E_j, E_i) = 2\sum_{2\le i<j\le 2r} R^{ker\varphi_*}(E_i, E_j, E_j, E_i) + 2\sum_{t=1}^{2n}\sum_{2\le i<j\le s}\left[\mathcal{T}^t_{ii}\mathcal{T}^t_{jj} - \left(\mathcal{T}^t_{ij}\right)^2\right].$$

From the last equality, (35) can be written as

$$2scal^{ker\varphi_*} \geq \frac{\alpha}{4}\left\{2r(2r-1+3\cos^2\theta)\right\} - 2r^2\|\mathcal{H}\|^2 + 2\sum_{2\leq i<j\leq 2r} R^{ker\varphi_*}(E_i,E_j,E_j,E_i) \\ -2\sum_{2\leq i<j\leq 2r} R^{H_M}(E_i,E_j,E_j,E_i). \tag{36}$$

Also, we know that

$$2scal^{ker\varphi_*} = 2\sum_{2\leq i<j\leq 2r} R^{ker\varphi_*}(E_i,E_j,E_j,E_i) + 2\sum_{j=1}^{2r} R^{ker\varphi_*}(E_1,E_j,E_j,E_1),$$

If we put the last equality in (36), then we have

$$2Ric^{ker\varphi_*}(E_1) \geq \frac{\alpha}{4}\left\{2r(2r-1+3\cos^2\theta)\right\} - 2r^2\|\mathcal{H}\|^2 - 2\sum_{2\leq i<j\leq 2r} R^{H_M}(E_i,E_j,E_j,E_i).$$

Since $H_M(\alpha)$ is a complex space form, curvature tensor R^{H_M} of H_M provides equation (3), therefore we acquire

$$Ric^{ker\varphi_*}(E_1) \geq \frac{\alpha}{4}(2r-1+3\cos^2\theta) - r^2\|\mathcal{H}\|^2.$$

Thus, we can give the following result:

Theorem 4.1. *Let $\varphi : H_M(\alpha) \to H_N$ be a slant submersion from a complex space form $(H_M(\alpha), g_M)$ onto a Riemannian manifold (H_N, g_N). Then we have*

$$Ric^{ker\varphi_*}(E_1) \geq \frac{\alpha}{4}(2r-1+3\cos^2\theta) - r^2\|\mathcal{H}\|^2$$

the equality status of the inequality satisfies if and only if

$$\mathcal{T}^t_{11} = \mathcal{T}^t_{22} + \dots + \mathcal{T}^t_{rr}$$

$$\mathcal{T}^t_{1j} = 0,\ j = 2,\dots,2r.$$

From (23), we have

$$2scal^{(ker\varphi_*)^\perp} = \frac{\alpha}{4}\{2n(2n+2)+6n+3tr(\omega B)\} - 3\sum_{i,j=1}^{2n} g_M(\mathcal{A}_{\csc\theta\omega e_i}\csc\theta\omega e_j, \mathcal{A}_{\csc\theta\omega e_i}\csc\theta\omega e_j).$$

Using (16) and (30), then we have

$$2scal^{(ker\varphi_*)^\perp} = \frac{\alpha}{4}\{2n(2n+2)+6n+3tr(\omega B)\} - 3\sum_{\alpha=1}^{2r}\sum_{i,j=1}^{2n}(\mathcal{A}^\alpha_{ij})^2. \tag{37}$$

From (6), then (37) turns into

$$2scal^{(ker\varphi_*)^\perp} = \frac{\alpha}{4}\{2n(2n+2)+6n+3tr(\omega B)\} - 6\sum_{\alpha=1}^{2r}\sum_{j=2}^{2n}(X_1)^2 - 6\sum_{\alpha=1}^{2r}\sum_{2\leq i<j\leq 2n}(\mathcal{A}^\alpha_{ij})^2. \tag{38}$$

Moreover, from (11), taking $X_1 = X_4 = \csc\theta\omega e_i, X_2 = X_3 = \csc\theta\omega e_j$ and using (30) we obtain

$$\begin{aligned} & 2\sum_{2\le i<j\le 2n} R^{H_M}(\csc\theta\omega e_i, \csc\theta\omega e_j, \csc\theta\omega e_j, \csc\theta\omega e_i) \\ = \;& 2\sum_{2\le i<j\le m} R^{(ker\varphi_*)^\perp}(\csc\theta\omega e_i, \csc\theta\omega e_j, \csc\theta\omega e_j, \csc\theta\omega e_i) + 6\sum_{\alpha=1}^{2r}\sum_{2\le i<j\le 2n}(\mathcal{A}_{ij}^{\alpha})^2 \end{aligned} \tag{39}$$

If we consider (34) in (33), then we obtain

$$\begin{aligned} 2scal^{(ker\varphi_*)^\perp} = \;& \frac{\alpha}{4}\{2n(2n+2)+6n+3tr(\omega B)\} - 6\sum_{\alpha=1}^{2r}\sum_{j=2}^{2n}(X_1)^2 \\ & -2\sum_{2\le i<j\le 2n} R^{H_M}(\csc\theta\omega e_i, \csc\theta\omega e_j, \csc\theta\omega e_j, \csc\theta\omega e_i) \\ & +2\sum_{2\le i<j\le 2n} R^{(ker\varphi_*)^\perp}(\csc\theta\omega e_i, \csc\theta\omega e_j, \csc\theta\omega e_j, \csc\theta\omega e_i) \end{aligned}$$

Since H_M is a complex space form, curvature tensor R^{H_M} of H_M satisfies (3), hence we have

$$2Ric^{(ker\varphi_*)^\perp}(\csc\theta\omega e_1) = \frac{\alpha}{4}\{(10(n+1)+6tr(\omega B))\} - 6\sum_{\alpha=1}^{2r}\sum_{j=2}^{2n}(\mathcal{A}_{1j}^{\alpha})^2.$$

Then, we can write

$$Ric^{(ker\varphi_*)^\perp}(\csc\theta\omega e_1) \le \frac{\alpha}{4}\{(5(n+1)+3tr(\omega B))\}.$$

Thus, we can give the following result:

Theorem 4.2. *Let $\varphi : H_M(\alpha) \to H_N$ be a slant submersion from a complex space form $(H_M(\alpha), g_M)$ onto a Riemannian manifold (H_N, g_N). Then we have*

$$Ric^{(ker\varphi_*)^\perp}(\csc\theta\omega e_1) \le \frac{\alpha}{4}\{(5(n+1)+3tr(\omega B))\}$$

the equality status of the inequality satisfies if and only if

$$\mathcal{A}_{1j} = 0,\; j = 2, ..., 2n.$$

Next, we can calculate the inequality of Chen-Ricci among the $ker\varphi_*$ and $(ker\varphi_*)^\perp$. The *scal* scalar curvature of $H_M(\alpha)$ is defined as

$$2scal = \sum_{t=1}^{2n} Ric(\csc\theta\omega e_t, \csc\theta\omega e_t) + \sum_{k=1}^{2r} Ric(E_k, \csc\theta\omega e_k),$$

$$\begin{aligned} 2scal = \;& \sum_{j,k=1}^{2r} R^{H_M}(E_j, E_k, E_k, E_j) + \sum_{i=1}^{2n}\sum_{k=1}^{2r} R^{H_M}(\csc\theta\omega e_i, E_k, E_k, \csc\theta\omega e_i) \\ & + \sum_{i,t=1}^{2n} R^{H_M}(\csc\theta\omega e_i, \csc\theta\omega e_t, \csc\theta\omega e_t, \csc\theta\omega e_i) \\ & + \sum_{t=1}^{2n}\sum_{j=1}^{2r} R^{H_M}(E_j, \csc\theta\omega e_t, \csc\theta\omega e_t, E_j). \end{aligned} \tag{40}$$

Since $H_M(\alpha)$ is a complex space form, using *(35)* and *(3)*, we have

$$2scal = \frac{\alpha}{4}\{2r(2r+2+3\cos^2\theta)+2n(2n+5)+8nr+9tr(\omega B)\}. \tag{41}$$

On the other hand, using the equations *(10)*,*(11)* and *(12)*, we obtain also the *scal* scalar curvature of $H_M(\alpha)$ as

$$\begin{aligned} 2scal &= 2scal^{ker\varphi_*}+2scal^{(ker\varphi_*)^\perp}+4r^2\|\mathcal{H}\|^2+\sum_{j,k=1}^{2r} g_M(\mathcal{T}_{E_k}E_j,\mathcal{T}_{E_k}E_j)\\ &+3\sum_{i,t=1}^{2n} g_M(\mathcal{A}_{\csc\theta\omega e_i}\csc\theta\omega e_t,\mathcal{A}_{\csc\theta\omega e_i}\csc\theta\omega e_t)-\sum_{i=1}^{2n}\sum_{k=1}^{2r} g_M((\nabla^{LC}\mathcal{T})_{E_k}E_k,\csc\theta\omega e_i)\\ &+\sum_{i=1}^{m}\sum_{k=1}^{s}\left\{\begin{array}{c} g_M(\mathcal{T}_{E_k}\csc\theta\omega e_i,\mathcal{T}_{E_k}\csc\theta\omega e_i)\\ -g_M(\mathcal{A}_{\csc\theta\omega e_i}E_k,\mathcal{A}_{\csc\theta\omega e_i}E_k)\end{array}\right\}-\sum_{t=1}^{2n}\sum_{j=1}^{2r} g_M((\nabla^{LC}\mathcal{T})_{E_j}E_j,\csc\theta\omega e_t)\\ &+\sum_{t=1}^{m}\sum_{j=1}^{s}\left\{\begin{array}{c} g_M(\mathcal{T}_{E_j}\csc\theta\omega e_t,\mathcal{T}_{E_j}\csc\theta\omega e_t)\\ -g_M(\mathcal{A}_{\csc\theta\omega e_t}E_j,\mathcal{A}_{\csc\theta\omega e_t}E_j)\end{array}\right\}.\end{aligned}$$

Using *(27)* and *(29)*, we obtain

$$\begin{aligned} 2scal &= 2scal^{ker\varphi_*}+2scal^{(ker\varphi_*)^\perp}\\ &+2r^2\|\mathcal{H}\|^2-\frac{1}{2}\sum_{t=1}^{2n}\left[\mathcal{T}^t_{11}-\mathcal{T}^t_{22}-\ldots-\mathcal{T}^t_{rr}\right]^2\\ &-2\sum_{t=1}^{2n}\sum_{j=2}^{2r}\left(\mathcal{T}^t_{1j}\right)^2+2\sum_{t=1}^{2n}\sum_{2\le j<k\le 2r}\left[\mathcal{T}^t_{jj}\mathcal{T}^t_{kk}-\left(\mathcal{T}^t_{jk}\right)^2\right]\\ &+6\sum_{\alpha=1}^{2r}\sum_{t=2}^{2n}(\mathcal{A}^\alpha_{1t})^2+6\sum_{\alpha=1}^{2r}\sum_{2\le i<t\le 2n}(\mathcal{A}^\alpha_{it})^2-2\delta(\mathcal{N})\\ &+\sum_{i=1}^{2n}\sum_{k=1}^{2r}\left\{\begin{array}{c} g_M(\mathcal{T}_{E_k}\csc\theta\omega e_i,\mathcal{T}_{E_k}\csc\theta\omega e_i)\\ -g_M(\mathcal{A}_{\csc\theta\omega e_i}E_k,\mathcal{A}_{\csc\theta\omega e_i}E_k)\end{array}\right\}\\ &+\sum_{t=1}^{2n}\sum_{j=1}^{2r}\left\{\begin{array}{c} g_M(\mathcal{T}_{E_j}\csc\theta\omega e_t,\mathcal{T}_{E_j}\csc\theta\omega e_t)\\ -g_M(\mathcal{A}_{\csc\theta\omega e_t}E_j,\mathcal{A}_{\csc\theta\omega e_t}E_j)\end{array}\right\}.\end{aligned} \tag{42}$$

Using *(31)*, *(39)* and *(41)* in the *(42)* then we have

$$\begin{aligned} \frac{\alpha}{2}\{3r+3n+4nr+3tr(\omega B)\} &= Ric^{ker\varphi_*}(E_1)+Ric^{(ker\varphi_*)^\perp}(\csc\theta\omega e_1)+2r^2\|\mathcal{H}\|^2\\ &-\frac{1}{4}\sum_{t=1}^{2n}\left[\mathcal{T}^t_{11}-\mathcal{T}^t_{22}-\ldots-\mathcal{T}^t_{rr}\right]^2-\sum_{t=1}^{2n}\sum_{j=2}^{2r}\left(\mathcal{T}^t_{1j}\right)^2\\ &+3\sum_{\alpha=1}^{2r}\sum_{t=2}^{2n}(\mathcal{A}^\alpha_{1t})^2-2\delta(\mathcal{N})+\|\mathcal{T}^{ker\varphi_*}\|^2-\|\mathcal{A}^{(ker\varphi_*)^\perp}\|^2\end{aligned}$$

where $\|\mathcal{T}^{ker\varphi_*}\|^2=\sum_{i=1}^{2n}\sum_{k=1}^{2r} g_M(\mathcal{T}_{E_k}\csc\theta\omega e_i,\mathcal{T}_{E_k}\csc\theta\omega e_i)$ and $\|\mathcal{A}^{(ker\varphi_*)^\perp}\|^2=\sum_{i=1}^{2n}\sum_{k=1}^{2r} g_M(\mathcal{A}_{\csc\theta\omega e_i}E_k,\mathcal{A}_{\csc\theta\omega e_i}E_k)$. Since $H_M(\alpha)$ is a complex space form, from *(3)*, we have following result readily:

Theorem 4.3. *Let $\varphi : H_M(\alpha) \to H_N$ be a slant submersion from a complex space form $(H_M(\alpha), g_M)$ onto a Riemannian manifold (H_N, g_N). Then we have*

$$\frac{\alpha}{2}\{3r + 3n + 4nr + 3tr(\omega B)\} \leq Ric^{ker\varphi_*}(E_1) + Ric^{(ker\varphi_*)^\perp}(\csc\theta\omega e_1) + \frac{1}{4}s^2\|\mathcal{H}\|^2 + 3\sum_{\alpha=1}^{s}\sum_{t=2}^{m}(\mathcal{A}^{\alpha}_{1t})^2$$
$$-\delta(\mathcal{N}) + \|\mathcal{T}^{ker\varphi_*}\|^2 - \|\mathcal{A}^{(ker\varphi_*)^\perp}\|^2$$

the equality status of the inequality satisfies if and only if

$$\mathcal{T}^t_{11} = \mathcal{T}^t_{22} + ... + \mathcal{T}^t_{rr}$$

$$\mathcal{T}^t_{1j} = 0, j = 2, ..., 2r.$$

Now, we give some examples which satisfy inequalities.

Example 4.4. *Let (R^6, g_{R^6}, J_{R^6}) be an almost Hermitian manifold and (R^3, g_{R^3}) a Riemannian manifold. Define a map $\varphi : R^6 \to R^3$ by $\varphi(u_1, u_2, u_3, u_4, u_5, u_6) = (u_1 \sin u_3 + u_2 \sin u_3, u_4, u_5)$ where $u_3 \in \left(0, \frac{\pi}{2}\right), u_1, u_2 \neq 0, \frac{u_1}{u_2} = \tan u_3$. Then, the map φ is a Riemannian submersion such that*

$$\ker\varphi_* = \langle V_1 = -\cos u_3\frac{\partial}{\partial u_1} + \sin u_3\frac{\partial}{\partial u_2}, V_2 = \frac{\partial}{\partial u_3}, V_3 = \frac{\partial}{\partial u_6}\rangle$$

and

$$(ker\varphi_*)^\perp = \langle Y_1 = \sin u_3\frac{\partial}{\partial u_1} + \cos u_3\frac{\partial}{\partial u_2}, Y_2 = \frac{\partial}{\partial u_4}, Y_3 = \frac{\partial}{\partial u_5}\rangle.$$

Moreover,, φ is a slant Riemannian submersion with slant angle $\theta = \frac{\pi}{2}$. Where $J_{R^6}(u_1, u_2, u_3, u_4, u_5, u_6) = (u_2, -u_1, u_4, -u_3, u_6, -u_5)$. By straightforward computations, we obtain $\mathcal{T}^{\ker\varphi_*}_{V_2}Y_1 = -V_1$ and $\mathcal{T}^{(\ker\varphi_*)^\perp}_{V_1}V_2 = Y_1$. Other components of operators $\mathcal{T}^{\ker\varphi_*}, \mathcal{T}^{(\ker\varphi_*)^\perp}, \mathcal{A}^{\ker\varphi_*}$ and $\mathcal{A}^{(\ker\varphi_*)^\perp}$ vanish identically. Moreover, $scal^{(ker\varphi_*)^\perp} = 0$, $Ric^{(\ker\varphi_*)^\perp}(Y_1) = 0$ and $Ric^{ker\varphi_*}(V_1) = 1$. Also, since $J(\ker\varphi_*) = (\ker\varphi_*)^\perp$, we have:

Remark 4.5. *It is clear that the Lagrangian submersion φ, in Example 4.4, satisfies the inequalities in Theorem 4.2 and Theorem 4.3.*

In order to have a Lagrangian submersion $\varphi : (M, J, g) \to N$ dimension must be related in the following way: $\dim(M) = 2\dim(N)$, the most natural examples of manifolds having this relation are given by the tangent bundle of $M = TN \to N$. In the seminal paper ([31]), Dombrowski introduces the almost complex structure J on the tangent bundle TN of a manifold N having a linear connection which is give by the conditions $J(X^{(\ker\varphi_*)^\perp}) = X^{\ker\varphi_*}; J(X^{\ker\varphi_*}) = -X^{(\ker\varphi_*)^\perp}$, $\ker\varphi_*$ and $(\ker\varphi_*)^\perp$ being the vertical and horizontal lifts. On the other hand, Sasaki ([32]) introduced the diagonal lift g^D, or Sasaki metric, over the tangent bundle of a Riemannian manifold (N, g), given by $g(X^{(\ker\varphi_*)^\perp}, Y^{(\ker\varphi_*)^\perp}) = g(X^{\ker\varphi_*}, Y^{\ker\varphi_*}) = g(X, Y); g(X^{(\ker\varphi_*)^\perp}, Y^{\ker\varphi_*}) = 0$. Thus, the tangent (TN, J, g^D) of a Riemannian manifold (N, g) is an almost Hermitian manifold. Then, one can easily gets:

Example 4.6. *With the above notation, $\varphi : (TN, J, g^D) \to (N, g)$ is a Lagrangian submersion ([33]).*

Remark 4.7. *Example 4.6 satisfies the inequality in Theorem 3.13.*

Let $\varphi : (H_M, J_M, g_M) \to (H_N, g_N)$ be a Lagrangian Riemannian submersion with totally geodesic from a Hermitian manifold to Riemannian manifold. Then, the total space (H_M, J_M, g_M) is Einstain if and only if the Riemannian manifold B and fibers of such a submersion are Einstain ([34]).

Example 4.8. *A Lagrangian Riemannian submersion $\varphi : (H_M, J_M, g_M) \to (H_N, g_N)$ with totally geodesic from a Hermitian manifold to Riemannian manifold, such that the total space H_M is an Einstain.*

Remark 4.9. *From the above notation, one can see that such a submersion $\varphi : H_M \to H_N$ satisfies the inequality in Corollary 3.3.*

Denote the equivalence class of a point $(z_0, ..., z_n) \in K^{n+1} \setminus \{0\}$ by $[z_0, ..., z_n]$ or $[z_0, ..., z_n]_K$; then we have a canonical projection

$$K^{n+1} \setminus \{0\} \to KP^n, (z_0, ..., z_n) \to [z_0, ..., z_n]_K . \tag{43}$$

For $K = \mathbb{C}$, we obtain the complex $n-$space $\mathbb{C}P^n$.

Example 4.10. *([36])(Maps to $\mathbb{C}P^n$) Let $n \in \{1, 2, ...\}$. For $K = \mathbb{C}$, the canonical projection (43) restricts to a map*

$$S^{2n+1} \to \mathbb{C}P^n, (z_0, ..., z_n) \to [z_0, ..., z_n]_{\mathbb{C}} , (z_i \in \mathbb{C}, \sum_{i=0}^{n} |z_i|^2 = 1)$$

called a Hopf fibration. We give $\mathbb{C}P^n$ the unique metric for which is a Riemannian submersion.

Remark 4.11. *Since the Hopf map in Example 4.10 has minimal (in fact totally geodesic) fibres, then Example 4.10 satisfies the inequality in Theorem 4.1.*

References

[1] M. A. Akyol and S. Beyendi, *Riemannian submersions endowed with a semi-symmetric non-metric connection,* Konuralp Journal of Mathematics, 6(1) (2018), 188–193.
[2] M. A. Akyol and B. Şahin *Conformal anti-invariant submersions from almost Hermitian manifolds*, Turk. J. Math., 40(2016),43-70.
[3] M. E. Aydın, A. Mihai and I. Mihai, *Some inequalities on submanifolds in statistical manifolds of constant curvature,* Filomat, 29:3 (2015), 465-477.
[4] H. Aytimur and C. Özgür, *Inequalities for submanifolds in statistical manifolds of quasi-constant curvature,* Ann. Pol. Math., 121:3 (2018), 197-215.
[5] B.-Y. Chen, *Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions,* Glasg. Math. J., 41(1999), 33-41.
[6] B.-Y. Chen, *A General optimal inequality for arbitrary Riemannian submanifolds,* Journal of Inequalities in Pure and Applied Mathematics, 6(3) (2005), Article ID: 77
[7] M. Falcitelli, S. Ianus and A. M. Pastore, *Riemannian Submersions and Related Topics,* World Scientific, 2004.
[8] A. Gray, *Pseudo-Riemannian almost product manifolds and submersions,* J. Math. Mech. 16 (1967), 715-737.
[9] M. Gülbahar, S. Eken Meriç, E. Kiliç, *Sharp inequalities involving the Ricci curvature for Riemannian submersions,* Kragujev. J. Math. 41(2) (2017), 279–293.
[10] Y. Gündüzalp, *Anti-invariant Pseudo-Riemannian Submersions and Clairaut Submersions from Paracosymplectic Manifolds.* Mediterr. J. Math. 16, 94 (2019).
[11] Y. Gündüzalp, *Slant submersions in paracontact geometry,* Hacet. J. Math. Stat. Volume 49 (2) (2020), 822-834.
[12] S. Ianuș, A. M. Ionescu, R. Mocanu, and G. E. Vilcu, *Riemannian submersions from almost contact metric Manifols,* Abh. Math. Semin. Univ. Hamburg 81(1), (2011), 101-114.
[13] S. Ianuș, R. Mazzocco and G. E. Vilcu, *Riemannian submersions from quaternionic manifolds,* Acta Appl. Math. 104 (2008), 83-89.
[14] K. Matsumoto, I. Mihai and Y. Tazawa, *Ricci tensor of slant submanifolds in complex space forms,* Kodai Math. J. 26 (2003), 85-94.
[15] B. O'Neill, *The fundamental equations of a submersion,* Michigan Math. J. 13 (1966), 459-469.
[16] C. Özgür, *B. Y. Chen Inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature,* Turk. J. Math., 35:3 (2011), 501-509.
[17] F. Özdemir, C. Sayar and H. M. Taștan, *Semi-invariant submersions whose total manifolds are locally product Riemannian,* Quaestiones Mathematicae, Vol. 49 No. 7 (2017), 909-926.
[18] B. Șahin, *Riemannian submersions, Riemannian maps in Hermitian Geometry, and their Applications,* Elsevir, Academic, Amsterdam, 2017.
[19] B. Șahin, *Anti-invariant Riemannian submersions from almost Hermitian manifolds,* Cent. Eur. J. Math. 8(3), (2010) 437-447.
[20] B. Șahin, *Chen's first inequality for Riemannian maps,* Ann. Pol. Math., 117:3 (2016) 249-258.
[21] G. E. Vilcu, *Slant submanifolds of quaternionic space forms,* Publ. Math. Debrecen 81:2-4 (2012), 397-413.
[22] B. Watson, *Almost Hermitian submersions,* J. Differential Geom. 11, (1976), 147-165.
[23] K. Yano and M. Kon, *Structures on Manifolds,* Singapore: World Scientific, 1984.
[24] D. W. Yoon, *Inequality for Ricci curvature of slant submanifolds in cosymplectic space forms,* Turk. J. Math., 30(2006), 43-56.
[25] B. Y. Chen, *Geometry of slant submanifolds*, Katholieke Universiteit Leuven, Leuven, 1990.
[26] B. Şahin, *Slant submersions from almost Hermitian manifolds*, Bull. Math. Soc. Sci. Math. Roumanie Tome 54(102), (2011), 93–105.
[27] Y.Gündüzalp and M.Polat, *Some inequalities of anti-invariant Riemannian submersions in complex space forms,* Miskolc Mathematical Notes, accepted, (2021).

[28] Y.Gündüzalp, *Slant submersions from almost product Riemannian manifolds*, Turk J Math, 37, (2013), 863 – 873.
[29] C. W. Lee, J. W. Lee, B. Şahin, and G.-E. Vilcu, *Optimal inequalities for Riemannian maps and Riemannian submersions involving Casorati curvatures,* Annali di Matematica Pura ed Applicata (1923 -), vol. 200, pp., , (2021) 1277–1295.
[30] J. W. Lee,C. W. Lee, B. Şahin, and G.-E. Vilcu, *Chen-Ricci inequalities for Riemannian maps and their applications*, ser. AMS book series Contemporary Mathematics, 2022, ch. Differential Geometry and Global Analysis, in honor of Tadashi Nagano, Edited by Chen, Bang-Yen and D. Brubaker, Nicholas and Sakai, Takashi and D. Suceava, Bogdan and Tanaka, Makiko Sumi and Tamaru, Hiroshi and Vajiac Mihaela B.
[31] P. Dombrowski, *On the geometry of tangent bundle*, J. Reine Angew. Math., 210, (1962), 73-88.
[32] S. Sasaki, *On the differential geometry of tangent bundles of Riemannian manifolds*, Tohoku. Math. J., 10 (1958), 338-354.
[33] H. M. Taştan, *On Lagrangian submersions*, Hacettepe Journal of Mathematics and Statistic, 43(6), (2014), 993-1000.
[34] J. Lee, J. H. Park, B. Şahin and D. Y. Song, *Einstain conditions for the base of anti-invariant Riemannian submersions and Clairaut submersions*, Taiwan. J. Math., 19 (4), (2015), 1145-1160.
[35] Ş. E. Meriç, E. kiliç and Y. Sagiroglu, *Scalar curvature of Lagrangian Riemannian submersions and their harmonicitiy*, Int. J. Geom. Methods Mod. Phys, 14 (16) (2017).
[36] P. Baird and J. C. Wood, *Harmonic morphisms between Riemannian manifolds*, Oxford University Press, (2003).